Optimal Timed Games

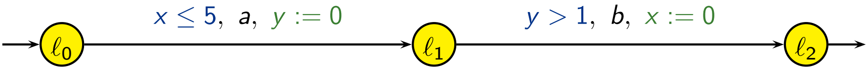
Patricia Bouyer

LSV – CNRS & ENS de Cachan – France

Timed automata

[Alur & Dill 90's]

x, y : clocks



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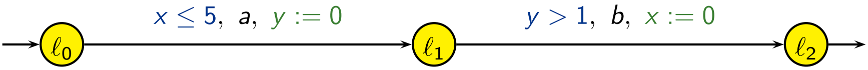

	ℓ_0	$\xrightarrow{\delta(4.1)}$	ℓ_0	$\xrightarrow{a}$	ℓ_1	$\xrightarrow{\delta(1.4)}$	ℓ_1	$\xrightarrow{b}$	ℓ_2
x	0		4.1		4.1		5.5		0
y	0		4.1		0		1.4		1.4

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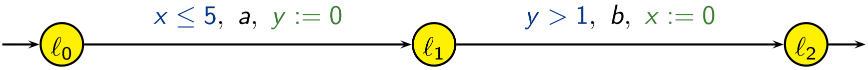

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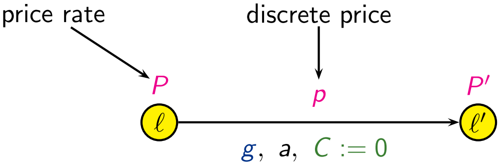
(clock) valuation

Model of priced timed automata [HSCC'01]



cost $\equiv$ price

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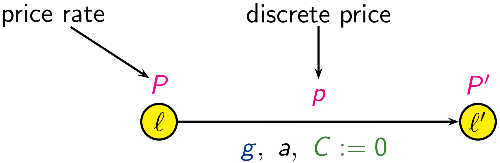

- a configuration: (ℓ, v)
- two kinds of transitions:

$$\begin{cases} (\ell, v) \xrightarrow{\delta(d)} (\ell, v+d) \\ (\ell, v) \xrightarrow{a} (\ell', v') \text{ where } \begin{cases} v \models g \\ v' = [C \leftarrow 0]v \end{cases} \text{ for some } \ell \xrightarrow{g, a, C := 0} \ell' \end{cases}$$

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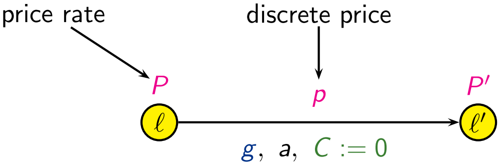

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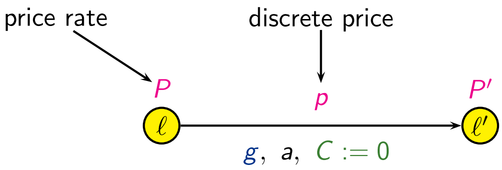
$$\text{Cost}\left((\ell, v) \xrightarrow{\delta(d)} (\ell, v+d)\right) = P.d \qquad \text{Cost}\left((\ell, v) \xrightarrow{a} (\ell', v')\right) = p$$

$$\text{Cost}(\rho) = \text{accumulated cost along run } \rho$$

Model of priced timed automata (cont.)



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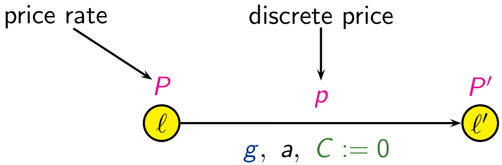

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  - reachability with an optimization criterium on the price [BFH+01a,BFH+01b,LBB+01,ALTP01]
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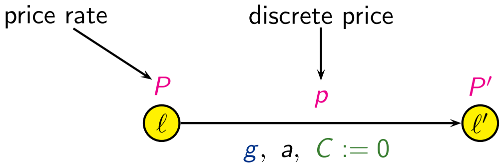

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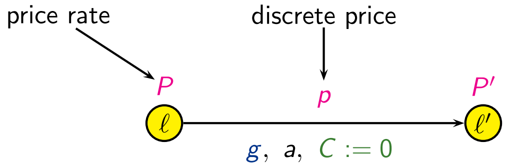

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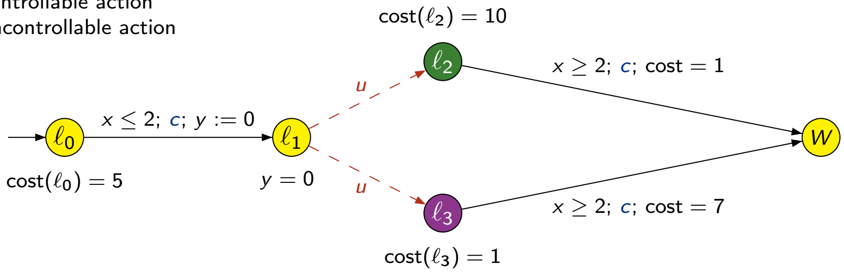
➜ **optimal reachability timed game**

An example

c: controllable action
u: uncontrollable action



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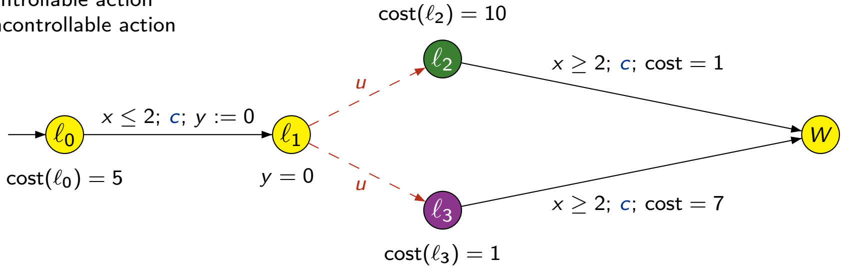

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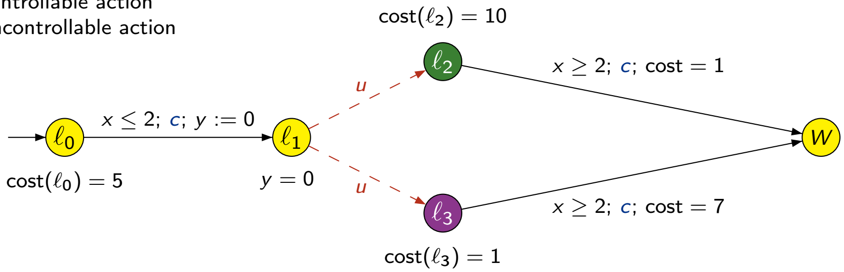

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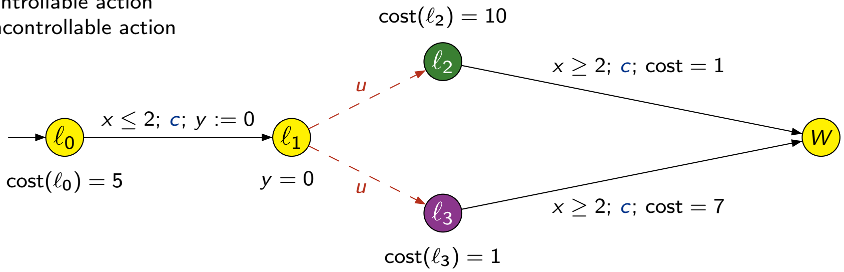

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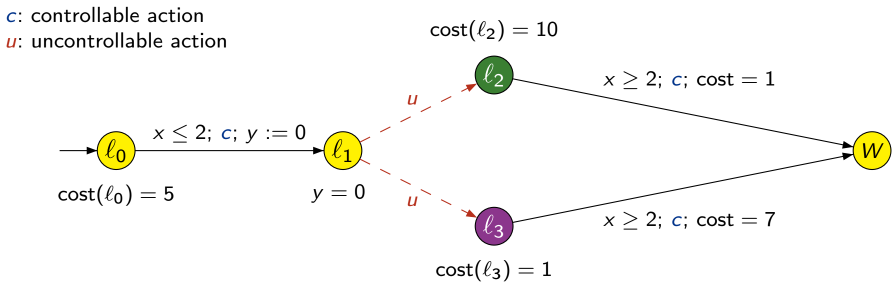

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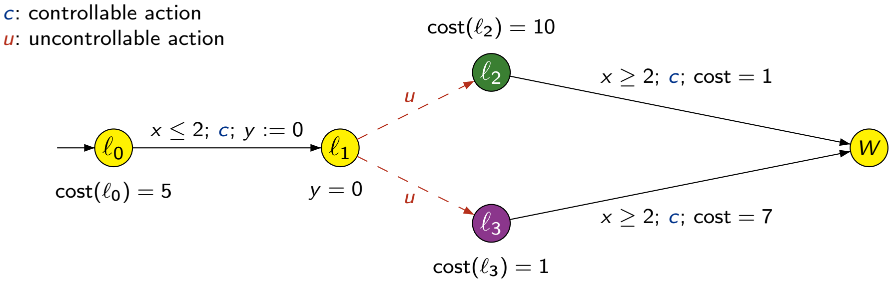

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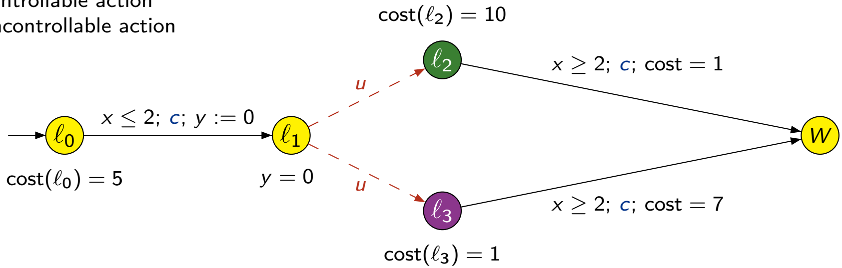

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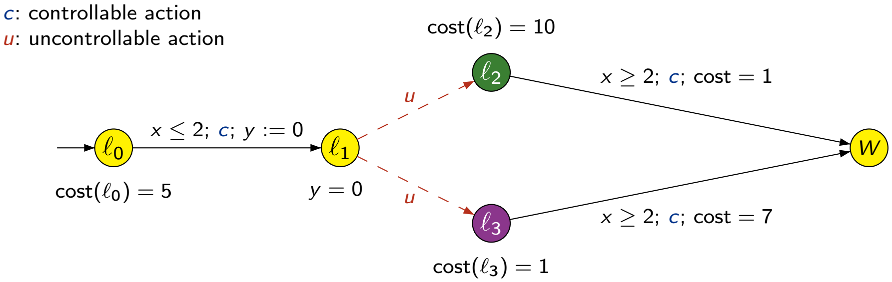

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- How to automatically compute such optimal prices?
- How to synthesize optimal strategies (if one exists)?

A hot topic!

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- [Bouyer, Brihaye, Markey – Submitted to IPL, 2005]:
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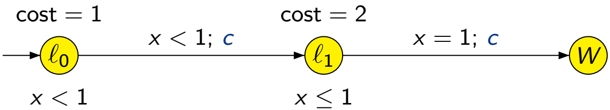
Do optimal strategies always exist?



$$\begin{cases} f(\ell_0, x < 1) = \lambda \\ \\ f(\ell_1, x < 1) = \lambda \\ f(\ell_1, x = 1) = c \end{cases}$$

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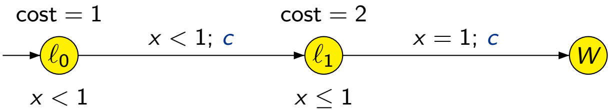

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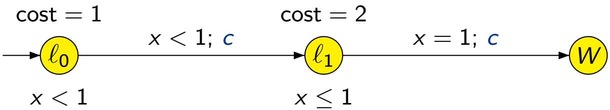

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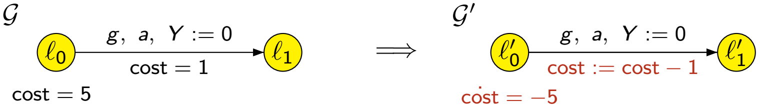
➜ **no optimal strategy exists**, but rather a family $(f_\varepsilon)_{\varepsilon>0}$ of ε-approximating strategies ($\text{cost}(f_\varepsilon) = 1+\varepsilon$)

An encoding

Idea: tranform the cost into a decreasing linear hybrid variable



Winning: W Winning: $W \wedge \text{cost} \geq 0$

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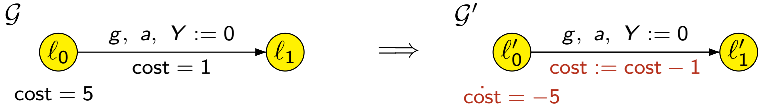

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Theorem

For priced timed games (under some hypotheses),

$$\left.\begin{array}{r} \exists f \text{ winning strategy in } \mathcal{G} \\ \text{s.t. } \text{cost}(f,(\ell,v)) \leq \gamma \end{array}\right\} \iff (\ell, v, \text{cost} = \gamma) \text{ winning in } \mathcal{G}'$$

+ constructive proof

An encoding (2)

The set of winning states in $\mathcal{G}'$ is upward-closed for the cost, *i.e.* of the form

$$\bigcup_{i \in I} (P_i \wedge cost \succ_i k_i) \qquad \text{(with } \succ_i \text{ either } > \text{ or } \geq\text{)}$$

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For priced timed games (under some hypotheses),

- *"reachable" optimal cost, or not* (cost $\geq \gamma$ *or* cost $> \gamma$*)*
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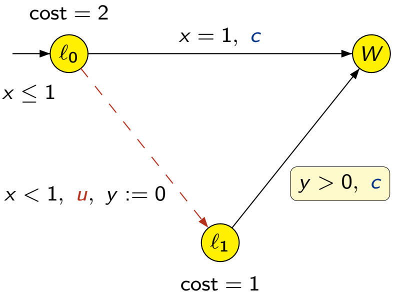
Nature of the strategy:

- state-based for the hybrid game, thus **cost-dependent** for the timed game
- cost-dependence is unavoidable in general!
- cost-independent strategies for syntactical restrictions of the games

  c: large constraints, u: strict constraints

Cost-dependence is unavoidable



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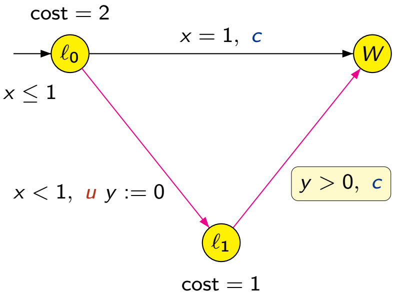

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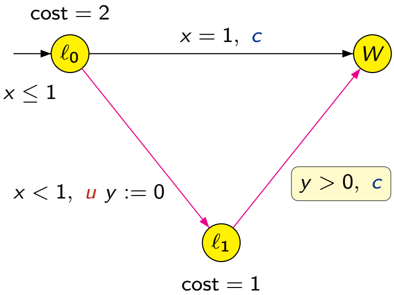

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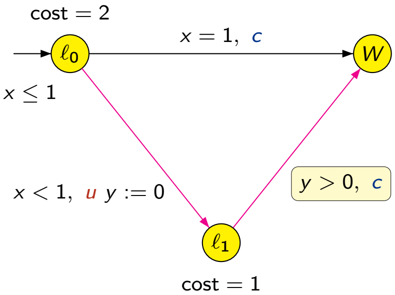

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$$2.d + d' \leq 2$$

$$(\text{accumulated cost}) + d' \leq 2$$

Hypotheses for termination

- all clocks are bounded (not restrictive)
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  → This condition is restrictive, but is decidable

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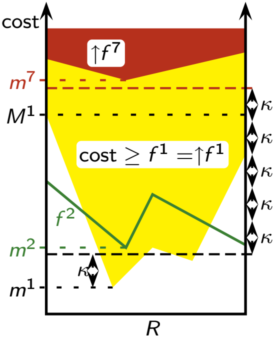

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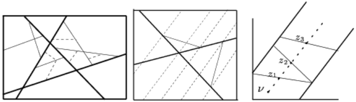

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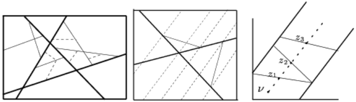

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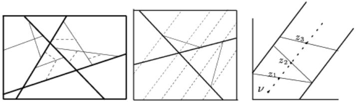

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Idea:

- environment chooses $r \in [0, 1]$,
- controller has to produce its binary encoding up to k digits

➜ Controller must have 2^k different strategies

Shape of the reduction

Original reduction: [Brihaye, Bruyère, Raskin 2005]
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- counter c_1 is encoded by a clock x_1 s.t. $x_1 = \frac{1}{2^{c_1}}$
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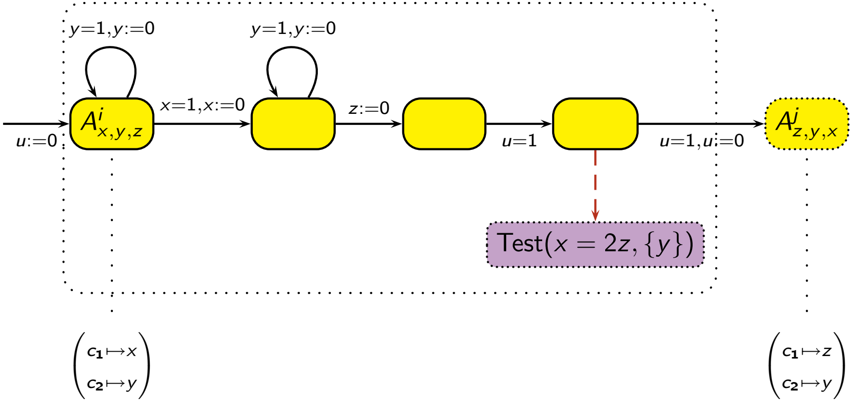
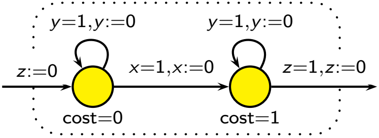
Player 1 has a winning strategy with cost ≤ 3 iff the two-counter machine halts

Simulation of an incrementation

Instruction i: $c_1 + +$; goto instruction j

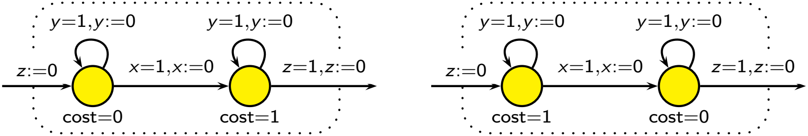


Adding x or $1 - x$ to the cost variable



The cost is increased by x_0

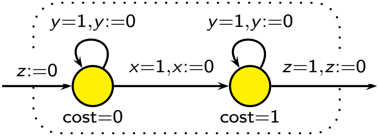
Adding x or $1 - x$ to the cost variable



The cost is increased by x_0 The cost is increased by $1 - x_0$

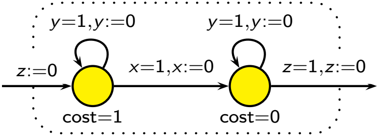
Adding x or $1 - x$ to the cost variable

$\mathrm{Add}^{+}(x, \{z\})$



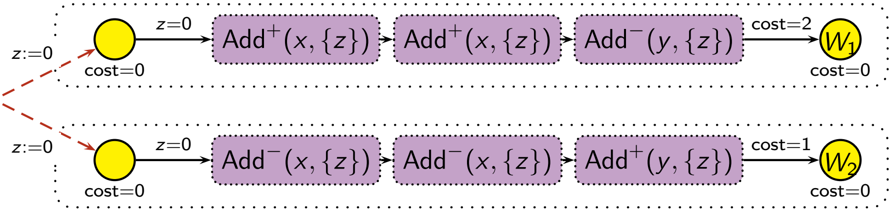
The cost is increased by x_0

$\mathrm{Add}^{-}(x, \{z\})$



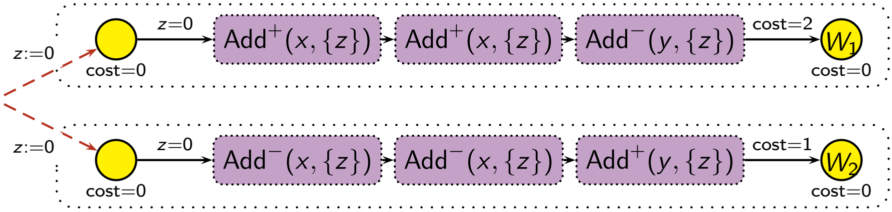
The cost is increased by $1 - x_0$

Checking $y = 2x$



In W_1, cost $= 2x_0 + (1 - y_0) + 2$.
In W_2, cost $= 2(1 - x_0) + y_0 + 1$.

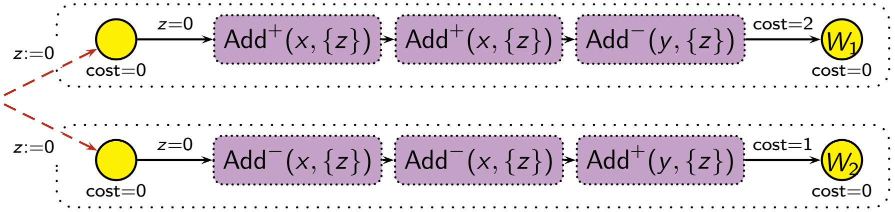
Checking $y = 2x$



In W_1, cost $= 2x_0 + (1 - y_0) + 2$.
In W_2, cost $= 2(1 - x_0) + y_0 + 1$.

- if $y_0 < 2x_0$, player 2 chooses the first branch: in W_1, cost > 3

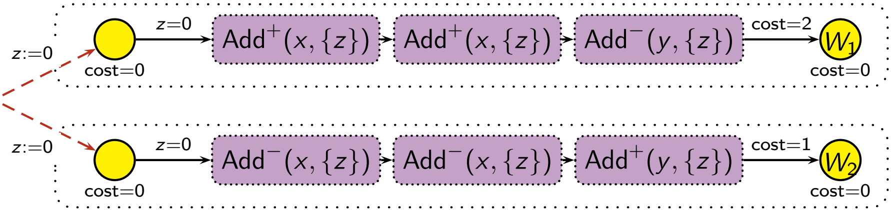
Checking $y = 2x$



In W_1, cost $= 2x_0 + (1 - y_0) + 2$.
In W_2, cost $= 2(1 - x_0) + y_0 + 1$.

- if $y_0 < 2x_0$, player 2 chooses the first branch: in W_1, cost > 3
- if $y_0 > 2x_0$, player 2 chooses the second branch: in W_2, cost > 3

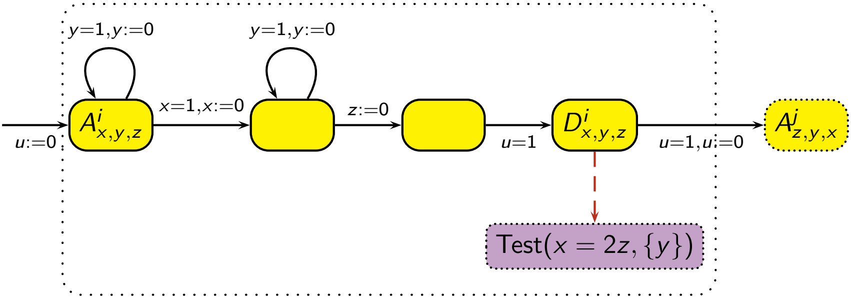
Checking $y = 2x$



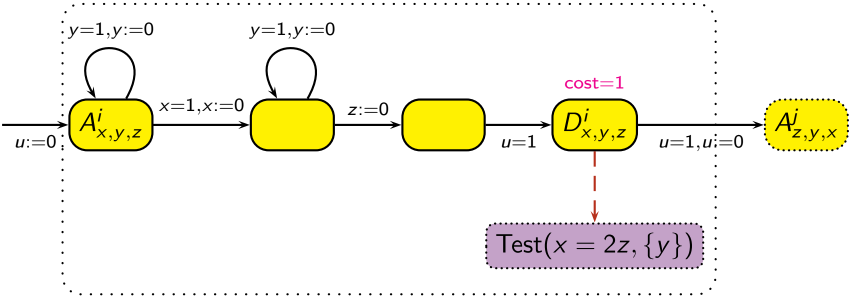
In W_1, cost $= 2x_0 + (1 - y_0) + 2$.
In W_2, cost $= 2(1 - x_0) + y_0 + 1$.

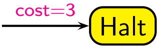
- if $y_0 < 2x_0$, player 2 chooses the first branch: in W_1, cost > 3
- if $y_0 > 2x_0$, player 2 chooses the second branch: in W_2, cost > 3
- if $y_0 = 2x_0$, in W_1 or in W_2, cost $= 3$.

How to get rid of tick clock u?



How to get rid of tick clock u?

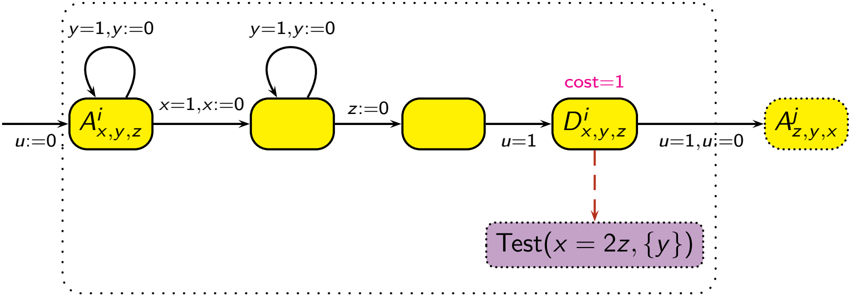




We will ensure that:

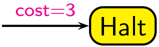
- no cost is accumulated in D-states

How to get rid of tick clock u?

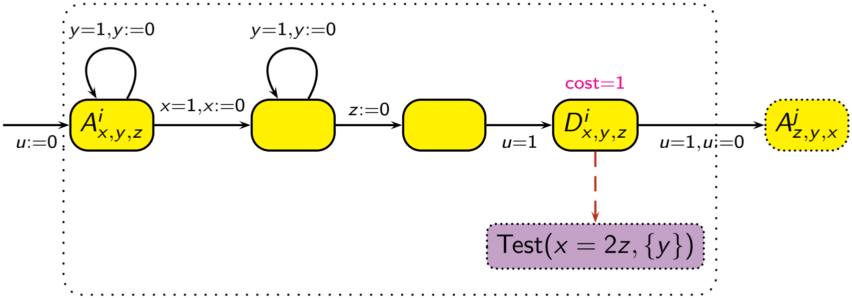


We will ensure that:

- no cost is accumulated in D-states
- the delay between the A-state and the D-state is 1 t.u.

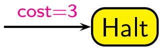


How to get rid of tick clock u?

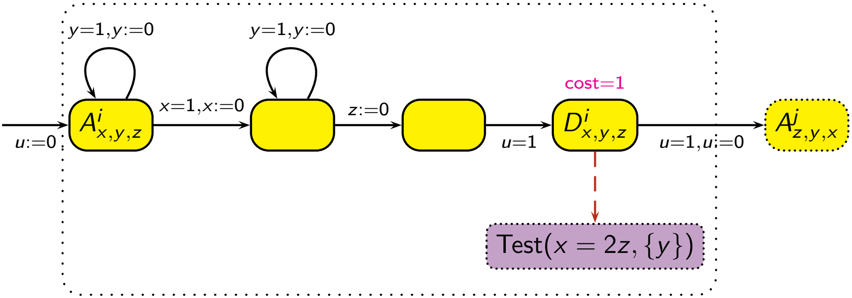


We will ensure that:

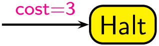
- no cost is accumulated in D-states
- the delay between the A-state and the D-state is 1 t.u.
  - the value of x in D is of the form $\frac{1}{2^n}$



How to get rid of tick clock u?

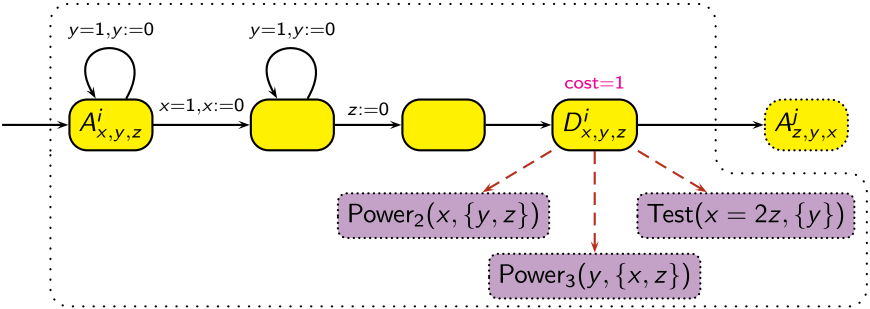


We will ensure that:

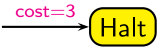


- no cost is accumulated in D-states
- the delay between the A-state and the D-state is 1 t.u.
  - the value of x in D is of the form $\frac{1}{2^n}$
  - the value of y in D is of the form $\frac{1}{3^m}$

How to get rid of tick clock u?

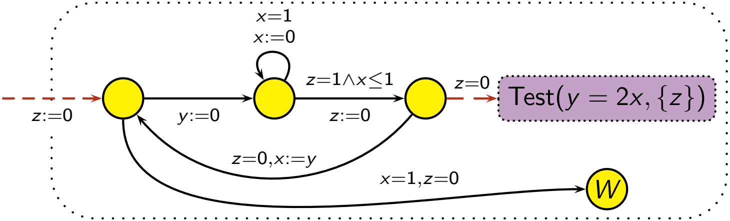


We will ensure that:



- no cost is accumulated in D-states
- the delay between the A-state and the D-state is 1 t.u.
  - the value of x in D is of the form $\frac{1}{2^n}$
  - the value of y in D is of the form $\frac{1}{3^m}$

Checking that x is of the form $\frac{1}{2^n}$



Conclusion

- Optimal cost is in general not computable in timed games.
- Under a strongly non-*zeno* hypothesis for the cost, optimal cost is computable.
- A much involved complexity bound for the number of splittings of regions.
- Properties of winning strategies.

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- Under a strongly non-*zeno* hypothesis for the cost, optimal cost is computable.
- A much involved complexity bound for the number of splittings of regions.
- Properties of winning strategies.

Further work

- Compute effectively ε-optimal winning strategies.
- Further understand this problem: provide decidable subclasses?
- And from an algorithmics point of view, what can be done? (integrate ideas from **[ABM04]** into encoding of **[BCFL04]**?)
- Adapt the forward algorithm of TiGA (game-extension of UPPAAL) **[CDFLL05]**.
- Mean-cost optimal safety timed games.

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[BCFL04] Bouyer, Cassez, Fleury, Larsen. Optimal Strategies in Priced Timed Game Automata. FSTTCS'04 (LNCS 3328).

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