# Complexité avancée - TD 1

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**Exercise 1: One-minute-long exercise**
Prove that any language $L \subset \{0,1\}^*$ that is neither empty nor $\{0,1\}^*$ is hard for $\mathsf{NL}$ for polynomial-time reductions.

**Exercice 2: Graph representation and why it does not matter.** Let $\Sigma = \{\mathtt{0}, \mathtt{1}, /, \bullet, \#\}$ with $\#$ the end-of-word symbol. For a directed graph $G = (V, E)$ with $V = [0, n-1]$ for some $n \in \mathbb{N}$ and $E \subseteq V \times V$, we consider the following two representations of $G$ by a word in $\Sigma^*$:

- By its adjacency matrix $m_G \in \Sigma^*$:
$$m_G \overset{\text{def}}{=} m_{0,0}m_{0,1}\ \ldots m_{0,n-1} \bullet \cdots \bullet m_{n-1,0} \ldots m_{n-1,n-1}\#$$
where for all $0 \leq i, j < n$, $m_{i,j}$ is $\mathtt{1}$ if $(i,j) \in E$, $\mathtt{0}$ otherwise.
- By its adjacency list $l_G \in \Sigma^*$:
$$l_g \overset{\text{def}}{=} k_0^0/\ldots/k_{m_1}^0 \bullet \cdots \bullet k_0^{n-1}/\ldots/k_{m_{n-1}}^{n-1}\#$$
where for all $0 \leq i < n$, $k_0^i, \ldots, k_{m_i}^i$ are binary words listing the (codes of) right neighbors of vertex $i$.

1. Show that it is possible to check in logarithmic space that a word $w \in \Sigma^*$ is a well-formed description of a graph (for any of the two representations).
2. Describe a logarithmic space bounded deterministic Turing machine taking as input a graph $G$, represented by its adjacency matrix, and computing the adjacency list representation of $G$.

**Exercise 3: A few $\mathsf{NL}$-complete problems**
Show that the following problems are $\mathsf{NL}$-complete for logspace reductions (you may use the fact that REACH is $\mathsf{NL}$-hard for logspace reductions):

1. Deciding if a non-deterministic automaton $\mathcal{A}$ accepts a word $w$.
2. Deciding if a directed graph has a cycle.

**Exercise 4: Inclusions of complexity classes**

**Definition 1** *A function $f : \mathbb{N} \to \mathbb{N}$ is said to be* space-constructible *if $\forall n \in \mathbb{N}$, $f(n) > log(n)$ and there exists a deterministic Turing machine that computes $f(|x|)$ in space $O(f(|x|))$ given $x$ as input.*

Show that for a space-constructible function $f$,
$$\mathsf{NSPACE}(f(n)) \subseteq \mathsf{DTIME}(2^{O(f(n))})$$

**Exercise 5: Restrictions in the definition of $\mathsf{SPACE}(f(n))$**

In the course, we restricted our attention to Turing machines that always halt, and whose computations are space-bounded on every input. In particular, remember that $\mathsf{SPACE}(f(n))$ is defined as the class of languages $L$ for which there exists some deterministic Turing machine $M$ that always halts (i.e. on every input), whose computations are $f(n)$ space-bounded (on every input), such that $M$ decides $L$.

Now, consider the following two classes of languages:

- $\mathsf{SPACE}'(f(n))$ is the class of languages $L$ such that there exists a deterministic Turing machine $M$, running in space bounded by $f(n)$, such that $M$ accepts $x$ iff $x \in L$. Note that if $x \notin L$, $M$ may not terminate.
- $\mathsf{SPACE}''(f(n))$ is the class of languages $L$ such that there exists a deterministic Turing machine $M$ such that $M$ accepts $x$ using space bounded by $f(n)$ iff $x \in L$ ($M$ may use more space and not even halt when $x \notin L$).

1. Show that for a space-constructible function $f = \Omega(logn)$, $\mathsf{SPACE}E'(f(n)) = \mathsf{SPACE}(f(n))$
2. Show that for a space-constructible function $f = \Omega(logn)$, $\mathsf{SPACE}''(f(n)) = \mathsf{SPACE}(f(n))$