Complexité avancée - Homework 3

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PSPACE and games The Geography game is played as follows:

- The game starts with a given name of a city, for instance *Cachan*;
- the first player gives the name of a city whose first letter coincides with the last letter of the previous city, for instance *Nice*;
- the second player gives then another city name, also starting with the last letter of the previous city, for instance *Evry*;
- the first player plays again, and so on – with the restriction that no player is allowed to give the name of a city already used in the game;
- the loser is the first player who does not find a new city name to continue.

This game can be described using a directed graph whose vertices represent cities and where an edge (X, Y) means that the last letter of the city X is the same as the first letter of the city Y. This graph has also a vertex marked as the initial vertex of the game (the initial city). Each player chooses a vertex of the graph, the first player choses first, and the two players alternate their moves. At each move, the sequence of vertices chosen by the two players must form a simple path in the graph (i.e.: a path with no cycles), starting from the distinguished initial vertex.

Player 1 wins the game if, after some number of moves, Player 2 has no valid move (that is no move that forms a simple path with the sequence of previous moves).

Generalized Geography (GG for short) is the following problem:

- INPUT: a directed graph G and an initial vertex s.
- QUESTION: does Player 1 have a winning strategy for a GG game played on G from s?

1. Show that GG is in PSPACE.
2. Exhibit a logarithmic space reduction tr from QBF to GG. Carefully prove that the reduction is logspace and the equivalence "$w \in \mathsf{QBF} \Leftrightarrow tr(w) \in \mathsf{GG}$".
3. What can you deduce about GG?

Solution:

1. We define a recursive function $\mathsf{win} : (G, s, F) \mapsto$ "True iff the 1st player has a winning strategy on the graph $G = (V, E)$, starting from $s \in V$, for a version of the game where it is forbidden to play a vertex in $F \subseteq V$". This is equivalent to: "the 1st player has a winning strategy on the restriction $G_{|V \setminus F}$." Note that in the specification of win, it is required that $s \notin F$.

   Now win is a simple recursive procedure:

   $$\mathsf{win}(G, s, F) = \exists (s, t) \in E \text{ s.t. } t \notin F' \wedge \mathsf{win}(G, t, F') = \mathsf{False} \text{ for } F' = F \cup \{s\}.$$

   An algorithm implementing this procedure explores every possible successors and calls itself recursively. It terminates since each call has a larger F and when F is equal to the whole set of vertices V, it stops. Therefore, there can be at most $|V|$ nested calls so the program will use a stack with a linear number of frames, each frame being of linear size (as one stores the set F and the current vertex s). This is polynomial space.

2. We construct a logarithmic space reduction tr from an instance of QBF to an instance of GG so that $\phi \in \mathsf{QBF} \Leftrightarrow tr(\phi) \in \mathsf{GG}$. Consider a QBF formula $\phi = Q_1 x_1 \cdot Q_2 x_2 \cdots Q_n x_n \cdot S$ where S is a propositional formula in CNF whose variable are in $\{x_1, \ldots x_n\}$ and $Q_i \in \{\exists, \forall\}$ for all $1 \leq i \leq n$. We first translate ϕ into an equivalent formula such that we have a strict alternation of existential and universal quantficators (it may require to add dummy variables that do not appear in ϕ). We obtain $\phi' = \exists y_1 \cdot \forall y_2 \cdots \exists y_{2k-1} \cdots \forall y_{2k} \cdot S$ for some $k \leq n$. Let $S = \wedge_{1 \leq j \leq m} C_j$ with $C_j = \vee_{1 \leq i \leq a_j} l_{i,j}$ where $l_{i,j}$ is a literal that is equal to y_l or $\neg y_l$ for some $1 \leq l \leq 2k$.

   We formally define $tr(\phi)$ as the graph $G_\phi = (V, E)$ where $V = \{s_i, y_i, \neg y_i, s'_i \mid 1 \leq i \leq 2k\} \cup \{C_j \mid 1 \leq j \leq m\} \cup \{l^c_{i,j} \mid 1 \leq j \leq m,\ 1 \leq i \leq 3\}$ and $E = \{(s_i, y_i), (s_i, \neg y_i), (y_i, s'_i), (\neg y_i, s'_i) \mid 1 \leq i \leq 2k\} \cup \{(s'_i, s_{i+1}) \mid 1 \leq i \leq 2k-1\} \cup \{(s'_{2k}, C_j) \mid 1 \leq j \leq m\} \cup \{(C_j, l^c_{i,j}) \mid 1 \leq j \leq m,\ 1 \leq i \leq a_j\} \cup \{(l^c_{i,j}, l_{i,j}) \mid 1 \leq j \leq m,\ 1 \leq i \leq a_j\}$. The initial vertex of the graph is s_1. Note that, for all i, Player 1 plays in the vertices s_i and s'_i if and only if i is odd. It is also Player 1's turn in vertex C_j for all $1 \leq j \leq m$.

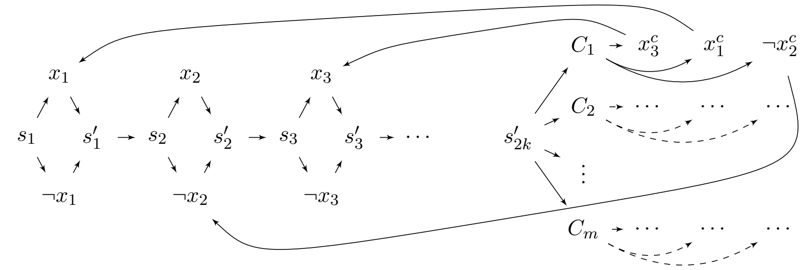

Figure 1: The graph G_ϕ for $\phi' = \exists x_1 \forall x_2 \cdots \forall x_{2k} \Big((x_3 \vee x_1 \vee \neg x_2) \wedge C_2 \wedge \cdots \wedge C_m \Big)$.

First note that this graph can be constructed in logarithmic space. Indeed, to do so, we need a fix number of pointers ranging over the variables and the clauses of the formula.

Now, let us prove that $\phi \in \mathsf{QBF}$ if and only if the first player has a winning strategy in the game of generalized geography G_ϕ. First, we define inductively the path corresponding to a valuation for the first $0 \leq l \leq 2k$ variables: $p(\emptyset) = \epsilon$ and $p(\nu' = \nu \cdot \{y_l \to \_\}) = p(\nu) \cdot s_l \cdot a \cdot s'_l$ with a the vertex/literal either equal to y_l or $\neg y_l$ ensuring $\nu' \models a$. Then, for a valuation ν of the first $1 \leq l \leq 2k$ variables, when we say that the game starts after the path $p(\nu)$ is seen, it means that the vertices in $p(\nu)$ are forbidden, the inital vertex is s_{l+1} and it is Player 1's turn if and only if $l+1$ is odd. In the case where $l = 2k$, then the game may start in any vertex c_j (according to Player 2's choice in s'_{2k}) at Player 1's turn.

We prove by induction the following property on $1 \leq i \leq 2k+1$, $\mathcal{P}(i)$: "for all valuation ν of the variables $y_1, \ldots, y_{i-1}$, we have $\nu \models Q_i y_i \cdots \forall y_{2k} \cdot S$ if and only if Player 1 wins in the game that starts after $p(\nu)$ is seen".

Let us first prove $\mathcal{P}(2k+1)$. Consider a valuation ν of the variables. Assume ν satisfies S and that the game starts at c_j for some $1 \leq j \leq m$ after $p(\nu)$ is seen. Let us prove that Player 1 wins. Since $\nu \models S$, there exists a literal $l_{i,j}$ such that $\nu \models l_{i,j}$ for some $1 \leq i \leq a_j$. Assume that Player 1 chooses $l^c_{i,j}$ as next vertex. Then, the only possible successor is $l_{i,j}$ and by definition of $p(\nu)$, since $\nu \models l_{i,j}$, it is forbidden in the graph. Hence, Player 1 wins. Now, assume that Player 1 wins from all vertices $\{c_j \mid 1 \leq j \leq m\}$ after $p(\nu)$ is seen. Let us prove that $\nu \models S$. Consider a clause C_j for some $1 \leq j \leq m$ and the choice made by Player 1 in vertex c_j. The vertex $l^c_{i,j}$ is reached for some $1 \leq i \leq a_j$. Since Player 1 wins, it follows that the vertex $l_{i,j}$ is forbidden, otherwise Player 2 would chose it as next vertex and Player 1 loses since, in any case, the successor of $l_{i,j}$, that is s'_j, is forbidden. By definition of $p(\nu)$, the fact that the vertex $l_{i,j}$ is forbidden means that the valuation ν ensures $\nu \models l_{i,j}$. It follows that $\nu \models C_j$. As this holds for all $1 \leq j \leq 2k$, we have $\nu \models S$.

Let us now prove that $\mathcal{P}(i+1) \Rightarrow \mathcal{P}(i)$ for all $1 \leq i \leq 2k$. Assume $\mathcal{P}(i+1)$ holds for some i. We deal with the case i is odd, the other is analogous. Consider a valuation ν of the variables $y_1, \ldots, y_{i-1}$ and the formula $\exists y_i \cdots \forall y_{2k} \cdot S$. It is Player 1's turn at vertex s_i. Assume Player 1 wins in the game after $p(\nu)$ is seen. Then consider his choice of next vertex $l_i = (\neg) y_i$. We consider the valuation $\nu' = \nu \cdot \{y_i \to \_\}$ such that $\nu' \models l_i$ (note that this is possible since the valuation ν does not deal with variable y_i). Then, since Player 1 wins after $p(\nu)$ is seen, he also wins after $p(\nu')$ is seen as $p(\nu')$ extends $p(\nu)$ by following his choice. Then, by $\mathcal{P}(i+1)$, $\nu' \models \forall y_{i+1} \cdots \forall y_{2k} \cdot S$. It follows that $\nu \models \exists y_i \cdots \forall y_{2k} \cdot S$. Now, assume that $\nu \models \exists y_i \cdots \forall y_{2k} \cdot S$ and let us prove that Player 1 wins in the game after $p(\nu)$ is seen. By definition of the semantics of the satisfiability of a quantified formula, $\nu \models \exists y_i \cdots \forall y_{2k} \cdot S$ means that there exists ν' extending ν to variable y_i such that $\nu' \models \forall y_{i+1} \cdots \forall y_{2k} \cdot S$. Then, by $\mathcal{P}(i+1)$, we have that Player 1 wins in the game after $p(\nu')$ is seen. Hence, in s_i, Player 1 can choose the next vertex so that it mimics the final part of the path $p(\nu')$. Thus, since Player 1 wins after $p(\nu')$ is seen, Player 1 also wins after $p(\nu)$ is seen. Overall, $\mathcal{P}(i)$ holds.

We can conclude that $\mathcal{P}(1)$ holds, which exactly corresponds to the equivalence $\phi \in \mathsf{QBF} \Leftrightarrow tr(\phi) \in \mathsf{GG}$.

3. QBF being PSPACE-complete for logarithmic space reductions, and since the composition of logspace reduction can be done in logspace, it follows that GG is PSPACE-hard. As it is also in PSPACE by question 1, we can deduce that GG is PSPACE-complete.