

# Complexité avancée - Homework 4

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Due at 8.30 a.m., October 21, 2020

**Closure under morphisms** Given a finite alphabet  $\Sigma$ , a function  $f : \Sigma^* \rightarrow \Sigma^*$  is a morphism if  $f(\Sigma) \subseteq \Sigma$  and for all  $a = a_1 \cdots a_n \in \Sigma^*$ ,  $f(a) = f(a_1) \cdots f(a_n)$  ( $f$  is uniquely determined by the value it takes on  $\Sigma$ ).

Show that  $P = NP$  if and only if  $P$  is closed under morphism.

## Solution:

- Assume that  $P = NP$ . Consider  $f$  a morphism and  $L \in P = NP$ . Let us show that  $f(L) \in NP = P$ . We consider a non-deterministic Turing machine  $M$  that, on an input  $w \in \Sigma^*$ , guesses a word  $a \in \Sigma^*$  such that  $|a| = |w|$  and then checks that  $f(a) = w$  and that  $a \in L$  in polynomial time. It follows that  $f(L) \in NP = P$  and  $P$  is closed under morphisms.
- Now, assume that  $P$  is closed under morphism. We show that  $SAT \in P$ , which proves that  $NP \subseteq P$  since  $SAT$  is NP-complete for logspace reductions and  $P$  is closed under logarithmic space reductions. Consider the following language:

$$L = \{(\phi, v) \mid v \text{ is a valuation satisfying } \phi\}$$

We have that  $L \in P$  as one can check in polynomial time that a valuation satisfies a boolean formula. Furthermore, we can assume that the alphabet  $\Sigma$  is equal to the disjoint union  $\Sigma_\phi \uplus \Sigma_v$  and the symbols used to encode  $\phi$  (resp.  $v$ ) are in  $\Sigma_\phi$  (resp.  $\Sigma_v$ ). Then, if we consider the morphism  $f$  that ensures  $f(a) = a$  for all  $a \in \Sigma_\phi$  and  $f(a) = 0$  for all  $a \in \Sigma_v$ . Then,

$$f(L) = \{(\phi, 0^n) \mid \phi \text{ has } n \text{ variables and is satisfiable}\}$$

By closure under morphism, it follows that  $f(L) \in P$ . Since, an instance of  $SAT$  can be reduced in polynomial time (in fact, in logarithmic space) to an instance of  $f(L)$ , it follows that  $SAT \in P$ . Hence,  $P = NP$ .