From timed to complex systems

— Stochastic timed games —

Patricia Bouyer-Decitre

LSV, CNRS & ENS Cachan, France

Based on joint works with Christel Baier, Nathalie Bertrand, Thomas Brihaye, Vojtěch Forejt, Marcus Größer and Nicolas Markey.

I am grateful to Vojtěch Forejt for some of the slides in this presentation.

Outline

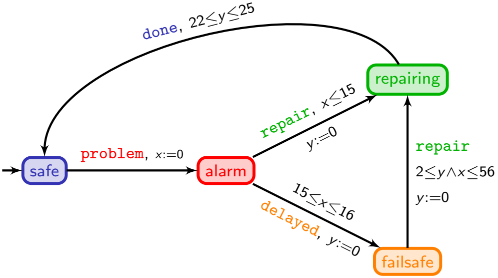
1. Timed automata

2. Timed games

3. A hint into stochastic timed games
   - Some informal description
   - A more formal view of the semantics
   - Summary of the results
   - Qualitative analysis of $\frac{1}{2}$-player games
   - Quantitative analysis of $2\frac{1}{2}$-player games
   - Quantitative analysis of $\frac{1}{2}$-player games

4. Conclusion

An example of a timed automaton



	safe	$\xrightarrow{23}$	safe	$\xrightarrow{\texttt{problem}}$	alarm	$\xrightarrow{15.6}$	alarm	$\xrightarrow{\texttt{delayed}}$	failsafe		
x	0		23		0		15.6		15.6	$\cdots$	
y	0		23		23		38.6		0		

	failsafe	$\xrightarrow{2.3}$	failsafe	$\xrightarrow{\texttt{repair}}$	repairing	$\xrightarrow{22.1}$	repairing	$\xrightarrow{\texttt{done}}$	safe
$\cdots$	15.6		17.9		17.9		40		40
	0		2.3		0		22.1		22.1

Verification

Emptiness problem

Is the language accepted by a timed automaton empty?

- basic reachability/safety properties (final states)
- basic liveness properties (ω-regular conditions)

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Theorem [AD90,AD94]

The emptiness problem for timed automata is decidable and PSPACE-complete.

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Method: construct a finite abstraction

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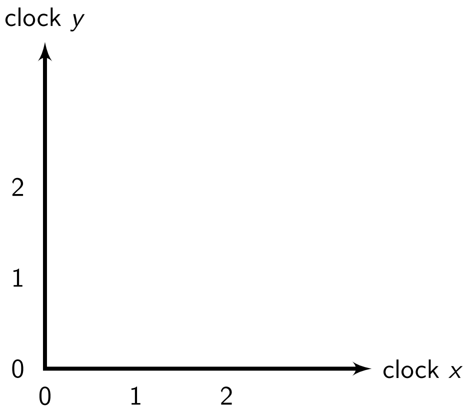
The region abstraction



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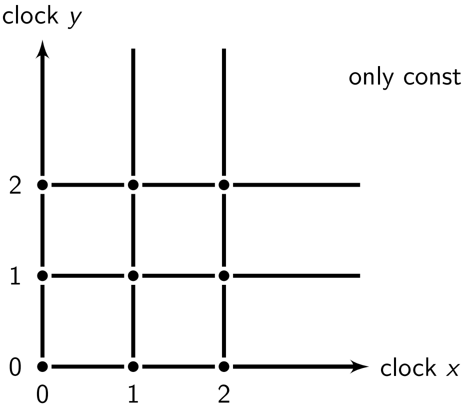

only constraints: $x \sim c$ with $c \in \{0, 1, 2\}$
$y \sim c$ with $c \in \{0, 1, 2\}$

- "compatibility" between regions and constraints

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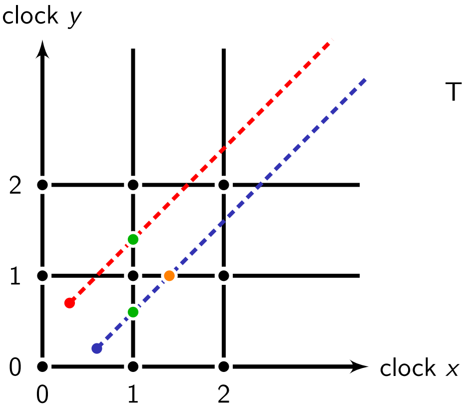

The path 

- can be fired from ●
- cannot be fired from ●

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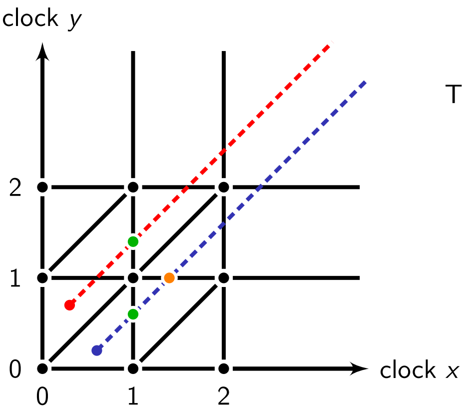

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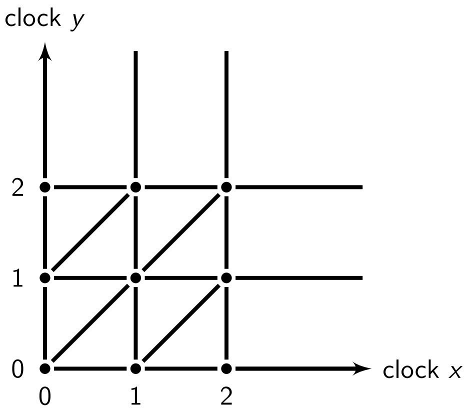

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$\rightsquigarrow$ an equivalence of finite index

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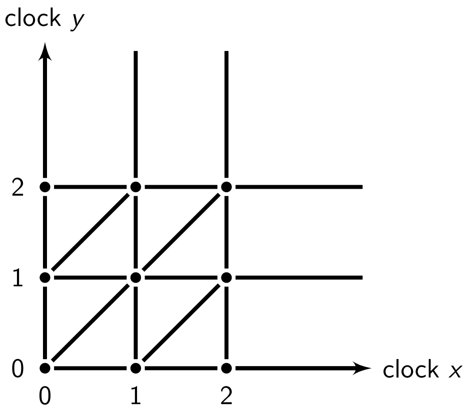

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$\rightsquigarrow$ an equivalence of finite index
a time-abstract bisimulation

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This is a relation between • and • such that:

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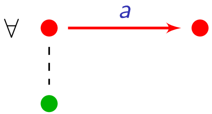

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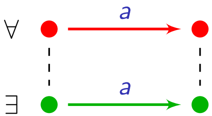

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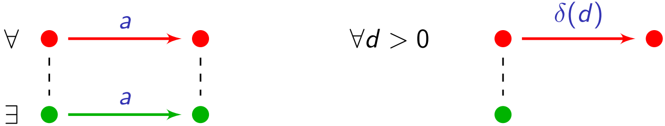

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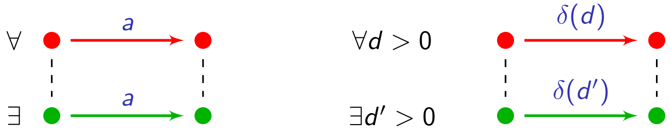

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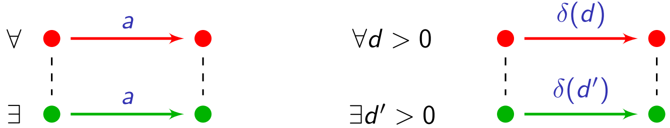
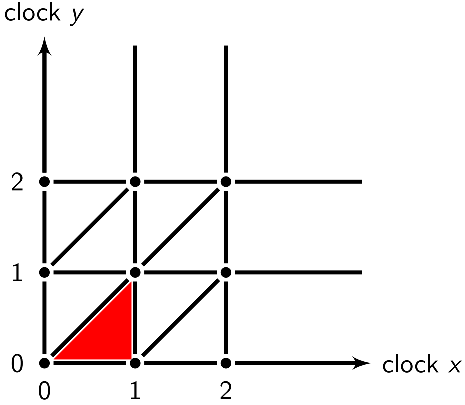


... and vice-versa (swap • and •).

The region abstraction (2)



- region R defined by:

$$\begin{cases} 0 < x < 1 \\ 0 < y < 1 \\ y < x \end{cases}$$

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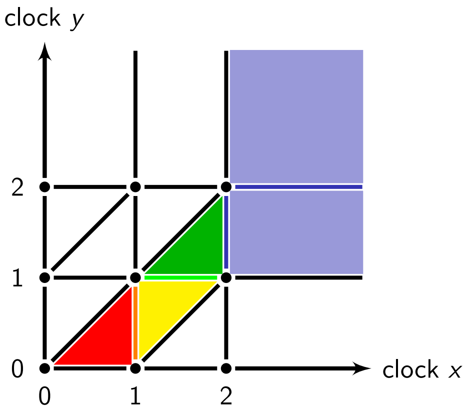

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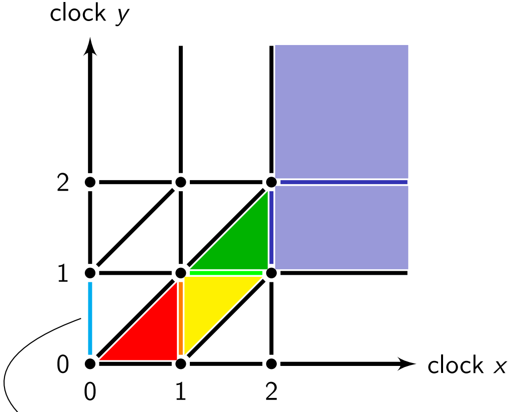

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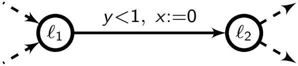
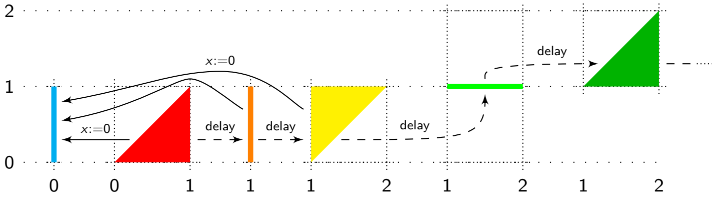
image of R when resetting clock x

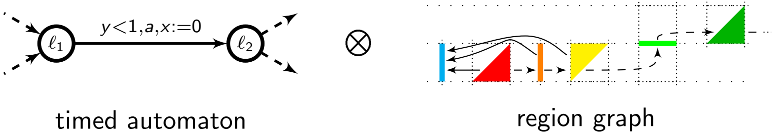
The construction of the region graph

It "mimics" the behaviours of the clocks.





Region automaton ≡ finite bisimulation quotient



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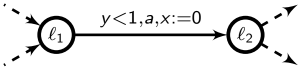
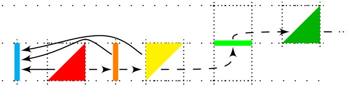


timed automaton

$\otimes$



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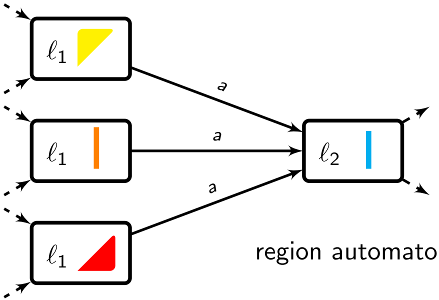

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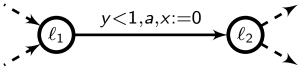

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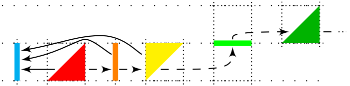

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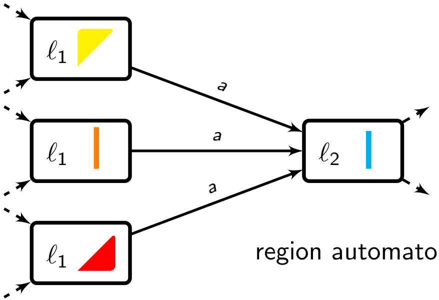
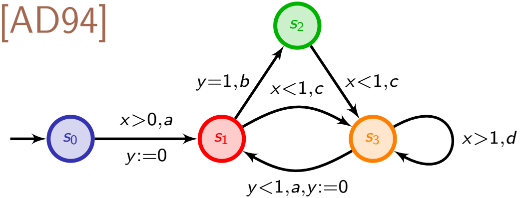


region automaton

$\mathcal{L}$(reg. aut.) $=$ UNTIME($\mathcal{L}$(timed aut.))

An example [AD94]



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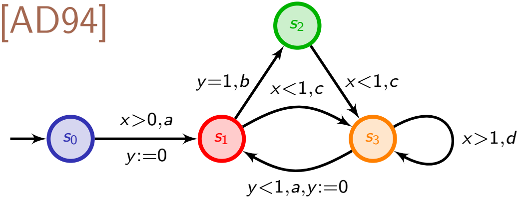

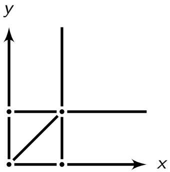

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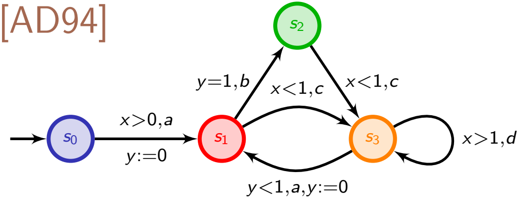
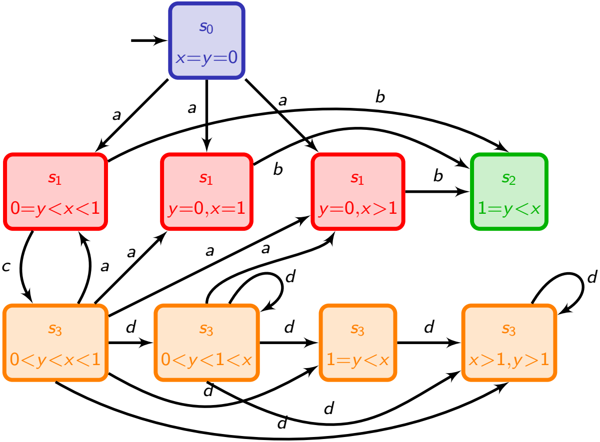
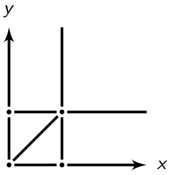
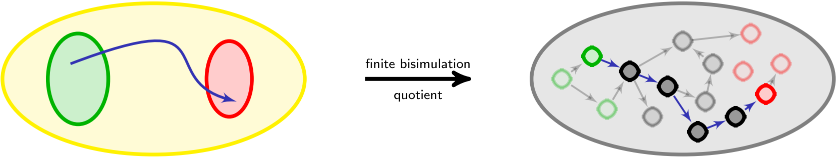


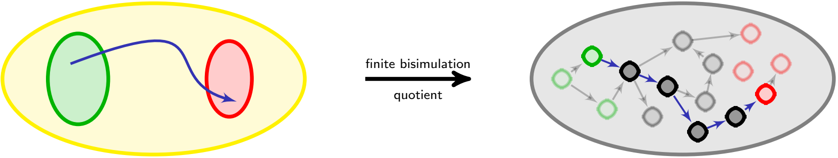






timed automaton

large (but finite) automaton
(region automaton)



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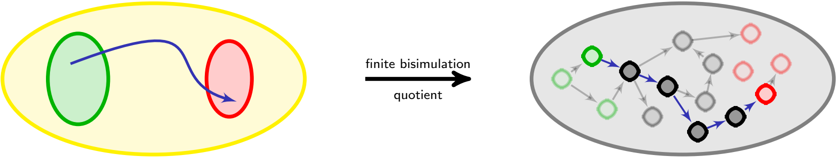
$$\prod_{x \in X} (2M_x + 2) \cdot |X|! \cdot 2^{|X|}$$



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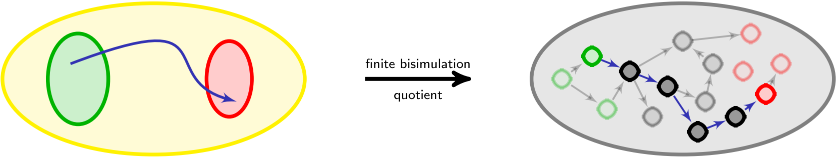

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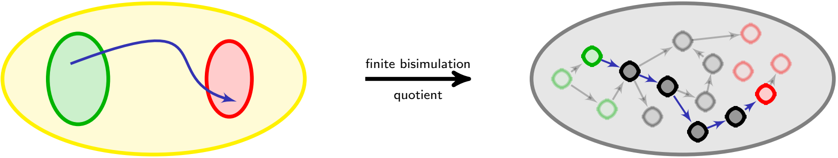

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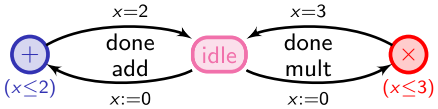
Why (timed) games?

- to model uncertainty

Example of a processor in the taskgraph example



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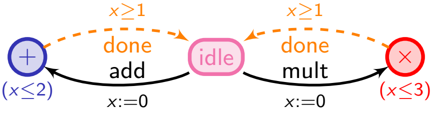

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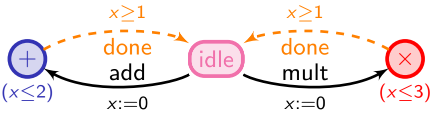
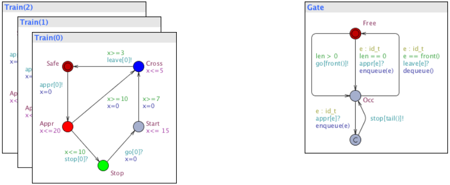


- to model an interaction with an environment

Example of the gate in the train/gate example



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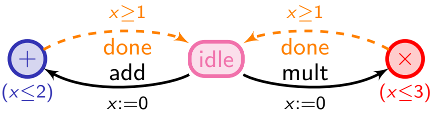

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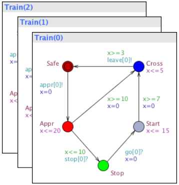

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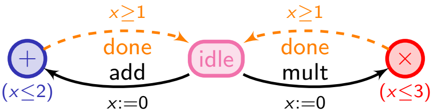

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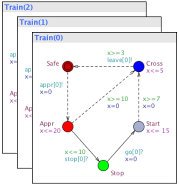

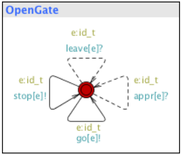

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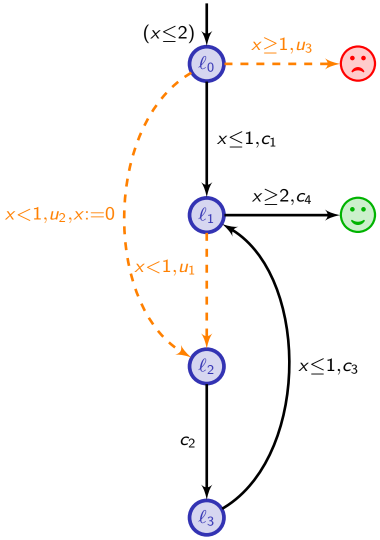

Rule of the game

- Aim: avoid ☹ and reach ☺

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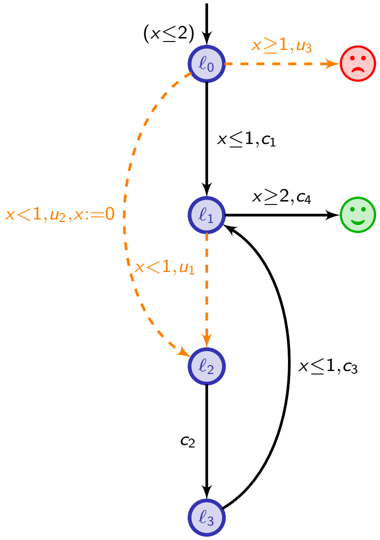

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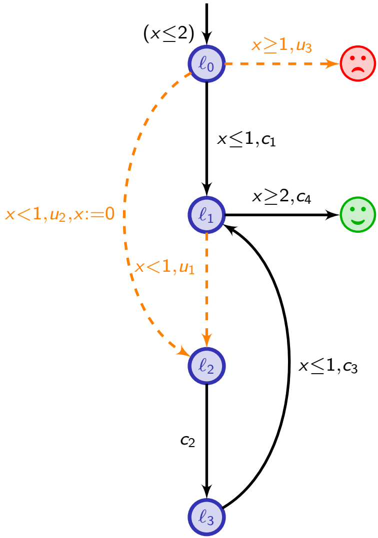

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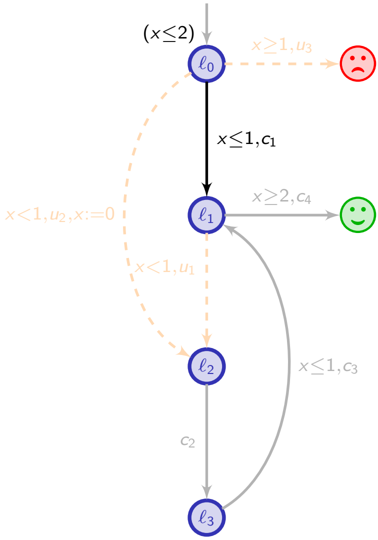

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- from $(\ell_0, 0)$, play $(0.5, c_1)$

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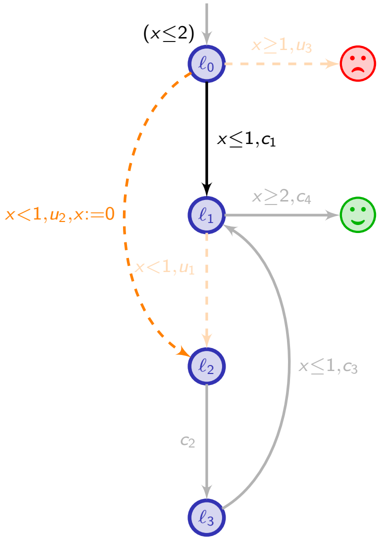

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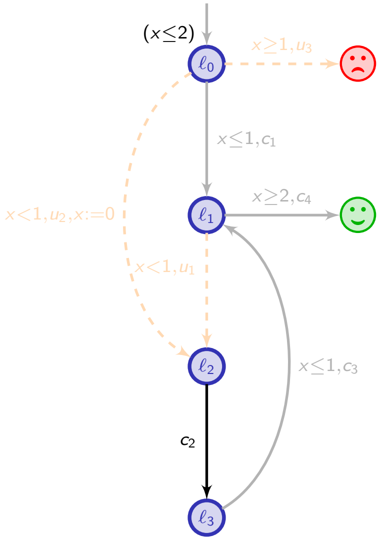

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- from $(\ell_2, \star)$, play $(1 - \star, c_2)$

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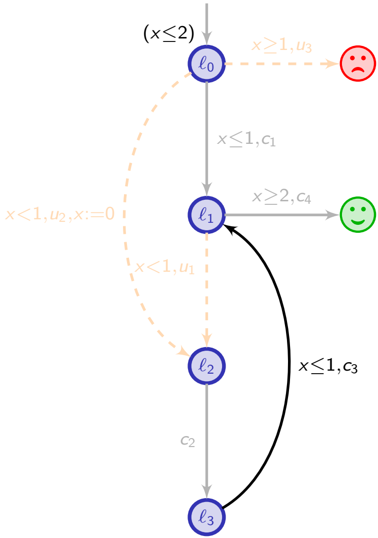

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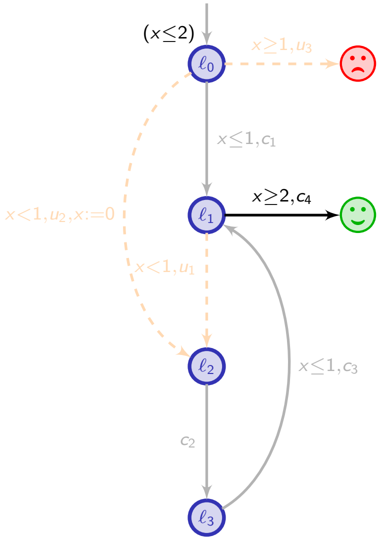
  $\rightsquigarrow$ can be preempted by u_2
- from $(\ell_2, \star)$, play $(1 - \star, c_2)$
- from $(\ell_3, 1)$, play $(0, c_3)$

An example of a timed game



Rule of the game

- Aim: avoid ☹ and reach ☺
- How do we play? According to a strategy:

$$f : \text{history} \mapsto (\text{delay}, \text{cont. transition})$$

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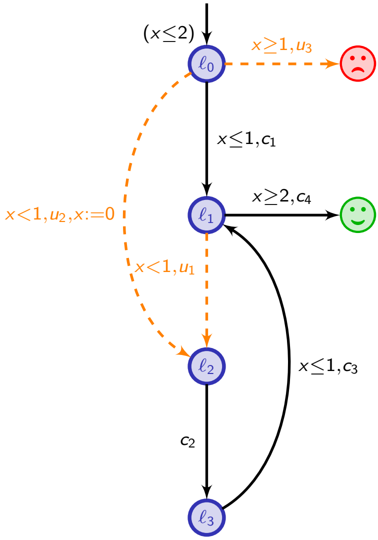

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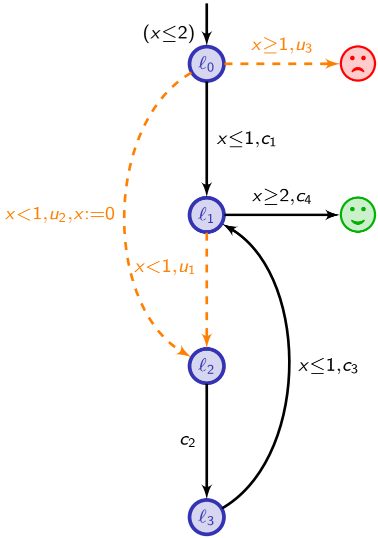

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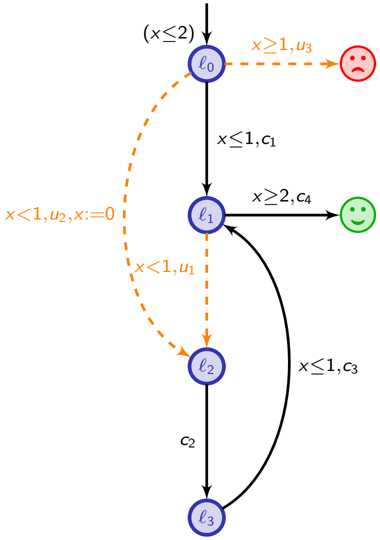

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Problems to be considered

- Does there exist a winning strategy?
- If yes, compute one (as simple as possible).

Decidability of timed games

Theorem [AMPS98,HK99]

Reachability and safety timed games are decidable and EXPTIME-complete. Furthermore memoryless and "region-based" strategies are sufficient.

[AMPS98] Asarin, Maler, Pnueli, Sifakis. Controller synthesis for timed automata *(HSSC'98)*.
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$\rightsquigarrow$ classical regions are sufficient for solving such problems

(one only needs to compute the so-called attractor)

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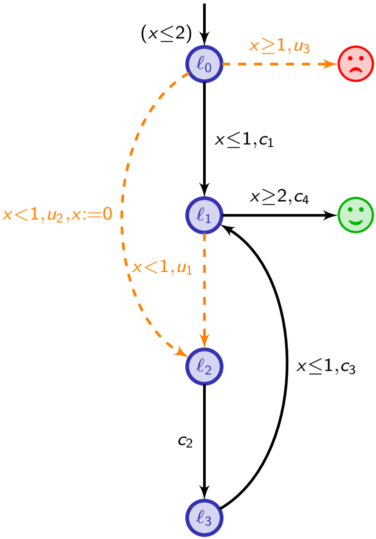

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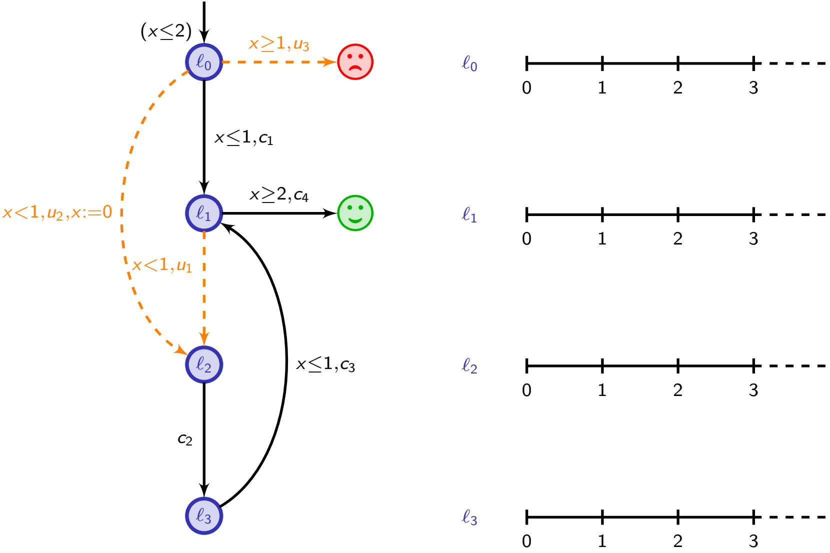

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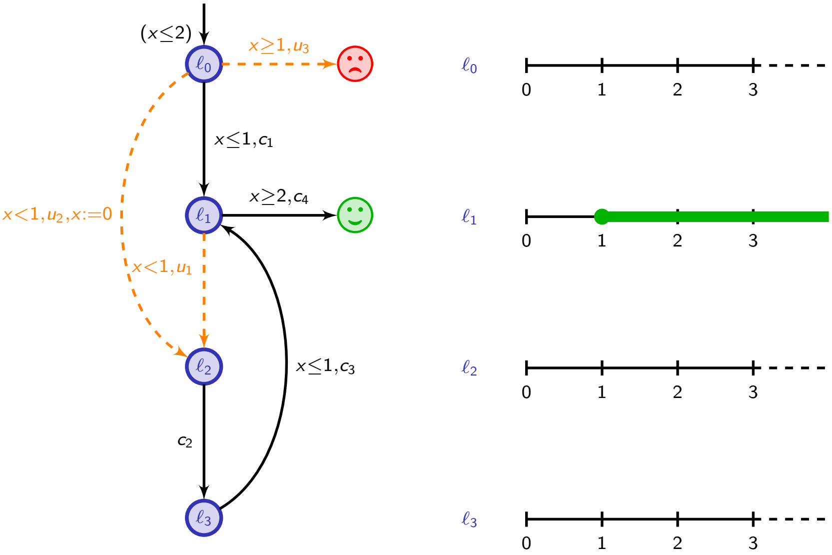

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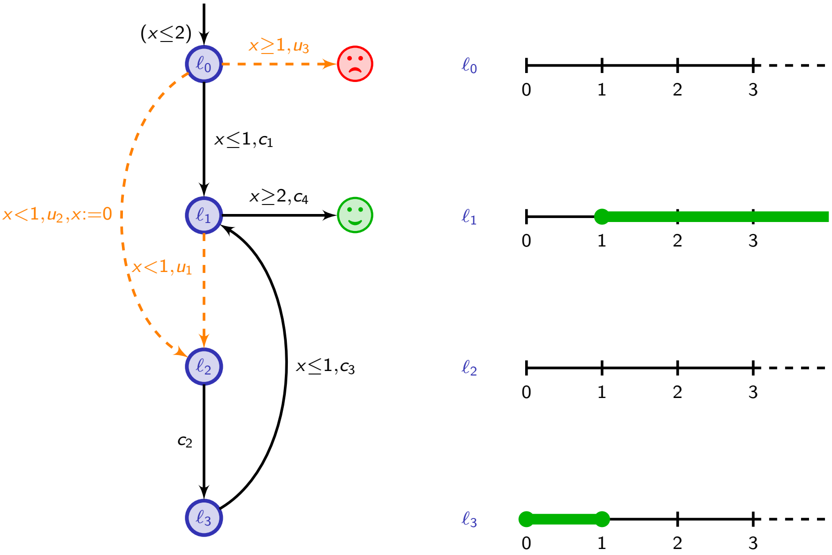

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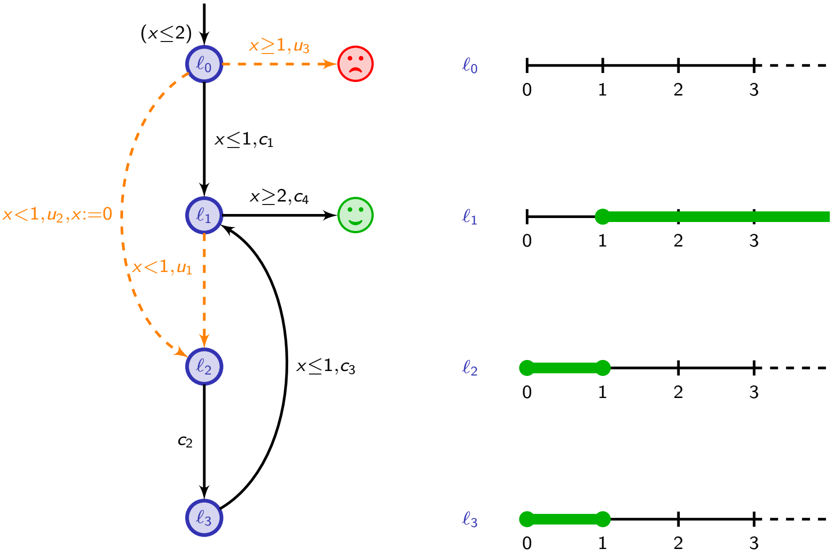

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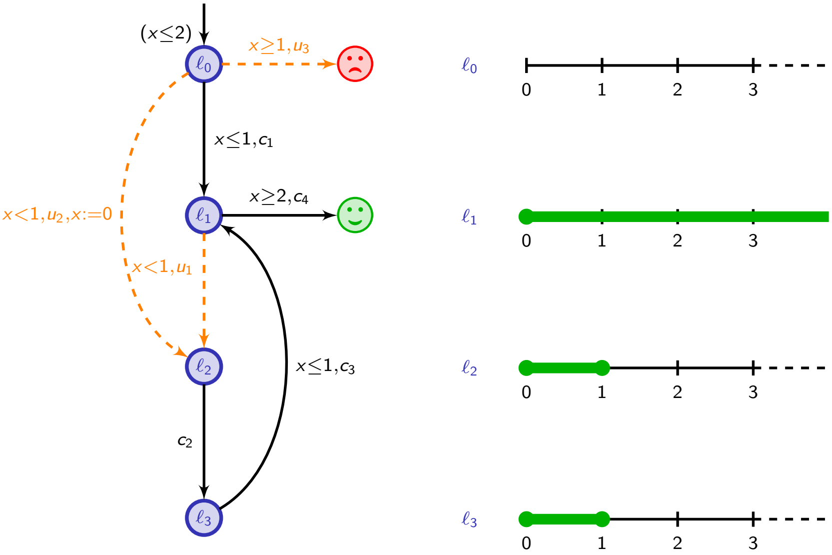

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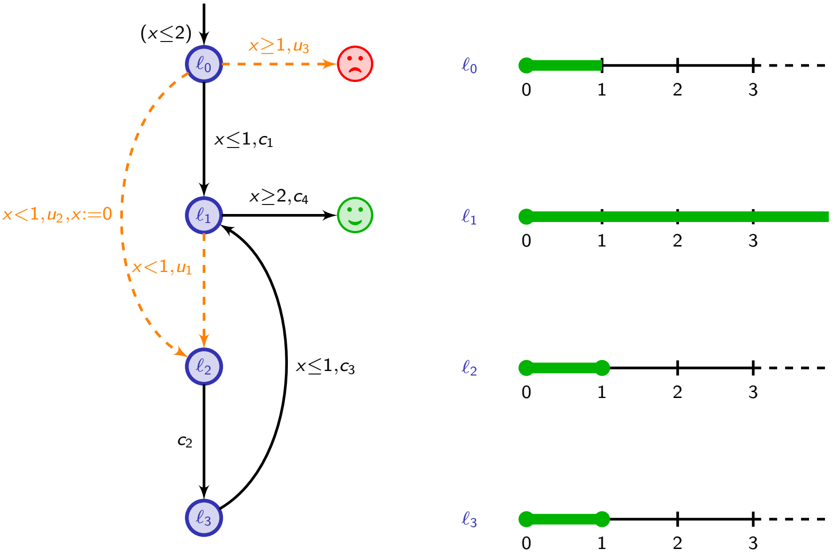

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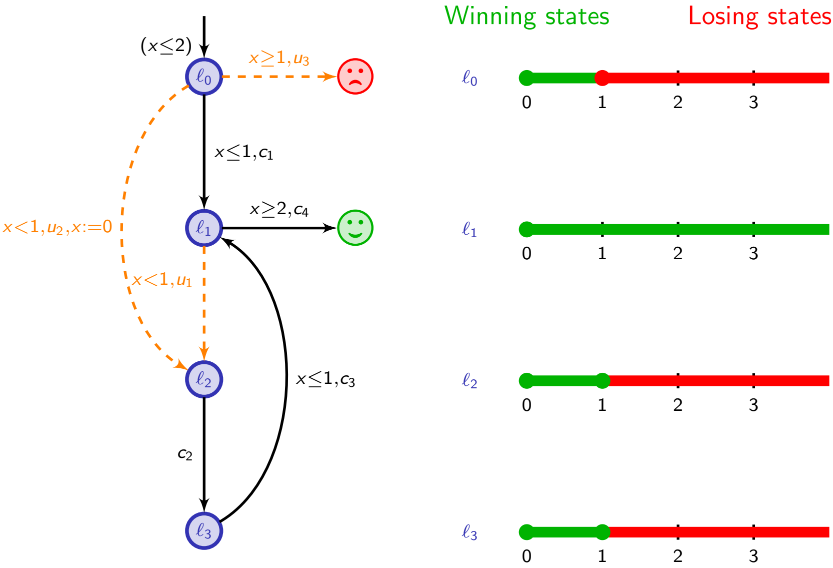

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4. Conclusion

Why add stochastic features? And how?

- to model probabilistic behaviours

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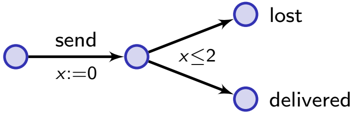
Example of losses when sending messages



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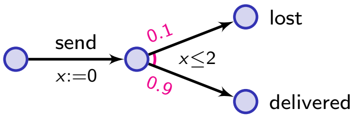

$\rightsquigarrow$ the probabilistic timed automata model used e.g. in PRISM and UPPAAL-PRO [KNSS02]

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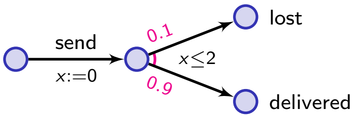

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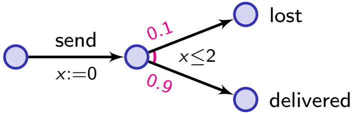

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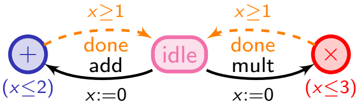
Example of a processor in the taskgraph example



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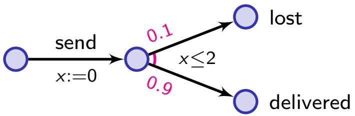

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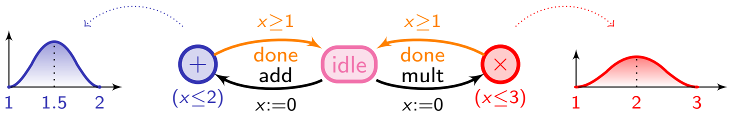

$\rightsquigarrow$ the stochastic timed automata model [BBB+08,BF09]

[KNSS02] Automatic verification of real-time systems with discrete probability distributions *(TCS)*.
[BBB+08] Baier, Bertrand, Bouyer, Brihaye, Größer. Almost-sure model checking of infinite paths in one-clock timed automata *(LICS'08)*.
[BF09] Bouyer, Forejt. Reachability in stochastic timed games *(ICALP'09)*.

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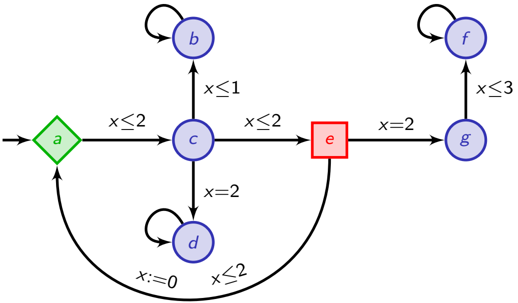
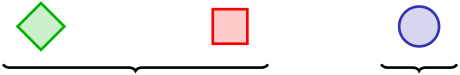
Stochastic timed game: an example



- Timed graph with vertices partitioned among three players:



classical players playing "turn-based"

the Nature stochastic player

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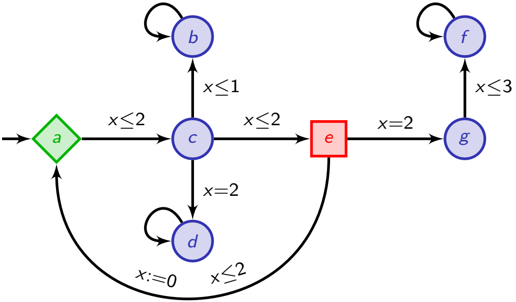

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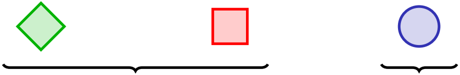

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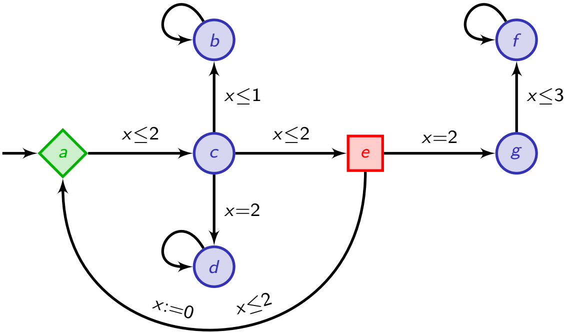
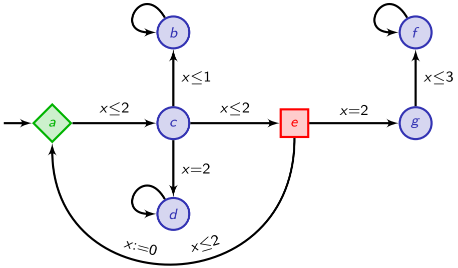
- There are prescribed probability distributions from vertices.

How is this game played?



- Players ◇ and □ play according to standard strategies
- Player ○ plays according to the prescribed probability distributions:
  - choose a delay according to some distribution
  - choose an action according to some discrete distribution

Play, an example



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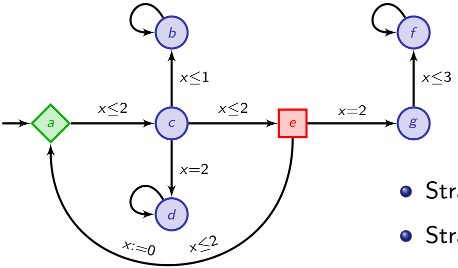

- Strategy for ◇: go to c when $x = 1$
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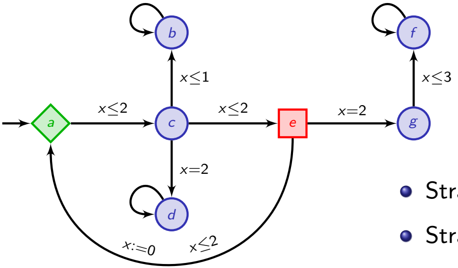

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- From the game and the strategies we obtain a Markov chain:

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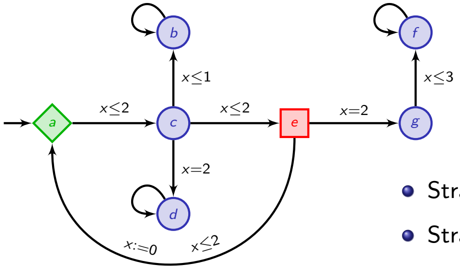

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$(a,0)$

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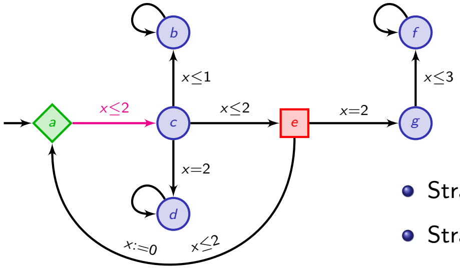

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$$(a,0) \longrightarrow (c,1)$$

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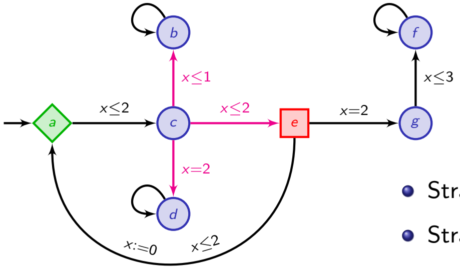

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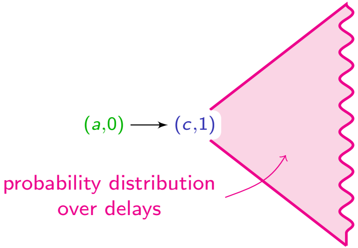

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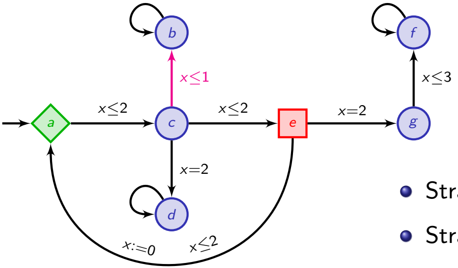

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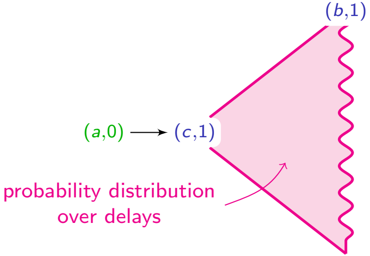

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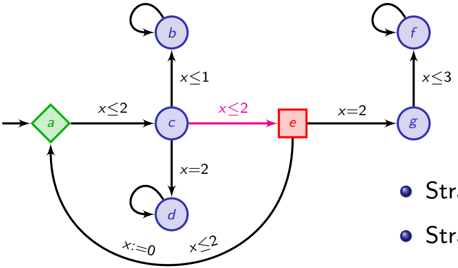

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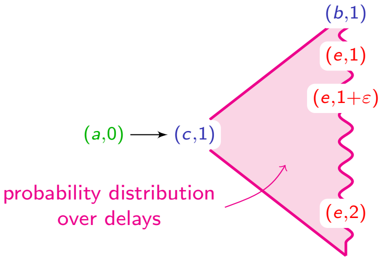

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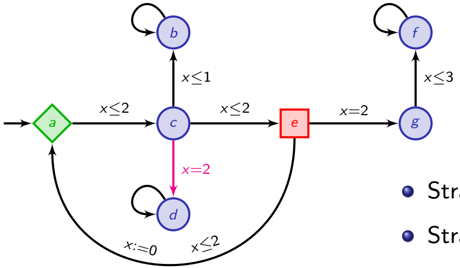

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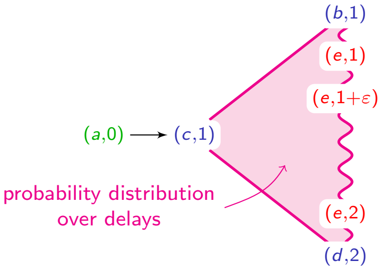

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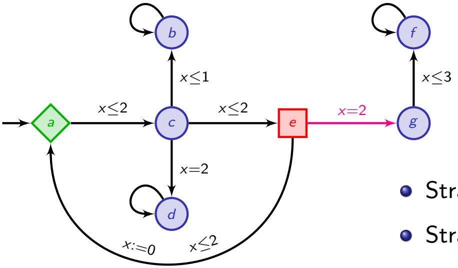

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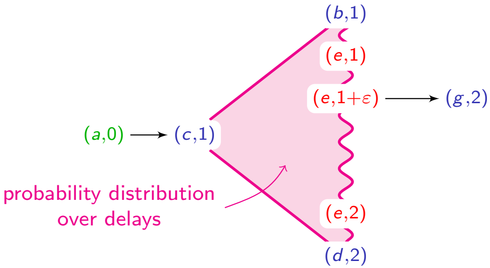

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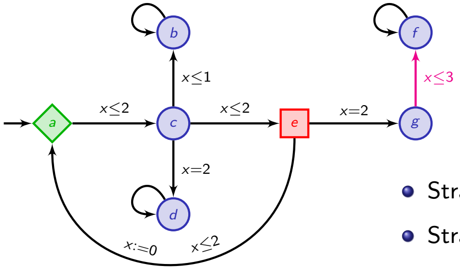

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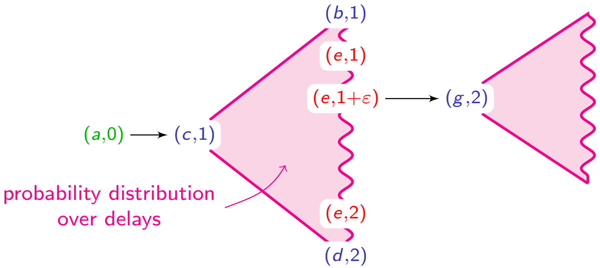

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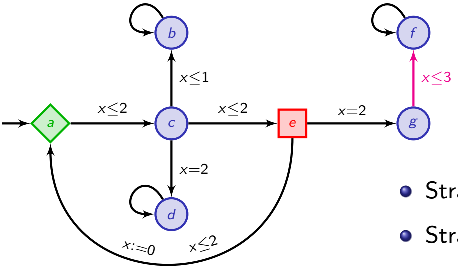

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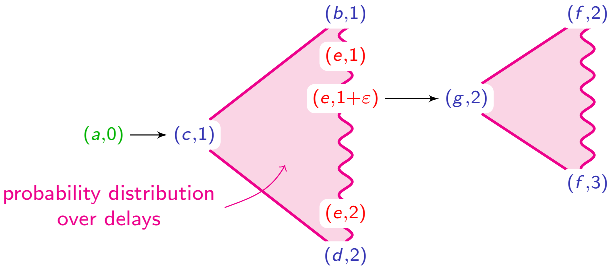

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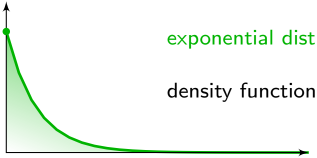
How can we attach probabilities to delays?

- The example of continuous-time Markov chains



exponential distribution

density function $t \mapsto \begin{cases} \lambda \cdot \exp(-\lambda t) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

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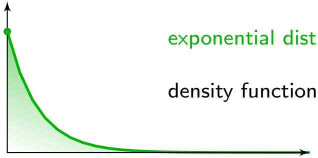

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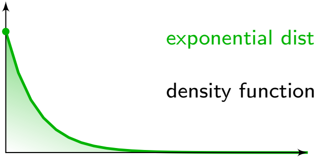

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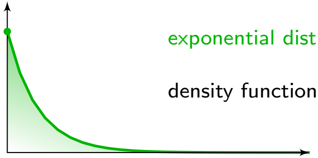

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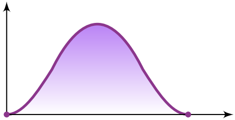

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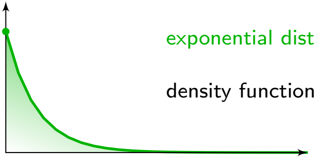

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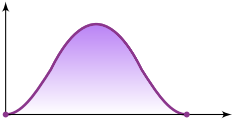

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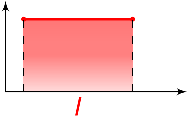

uniform distribution
density function $t \mapsto \begin{cases} \frac{1}{|I|} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

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- We will then extend it using standard strategies for the two other players, which need however satisfy some measurability assumption

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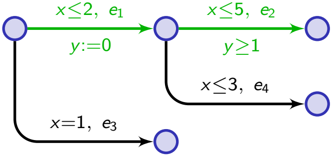

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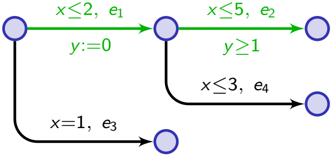

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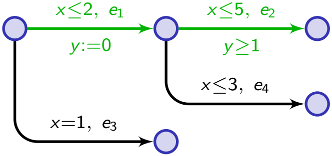

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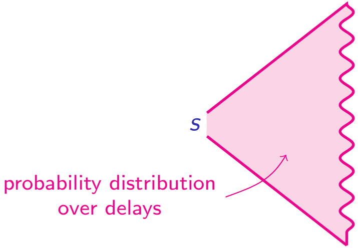

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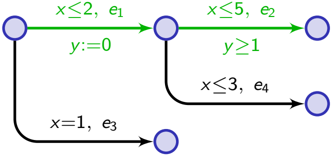

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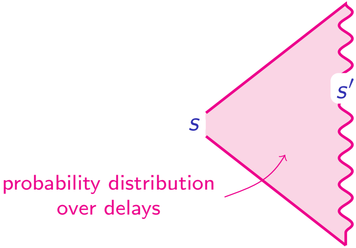

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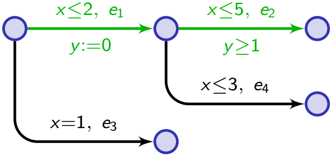

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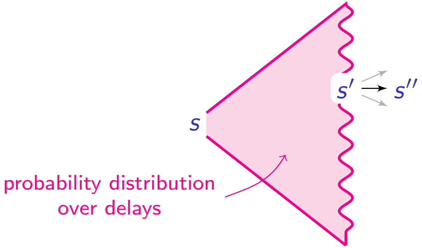

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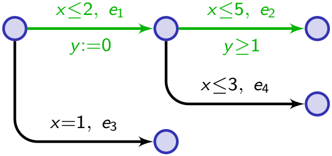

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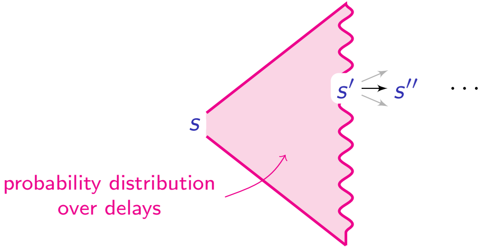

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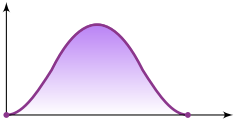

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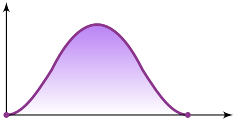

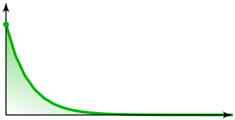

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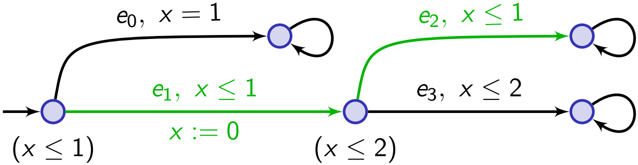
- Example:
  - $\mathsf{Zeno}(s) \;=\; \bigcup_{M \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \bigcup_{(e_1,\cdots,e_n) \in E^n} \mathsf{Cyl}(\pi_{\Sigma_i \tau_i \leq M}(s \xrightarrow{e_1} \cdots \xrightarrow{e_n}))$

An example of computation (with uniform distributions)



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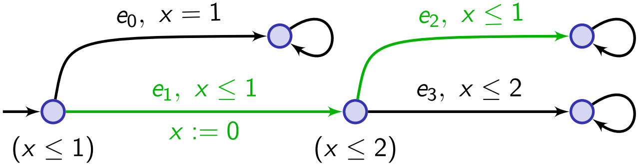

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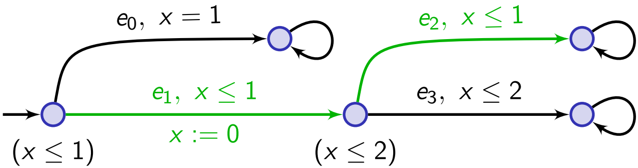

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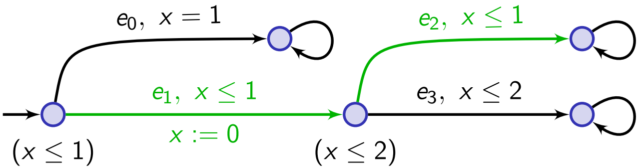

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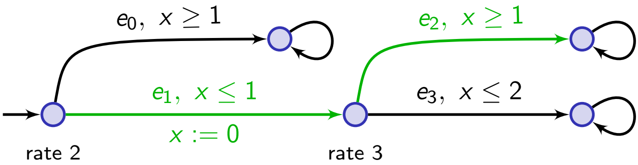
&= \int_0^1 \int_0^1 \left(\frac{\mathbb{P}\big(\pi(s_2)\big)}{2}\mathrm{d}\mu_{s_1}(u)\right) \mathrm{d}\mu_{s_0}(t) \\
&= \int_0^1 \int_0^1 \left(\frac{1}{2}\,\frac{\mathrm{d}u}{2}\right) \mathrm{d}t \quad = \frac{1}{4}
\end{aligned}
$$

An example of computation (with exponential distrib.)



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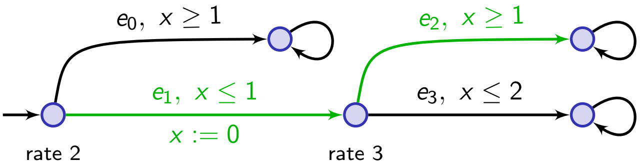

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\mathbb{P}\Big(\pi(s_0 \xrightarrow{e_1}\xrightarrow{e_2}\ )\Big) &= \int_0^1 \mathbb{P}\Big(\pi(s_1 \xrightarrow{e_2}\ )\Big)\mathrm{d}\mu_{s_0}(t) = \int_0^1 \mathbb{P}\Big(\pi(s_1 \xrightarrow{e_2}\ )\Big) 2\exp(-2t)\mathrm{d}t \\
&= \int_0^1 \left(\int_1^{+\infty} 3\exp(-3u)\mathrm{d}u\right) 2\exp(-2t)\mathrm{d}t \\
&= [-\exp(-2t)]_{t=0}^1 \cdot [-\exp(-3u)]_{u=1}^{+\infty} \\
&= (1 - e^{-2}) \cdot e^{-3} = e^{-3} - e^{-5}
\end{aligned}$$

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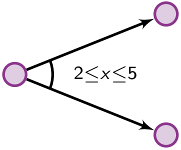
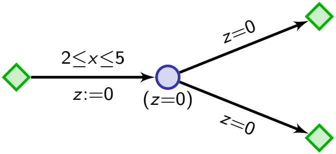
- Probabilistic timed automata = a subclass of the $1\frac{1}{2}$-player games





The synthesis problem

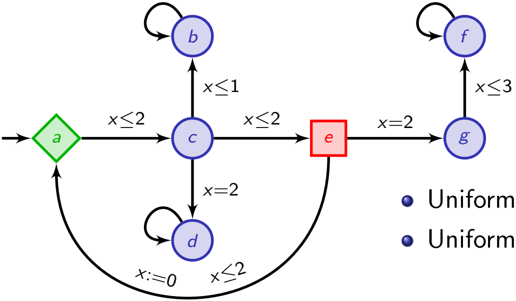
Problem statement

Given a game G, a (linear-time) property φ, a rational threshold $\bowtie r$,

is there a strategy $f_{\diamond}$ for player $\diamond$ s.t.

for all strategies $f_{\square}$ of player $\square$, $\mathbb{P}\Big(G_{f_{\diamond},f_{\square}} \models \varphi\Big) \bowtie r$?

Reachability problem – Example



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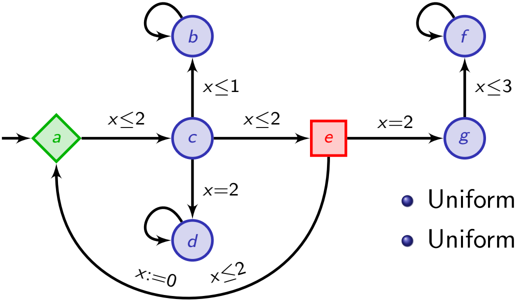

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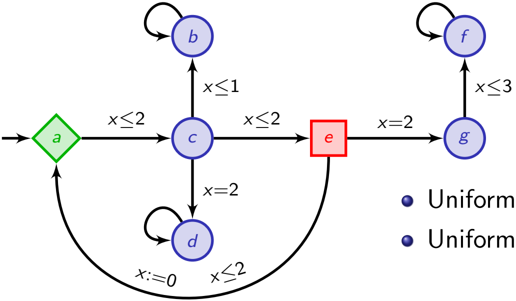

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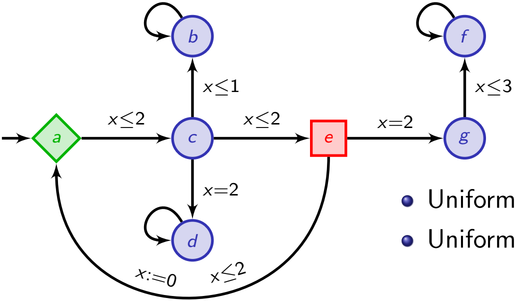

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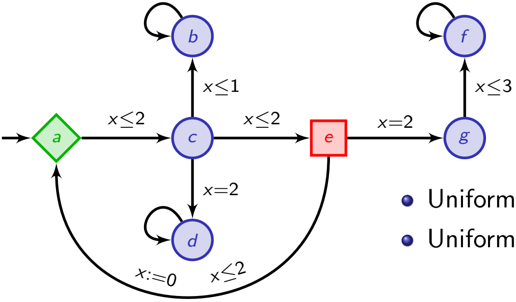

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What kind of games will we play?

Number of players

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- $1\frac{1}{2}$-player games: ("Markov decision process")
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Winning objective

The winning objective will be an ω-regular condition, or some LTL property, or some more restrictive condition like a reachability condition.

Outline

1. Timed automata

2. Timed games

3. A hint into stochastic timed games
   - Some informal description
   - A more formal view of the semantics
   - Summary of the results
   - Qualitative analysis of $\frac{1}{2}$-player games
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   - Quantitative analysis of $\frac{1}{2}$-player games

4. Conclusion

Rough summary of the results

Model		Qualitative	Quantitative
$\frac{1}{2}$-player game	1 clock	decidable [BBB+08]	decidable[1] [BBBM08]
	n clocks	decidable?[2]	?
$1\frac{1}{2}$-player game	1 clock	decidable[3] [BF09]	?
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	n clocks	?	undecidable[4] [BF09]

under some assumptions...

1. reactive automata $I(s) = \mathbb{R}_+$, exponential distributions and resets on every cycle
2. reactive automata $I(s) = \mathbb{R}_+$ and exponential distributions
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Almost-sure model-checking

The qualitative synthesis problem reduces to the so-called "almost-sure model-checking problem"

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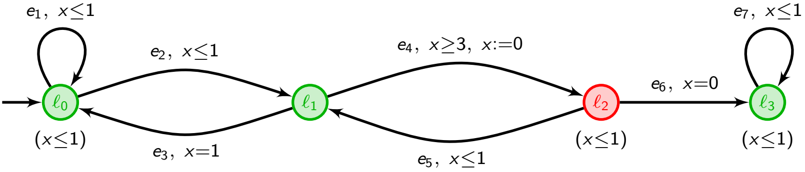
There are only ◯ vertices, but we will use extra colors to represent atomic propositions.

An example



$e_1, x \leq 1$
$e_2, x \leq 1$
$e_4, x \geq 3, x := 0$
$e_7, x \leq 1$
$e_6, x = 0$
ℓ_0
ℓ_1
ℓ_2
ℓ_3
$(x \leq 1)$
$(x \leq 1)$
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$e_3, x = 1$
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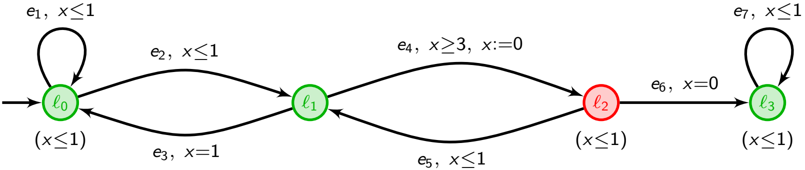

$$\mathcal{A} \not\models \mathbf{G}\,(\text{green} \Rightarrow \mathbf{F}\,\text{red})$$

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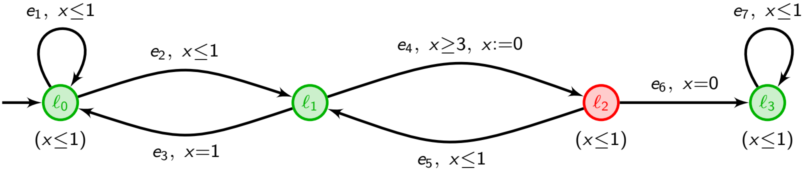

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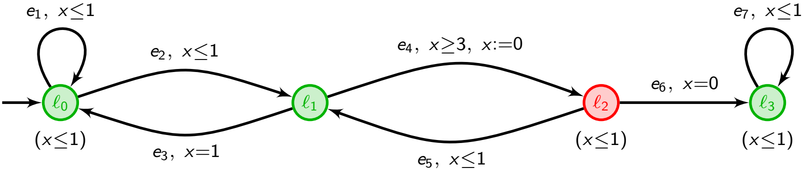

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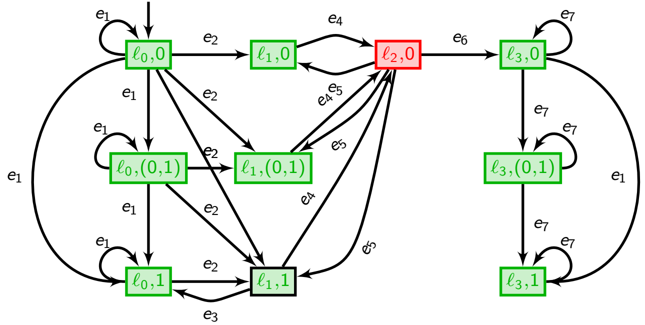
Indeed, almost surely, paths are of the form $e_1^* e_2 \Big(e_4 e_5\Big)^\omega$

The classical region automaton



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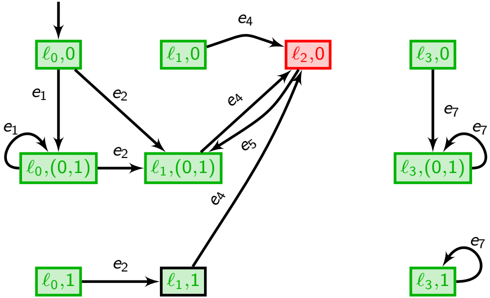

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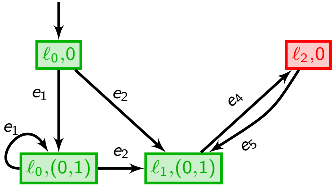

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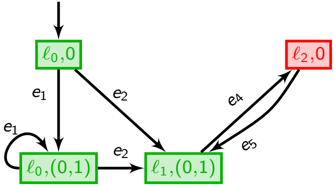

... viewed as a finite Markov chain $MC(\mathcal{A})$

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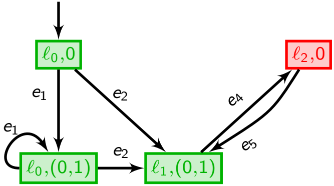

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Proposition

For single-clock timed automata,

$$\mathbb{P}\Big(\mathcal{A} \models \varphi\Big) = 1 \quad \text{iff} \quad \mathbb{P}\Big(MC(\mathcal{A}) \models \varphi\Big) = 1$$

(this is independent of the choice of the distributions...)

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Theorem [BBB+08]

For single-clock timed automata, the almost-sure model-checking

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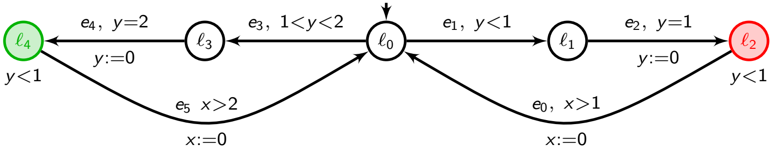

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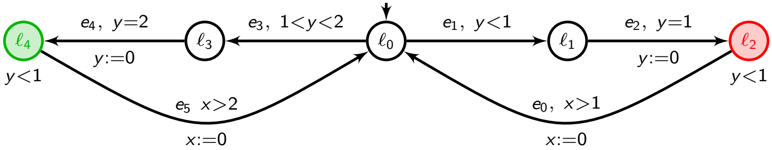

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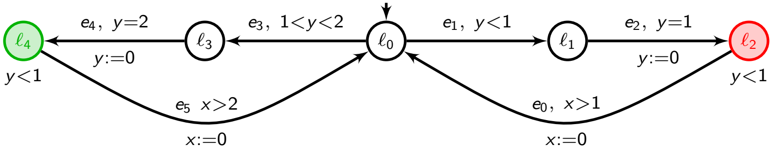

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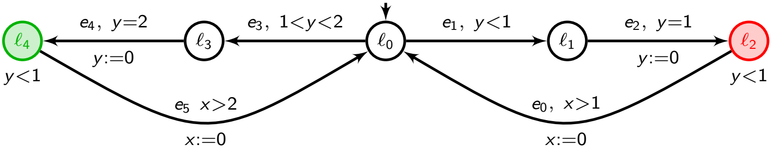

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- However, we can prove that $\mathbb{P}\Big(\mathbf{G}\,\neg\text{red}\Big) > 0$
- There is a *strange* convergence phenomenon: along an execution, if $\delta_i > 0$ is the delay in locations ℓ_2 or ℓ_4, then we have that $\sum_i \delta_i \leq 1$

A note on Zeno behaviours

- The set of Zeno behaviours is measurable:

$$\mathrm{Zeno}(s) \;=\; \bigcup_{M \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \bigcup_{(e_1, \cdots, e_n) \in E^n} \mathrm{Cyl}(\pi(s \xrightarrow{e_1} \cdots \xrightarrow{e_n}))$$

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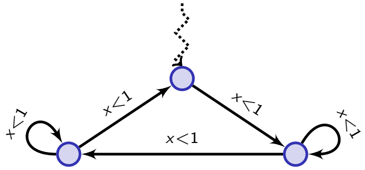

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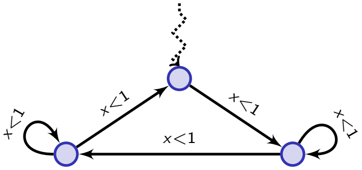

  - an interesting notion of non-Zeno timed automata

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Summary of the results

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$\frac{1}{2}$-player game	1 clock	decidable [BBB+08]	decidable [BBBM08]
	n clocks	decidable?	?
$1\frac{1}{2}$-player game	1 clock	decidable [BF09]	?
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Theorem [BF09]

The reachability problem for stochastic timed games ($2\frac{1}{2}$ players) is undecidable.

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- Holds for uniform and exponential distributions on delays.
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- Proof by reduction from halting problem of two-counter machine to the reachability with probability precisely $\frac{1}{2}$:
  - $\diamond$ simulates a computation of the two-counter machine and encodes counter values in clock values
  - $\diamond$ stores counter values c_1 and c_2 as $\frac{1}{2^{c_1} 3^{c_2}}$
  - $\square$ will check that $\diamond$ is not cheating using the power of the probabilities

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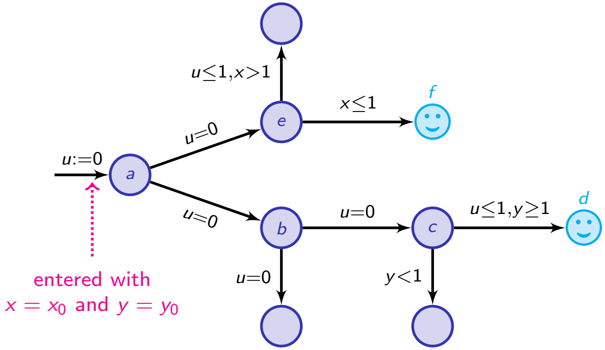
Undecidability – Comparing counter values

- Check clock y stores $(c_1 + 1, c_2)$, assuming that x stores (c_1, c_2)



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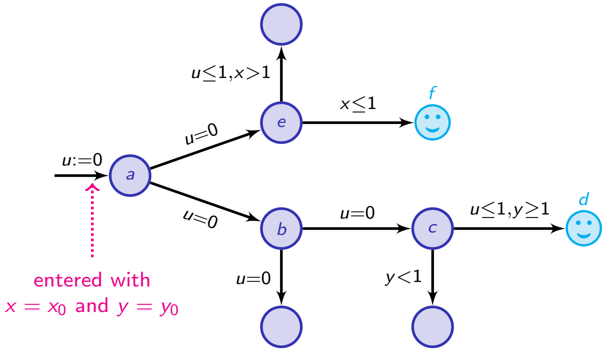

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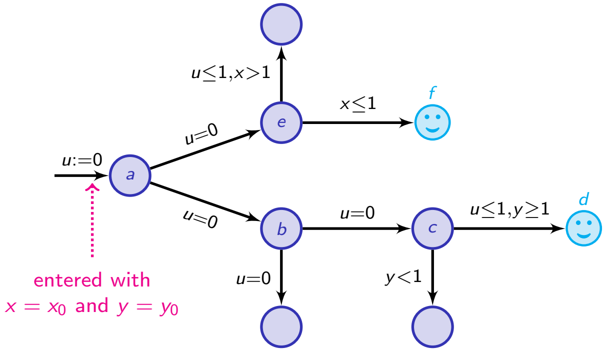

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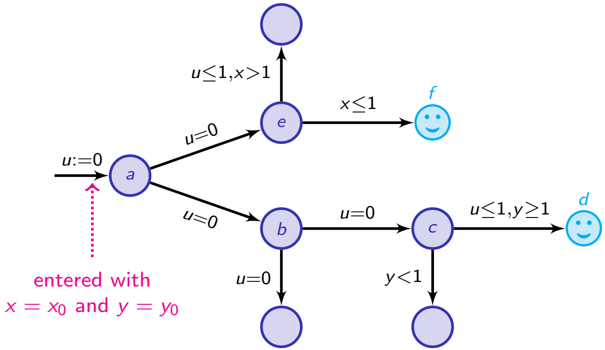

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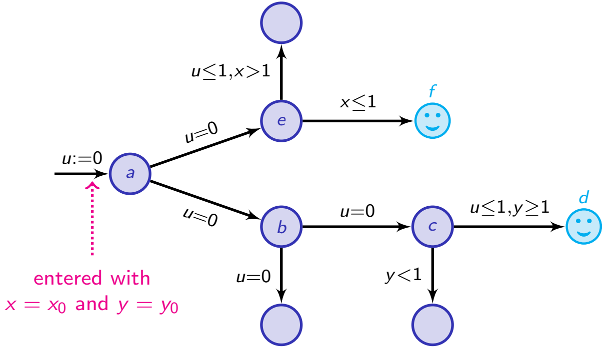

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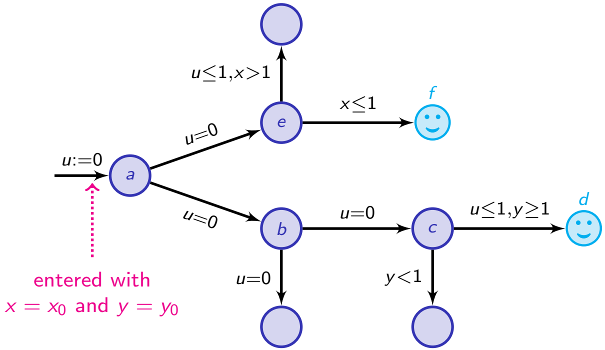

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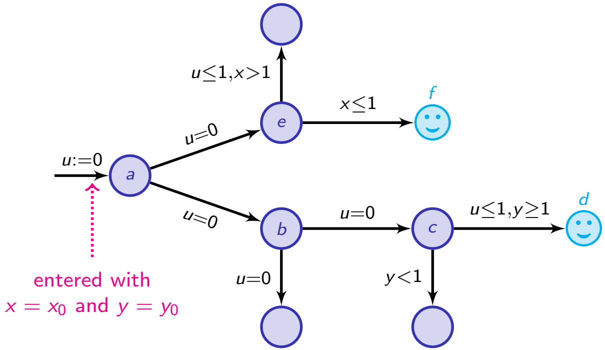

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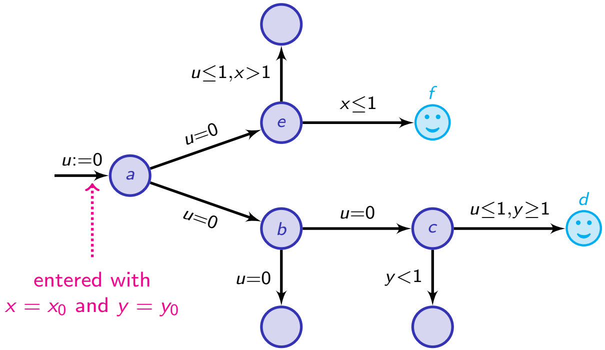

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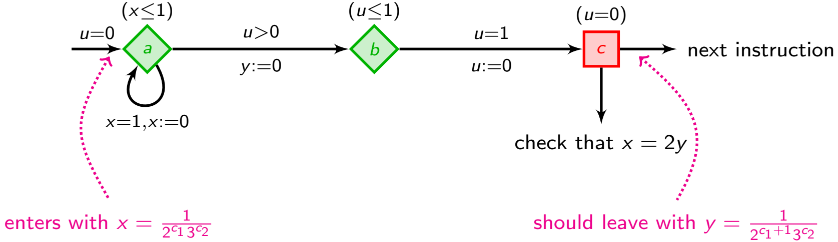
How do we properly increment the first counter?



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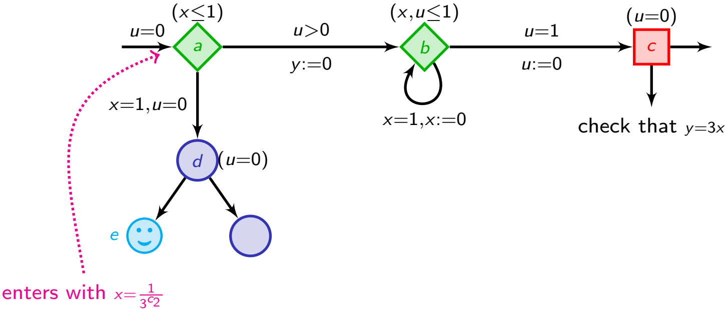

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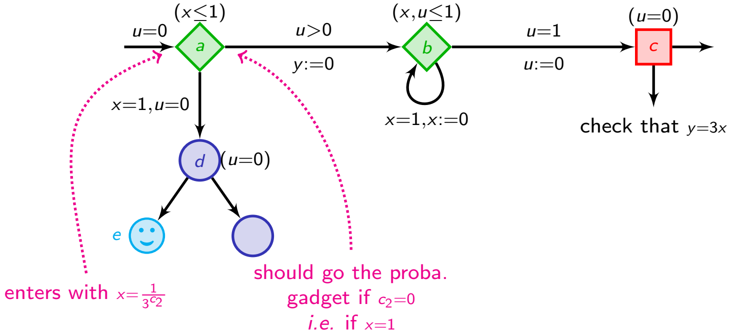

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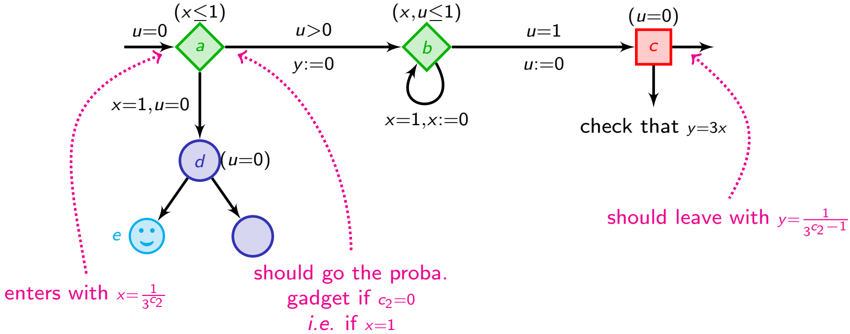

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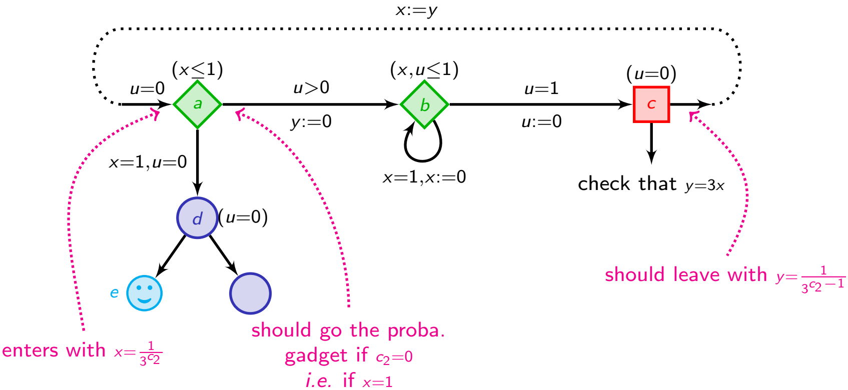

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  - we will be able to decide the threshold problem:
    "Given $\mathcal{A}$, φ, $c \in \mathbb{Q}$, and $\sim \in \{<, \leq, =, \geq, >\}$,
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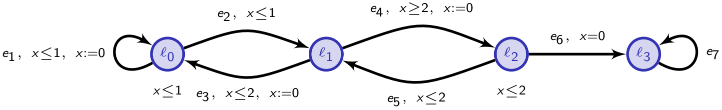
An example



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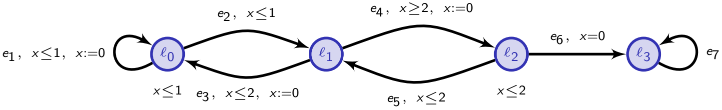

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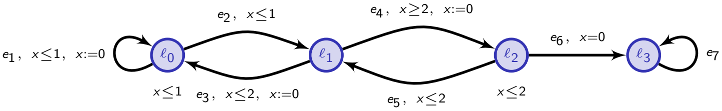

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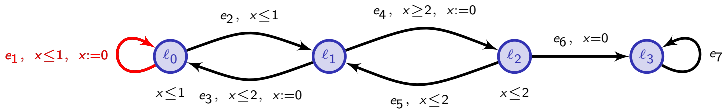

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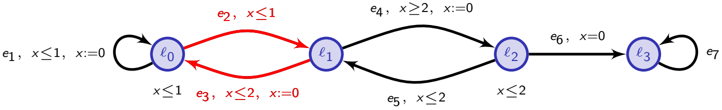

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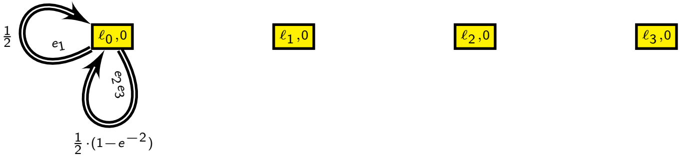

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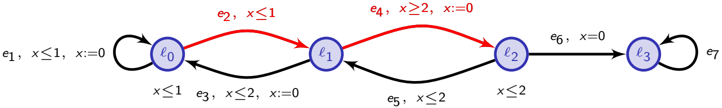

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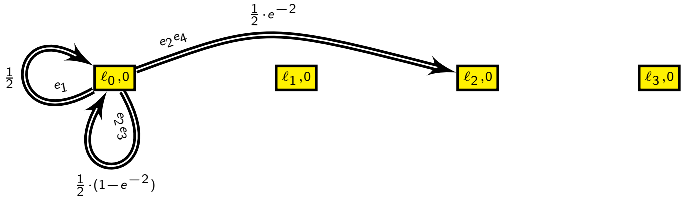

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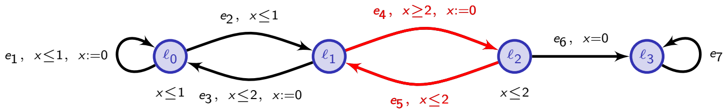

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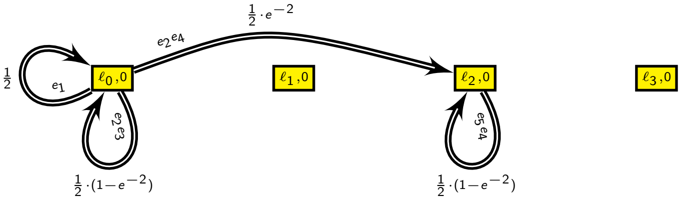

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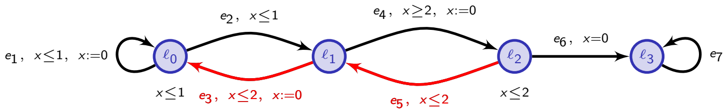

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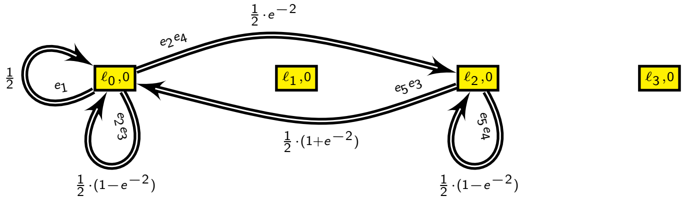

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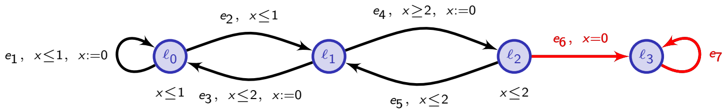

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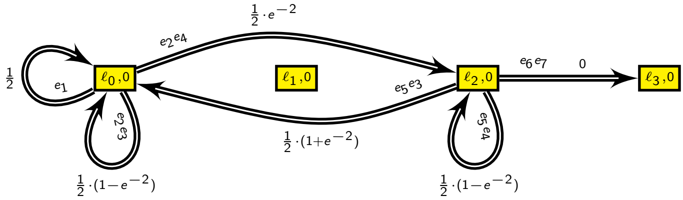

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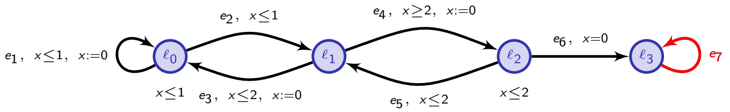

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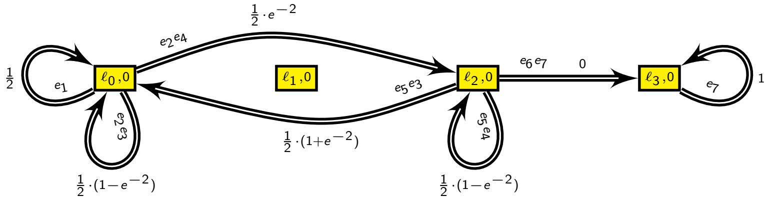

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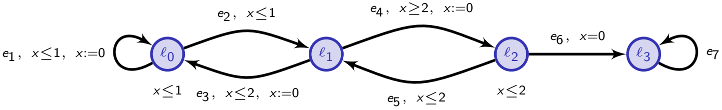

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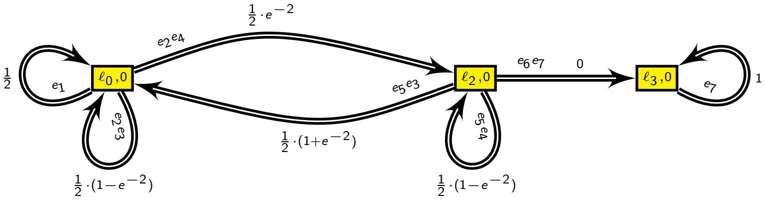

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Theorem

Under some hypotheses, for single-clock automaton $\mathcal{A}$ and property φ,

$$\mathbb{P}_{\mathcal{A}}(s_0 \models \varphi) = \mathbb{P}_{MC'(\mathcal{A})}(s_0 \models \Diamond F_\varphi)$$

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- **Limits of the abstraction:** there may be no closed form for the values labelling the edges of $MC'(\mathcal{A})$.

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Under those hypotheses, $\mathbb{P}\Big(s_0 \models \varphi\Big)$ can be expressed as $f\,(e^{-r})$ where r is a rational number, and $f \in \mathbb{Q}(X)$ is a rational function.

$\rightsquigarrow$ Note: the hypothesis "reset all bounded cycles" is necessary to get this form.

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- When $(a_N, b_N) \subseteq (\alpha, \beta)$, the two sequences $(f(a_i))_{i \geq N}$ and $(f(b_i))_{i \geq N}$ are monotonic and converge to $f\left(e^{-r}\right)$

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Under the previous hypotheses, the threshold problem is decidable.

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Outline

1. Timed automata

2. Timed games

3. A hint into stochastic timed games
   - Some informal description
   - A more formal view of the semantics
   - Summary of the results
   - Qualitative analysis of $\frac{1}{2}$-player games
   - Quantitative analysis of $2\frac{1}{2}$-player games
   - Quantitative analysis of $\frac{1}{2}$-player games

4. Conclusion

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- Probabilistic timed automata (PRISM and UPPAAL-PRO model)
  - the questions considered in this presentation can be "trivially" answered (because they reduce to similar questions on discrete-time Markov decision processes)
  - quantitative objectives should be investigated