ENS Paris-Saclay

Informatique, Logique L3

Ehrenfeucht-Fraïssé Games

Revision 2*

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Abstract

We introduce a classic technique for showing a refinement of elementary equivalence, with applications to definability and completeness.

We have already obtained some expressivity results as corollaries of completeness results. For example, we have shown that well-foundedness is not expressible in the theory of discrete orders. Indeed, we have shown by quantifier elimination that this theory is complete, so all its models are elementarily equivalent. Moreover, the theory admits models where the order is well-founded but also models where this is not the case.

We would like to obtain similar results for theories that are not complete. For example, show that well-foundedness is not definable in arithmetic: there is no formula ϕ such that a model $\mathcal{S}$ of arithmetic has a well-founded order iff $\mathcal{S} \models \phi$. Or, more simply, show that well-foundedness is not definable at all: there is no formula whose models are exactly the structures where some binary predicate is interpreted as a well-founded relation. Ehrenfeucht-Fraïssé games are a tool for answering such questions.

*Modifications made since the original version of the document are indicated in margins with their corresponding revision.

In all of this document we assume that equality is part of the language and that all structures interpret it as the identity relation over their domain: for all $\mathcal{S}$ and for all $a, b \in \mathrm{dom}(\mathcal{S})$ we have

$$a \mathrel{\hat{=}_{\mathcal{S}}} b \quad \text{iff} \quad a = b.$$

We can express that a structure has at least n elements: the formula

$$\exists x_1, \ldots, x_n. \bigwedge_{i \neq j} x_i \neq x_j$$

is satisfied by $\mathcal{S}$ iff the domain of $\mathcal{S}$ has more than n elements. Similarly, we can express that there are more than m elements. However, as we shall see, we cannot express that there are finitely many elements without specifying their exact number.

1 Preliminaries

Recall that two structures are said to be elementarily equivalent when $\mathcal{S}_1 \models \phi$ iff $\mathcal{S}_2 \models \phi$ for any closed formula ϕ. This equivalence is noted $\mathcal{S}_1 \equiv \mathcal{S}_2$.

Definition 1.1 (Partial isomorphism)**.** Given two structures $\mathcal{S}$ and $\mathcal{S}'$, a partial isomorphism from one to the other is a partial mapping $h : \mathrm{dom}(\mathcal{S}) \to \mathrm{dom}(\mathcal{S}')$ that is injective and such that:

(1) for any $f \in \mathcal{F}$ of arity n, for any $a_1, \ldots, a_n, a$ in the domain of h,

$$\hat{f}_{\mathcal{S}}(a_1, \ldots, a_n) = a \quad \text{iff} \quad \hat{f}_{\mathcal{S}'}(h(a_1), \ldots, h(a_n)) = h(a);$$

(2) for any $P \in \mathcal{P}$ of arity n, for any $a_1, \ldots, a_n$ in the domain of h,

$$(a_1, \ldots, a_n) \in \hat{P}_{\mathcal{S}} \quad \text{iff} \quad (h(a_1), \ldots, h(a_n)) \in \hat{P}_{\mathcal{S}}.$$

Note that condition (2) needs not be imposed on the equality predicate, because the injectivity condition on h amounts to it when P is the equality predicate symbol: $(a_1, a_2) \in \hat{=}_{\mathcal{S}}$ iff $a_1 = a_2$.

Example 1.2. With $\mathcal{F} = \mathcal{P} = \emptyset$, $\mathcal{F}, \mathcal{P}$-structures are reduced to their domain: there is nothing to interpret. Partial isomorphisms are just injective partial maps.

Example 1.3. Assume $\mathcal{F} = \{f\}$ and $\mathcal{P} = \emptyset$. Let $\mathcal{S}$ be an $\mathcal{F}, \mathcal{P}$-structure of domain $\{a\}$ with $f_{\mathcal{S}}(a) = a$. Let $\mathcal{S}'$ be an $\mathcal{F}, \mathcal{P}$-structure of domain $\{0, 1\}$ with $f_{\mathcal{S}'}(x) = 1 - x$. There is no partial isomorphism from $\mathcal{S}$ to $\mathcal{S}'$ that is defined on $\mathrm{dom}(\mathcal{S})$.

Example 1.4. Consider the canonical $\mathcal{F}, \mathcal{P}$-structures $\mathbb{Q}$ and $\mathbb{R}$ over $\mathcal{P} = \{\leq\}$ and $\mathcal{F} = \{0, 1, +, \times\}$. For any subset $S \subseteq \mathbb{Q}$, the identity function on S is a partial isomorphism from $\mathbb{Q}$ to $\mathbb{R}$, but also from $\mathbb{R}$ to $\mathbb{Q}$. There is no partial isomorphism from $\mathbb{R}$ to $\mathbb{Q}$ whose domain of definition is $\{\sqrt{2}, 2, 1\}$. Rev. 2

We can reformulate the conditions on partial isomorphisms as formulas in a specific class.

Definition 1.5 (Flat formulas)**.** A formula is flat if its atoms are of the form $x = y$, $x = f(x_1, \ldots, x_n)$ or $P(x_1, \ldots, x_n)$ where x, y and the x_i are variables. In other words, a formula is flat when its atoms contain at most one symbol from $\mathcal{F} \cup \mathcal{P}$. Rev. 1

Proposition 1.6. A partial mapping $h : \mathcal{S} \to \mathcal{S}'$ of domain $\{a_1, \ldots, a_n\}$ is a partial isomorphism iff we have, for any flat atomic formula with free variables among $x_1, \ldots, x_n$,

$$\mathcal{S}, \{x_i \mapsto a_i\}_i \models \phi \quad \text{iff} \quad \mathcal{S}', \{x_i \mapsto h(a_i)\}_i \models \phi.$$

Proof. Let $\sigma = \{x_i \mapsto a_i\}_{1 \leq i \leq n}$ and $\sigma' = \{x_i \mapsto h(a_i)\}_{1 \leq i \leq n}$. The result is due to the fact that the three kinds of conditions imposed in definition 1.1 and in the statement of proposition 1.6 are equivalent. On one side we have the injectivity condition, and conditions (1) and (2). On the other side we have equisatisfiability conditions for atoms of the form $x = y$, $x = f(x_1, \ldots, x_k)$ and $P(x_1, \ldots, x_k)$.

The injectivity condition on h is equivalent to

$$\mathcal{S}, \sigma \models x_i = x_j \text{ iff } \mathcal{S}', \sigma' \models x_i = x_j \text{ for all } 1 \leq i \leq j \leq n.$$

Condition (1) of definition 1.1 for some function symbol f of arity k requires that, for all elements $b_1, \ldots, b_k, b$ of $\mathrm{dom}(h)$, we have

$$f_{\mathcal{S}'}(h(b_1), \ldots, h(b_k)) = h(b) \text{ iff } f_{\mathcal{S}}(b_1, \ldots, b_k) = b.$$

This can be rephrased equivalently as

$$\mathcal{S}, \sigma \models f(x_{i_1}, \ldots, x_{i_k}) = x_i \text{ iff } \mathcal{S}', \sigma' \models f(x_{i_1}, \ldots, x_{i_k}) = x_i$$

where i_p is such that $a_{i_p} = b_p$, and i is such that $a_i = b$.

Similarly, condition (2) for some predicate symbol P of arity k can be equivalently reformulated as

$$\mathcal{S}, \sigma \models P(x_{i_1}, \ldots, x_{i_k}) \text{ iff } \mathcal{S}', \sigma' \models P(x_{i_1}, \ldots, x_{i_k}).$$

$\square$

We finally define two measures over formulas that will be useful for our technical development.

Definition 1.7 (Size, $\equiv_n$)**.** The size of a formula counts the number of logical connectives, predicate symbols, function symbols and variables in the formula. In other words it is the number of nodes of the abstract syntax tree (AST) of the formula and its terms. We say that two structures are elementarily equivalent up to size n, noted $\mathcal{S}_1 \equiv_n \mathcal{S}_2$, when $\mathcal{S}_1 \models \phi$ iff $\mathcal{S}_2 \models \phi$ for any closed formula ϕ of size at most n.

Definition 1.8 (Rank, $\simeq_m$)**.** The rank of a formula is the maximal number of nested quantifiers in it. In other words, it is maximum number of quantifier nodes on a path from the root to a leaf of the formula AST. We say that two structures are elementarily equivalent up to rank m, noted $\mathcal{S}_1 \simeq_m \mathcal{S}_2$, when $\mathcal{S}_1 \models \phi$ iff $\mathcal{S}_2 \models \phi$ for any closed formula ϕ of rank at most m.

Note that $\mathcal{S} \equiv \mathcal{S}'$ iff ($\mathcal{S} \equiv_n \mathcal{S}'$ for all n) iff ($\mathcal{S} \simeq_m \mathcal{S}'$ for all m).

Proposition 1.9. Any formula of size less than n is logically equivalent to a flat formula of rank less than $2n$.

Proof. We modify the formula by introducing fresh intermediate variables:

- $f(\vec{u}) = g(\vec{v})$ becomes $\exists x.\ f(\vec{u}) = x \wedge x = g(\vec{v})$;
- $x = f(x_1, \ldots, x_k, g(\vec{u}), \vec{v})$ becomes $\exists y.\ x = f(x_1, \ldots, x_k, y, \vec{v}) \wedge y = g(\vec{u})$;
- $P(x_1, \ldots, x_k, f(\vec{u}), \vec{v})$ becomes $\exists y.\ P(x_1, \ldots, x_k, y, \vec{v}) \wedge y = f(\vec{u})$.

Each elementary transformation introduces one new quantifier, and there cannot be more than n transformation steps, after which the resulting formula is flat. Since the initial rank is bounded by n, the final one is bounded by $2n$. $\square$

2 Ehrenfeucht-Fraïssé games

An Ehrenfeucht-Fraïssé game is played on two $\mathcal{F}, \mathcal{P}$-structures $\mathcal{S}_1$ and $\mathcal{S}_2$, between two players called Spoiler (S) and Duplicator (D). At the beginning of the game, the number n of rounds is announced. The play between Spoiler and Duplicator leads to the construction of some sequence $(a_i, b_i)_{1 \leq i < k}$ of semantic values, with $a_i \in \mathrm{dom}(\mathcal{S}_1)$ and $b_i \in \mathrm{dom}(\mathcal{S}_2)$ for all $1 \leq i < k$.

At round k, some sequence $(a_i, b_i)_{1 \leq i < k}$ has already been constructed. Spoiler plays first and chooses either some $a_k \in \mathrm{dom}(\mathcal{S}_1)$ or $b_k \in \mathrm{dom}(\mathcal{S}_2)$. Duplicator chooses the other value.

At the end of the game, Duplicator wins iff $(a_i \mapsto b_i)_{1 \leq i \leq n}$ defines a partial isomorphism.

Example 2.1. Take $\mathcal{F} = \mathcal{P} = \emptyset$ and two $\mathcal{F}, \mathcal{P}$-structures $\mathcal{S}_1$ and $\mathcal{S}_2$. Duplicator has a winning strategy for the Ehrenfeucht-Fraïssé game between $\mathcal{S}_1$ and $\mathcal{S}_2$ with n rounds iff $\min(|\mathcal{S}_1|, n) = \min(|\mathcal{S}_2|, n)$. Indeed, at round $k \leq n$, Duplicator only has to ensure that, for all $1 \leq i < k$, $a_k = a_i$ iff $b_k = b_i$. This is possible iff the condition on cardinals is met.

Example 2.2. With the two structures of example 1.3, Spoiler has a winning strategy for one round: he simply has to choose $a_1 := a \in \mathrm{dom}(\mathcal{S}_1)$.

Exercise 2.3. Give two structures for which Duplicator wins the game for two rounds but not three.

2.1 An extended example

Let $\mathcal{F} = \emptyset$ and $\mathcal{P} = \{R\}$ with R of arity 2. Let n be some natural number.

Let S_1 be the $\mathcal{F}, \mathcal{P}$-structure of domain $\{0, \ldots, 2^n - 1\}$ with Rev. 1

$$R_{\mathcal{S}_1} = \{(k, k+1 \; mod \; 2^n) \mid k \in [0; 2^n - 1]\}.$$

Let S_2 be the $\mathcal{F}, \mathcal{P}$-structure of domain $\{(i, 0), \ldots, (i, 2^n - 1) \mid i \in \{0, 1\}\}$ with

$$R_{\mathcal{S}_2} = \{((i, k), (i, k+1 \; mod \; 2^n)) \mid k \in [0; 2^n - 1], \; i \in \{0, 1\}\}.$$

In other words, $\mathcal{S}_1$ is an oriented loop of size 2^n and $\mathcal{S}_2$ is two disconnected copies of $\mathcal{S}_1$. We will show that Duplicator has a winning strategy on $\mathcal{S}_1$ and $\mathcal{S}_2$ for n rounds.

For a, a' in the domain of some $\mathcal{S}_i$, define the distance $d(a, a')$ from a to a' as the minimum number of $R_{\mathcal{S}_i}$ steps from a to a'. In particular $d(a, a') = 0$ iff $a = a'$ and $d(a, a') = \infty$ if a' is not reachable from a. At round k, Duplicator will maintain the following invariant:

> for all $i, j \in [1; k]$,
> if $d(a_i, a_j) \leq 2^{n-k}$ or $d(b_i, b_j) \leq 2^{n-k}$ then $d(a_i, a_j) = d(b_i, b_j)$.

We first observe that, if this can be realized, the Duplicator will indeed win: for $k = n$ the invariant ensures that the sequence of pairs defines an injective mapping from $\mathcal{S}_1$ to $\mathcal{S}_2$ and that $(a_i, a_j) \in R_{\mathcal{S}_1}$ iff $(b_i, b_j) \in R_{\mathcal{S}_2}$, i.e. we have a partial isomorphism.

Let us now verify that this invariant can be maintained by Duplicator.

- If Spoiler plays a_k such that $d(a_k, a_i) \leq 2^{n-k}$ for some $i < k$ then Duplicator must play the unique b_k such that $d(b_k, b_i) = d(a_k, a_i)$. Let us verify that this choice maintains the invariant.
  - Obviously, the condition of $d(a_i, a_j)$ for $i, j \in [1; k-1]$ at round $k-1$ implies the condition at round k.
  - Assume $d(a_k, a_j) \leq 2^{n-k}$ for some $j < k$ distinct from i. We have either $d(a_k, a_j) = d(a_k, a_i) + d(a_i, a_j)$ or $d(a_k, a_i) = d(a_k, a_j) + d(a_j, a_i)$. In the first case we conclude that $d(a_i, a_j) \leq 2^{n-k}$, thus $d(b_i, b_j) = d(a_i, a_j)$ and the choice of b_k made above wrt. b_i also ensures $d(b_k, b_j) = d(b_k, b_i)$. In the second case we have $d(a_j, a_i) \leq 2^{n-k}$, hence $d(b_j, b_i) = d(a_j, a_i)$ and we conclude similarly. Rev. 1
  - A similar argument applies for the case where $d(a_j, a_k) \leq 2^{n-k}$. This time we have $d(a_j, a_i) = d(a_j, a_k) + d(a_k, a_i)$. Since the two distances are less than 2^{n-k} we conclude that $d(a_j, a_i) \leq 2^{n-k+1}$ and, by the invariant at round $k-1$, we have $d(b_j, b_i) = d(a_j, a_i)$. We conclude that $d(b_j, b_k) = d(b_j, b_i) - d(b_k, b_j) = d(a_j, a_i) - d(a_k, a_j) = d(a_j, a_k)$.
- Otherwise, if Spoiler chooses a_k such that $d(a_i, a_k) \leq 2^{n-k}$, we proceed similarly. The cases where Spoiler chooses b_k within 2^{n-k} steps of some b_i are also similar.
- Otherwise, Spoiler chooses some a_k that is more than 2^{n-k} steps away from all $(a_i)_{1 \leq i < k}$ (or symmetrically with b_k). Duplicator needs to find some b_k that is more than 2^{n-k} steps aways from all $(b_i)_{1 \leq i < k}$. It is always possible because we have at most $k-1$ elements on a cycle of length 2^n, and

the distances between two consecutive elements cannot all be $\leq 2^{n-k+1}$ because $(k-1)2^{n-k+1} < 2^n$.

Exercise 2.4. Show that the construction is tight: Spoiler wins the game for $n = 2$ if $\mathcal{S}_1$ and $\mathcal{S}_2$ consist of cycles of length $2^n - 1$.

2.2 Main theorem

We now prove the main result about Ehrenfeucht-Fraïssé games. In this section we assume that $\mathcal{F}$ and $\mathcal{P}$ are finite.

Theorem 2.5. Two $\mathcal{F}, \mathcal{P}$-structures $\mathcal{S}_1$ et $\mathcal{S}_2$ are elementarily equivalent iff Duplicator has a winning strategy for any number of rounds.

Note that, in the above result, the strategy does not have to be uniform in the number of rounds: as was the case in the previous example, the winning strategy can be adapted to the remaining number of rounds.

Ehrenfeucht-Fraïssé games can still be used to establish that two structures are elementarily equivalent when $\mathcal{F}, \mathcal{P}$ is infinite: it suffices to verify that they are elementarily equivalent for any finite subsets $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{P}' \subseteq \mathcal{P}$.

We show each direction separately, working on the size of formulas in one direction and on their rank in the other. More precisely, we relate the existence of winning strategies for Duplicator in n rounds to elementary equivalence for flat formulas of rank at most n, i.e. $\equiv_n$.

Lemma 2.6. If Duplicator has a winning strategy for the game in $2n$ rounds on $\mathcal{S}_1$ and $\mathcal{S}_2$, then $\mathcal{S}_1 \equiv_n \mathcal{S}_2$.

Proof. It suffices to show that $\mathcal{S}_1 \models \phi$ and $\mathcal{S}_2 \models \phi$ are equivalent for all closed flat formulas ϕ of rank at most $2n$. Intuitively, the winning strategy for Duplicator will tell us how to map the values used for the satisfaction of the quantifiers of ϕ from $\mathcal{S}_1$ to $\mathcal{S}_2$ and vice versa; since the formula has rank $2n$ a game in $2n$ rounds is exactly what we need.

Formally, we show more generally that, for any flat formula ϕ of rank $2n - k$ and such that $\mathrm{fv}(\phi) \subseteq \{x_1, \ldots, x_k\}$, and for any partial isomorphism $h : \mathcal{S}_1 \to \mathcal{S}_2$ of domain $a_1, \ldots, a_k$, we have:

$$\mathcal{S}_1, \{x_i \mapsto a_i\}_{1 \leq i \leq k} \models \phi \quad \text{iff} \quad \mathcal{S}_2, \{x_i \mapsto h(a_i)\}_{1 \leq i \leq k} \models \phi$$

We proceed by induction over ϕ. Propositional cases are straightforward. The cases of flat atoms follow from proposition 1.6.

Assume ϕ is of the form $\forall x_{k+1}.\phi'$, and we have $\mathcal{S}_1, \{x_i \mapsto a_i\}_{1\leq i\leq k} \models \phi$. We need to show, for any $b_{k+1} \in \mathrm{dom}(\mathcal{S}_2)$, that $\mathcal{S}_2, \{x_i \mapsto b_i\}_{1\leq i\leq k+1} \models \phi'$. Let a_{k+1} be the element that Duplicator would choose using its winning strategy in the Ehrenfeucht-Fraïssé game at round $k+1$ when Spoiler choose b_{k+1}. By definition of the game, the mapping h extended with $h(a_{k+1}) = b_{k+1}$ is still a partial isomorphism. We conclude by induction hypothesis using this extended partial isomorphism and ϕ', which is now of rank $2n-k-1$. Rev. 1

The case of the existential quantifier is analogous: we leave it as an exercise – it is of course also possible to deal with this case using de Morgan laws for quantifiers, but that would not be very informative. □

Lemma 2.7. If $\mathcal{S}_1 \simeq_n \mathcal{S}_2$ then Duplicator has a winning strategy for n rounds on $\mathcal{S}_1$ and $\mathcal{S}_2$.

Proof. The idea is to construct a formula of rank n whose satisfaction expresses that Duplicator has a winning strategy. For $0 \leq k \leq n$, the formula $\phi_k^{b_1,\ldots,b_{n-k}}$ of rank k and free variables $x_1, \ldots, x_{n-k}$ will intuitively express that there exists a winning strategy for k rounds, assuming that $b_1, \ldots, b_{n-k}$ have been chosen so far in $\mathcal{S}_2$. More precisely, we will construct this formula in such a way that the following always holds:

$$\mathcal{S}_2, \{x_i \mapsto b_i\}_{1\leq i\leq n-k} \models \phi_k^{b_1,\ldots,b_{n-k}} \text{ for all } k, b_1, \ldots, b_{n-k} \tag{1}$$

And we will show that Duplicator has a winning strategy for k more rounds assuming that $(a_i, b_i)_{1\leq i\leq n-k}$ have been chosen so far when the symmetric satisfaction holds:

$$\mathcal{S}_1, \{x_i \mapsto a_i\}_{1\leq i\leq n-k} \models \phi_k^{b_1,\ldots,b_{n-k}} \tag{2}$$

Given $b_1, \ldots, b_n \in \mathrm{dom}(\mathcal{S}_2)$, we define $\mathcal{A}(b_1, \ldots, b_n)$ as the set of flat literals A over $x_1, \ldots, x_n$ such that $\mathcal{S}_2, \{x_i \mapsto b_i\}_{1\leq i\leq n} \models A$. We then define:

$$\begin{aligned}
\phi_0^{b_1,\ldots,b_n} &\stackrel{def}{=} \bigwedge_{A\in\mathcal{A}(b_1,\ldots,b_n)} A \\
\phi_{k+1}^{b_1,\ldots,b_{n-k-1}} &\stackrel{def}{=} (\forall x_{n-k}. \bigvee_{b\in\mathcal{S}_2} \phi_k^{b_1,\ldots,b_{n-k-1},b}) \\
&\quad \wedge (\bigwedge_{b\in\mathcal{S}_2} \exists x_{n-k}.\ \phi_k^{b_1,\ldots,b_{n-k-1},b})
\end{aligned}$$

For these formulas to be well-defined, we need to make sure that we are forming conjunctions and disjunctions over finite sets: there are a priori infinitely many

$b \in \mathcal{S}_2$, but only finitely many $\phi_k^{b_1,\ldots,b_n-k-1,b}$. This is shown by induction on k: for $k = 0$ we have as many formulas as there are subsets of the finite set A; for $k+1$ the number N_{k+1} of formulas $\phi_{k+1}^{b_1,\ldots,b_{n-k-1}}$ that can be obtained for the various choices of $(b_i)_{1 \le i \le n-k-1}$ is bounded by the number N_k of possible formulas for rank k (we have $N_{k+1} \le 2^{N_k+1}$).

Now that our formulas are defined, let us check that they have the expected meaning. Equation (1) can be verified easily by induction on k: we leave it as an exercise.

In particular, we have $\mathcal{S}_2 \models \phi_n$ and, since $\mathcal{S}_1 \simeq_n \mathcal{S}_2$, we also have $\mathcal{S}_1 \models \phi_n$. We will show that this implies the existence of a winning strategy for Duplicator for n rounds.

We show more generally, for all $0 \le k \le n$ and $(a_i, b_i)_{1 \le i \le n-k}$ such that eq. (2) holds, that Duplicator has a a winning strategy for k rounds when $(a_i, b_i)_i$ have previously been chosen. This is done by induction on k.

- For $k = 0$ the game is over, and Duplicator wins because we have a partial isomorphism: by definition of $\phi_0^{b_1,\ldots,b_n}$ the two structures satisfy the same flat atoms.

- For $k+1$ we define the Duplicator strategy as follows:

  - Assume Spoiler plays a_{n-k} in $\mathcal{S}_1$. Because the first conjunct of $\phi_{k+1}^{b_1,\ldots,b_{n-k-1}}$ is satisfied in $\mathcal{S}_1$, we can choose the value of the universally quantified variable x_{n-k} to be a_{n-k} and we have some $b \in \mathcal{S}_2$ (call it b_{n-k}) such that
  $$\mathcal{S}_1, \{x_i \mapsto a_i\}_{1 \le i \le n-k} \models \phi_k^{b_1,\ldots,b_{n-k}}.$$
  Duplicator chooses this value, and keeps playing using the winning strategy obtained by induction hypothesis on k.

  - If Spoiler plays b_{n-k} in $\mathcal{S}_2$, by the second conjunct we have some a_{n-k} such that
  $$\mathcal{S}_1, \{x_i \mapsto a_i\}_{1 \le i \le n-k} \models \phi_k^{b_1,\ldots,b_{n-k}}.$$
  Again, Duplicator chooses this value and keeps playing using the winning strategy for k.

□

3 Applications

The primary application of these games is to show that some property of structures cannot be defined as a first-order formula. For instance, the winning strategy of section 2.1 shows that connectedness cannot be defined. Indeed, the existence of a formula ϕ such that $\mathcal{S} \models \phi$ iff $\mathcal{S}$ is connected would contradict $\mathcal{S}_1 \simeq_n \mathcal{S}_2$ when n is the rank of ϕ.

Exercise 3.1. With $\mathcal{F} = \mathcal{P} = \emptyset$, show that there cannot be a closed formula ϕ such that $\mathcal{S} \models \phi$ iff $\mathrm{dom}(\mathcal{S})$ is finite.

Ehrenfeucht-Fraïssé games can also be used to prove completeness results. For $\alpha \in \mathbb{N} \cup \{\infty\}$, define the theory T_R^α generated by the following axioms, with an axiom (NoLoop_i) for each $i < \alpha$:

$$\begin{array}{rll}
\forall x. & \exists y. R(x, y) & (\mathrm{Tot}) \\
\forall x. & \exists y. R(y, x) & (\mathrm{Tot}^{-1}) \\
\forall x, y, z. & R(x, y) \wedge R(x, z) \Rightarrow y = z & (\mathrm{Uniq}) \\
\forall x, y, z. & R(y, x) \wedge R(z, x) \Rightarrow y = z & (\mathrm{Uniq}^{-1}) \\
\forall x_1, \ldots, x_i. & \neg(R(x_1, x_2) \wedge \ldots \wedge R(x_{i-1}, x_i) \wedge R(x_i, x_1)) & (\mathrm{NoLoop}_i)
\end{array}$$

Section 2.1 can be adapted to show the existence of a winning strategy for Duplicator on any two models of T_R^∞. More precisely, Duplicator wins for n rounds on any two models of T_R^n. Hence, any two models of T_R^∞ are elementarily equivalent, i.e. the theory is complete.

Exercise 3.2. Let $n \in \mathbb{N}$. Show that finiteness is not first-order definable in T_R^n: there cannot exist a formula ϕ such that, for all models $\mathcal{S}$ of T_R^n, we have $\mathcal{S} \models \phi$ iff $\mathrm{dom}(\mathcal{S})$ is finite. Show that finiteness is however first-order definable in T_R^∞.