Playing (Almost-)Optimally in Concurrent Büchi and co-Büchi Games

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Context

- Two-player games on graphs as a tool for formal verification (e.g. controller synthesis)
- Win/lose games: the objectives of the two players are opposite
- **Concurrent games**, as opposed to turn-based games
Concurrent games

$q_0, \begin{bmatrix} \bot & \top \end{bmatrix}$

« Matching-penny game »
Concurrent games

- Player A chooses a row

Matching-penny game
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- Player B chooses a column

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- Objective for Player A: $W \subseteq Q^\omega$

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- Objective for Player B: \( W^c \)

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- Winning strategy $\sigma_A$: all outcomes of $\sigma_A$ belong to $W$

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Major difference with turn-based games
Concurrent games

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Concurrent games

- Need for randomization!
- Randomized strategy: choose rows/columns according to a distribution

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Concurrent games

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- Randomized strategy: choose rows/columns according to a distribution
- Given randomized strategies $\sigma_A$ and $\sigma_B$, the **payoff** (for A) is the probability $\mathbb{P}_{\sigma_A,\sigma_B}(W)$
- Optimal strategy for A: $\sigma_A$ that maximizes $\inf_{\sigma_B} \mathbb{P}_{\sigma_A,\sigma_B}(W)$

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- There are optimal strategies for both players:
  - Player A: chooses uniformly at random a row

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- There are optimal strategies for both players:
  - Player A: chooses uniformly at random a row
  - Player B: chooses uniformly at random a column
- Value of the game: $\frac{1}{2}$
Properties of concurrent games

Martin’s determinacy theorem for Blackwell games

Concurrent games with Borel objectives have values:

\[ v(q) = \sup_{\sigma_A} \inf_{\sigma_B} \mathbb{P}_{\sigma_A, \sigma_B}(W) = \inf_{\sigma_B} \sup_{\sigma_A} \mathbb{P}_{\sigma_A, \sigma_B}(W) \]
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- Optimal strategies might not exist in general (except for safety objectives)
- (Infinite) Memory is sometimes needed by optimal and almost-optimal strategies
  - Parity games require infinite memory for both optimal and almost-optimal strategies
- Note: this is specific to concurrent games! (as compared to turn-based)
An example of a Büchi game

"The snowball game"

[AH00] L. De Alfaro, T. Henzinger. Concurrent omega-regular games (LICS’00)
An example of a Büchi game

- Objective is to visit $T$ infinitely often

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An example of a Büchi game

- Objective is to visit $T$ infinitely often
- Value of the game is $1$
- Player A (rows) has no optimal strat.
- Every finite-memory strat. has value $0$
- Player A needs infinite memory to play $\varepsilon$-optimal for every $\varepsilon > 0$:
  - Play first row with probability $1 - \varepsilon_k$ and second row with probability $\varepsilon_k$
  - $k$ is the number of visits to $T$
  - $(\varepsilon_k)_k$ quickly decreases to $0$

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- We are interested in **low memory requirements** for optimal and almost-optimal strategies in concurrent games with parity objectives in general, and more specifically Büchi and co-Büchi objectives
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- $\sigma_A$ is positional if it depends only on the last visited state.
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- Low memory requirement = **positional** strategies

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Our approach: focus on interactions, and characterize well-behaved interactions.
A tool to apprehend concurrent interactions: game forms
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Game form

\[
\begin{bmatrix}
x & y \\
y & z
\end{bmatrix}
\]
A tool to apprehend concurrent interactions: game forms

Elementary brick
A tool to apprehend concurrent interactions: game forms

Nice constructions ← Elementary brick → Nice bricks
A tool to apprehend concurrent interactions: game forms

Games on graphs with good properties

Game forms with good properties

Game form:

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
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Approach in this work
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- $\mathcal{I}$ set of game forms
Approach in this work

- \( \mathcal{I} \) set of game forms
- Identify properties of \( \mathcal{I} \) so that all concurrent games built using game forms \( \mathcal{I} \) behave well
Approach in this work

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Behave well = positional (almost-)optimal strategies are sufficient
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Behave well = positional (almost-)optimal strategies are sufficient
Previous works with a similar methodology

[BBL21] Bordais, Bouyer, Le Roux. From local to global determinacy in concurrent graph games (FSTTCS’21)
[BBL22] Bordais, Bouyer, Le Roux. Optimal Strategies in Concurrent Reachability Games (CSL’22)
Previous works with a similar methodology

- Determinacy of deterministic games [BBL21]
  - The matching-penny is not a good game form
  - Local determinacy condition on game forms

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- Reachability objectives [BBL22]
  - Optimal and almost-optimal strategies can be chosen positional (when they exist)
  - Local condition (called RM) on game forms to ensure existence (and therefore positionality) of optimal strategies everywhere

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What game theory tells us

- One can associate to each state $q$ of the game its value $v(q)$, and these values satisfy local optimality equations.
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![Diagram of a game in normal form]

Game in normal form:

$$\begin{bmatrix}
\frac{1}{2} & 1/4 \\
\frac{1}{2} & 3/4
\end{bmatrix}$$
What game theory tells us

- One can associate to each state $q$ of the game its value $v(q)$, and these values satisfy local optimality equations.

- Both players have (local) optimal strategies in this game in normal form.

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MinMax theorem (van Neumann)
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- Locally optimal strategies may not be globally optimal (in the graph).

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MinMax theorem (van Neumann)
Example

$q_0, \begin{bmatrix} q_0 & T & \perp \end{bmatrix}$
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Locally optimal strategy $\sigma_A$:
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Locally optimal strategy $\sigma_A$:
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What is wrong?
- In the MDP generated by $\sigma_A$, there is an end-component which is losing
Our contributions
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Characterize positional (almost-)optimal strategies using locally (almost-)optimal strategies (applies to tail objectives)
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Characterize positional (almost-)optimal strategies using locally (almost-)optimal strategies (applies to tail objectives)

Büchi objectives

- Optimal strategies may not exist (known)
- When optimal strategies exist from all states, they can be chosen positional (inherited from reachability games)
- Almost-optimal strategies may require infinite memory (known)
- Characterization of nice game forms (aBM) for ensuring:
  - Positional almost-optimal strategies
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Characterize positional (almost-)optimal strategies using locally (almost-)optimal strategies (applies to tail objectives)

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co-Büchi objectives

- Optimal strategies may not exist (known)
- When optimal strategies exist, they may require infinite memory
- Almost-optimal strategies can be chosen positional (known [CDAH06])
- Characterization of nice game forms (coBM) for ensuring:
  - Positional optimal strategies

Our contributions

Characterize positional (almost-)optimal strategies using locally (almost-)optimal strategies (applies to tail objectives)

**Büchi objectives**
- Optimal strategies may not exist (known)
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**co-Büchi objectives**
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Recall: there exist Büchi games where infinite memory is required to play $\varepsilon$-optimally.
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How should we restrict interactions to avoid this phenomenon?
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$$\mathcal{F} = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$$
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$q_0, \begin{bmatrix} q_0 & T & \bot \\ T & \bot & \bot \end{bmatrix}$

$\mathcal{F} = \begin{bmatrix} \bot & \bot & \bot \\ \bot & \bot & \bot \end{bmatrix}$
Focus on Büchi conditions

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x & y \\
y & z
\end{bmatrix}
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Focus on Büchi conditions

\[
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Several local « environments »
Focus on Büchi conditions

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\begin{bmatrix}
  x & y \\
  y & z
\end{bmatrix}
\]

Several local « environments »

\[
\begin{bmatrix}
  \overline{T} & T \\
  T & 0
\end{bmatrix}
\]

Target

Not target
Focus on Büchi conditions

\[
\begin{bmatrix}
x & y \\
y & z
\end{bmatrix}
\]

Several local «environments»

Payoff if the game proceeds to here

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Not target
Focus on Büchi conditions

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Several local « environments »

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Several local « environments »

Payoff if the game proceeds to here

Target

Not target
Focus on Büchi conditions

Several local « environments »

Local environment

- $O$ set of variables ($\{x, y, z\}$ in the example)
- One small game
  - for every $E \subseteq O$, $p_T : E \rightarrow [0,1]$, and
  - for every $\alpha : O \setminus E \rightarrow [0,1]$

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Payoff if the game proceeds to here
Definition of aBM (almost-Büchi maximizable)

- A game form $\mathcal{F}$ is aBM whenever every embedding of $\mathcal{F}$ into a local environment admits a positional $\varepsilon$-optimal strategy for every $\varepsilon > 0$. 
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An aBM game form can be characterized and decided (it can be encoded as a formula of the first-order theory of the reals).
Characterization

Definition of aBM (almost-Büchi maximizable)

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Characterization

- If all game forms used in a concurrent game $\mathcal{G}$ are aBM, then $\mathcal{G}$ admits positional $\varepsilon$-optimal strategies for every $\varepsilon > 0$.
- If a game form is not aBM, then there is a concurrent game which does not admit a positional $\varepsilon$-optimal strategy for some $\varepsilon > 0$. 
How to ensure positional (almost-)optimal strategies?

Existence of positional optimal or $\epsilon$-optimal strategies under the following restrictions on game forms:

<table>
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<tr>
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<td>No restr.</td>
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If game forms satisfy the properties below, then positional strategies exist and can be chosen positional.
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If game forms at states not in target are coBM and in targets are RM, then optimal strategies exist and can be chosen positional.
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- If game forms at states not in target are coBM and in targets are RM, then optimal strategies exist and can be chosen positional.
- If game forms at states not in target are aBM then $\varepsilon$-optimal strategies can be chosen positional.
How to ensure positional (almost-)optimal strategies?

Existence of positional optimal or $\varepsilon$-optimal strategies under the following restrictions on game forms:

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If game forms satisfy the properties below, then positional strategies exist and can be chosen positional.

- If game forms at states not in target are coBM and in targets are RM, then optimal strategies exist and can be chosen positional.
- $\varepsilon$-optimal strategies can always be chosen positional.
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- coBM ⊆ RM ⊆ aBM
Properties of game forms

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- coBM \( \subseteq \) RM \( \subseteq \) aBM

- These game forms are coBM:
  - « Turn-based » game forms:
    
    \[
    \begin{bmatrix}
    x & y & z \\
    x & y & z \\
    \end{bmatrix}
    \]

  - Two-variable game forms:
    
    \[
    \begin{bmatrix}
    x & y & x \\
    y & x & x \\
    \end{bmatrix}
    \]

  - Permutation game forms:
    
    \[
    \begin{bmatrix}
    x & y & z \\
    z & x & y \\
    y & z & x \\
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    \]
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- Going further:
  - Understand beyond (co-)Büchi conditions, e.g. parity conditions
  - (Ongoing work) A different approach, which should be able to deal with parity conditions