Homework (Solving generalized reachability games). In the two-player turn-based setting:

1. A generalized reachability condition is the following: given several target sets of states $T_1, \ldots, T_k \subseteq V$, Player $A$ wins if and only if, for all $1 \leq i \leq k$, a state in $T_i$ is seen at some point. The difference with the $k$-generalized reachability objective is that the number of targets is not bounded a priori. Show that deciding the winner of a generalized reachability game is $\text{PSPACE}$-complete.

   Hint: for the $\text{PSPACE}$ membership, you can use without a proof that if Player $A$ wins with $k$ target sets of states, she can win in at most $k \cdot n$ steps where $n := |V|$. For the hardness, you may reduce from $\text{TQBF}$ where the formula is in $\text{CNF}$.

2. Show that deciding the winner of a generalized reachability game when $V_B = \emptyset$ (i.e. only Player $A$ is playing) is $\text{NP}$-hard.

Solution. 1. Let us denote by $\text{GenReach}$ this decision problem. Let us first show that it is in $\text{PSPACE}$. Then, for all vertices $v \in V$, we denote by $t(x) \subseteq \{1, \ldots, k\}$ the set of indexes of target sets to which it belongs: $t(x) := \{i \leq k \mid x \in T_i\}$. Now, we define a recursive function $\text{win} : (v, \text{visit}, n) \mapsto \text{true}$ if and only if Player $A$ wins in the graph $G = (V,E)$, starting from $v \in V$, for a version of the game where the targets in $\text{visit}$ have already been seen the remaining target are to be seen in at most $n$ steps”. Now $\text{win}$ is a simple recursive procedure:

$$\text{win}(v, \text{visit}, n) = \begin{cases} \text{true} & \text{if visit} = \{1, \ldots, k\} \\ \text{false} & \text{otherwise, if } n = 0 \\ \exists (v, v') \in E, \text{win}(v', \text{visit} \cup h(v), n - 1) = \text{true} & \text{otherwise, if } v \in V_A \\ \forall (v, v') \in E, \text{win}(v', \text{visit} \cup h(v), n - 1) = \text{true} & \text{otherwise, if } v \in V_B \end{cases}$$

Then, Player $A$ wins from $v_0$ if and only if $\text{win}(v_0, \emptyset, n \cdot k) = \text{true}$.

Then, an algorithm implementing this procedure can run in polynomial space: it explores every successors and calls itself recursively. The number of nested calls is at most $n \cdot k$ and each call takes polynomial space to store (the current vertex, the set of targets already seen and the index $n$).

Consider now the $\text{PSPACE}$-hardness. Let $\Phi = Q_1 x_1, Q_2 x_2, \ldots, Q_n x_n, \phi$ be a $\text{QBF}$ formula with $Q_i \in \{\forall, \exists\}$ for all $i \leq n$ and $\phi = \bigwedge_{1 \leq j \leq k} C_j$ a $\text{CNF}$ formula. We construct the following generalized reachability game:

- $G_\Phi := (V,E)$;
- $V := \{x_i, \neg x_i \mid 1 \leq i \leq n\} \cup \{s_j \mid 1 \leq j \leq n + 1\}$;
2. This decision problem is now in $\text{NP}$.

First, note that this can be computed in logspace as it only amounts to loop on the variables and clauses of the input to produce the output.

Let us show that $\Phi \in \text{QBF} \iff G_\Phi \in \text{GenReach}$. We prove it by induction with a partial valuation of the variables. For all $k \leq n$, a $k$-partial valuation is a valuation $v_k : \{x_1, \ldots, x_n\} \rightarrow \{\top, \bot\}$ of the variables $x_1, \ldots, x_k$. To each partial valuation, we can associate the corresponding path $p_{v_k}$ in the graph $G_\Phi$ from $s_1$ to $s_{k+1}$ that visits the variables $x_i$ if $v_k(x_i) = \top$ and visits $\neg x_i$ otherwise. We denote by $\Phi_{v_k}$ the resulting QBF-formula:

$$\Phi_{v_k} = Q_{k+1}x_{k+1}, \ldots, Q_nx_n, \phi[v_k]$$

where $\phi[v_k]$ is the formula $\phi$ where each variable in $\{x_1, \ldots, x_k\}$ has been replaced by its value w.r.t. $v_k$. Similarly, we denote by $G_{\phi}^{p_{v_k}}$ the game that starts in $s_{k+1}$ with the path $p_{v_k}$ already seen. Then, let us prove the following property on $0 \leq k \leq n$ by induction $P(k)$: for all $k$-partial valuations $v_k$, we have $\Phi_{v_k} \in \text{QBF} \iff G_{\phi}^{p_{v_k}} \in \text{GenReach}$.

First, $P(n)$ holds by definition of the target sets $T_i$, they exactly correspond to the clauses in the formula. Hence, all the clauses are satisfied by a valuation $v_n$ if and only if all the target sets are visited by the path $p_{v_n}$. Now, assume that the property $P$ holds for some $0 < k + 1 \leq n$. Consider a $k$-partial valuation $v_k$, the QBF-formula $\Phi_{v_k}$ and the corresponding game $G_{\phi}^{p_{v_k}}$. Assume that $Q_k = \exists$. Then, we have the following equivalence:

$$\Phi_{v_k} \in \text{QBF} \iff \Phi_{v_k}[x_{k+1} \rightarrow \top] \in \text{QBF} \lor \Phi_{v_k}[x_{k+1} \rightarrow \bot] \in \text{QBF}$$

$$\iff G_{\phi}^{p_{v_k}, x_{k+1}, s_{k+2}} \in \text{GenReach} \lor G_{\phi}^{p_{v_k}, (\neg x_{k+1}), s_{k+2}} \in \text{GenReach}$$

$$\iff G_{\phi}^{p_{v_k}} \in \text{GenReach}$$

The case $Q_k = \forall$ is analogous. That is $P(k)$ holds. In, fact, it holds for all $0 \leq k \leq n$. In particular, $P(0)$ holds, which exactly corresponds to the equivalence $\Phi \in \text{QBF} \iff G_\Phi \in \text{GenReach}$.

2. This decision problem is now in $\text{NP}$: one can guess a path of length at most $k \cdot n$ and then check in polynomial time that indeed all target sets appear in that path. Furthermore, the previous reduction also applies here: indeed, $\text{SAT}$ is a special case of $\text{QBF}$ where there are no universal quantifiers. In the reduction, it induces that the set of states $V_B$ is empty, which corresponds to the restriction of the problem we are considering. Hence, this decision problem can be reduced in logspace to $\text{SAT}$ and is therefore $\text{NP}$-hard.