Complexité avancée - TD 6

Benjamin Bordais

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We recall the definition of RP, coRP and BPP. A language L is in RP if there exists a Turing machine $\mathcal{M}$ running in polynomial time $p(n)$ on all input x such that $|x| = n$ and random tape r of size $p(n)$ such that:

- If $x \in L$, then $Pr_r[\mathcal{M}(x,r) = \top] \geq 1/2$;
- If $x \notin L$, then $Pr_r[\mathcal{M}(x,r) = \top] = 0$.

Similarly, a language L is in coRP if there exists such a Turing machine $\mathcal{M}$ ensuring:

- If $x \in L$, then $Pr_r[\mathcal{M}(x,r) = \top] = 1$;
- If $x \notin L$, then $Pr_r[\mathcal{M}(x,r) = \top] \leq 1/2$.

Finally, a language L is in BPP if there exists such a Turing machine $\mathcal{M}$ ensuring:

- If $x \in L$, then $Pr_r[\mathcal{M}(x,r) = \top] \geq 2/3$;
- If $x \notin L$, then $Pr_r[\mathcal{M}(x,r) = \top] \leq 1/3$.

Exercise 1 One-Minute Long Exercise

Between RP and coRP which language is "No-means-No" which language is "Yes-means-Yes"?

Exercise 2 Expected Running Time

Given a probabilistic Turing Machine M, not necessarily halting, let $T_M(x,r)$ be the random variable describing the running time of M on input x and random tape r (take $T_M(x,r) = +\infty$ if M does not halt on x,r). That is for all x, $Pr[T_M(x,r) = T]$ is the probability, taken over all possible (infinite) random tape contents, that M on input x halts after exactly T steps.

The expected running time of M on input x is the expectation $E[T_M(x,r)]$.

Consider the definitions of RP and BPP: here the Turing machines considered are required to halt in time at most n^c steps for some $c \geq 1$ on all inputs and for all possible random tape strings (worst case running time). Define RP^E and BPP^E as RP and BPP, but replacing the worst case running time with the expected running time.

Formally:

- $\mathsf{RP}^E = \bigcup_{c \in \mathbb{N}} \mathsf{RTIME}^E(n^c, 0, 1/2)$
- $\mathsf{BPP}^E = \bigcup_{c \in \mathbb{N}} \mathsf{RTIME}^E(n^c, 1/3, 2/3)$

where $\mathsf{RTIME}^E(n^c, p_{acc}, p_{rej})$ is the class of languages L for which there exists a probabilistic Turing machine M (which may not halt) such that, for each input x of size n:

- $Pr[T_M(x,r) = +\infty] = 0$;
- $E[T_M(x,r)] \leq |x|^c$;
- if $x \in L$ then $Pr[M(x,r)] = \top] \geq p_{rej}$;
- if $x \notin L$ then $Pr[M(x,r)] = \top] \leq p_{acc}$.

Show that $\mathsf{RP}^E = \mathsf{RP}$ and $\mathsf{BPP}^E = \mathsf{BPP}$.

Exercise 3 BPP and PSPACE

- Argue that $\mathsf{BPP}(1/2) = \{$ all languages $\}$ and $\mathsf{BPP} = \mathsf{coBPP}$.
- Give a direct proof that $\mathsf{BPP} \subseteq \mathsf{PSPACE}$.

Exercise 4 Probabilistic Logarithmic Space

Propose a definition of $\mathsf{RSPACE}(f(n), p_{acc}, p_{rej})$.
Consider $\mathsf{RL} = \bigcup_{k \in \mathbb{N}} \mathsf{RSPACE}(k \cdot log(n), 0, 1/2)$ the class of languages that can be decided in probabilistic logarithmic space (the machine does not necessarily halt).

Show that:

1. Consider L in RL and $\mathcal{M}$ a probabilistic Turing machine which decides L. If $x \notin L$, then $\forall r$, $M(x,r) \neq \top$
2. $\mathsf{RL} \subseteq \mathsf{NL}$
3. $\mathsf{RL} \subseteq \mathsf{RP}$

Exercise 5 Dunno Machine

Define a ?-probabilistic Turing machine as a probabilistic Turing machine that halts on all inputs but with three final states: an accepting state, a rejecting state and a dunno state. Given x an input and r a random tape content, we note $M(x,r) = \top$ (resp. $\perp$, resp. ?) if the computation of M on x with random tape r accepts (resp. rejects, resp. ends in the dunno state).

We define the probabilistic complexity class $?\mathsf{PP}$ as follows:

$L \in ?\mathsf{PP}$ if and only if there exists a ?-probabilistic Turing machine M working in (worst case) time $p(n)$, with random tape size $p(n)$ (for some polynomial p) and such that:

- for all x, $Pr[M(x,r) = ?] \leq \frac{1}{2}$
- if $x \in L$ then $Pr[M(x,r) = \perp] = 0$
- if $x \notin L$ then $Pr[M(x,r) = \top] = 0$

How does this class relate to the classical probabilistic complexity classes?