Closure under morphisms

Given a finite alphabet $\Sigma$, a function $f : \Sigma^* \rightarrow \Sigma^*$ is a morphism if $f(\Sigma) \subseteq \Sigma$ and for all $a = a_1 \cdots a_n \in \Sigma^*$, $f(a) = f(a_1) \cdots f(a_n)$ ($f$ is uniquely determined by the value it takes on $\Sigma$).

Show that $P = \text{NP}$ if and only if $P$ is closed under morphism.

Solution:

• Assume that $P = \text{NP}$. Consider $f$ a morphism and $L \in P = \text{NP}$. Let us show that $f(L) \in \text{NP} = P$. We consider a non-deterministic Turing machine $M$ that, on an input $w \in \Sigma^*$, guesses a word $a \in \Sigma^*$ such that $|a| = |w|$ and then checks that $f(a) = w$ and that $a \in L$ in polynomial time. It follows that $f(L) \in \text{NP} = P$ and $P$ is closed under morphisms.

• Now, assume that $P$ is closed under morphism. We show that $\text{SAT} \in P$, which proves that $\text{NP} \subseteq P$ since $\text{SAT}$ is $\text{NP}$-complete for logspace reductions and $P$ is closed under logarithmic space reductions. Consider the following language:

$$L = \{(\phi, v) \mid v \text{ is a valuation satisfying } \phi\}$$

We have that $L \in P$ as one can check in polynomial time that a valuation satisfies a boolean formula. Furthermore, we can assume that the alphabet $\Sigma$ is equal to the disjoint union $\Sigma_\phi \cup \Sigma_v$ and the symbols used to encode $\phi$ (resp. $v$) are in $\Sigma_\phi$ (resp. $\Sigma_v$). Then, if we consider the morphism $f$ that ensures $f(a) = a$ for all $a \in \Sigma_\phi$ and $f(a) = 0$ for all $a \in \Sigma_v$. Then,

$$f(L) = \{(\phi, 0^n) \mid \phi \text{ has } n \text{ variables and is satisfiable}\}$$

By closure under morphism, it follows that $f(L) \in P$. Since, an instance of $\text{SAT}$ can be reduced in polynomial time (in fact, in logarithmic space) to an instance of $f(L)$, it follows that $\text{SAT} \in P$. Hence, $P = \text{NP}$.