On the optimal reachability problem in weighted timed games

Patricia Bouyer-Decitre

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Based on former works with Thomas Brihaye, Kim G. Larsen, Nicolas Markey, etc...
And on recent work with Samy Jaziri and Nicolas Markey

Outline

1. Introduction

2. Overview of “old” results
   - Weighted timed automata
   - Timed games
   - Weighted timed games

3. Some recent developments
   - Undecidability of the value problem
   - Approximation of the optimal cost
   - Back to the undecidability

4. Conclusion

Time-dependent systems

- We are interested in timed systems

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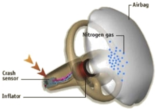

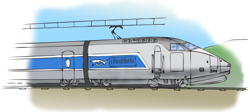

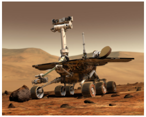

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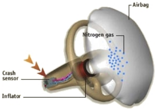
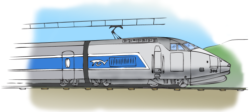
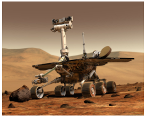




- ... and in their analysis and control

An example: The task graph scheduling problem

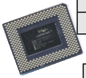
Compute $D\times(C\times(A+B))+(A+B)+(C\times D)$ using two processors:

P_1 (fast):

time	
+	2 picoseconds
×	3 picoseconds

energy	
idle	10 Watt
in use	90 Watts

P_2 (slow):



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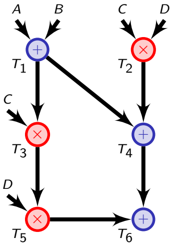

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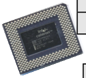

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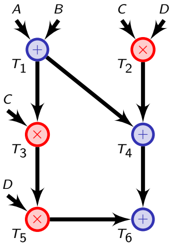

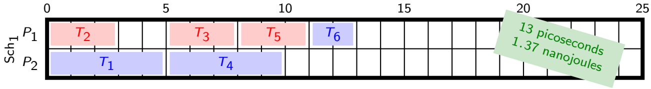

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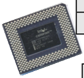

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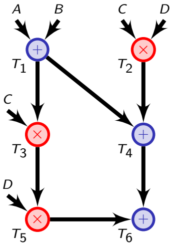

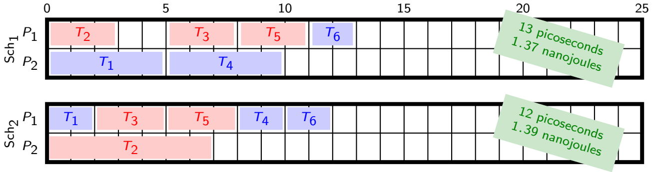

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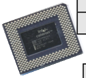

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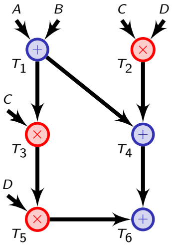

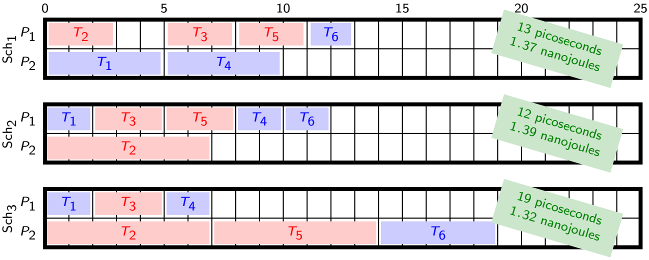

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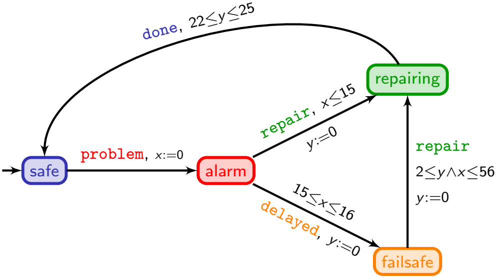
The model of timed automata



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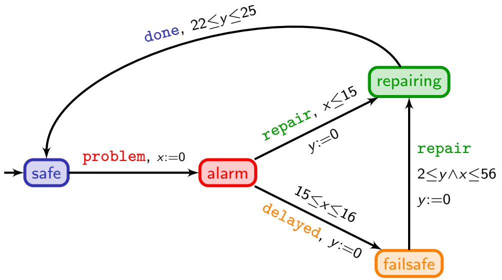

$$
\begin{array}{ccccccccccc}
 & \text{safe} & \xrightarrow{23} & \text{safe} & \xrightarrow{\texttt{problem}} & \text{alarm} & \xrightarrow{15.6} & \text{alarm} & \xrightarrow{\texttt{delayed}} & \text{failsafe} & \\
x & 0 & & 23 & & 0 & & 15.6 & & 15.6 & \cdots \\
y & 0 & & 23 & & 23 & & 38.6 & & 0 &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & \text{failsafe} & \xrightarrow{2.3} & \text{failsafe} & \xrightarrow{\texttt{repair}} & \text{repairing} & \xrightarrow{22.1} & \text{repairing} & \xrightarrow{\texttt{done}} & \text{safe} \\
\cdots & 15.6 & & 17.9 & & 17.9 & & 40 & & 40 \\
 & 0 & & 2.3 & & 0 & & 22.1 & & 22.1
\end{array}
$$

Modelling the task graph scheduling problem

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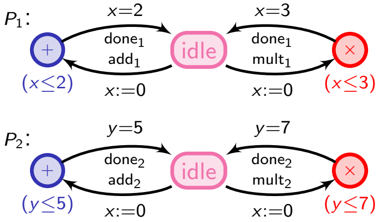
- Processors



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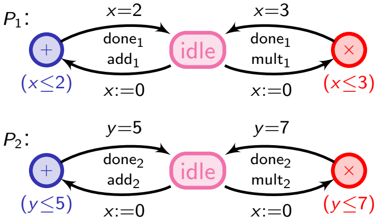
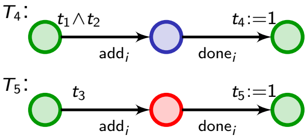
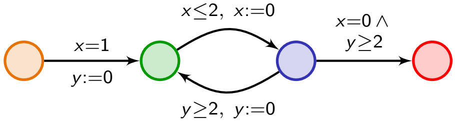


- Tasks



A schedule is a path in the product automaton

Analyzing timed automata



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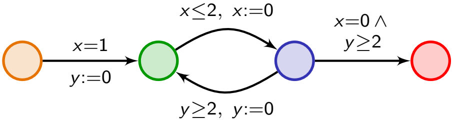

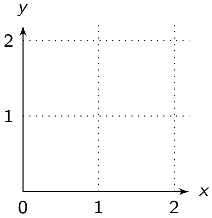

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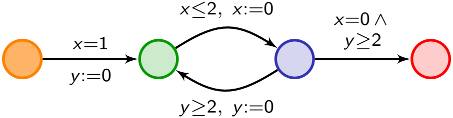

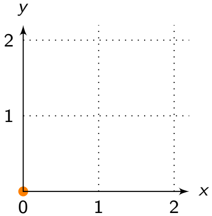

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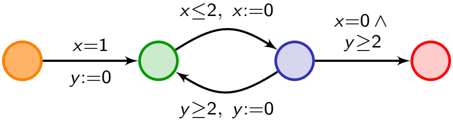

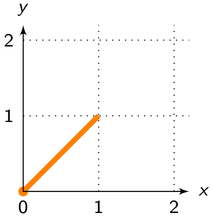

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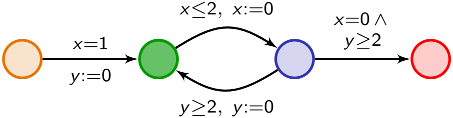

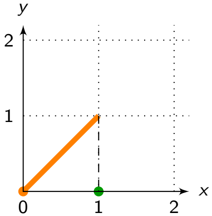

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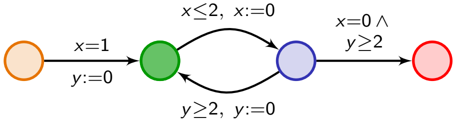

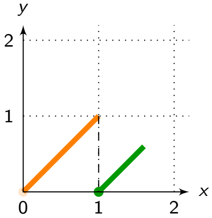

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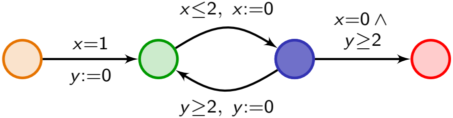

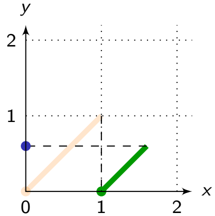

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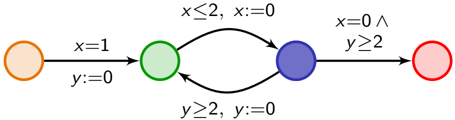

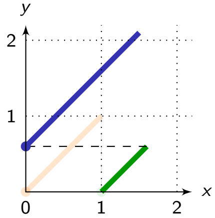

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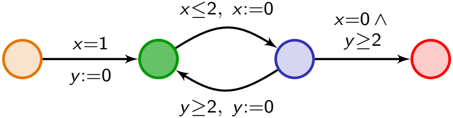

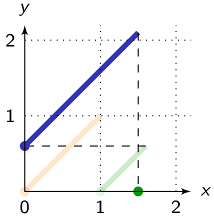

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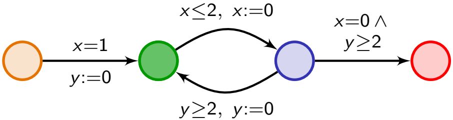

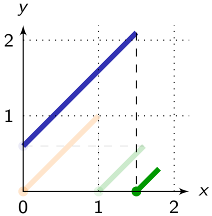

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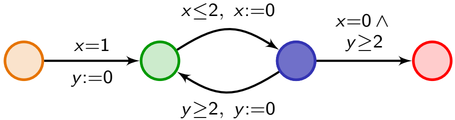

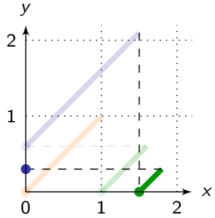

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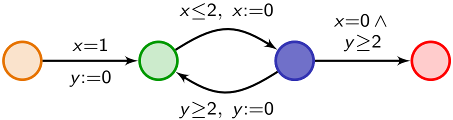

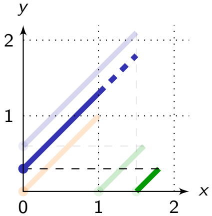

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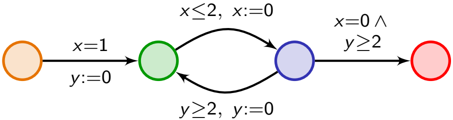

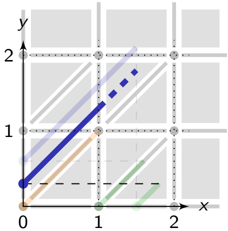

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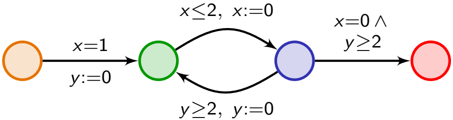
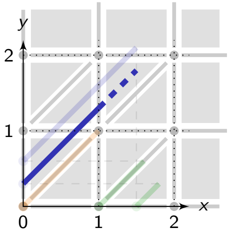


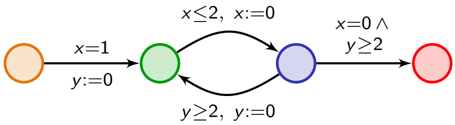


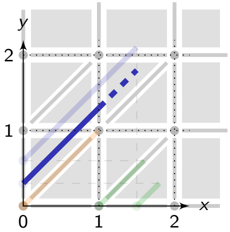
Theorem [AD94]

Reachability in timed automata is decidable (as well as many other important properties). It is PSPACE-complete.

- Technical tool: region abstraction

Analyzing timed automata





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- Technical tool: region abstraction
- Efficient symbolic technics based on zones, implemented in tools

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- A possible solution: use hybrid automata

$$\begin{array}{ll} & \text{a discrete control (the mode of the system)} \\ + & \text{continuous evolution of the variables within a mode} \end{array}$$

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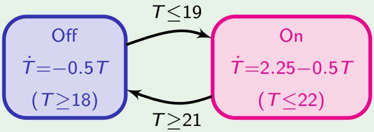

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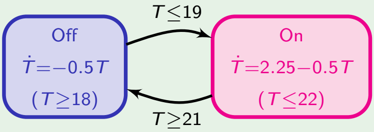

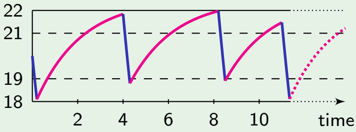

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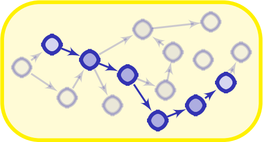

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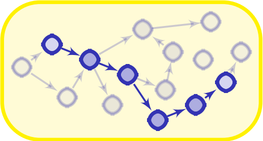

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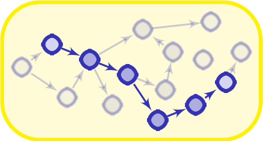

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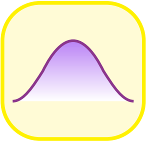

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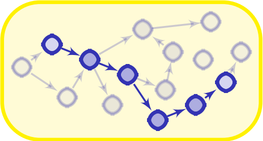

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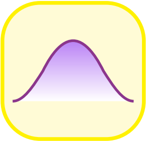
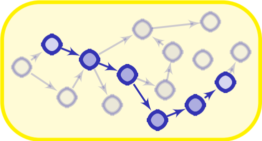
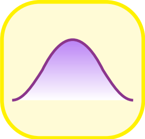


Easy...

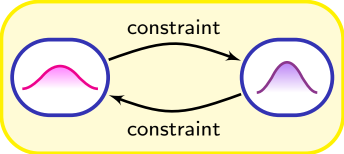
Ok... but?



Easy...



Easy...



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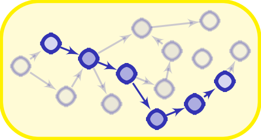
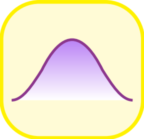
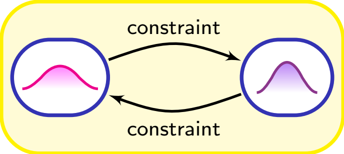


Easy...



Easy...



Hard!

Modelling resources in timed systems

- System resources might be relevant and even crucial information
  - energy consumption,
  - memory usage,
  - ...
  - price to pay,
  - bandwidth,

  $\rightsquigarrow$ timed automata are not powerful enough!

- A possible solution: use hybrid automata

Theorem [HKPV95]

The reachability problem is undecidable in hybrid automata. Even for the simplest, the so-called stopwatch automata (clocks can be stopped).

[HKPV95] Henzinger, Kopke, Puri, Varaiya. What's decidable wbout hybrid automata? *(SToC'95)*.

Modelling resources in timed systems

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Theorem [HKPV95]

The reachability problem is undecidable in hybrid automata. Even for the simplest, the so-called stopwatch automata (clocks can be stopped).

- An alternative: weighted/priced timed automata [ALP01,BFH+01]
  $\rightsquigarrow$ hybrid variables do not constrain the system
  hybrid variables are observer variables

[HKPV95] Henzinger, Kopke, Puri, Varaiya. What's decidable wbout hybrid automata? *(SToC'95)*.
[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
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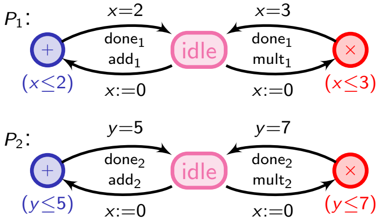
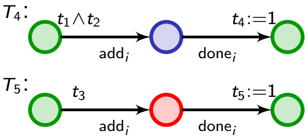
Modelling the task graph scheduling problem

- Processors

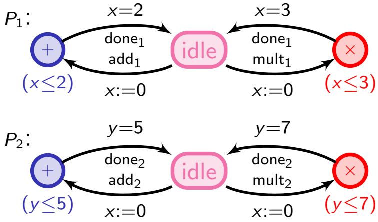


- Tasks



Modelling the task graph scheduling problem

- Processors



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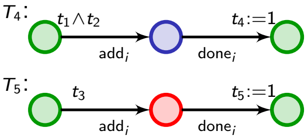
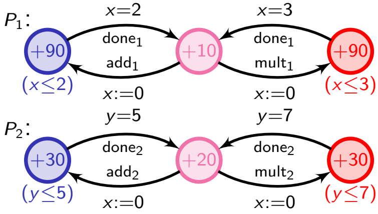
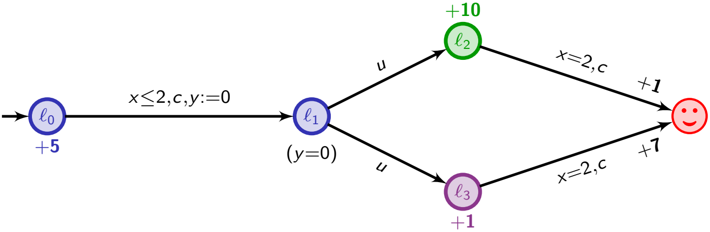


- Modelling energy



A good schedule is a path in the product automaton with a low cost

Weighted/priced timed automata [ALP01,BFH+01]



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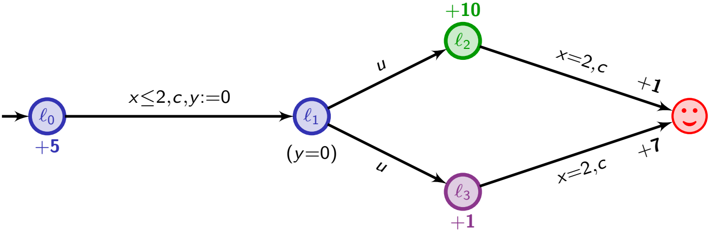
Weighted/priced timed automata [ALP01,BFH+01]



	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

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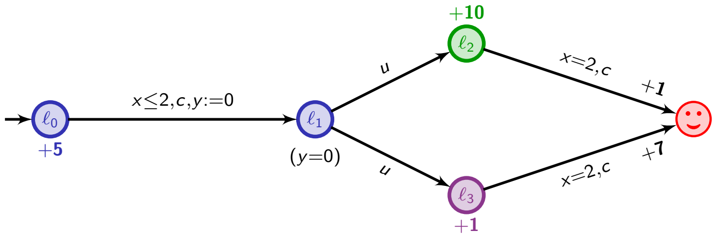

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost :

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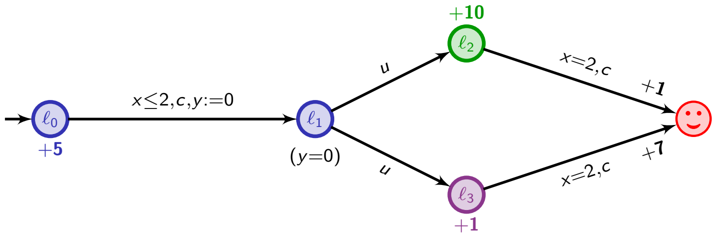
Weighted/priced timed automata [ALP01,BFH+01]



	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5

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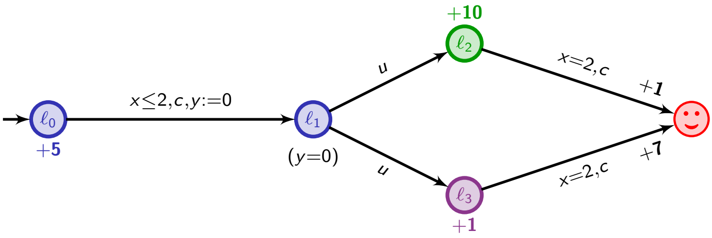

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5 + 0

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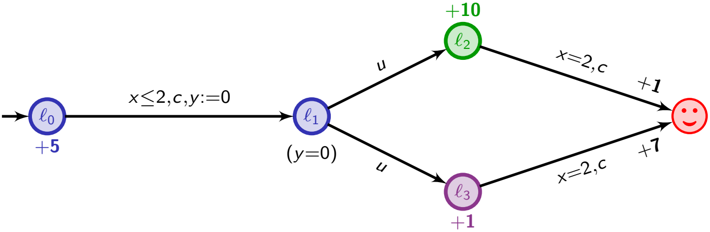

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x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5 + 0 + 0

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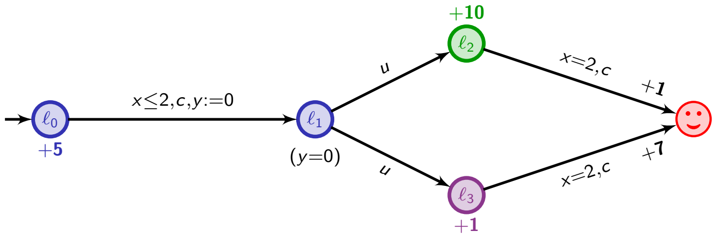

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x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5 + 0 + 0 + 0.7

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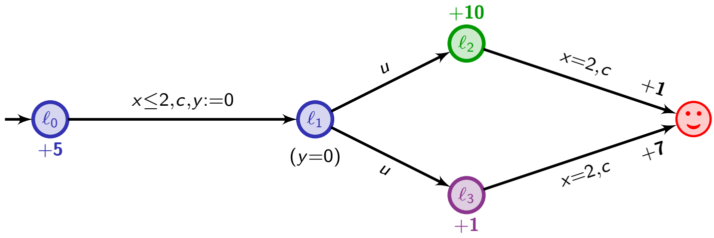

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : 6.5 + 0 + 0 + 0.7 + 7

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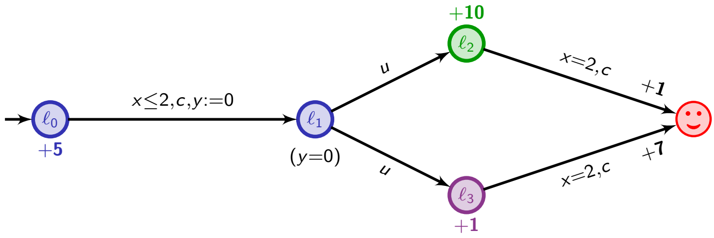

	ℓ_0	$\xrightarrow{1.3}$	ℓ_0	$\xrightarrow{c}$	ℓ_1	$\xrightarrow{u}$	ℓ_3	$\xrightarrow{0.7}$	ℓ_3	$\xrightarrow{c}$	☺
x	0		1.3		1.3		1.3		2		
y	0		1.3		0		0		0.7		

cost : $6.5 + 0 + 0 + 0.7 + 7 = 14.2$

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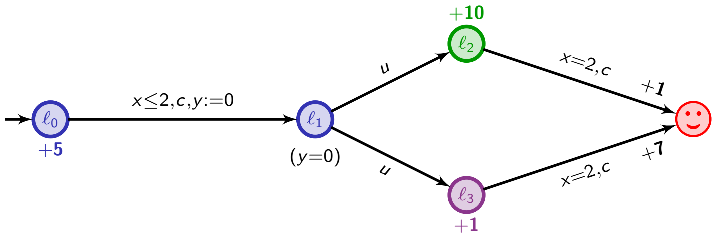
Weighted/priced timed automata [ALP01,BFH+01]



Question: what is the optimal cost for reaching ☺ ?

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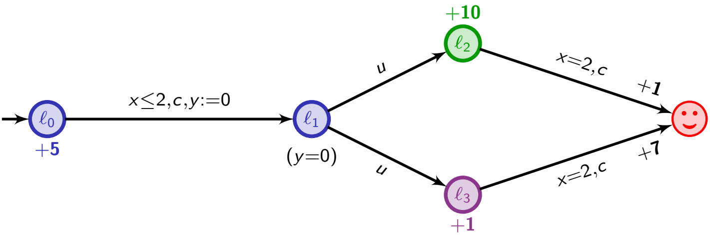

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$$5t + 10(2-t) + 1$$

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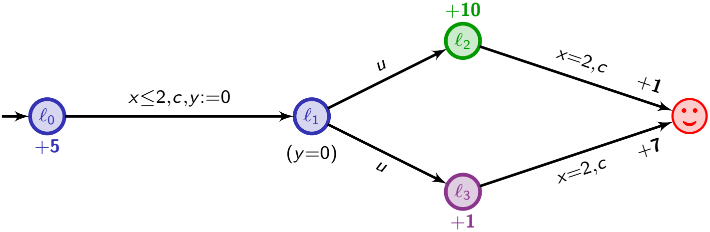

Question: what is the optimal cost for reaching ☺ ?

$$5t + 10(2-t) + 1 \;,\; 5t + (2-t) + 7$$

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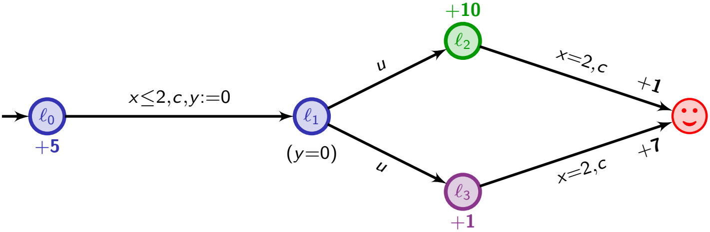
Weighted/priced timed automata [ALP01,BFH+01]



Question: what is the optimal cost for reaching 🙂?

$$\min\left(\,5t+10(2-t)+1\,,\;5t+(2-t)+7\,\right)$$

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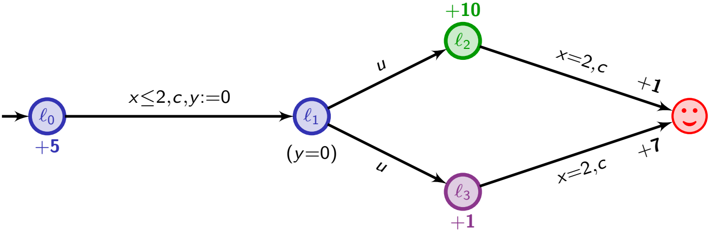
Weighted/priced timed automata [ALP01,BFH+01]



Question: what is the optimal cost for reaching ☺?

$$\inf_{0\leq t\leq 2} \min\left(5t+10(2-t)+1 \,,\; 5t+(2-t)+7 \right) = 9$$

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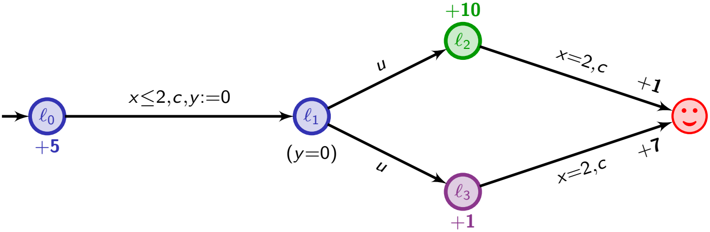


Question: what is the optimal cost for reaching ☺?

$$\inf_{0 \leq t \leq 2} \min \left(5t + 10(2-t) + 1 \ , \ 5t + (2-t) + 7 \right) = 9$$

$\rightsquigarrow$ *strategy:* leave immediately ℓ_0, go to ℓ_3, and wait there 2 t.u.

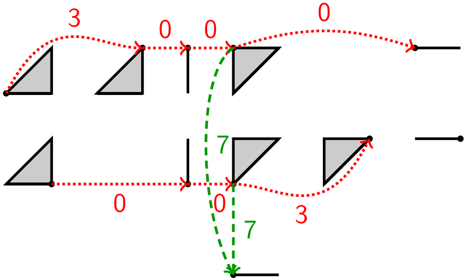
[ALP01] Alur, La Torre, Pappas. Optimal paths in weighted timed automata *(HSCC'01)*.
[BFH+01] Behrmann, Fehnker, Hune, Larsen, Pettersson, Romijn, Vaandrager. Minimum-cost reachability in priced timed automata *(HSCC'01)*.

Optimal-cost reachability

Theorem [ALP01,BFH+01,BBBR07]

In weighted timed automata, the optimal cost is an integer and can be computed in PSPACE.

- Technical tool: a refinement of the regions, the corner-point abstraction



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[BBBR07] Bouyer, Brihaye, Bruyère, Raskin. On the optimal reachability problem *(Formal Methods in System Design)*.

From timed to discrete behaviours

Optimal reachability as a linear programming problem

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\circ \xrightarrow{t_1} \circ \xrightarrow{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots$$

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\circ \xrightarrow{t_1} \circ \xrightarrow[x \leq 2]{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots \quad \left\{ \begin{array}{l} t_1 + t_2 \leq 2 \\ \end{array} \right.$$

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\circ \xrightarrow[y:=0]{t_1} \circ \xrightarrow[x\leq 2]{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow[y\geq 5]{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots \quad \begin{cases} t_1+t_2\leq 2 \\ t_2+t_3+t_4\geq 5 \end{cases}$$

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\begin{array}{ccccc} T_1 & T_2 & T_3 & T_4 & T_5 \end{array}$$

$$\circ \xrightarrow[y:=0]{t_1} \circ \xrightarrow[x\leq 2]{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow[y\geq 5]{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots \quad \left\{ \begin{array}{ll} t_1+t_2\leq 2 & T_2\leq 2 \\ t_2+t_3+t_4\geq 5 & T_4-T_1\geq 5 \end{array} \right.$$

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\begin{array}{cccccc} & T_1 & T_2 & T_3 & T_4 & T_5 \end{array}$$

$$\circ \xrightarrow[y:=0]{t_1} \circ \xrightarrow[x\leq 2]{t_2} \circ \xrightarrow{t_3} \circ \xrightarrow[y\geq 5]{t_4} \circ \xrightarrow{t_5} \circ \quad \cdots \quad \begin{cases} t_1+t_2\leq 2 & T_2\leq 2 \\ t_2+t_3+t_4\geq 5 & T_4-T_1\geq 5 \end{cases}$$

Lemma

Let Z be a bounded zone and f be a function

$$f : (T_1, \dots, T_n) \mapsto \sum_{i=1}^{n} c_i T_i + c$$

well-defined on $\overline{Z}$. Then $inf_Z f$ is obtained on the border of $\overline{Z}$ with integer coordinates.

From timed to discrete behaviours

Optimal reachability as a linear programming problem

$$\begin{array}{cccccccccccc} & T_1 & & T_2 & & T_3 & & T_4 & & T_5 & & \\ \circ & \xrightarrow[y:=0]{t_1} & \circ & \xrightarrow[x\leq 2]{t_2} & \circ & \xrightarrow{t_3} & \circ & \xrightarrow[y\geq 5]{t_4} & \circ & \xrightarrow{t_5} & \circ & \cdots \end{array} \quad \begin{cases} t_1+t_2\leq 2 & T_2\leq 2 \\ t_2+t_3+t_4\geq 5 & T_4-T_1\geq 5 \end{cases}$$

Lemma

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well-defined on $\overline{Z}$. Then $inf_Z f$ is obtained on the border of $\overline{Z}$ with integer coordinates.

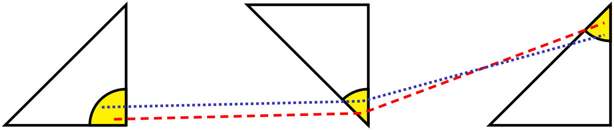
$\rightsquigarrow$ for every finite path π in $\mathcal{A}$, there exists a path Π in $\mathcal{A}_{\text{cp}}$ such that

$$\text{cost}(\Pi) \leq \text{cost}(\pi)$$

[Π is a "corner-point projection" of π]

From discrete to timed behaviours

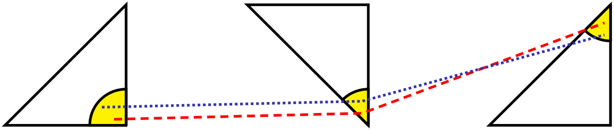
Approximation of abstract paths:



For any path Π of $\mathcal{A}_{\mathsf{cp}}$,

From discrete to timed behaviours

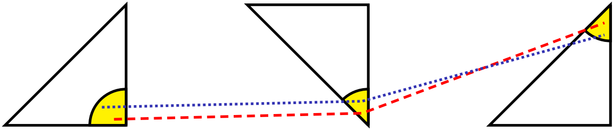
Approximation of abstract paths:



For any path Π of $\mathcal{A}_{\mathsf{cp}}$, for any $\varepsilon > 0$,

From discrete to timed behaviours

Approximation of abstract paths:

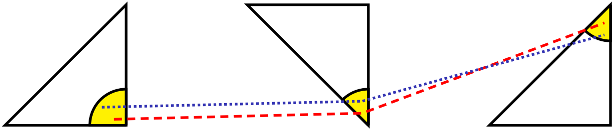


For any path Π of $\mathcal{A}_{\mathsf{cp}}$, for any $\varepsilon > 0$, there exists a path π_ε of $\mathcal{A}$ s.t.

$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon$$

From discrete to timed behaviours

Approximation of abstract paths:



For any path Π of $\mathcal{A}_{\mathsf{cp}}$, for any $\varepsilon > 0$, there exists a path π_ε of $\mathcal{A}$ s.t.

$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon$$

For every $\eta > 0$, there exists $\varepsilon > 0$ s.t.

$$\|\Pi - \pi_\varepsilon\|_\infty < \varepsilon \Rightarrow |\mathsf{cost}(\Pi) - \mathsf{cost}(\pi_\varepsilon)| < \eta$$

Note on the corner-point abstraction

It is a very interesting abstraction, that can be used in several other contexts:

- for mean-cost optimization [BBL04,BBL08]
- for discounted-cost optimization [FL08]
- for all concavely-priced timed automata [JT08]
- for deciding frequency objectives [BBBS11,Sta12]
- ...

[BBL04] Bouyer, Brinksma, Larsen. Staying Alive As Cheaply As Possible *(HSCC'04)*.
[BBL08] Bouyer, Brinksma, Larsen. Optimal infinite scheduling for multi-priced timed automata *(Formal Methods in System Designs)*.
[FL08] Fahrenberg, Larsen. Discount-optimal infinite runs in priced timed automata *(INFINITY'08)*.
[JT08] Judziński, Trivedi. Concavely-priced timed automata *(FORMATS'08)*.
[BBBS11] Bertrand, Bouyer, Brihaye, Stainer. Emptiness and universality problems in timed automata with positive frequency *(ICALP'11)*.
[Sta12] Stainer. Frequencies in forgetful timed automata *(FORMATS'12)*.

Outline

1. Introduction
2. Overview of “old” results
   - Weighted timed automata
   - Timed games
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3. Some recent developments
   - Undecidability of the value problem
   - Approximation of the optimal cost
   - Back to the undecidability
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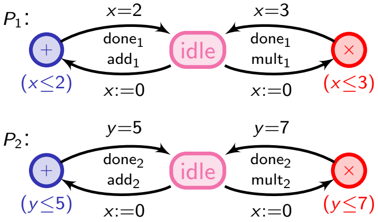
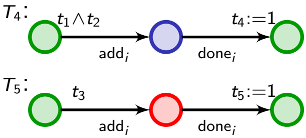
Modelling the task graph scheduling problem

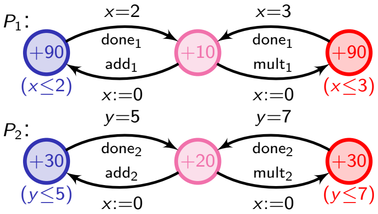
- Processors



- Tasks



- Modelling energy



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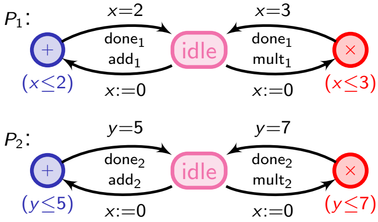

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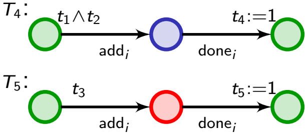

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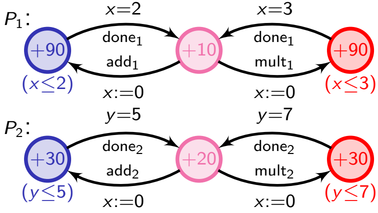
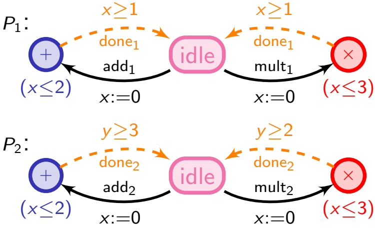


- Modelling uncertainty



Modelling the task graph scheduling problem

- Processors

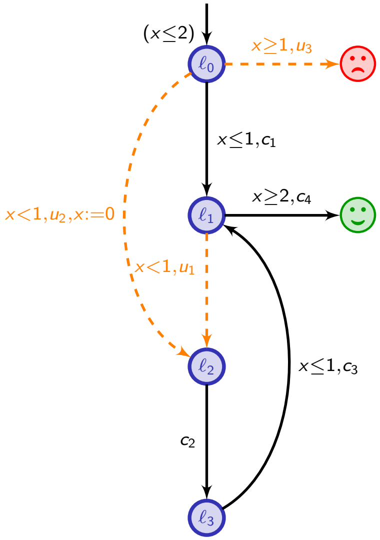
- Tasks

A (good) schedule is a strategy in the product game (with a low cost)

- Modelling energy

- Modelling uncertainty

An example of a timed game



Rule of the game

- Aim: avoid ☹ and reach ☺

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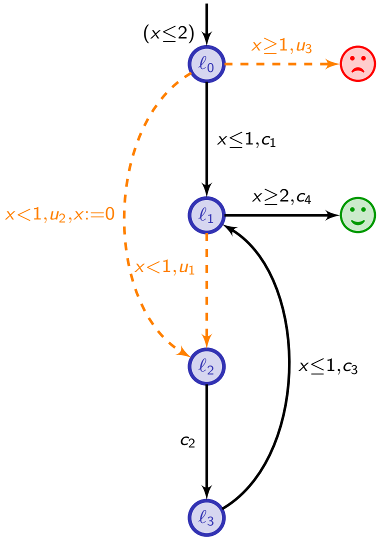

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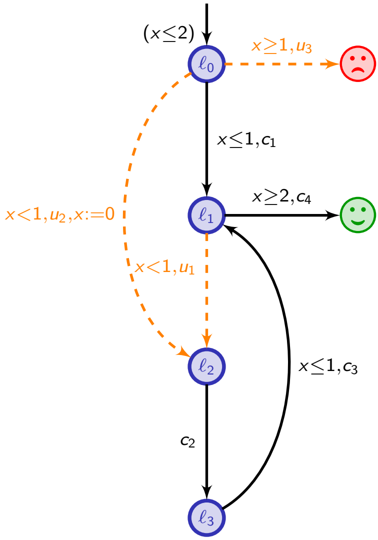

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$$f : \text{history} \mapsto (\text{delay}, \text{cont. transition})$$

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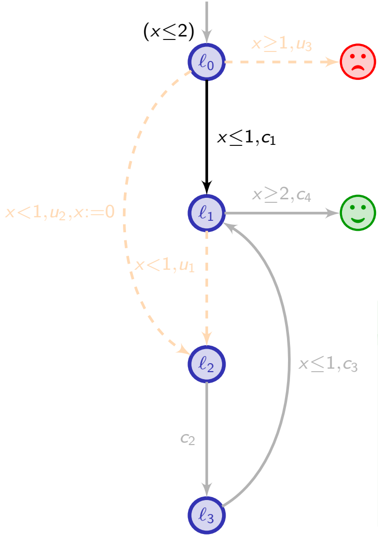

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A (memoryless) winning strategy

- from $(\ell_0, 0)$, play $(0.5, c_1)$

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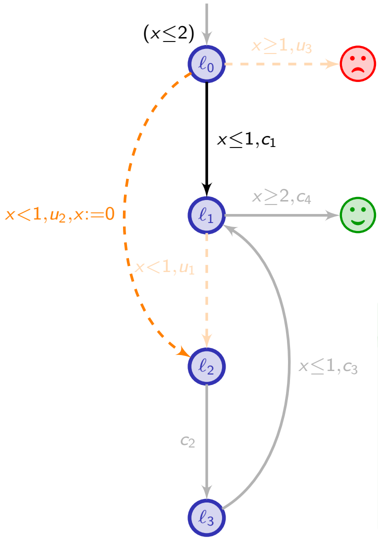

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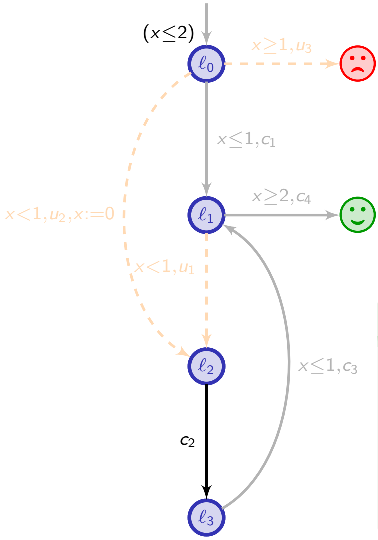

Rule of the game

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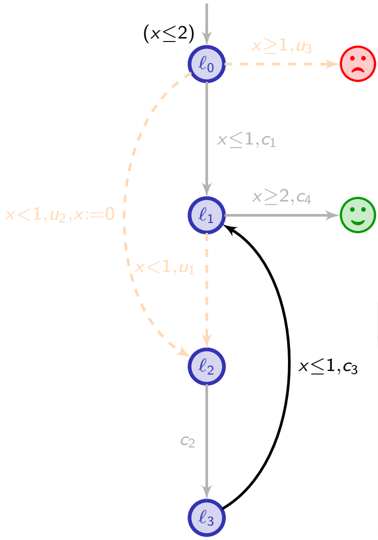

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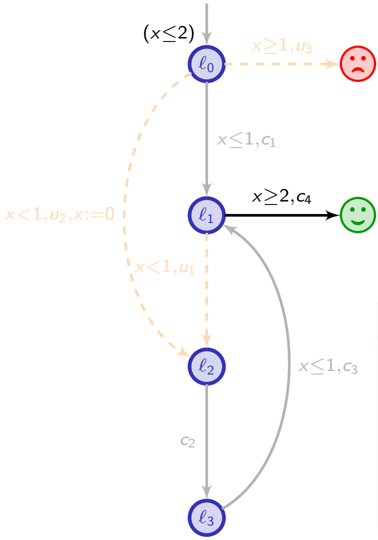

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- from $(\ell_1, 1)$, play $(1, c_4)$

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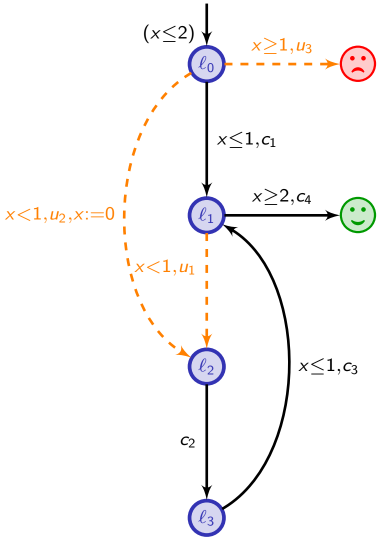

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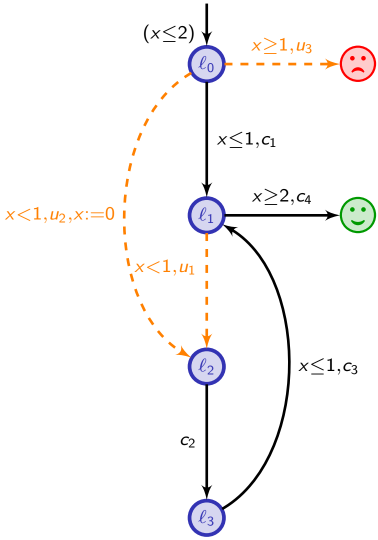

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Problems to be considered

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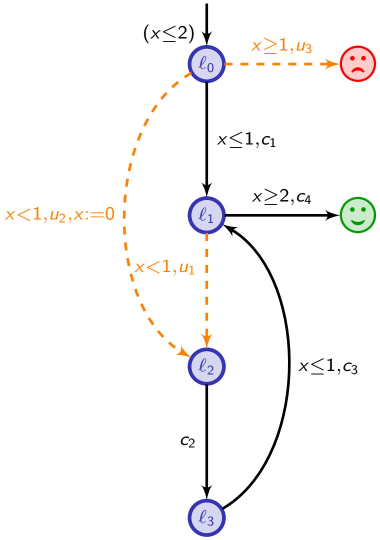

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Problems to be considered

- Does there exist a winning strategy?
- If yes, compute one (as simple as possible).

Decidability of timed games

Theorem [AMPS98,HK99]

Reachability and safety timed games are decidable and EXPTIME-complete. Furthermore memoryless and "region-based" strategies are sufficient.

[AMPS98] Asarin, Maler, Pnueli, Sifakis. Controller synthesis for timed automata *(SSC'98)*.
[HK99] Henzinger, Kopke. Discrete-time control for rectangular hybrid automata *(Theoretical Computer Science)*.

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Theorem [AM99,BHPR07,JT07]

Optimal-time reachability timed games are decidable and EXPTIME-complete.

[AM99] Asarin, Maler. As soon as possible: time optimal control for timed automata *(HSCC'99)*.
[BHPR07] Brihaye, Henzinger, Prabhu, Raskin. Minimum-time reachability in timed games *(ICALP'07)*.
[JT07] Jurdziński, Trivedi. Reachability-time games on timed automata *(ICALP'07)*.

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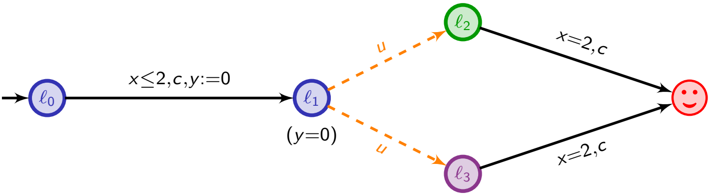

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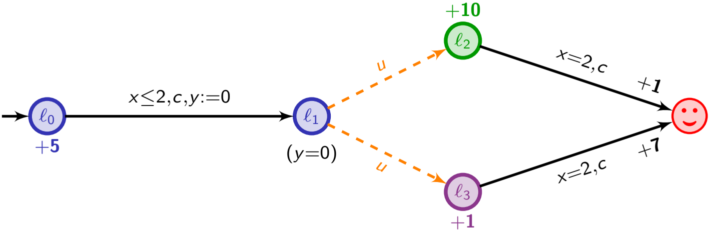

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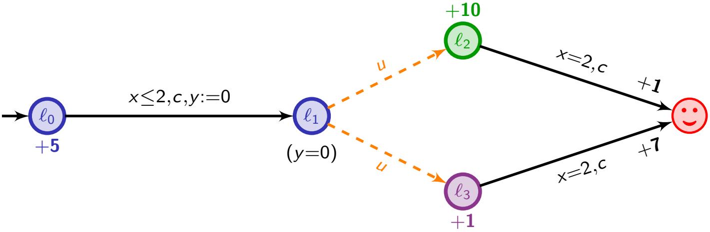
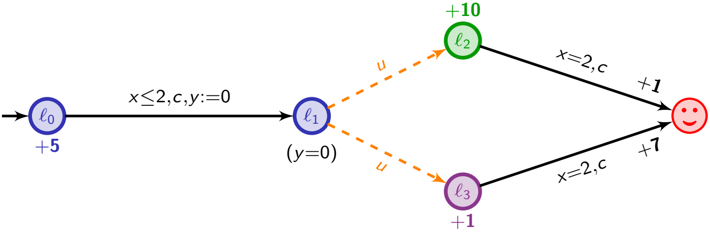


Question: what is the optimal cost we can ensure while reaching 🙂?

A simple weighted timed game



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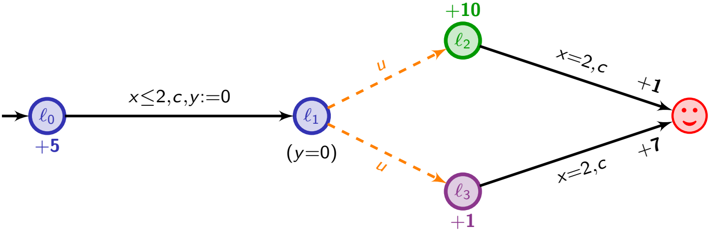
$$5t + 10(2 - t) + 1$$

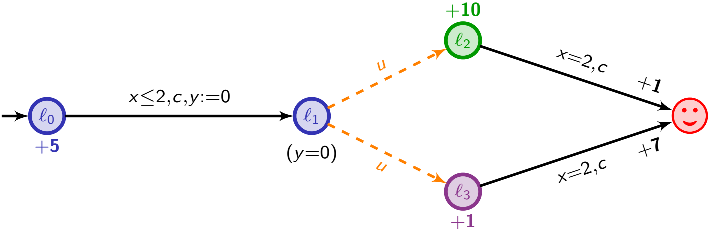
A simple weighted timed game



Question: what is the optimal cost we can ensure while reaching ☺ ?

$$5t + 10(2-t) + 1 \;,\; 5t + (2-t) + 7$$

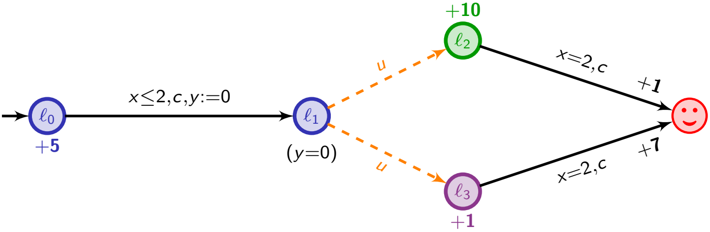
A simple weighted timed game



Question: what is the optimal cost we can ensure while reaching 🙂 ?

$$\max\ (\ 5t + 10(2-t) + 1\ ,\ 5t + (2-t) + 7\ )$$

A simple weighted timed game



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$$\inf_{0\leq t\leq 2} \max\big(\, 5t+10(2-t)+1 \,,\, 5t+(2-t)+7 \,\big) = 14+\frac{1}{3}$$

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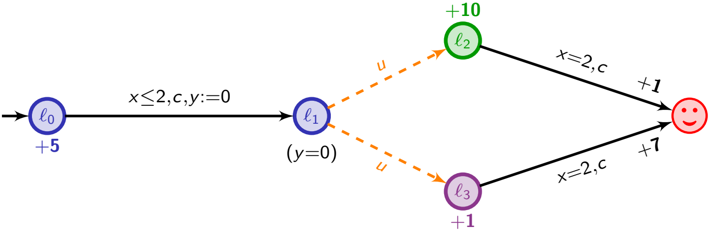

Question: what is the optimal cost we can ensure while reaching 🙂?

$$\inf_{0\leq t\leq 2} \max\left(5t+10(2-t)+1\ ,\ 5t+(2-t)+7\right) = 14+\frac{1}{3}$$

$\rightsquigarrow$ *strategy:* wait in ℓ_0, and when $t=\frac{4}{3}$, go to ℓ_1

Optimal reachability in weighted timed games (1)

This topic has been fairly hot these last fifteen years...

[LMM02,ABM04,BCFL04,BBR05,BBM06,BLMR06,Rut11,HIM13,BGK+14]

[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.
[ABM04] Alur, Bernardsky, Madhusudan. Optimal reachability in weighted timed games *(ICALP'04)*.
[BCFL04] Bouyer, Cassez, Fleury, Larsen. Optimal strategies in priced timed game automata *(FSTTCS'04)*.
[BBR05] Brihaye, Bruyère, Raskin. On optimal timed strategies *(FORMATS'05)*.
[BBM06] Bouyer, Brihaye, Markey. Improved undecidability results on weighted timed automata *(Information Processing Letters)*.
[BLMR06] Bouyer, Larsen, Markey, Rasmussen. Almost-optimal strategies in one-clock priced timed automata *(FSTTCS'06)*.
[Rut11] Rutkowski. Two-player reachability-price games on single-clock timed automata *(QAPL'11)*.
[HIM13] Hansen, Ibsen-Jensen, Miltersen. A faster algorithm for solving one-clock priced timed games *(CONCUR'13)*.
[BGK+14] Brihaye, Geeraerts, Krishna, Manasa, Monmege, Trivedi. Adding Negative Prices to Priced Timed Games *(CONCUR'14)*.

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[LMM02]

Tree-like weighted timed games can be solved in 2EXPTIME.

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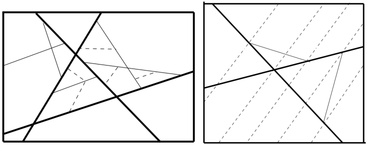
[LMM02]

Tree-like weighted timed games can be solved in 2EXPTIME.

[ABM04,BCFL04]

Depth-k weighted timed games can be solved in EXPTIME. There is a symbolic algorithm to solve weighted timed games **with a strongly non-Zeno cost**.



Optimal reachability in weighted timed games (2)

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In weighted timed games, the optimal cost cannot be computed, as soon as games have three clocks or more.

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Turn-based optimal timed games are decidable in EXPTIME (resp. PTIME) when automata have a single clock (resp. with two rates). They are PTIME-hard.

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Turn-based optimal timed games are decidable in EXPTIME (resp. PTIME) when automata have a single clock (resp. with two rates). They are PTIME-hard.

- Key: resetting the clock somehow resets the history...

Computing the optimal cost: why is that hard?

Given two clocks x and y, we can check whether $y = 2x$.

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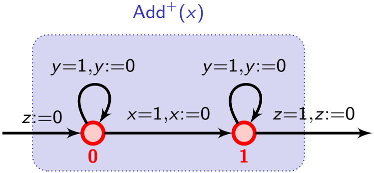
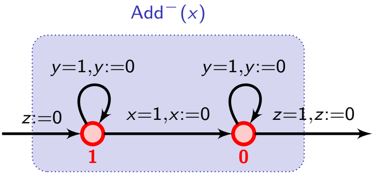


The cost is increased by x_0



The cost is increased by $1-x_0$

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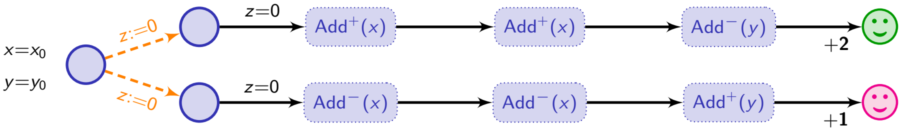

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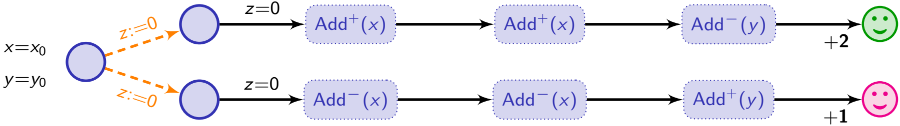

- In ☺, cost $= 2x_0 + (1 - y_0) + 2$

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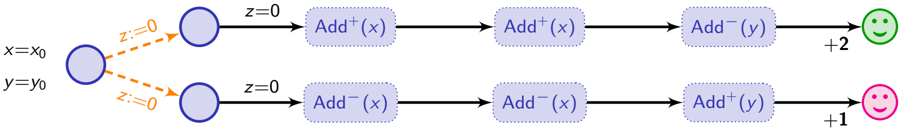

- In 🙂, cost $= 2x_0 + (1 - y_0) + 2$
  In 🙂, cost $= 2(1 - x_0) + y_0 + 1$

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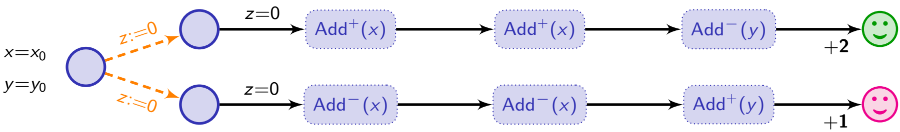

- In , cost $= 2x_0 + (1 - y_0) + 2$
  In , cost $= 2(1 - x_0) + y_0 + 1$
- if $y_0 < 2x_0$, player 2 chooses the first branch: cost > 3

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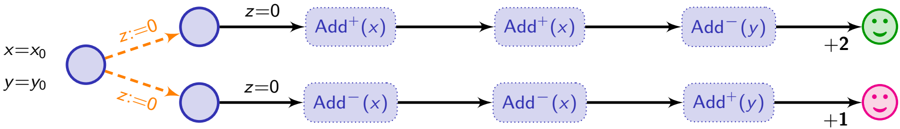

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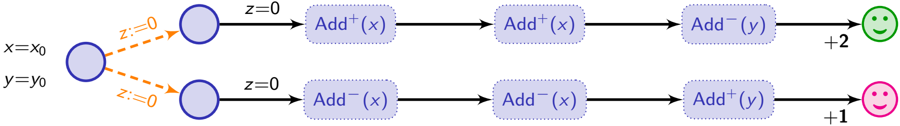

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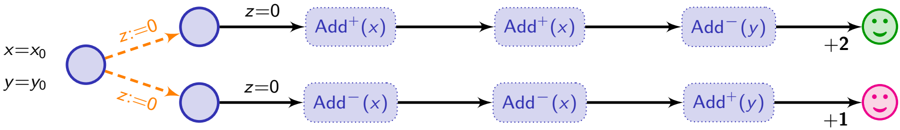

- In 🙂 (green), cost $= 2x_0 + (1 - y_0) + 2$
  In 🙂 (pink), cost $= 2(1 - x_0) + y_0 + 1$
- if $y_0 < 2x_0$, player 2 chooses the first branch: cost > 3
  if $y_0 > 2x_0$, player 2 chooses the second branch: cost > 3
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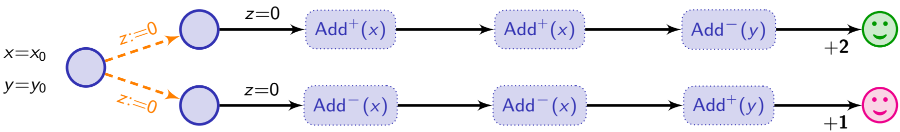
$\rightsquigarrow$ player 2 can enforce cost $3 + |y_0 - 2x_0|$

Computing the optimal cost: why is that hard?

Given two clocks x and y, we can check whether $y = 2x$.



- In 🙂 (green), cost $= 2x_0 + (1 - y_0) + 2$

  In 🙂 (pink), cost $= 2(1 - x_0) + y_0 + 1$

- if $y_0 < 2x_0$, player 2 chooses the first branch: cost > 3

  if $y_0 > 2x_0$, player 2 chooses the second branch: cost > 3

  if $y_0 = 2x_0$, in both branches, cost $= 3$

  $\rightsquigarrow$ player 2 can enforce cost $3 + |y_0 - 2x_0|$

- Player 1 has a winning strategy with cost ≤ 3 iff $y_0 = 2x_0$

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Player 1 will simulate a two-counter machine:

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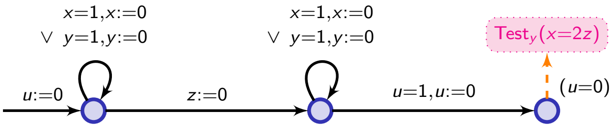
Globally, $(x\leq 1, y\leq 1, u\leq 1)$



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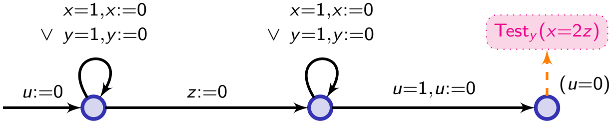

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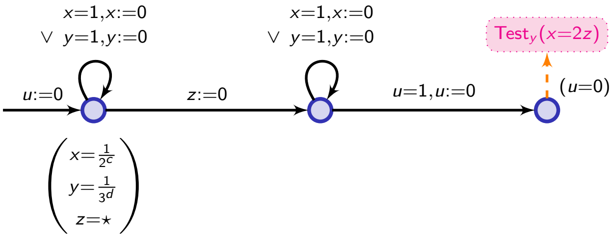

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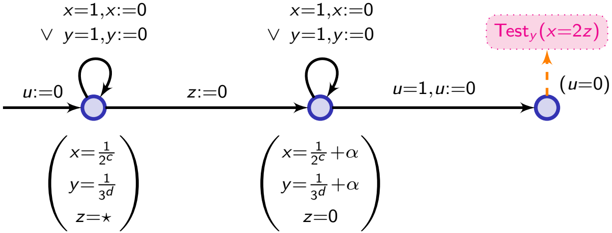

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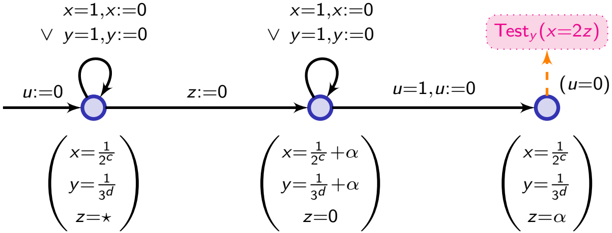

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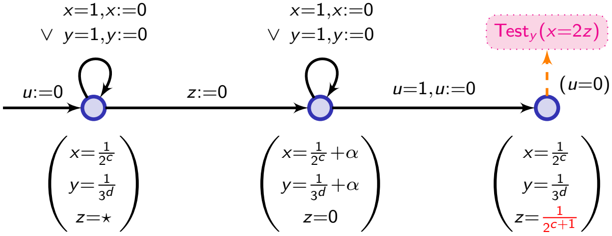

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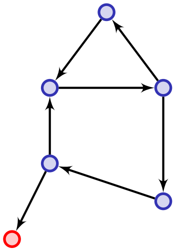

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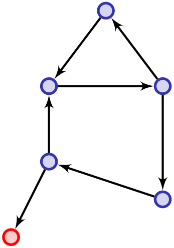

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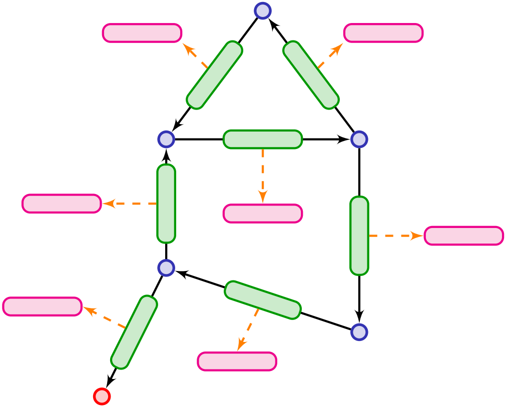

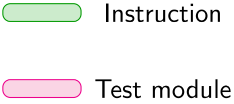

Are we done?

Outline

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Note: These problems are distinct...

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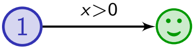
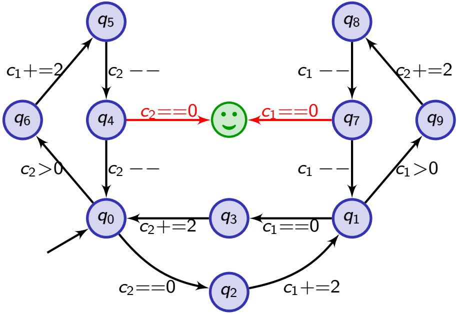


The value of the game is 3, but no strategy has cost 3.



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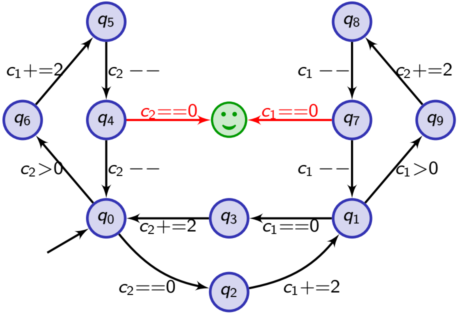

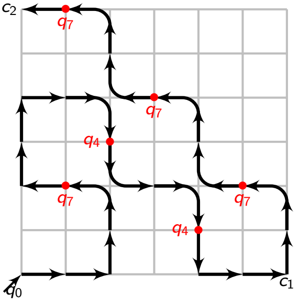

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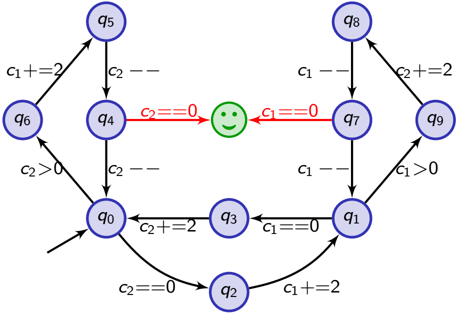
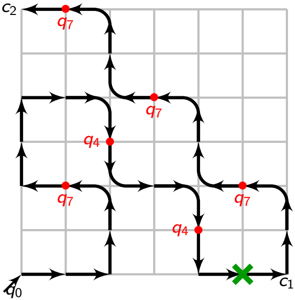
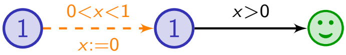




The value of the game is 1, but there is a strategy that secures cost < 1.



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The existence problem is undecidable in weighted timed games.

Outline of the rest of the talk

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[BMM14] Bouyer, Markey, Matteplackel. Averaging in LTL *(CONCUR'14)*.

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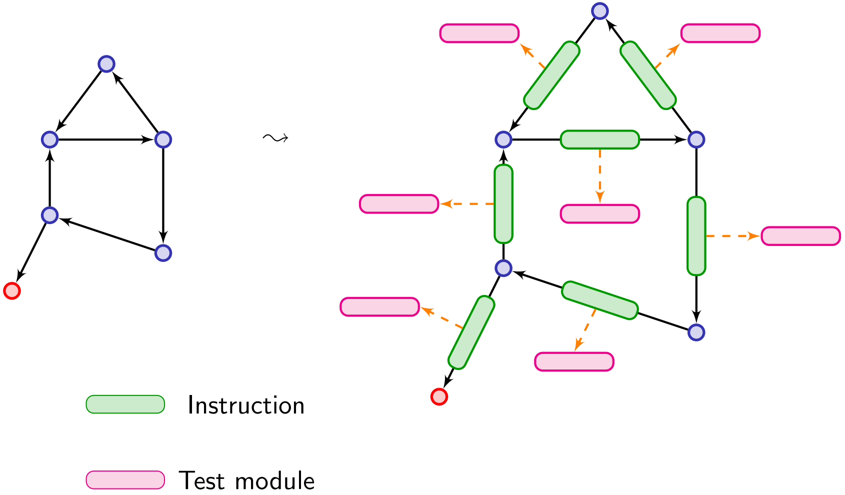
2. Propose an approximation algorithm for a large class of weighted timed games (that comprises the class of games used for proving the above undecidability)
   - Almost-optimality in practice should be sufficient
   - Even when we know how to compute the value, we are only able to synthesize almost-optimal strategies...

Outline

A snapshot on the undecidability proof



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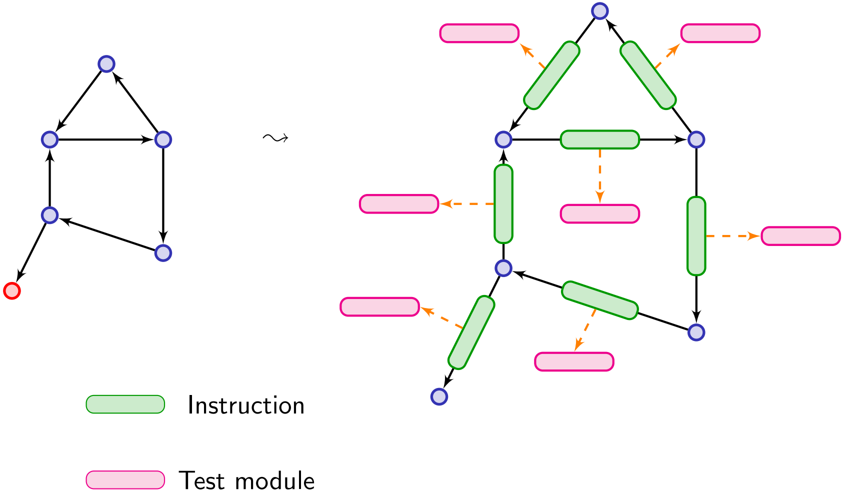

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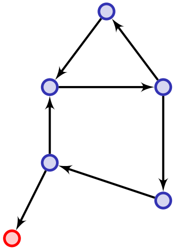

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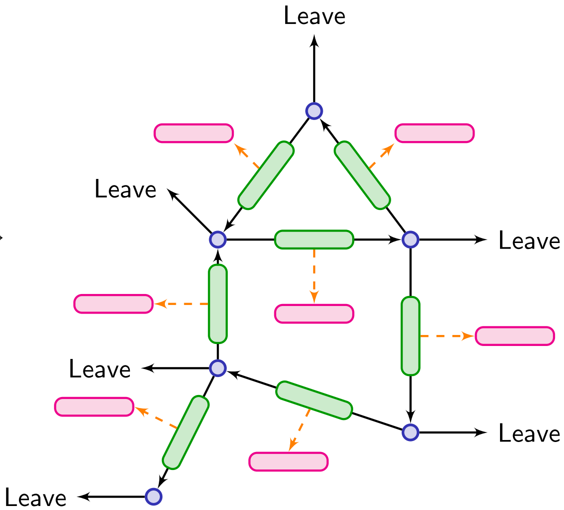

Leave with cost $3 + 1/2^n$ (n: length of the path)

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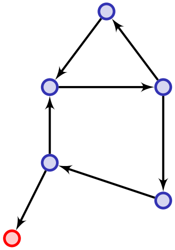
$\mathcal{M}$ does not halt iff the value of $\mathcal{G}_{\mathcal{M}}$ is 3



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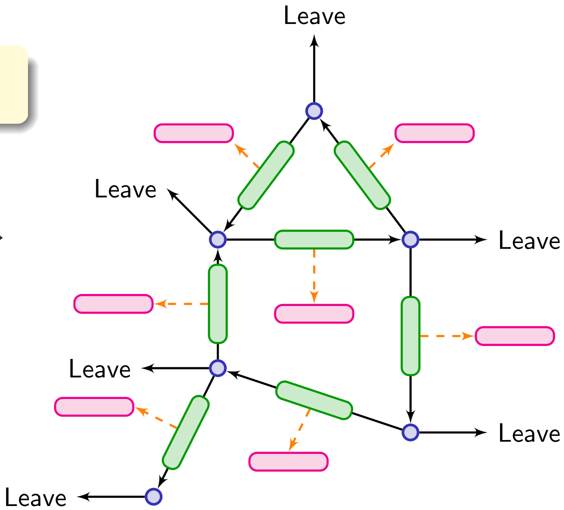

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Theorem [BJM15]

The value problem is undecidable in weighted timed games (with four clocks or more).

- Remark on the reduction:
  - Cost 0 within the core of the game
  - The rest of the game is acyclic

[BJM15] Bouyer, Jaziri, Markey. On the value problem in weighted timed games.

Outline

1. Introduction
2. Overview of "old" results
   - Weighted timed automata
   - Timed games
   - Weighted timed games
3. Some recent developments
   - Undecidability of the value problem
   - Approximation of the optimal cost
   - Back to the undecidability
4. Conclusion

Optimal cost is computable...

... when cost is strongly non-zeno. [AM04,BCFL04]

That is, there exists $\kappa > 0$ such that for every region cycle C, for every real run ϱ read on C,

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Let $\mathcal{G}$ be a weighted timed game, in which the cost is almost-strongly non-zeno. For every $\epsilon > 0$, one can compute:

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$\rightsquigarrow$ This is not possible here
There might be runs with prefixes of arbitrary length and cost 0 (e.g. the game of the undecidability proof)

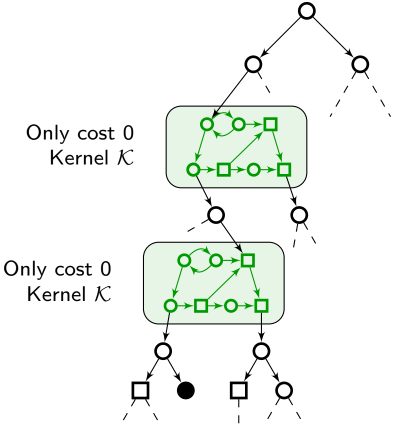
Idea for approximation

Idea

Only partially unfold the game:

- Keep components with cost 0 untouched – we call it the kernel
- Unfold the rest of the game

Semi-unfolding



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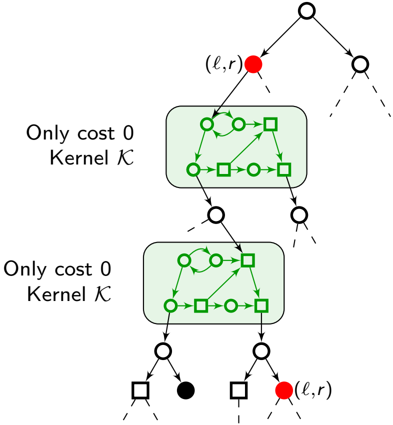

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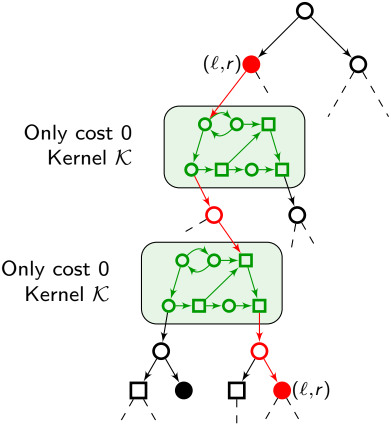
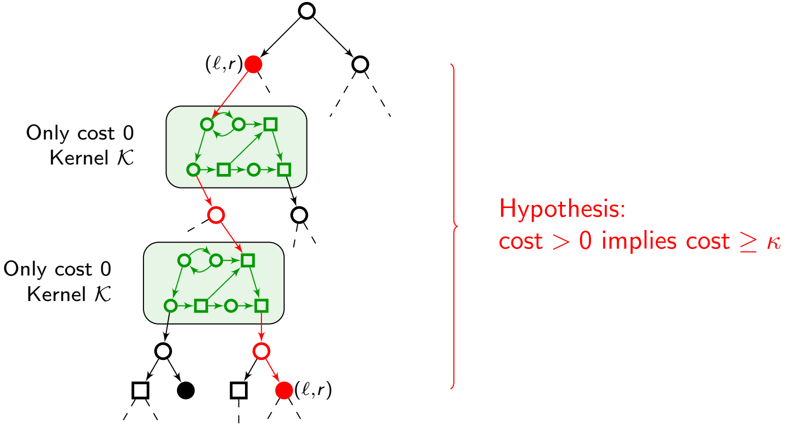


Semi-unfolding



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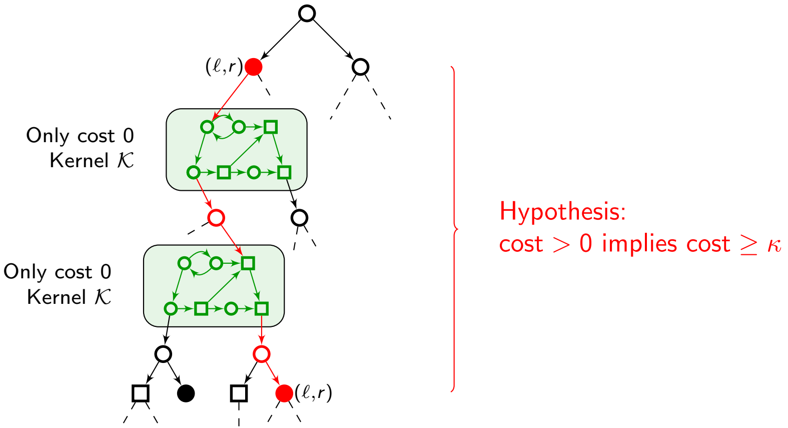
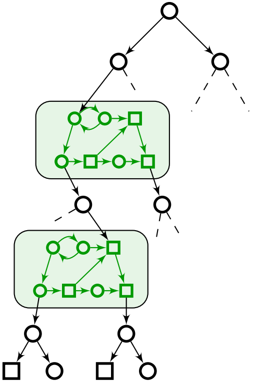
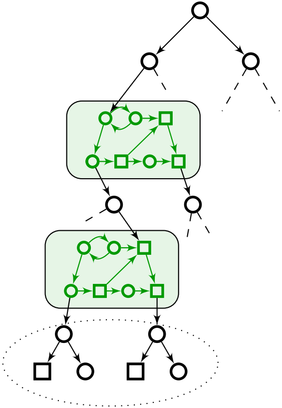


Conclusion: we can stop unfolding the game after N steps
(e.g. $N = (M+2) \cdot |\mathcal{R}(\mathcal{A})|$, where M is a pre-computed bound on $\mathrm{optcost}_{\mathcal{G}}$)

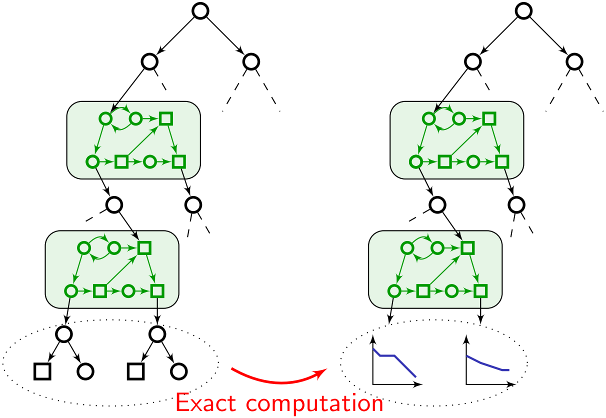
Approximation scheme



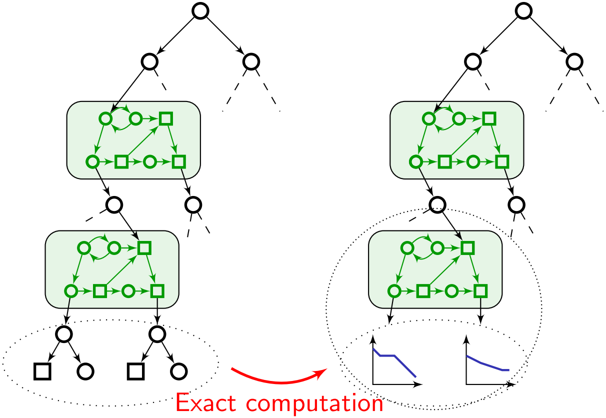
Approximation scheme



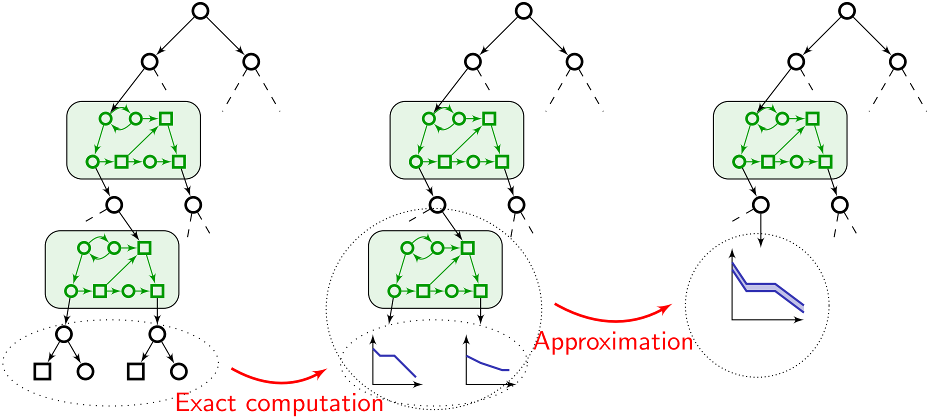
Approximation scheme



Approximation scheme



Approximation scheme



First step: Tree-like parts

$\rightsquigarrow$ Goes back to [LMM02]

[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.

First step: Tree-like parts

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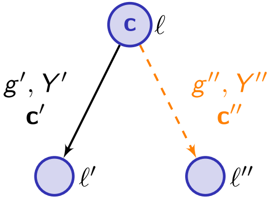
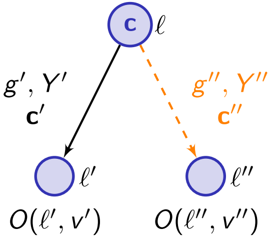


[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.

First step: Tree-like parts

$\rightsquigarrow$ Goes back to [LMM02]



$O(\ell, v) =$

[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.

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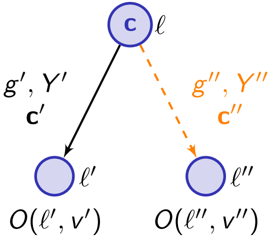

$$O(\ell, v) = \inf_{t' \mid v+t' \models g'}$$

[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.

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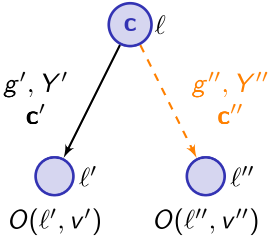

$$O(\ell, v) = \inf_{t' \mid v+t' \models g'} \max(\quad , \quad)$$

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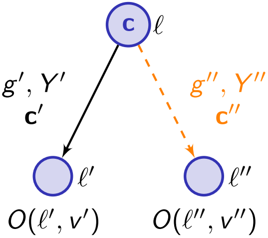


$$O(\ell, v) = \inf_{t' \mid v+t' \models g'} \max\big((\alpha), \quad\big)$$

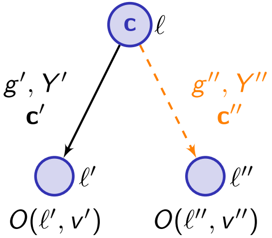
$$(\alpha) = t'\mathbf{c} + \mathbf{c}' + O(\ell', v')$$

$v' = [Y' \leftarrow 0](v+t')$

[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.

First step: Tree-like parts

$\rightsquigarrow$ Goes back to [LMM02]



$$O(\ell, v) = \inf_{t' \mid v+t' \models g'} \max\big((\alpha), (\beta)\big)$$

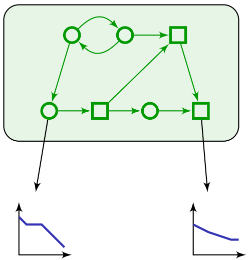
$$(\alpha) = t'\mathbf{c} + \mathbf{c}' + O(\ell', v')$$

$$(\beta) = \sup_{t'' \leq t' \mid v+t'' \models g''} t''\mathbf{c} + \mathbf{c}'' + O(\ell'', v'')$$

$$v' = [Y' \leftarrow 0](v+t')$$
$$v'' = [Y'' \leftarrow 0](v+t'')$$

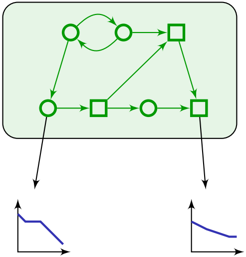
[LMM02] La Torre, Mukhopadhyay, Murano. Optimal-reachability and control for acyclic weighted timed automata *(TCS@02)*.

Second step: Kernels



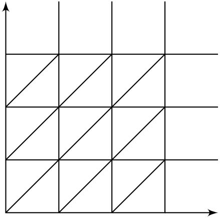
Output cost functions f

Second step: Kernels



Output cost functions f

1. Refine the regions such that f differs of at most ϵ within a small region



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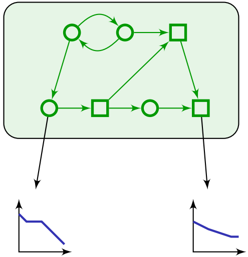

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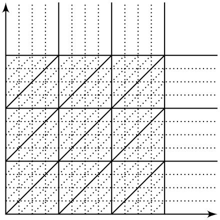

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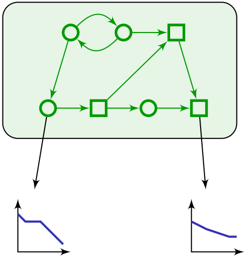

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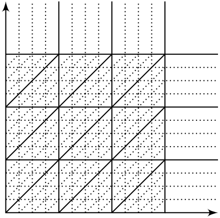

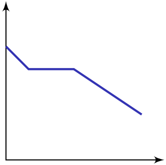

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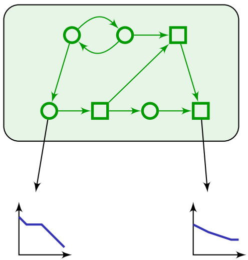

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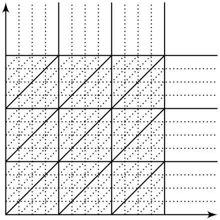
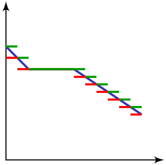
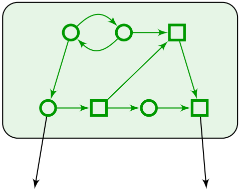


2. Under- and over-approximate by piecewise constant functions f_ϵ^- and f_ϵ^+



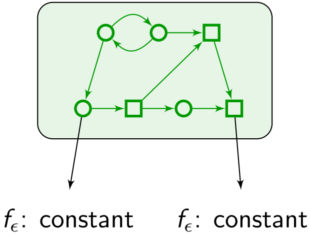
Second step: Kernels

3. Refine/split the kernel along the new small regions and fix f_ϵ^- or f_ϵ^+, write f_ϵ



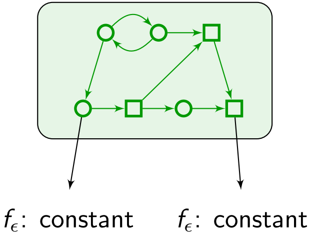
f_ϵ: constant f_ϵ: constant

Second step: Kernels



3. Refine/split the kernel along the new small regions and fix f_ϵ^- or f_ϵ^+, write f_ϵ
4. Since cost is 0 everywhere, the resulting game is nothing more than a reachability timed game with an order on target (output) edges (given by f_ϵ)

Second step: Kernels



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5. Those can be solved using standard technics based on attractors: small regions are sufficient, and the local optimal cost (for output f_ϵ) is constant within a small region

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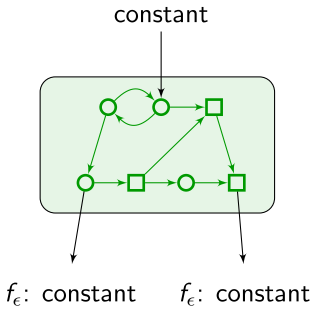

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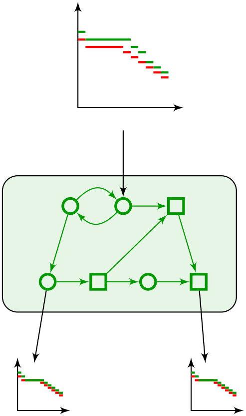

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$\rightsquigarrow$ We have computed ϵ-approximations of the optimal cost, which are constant within small regions. Corresponding strategies can be inferred

Outline

Consequence of the approximation algorithm

Theorem

The value problem is co-recursively enumerable (for almost-strongly non-zeno weighted timed games), but not recursively enumerable.

Outline

Conclusion

Summary of the talk

- Quick overview of results concerning the optimal reachability problem in weighted timed games
- New insight into the value problem for this model:
  - Undecidability of this problem
  - Approximability of the optimal cost (under some conditions)

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Future work

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- Extend to the whole class of weighted timed games? Or understand why it is not possible
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- Is the value of any game a rational number?
- Understand the multiplayer setting