Exercise 1 Space hierarchy theorem

Consider two space-constructible functions $f$ and $g$ such that $f(n) = o(g(n))$. Prove that $\text{DSPACE}(f) \subsetneq \text{DSPACE}(g)$.

Hint: You may consider a language $L = \{(M, w') \mid \text{the simulation of } M \text{ on } (M, w') \text{ rejects}\}$ with an appropriate restriction on the simulation of $M$.

Exercise 2 Polylogarithmic space

1. Let $\text{polyL} = \bigcup_{k \in \mathbb{N}} \text{SPACE}(\log^k)$. Show that $\text{polyL}$ does not have a complete problem for logarithmic space reduction.\(^1\)

2. We recall that $\text{PSPACE} = \bigcup_{k \in \mathbb{N}} \text{SPACE}(n^k)$. Does $\text{PSPACE}$ have a complete problem for logarithmic space reduction? Why doesn’t the proof of the previous question apply to $\text{PSPACE}$?

Exercise 3 Padding argument

1. Show that if $\text{DSPACE}(n^c) \subseteq \text{NP}$ for some $c > 0$, then $\text{PSPACE} \subseteq \text{NP}$.

   Hint: for $L \in \text{DSPACE}(n^k)$ one may consider the language $\tilde{L} = \{(x, 1|x|^{k/c}) \mid x \in L\}$.

2. Deduce that $\text{DSPACE}(n^c) \neq \text{NP}$.

\(^1\)From this, we can deduce that $\text{polyL} \neq \text{P}$.