Intuitionistic Logic

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1 Kripke semantics

We first define Kripke semantics for (unsorted) first-order intuitionistic logic.

Definition 1.1 (Kripke structure)**.** A *Kripke structure* is given by:

- a set $\mathcal{W}$ of worlds;
- an order $\leq$ on worlds, often called *accessibility relation*;
- a mapping $\mathcal{D}$ which associates, to each world w, a non-empty domain $\mathcal{D}_w$;
- a mapping α which associates, to each world w and predicate symbol $p \in \mathcal{P}$ or arity n, a subset of $\mathcal{D}_w^n$;
- for each function symbol $f \in \mathcal{F}$ of arity n, and for each world $w \in \mathcal{W}$, an interpretation $\hat{f}_w : \mathcal{D}_w^n \to \mathcal{D}$.

Furthermore, $\mathcal{D}$, α and $\hat{f}$ must be monotonic with respect to the accessibility relation: for all $w \leq w'$ we require that

- $D_w \subseteq D_{w'}$;
- $\alpha(w, p) \subseteq \alpha(w', p)$ for all $p \in \mathcal{P}$;
- $\hat{f}_w(v_1, \ldots, v_n) = \hat{f}_{w'}(v_1, \ldots, v_n)$ for all f of arity n and $(v_1, \ldots, v_n) \in \mathcal{D}_w^n$.

When $\mathcal{K}$ is a Kripke structure, we shall denote its set of worlds by $\mathcal{W}(\mathcal{K})$. In most cases, we actually simply write $\mathcal{W}$, $\mathcal{D}$, etc. since the underlying Kripke structure is clear from the context.

Definition 1.2 (Satisfaction)**.** Given a Kripke structure $\mathcal{K}$, a world $w \in \mathcal{W}$, a formula ϕ and an assignemnt $\sigma : \mathsf{fv}(\phi) \to \mathcal{D}_w$, the *satisfaction* relation is defined by induction on ϕ:

- $\mathcal{K}, w, \sigma \models p(t_1, \ldots, t_n)$ iff $([\![t_1]\!]_\sigma, \ldots, [\![t_n]\!]_\sigma) \in \alpha(w, p)$;
- $\mathcal{K}, w, \sigma \models \top$ always holds;
- $\mathcal{K}, w, \sigma \models \bot$ never holds;

- $\mathcal{K}, w, \sigma \models \phi \wedge \psi$ iff $\mathcal{K}, w, \sigma \models \phi$ and $\mathcal{K}, w, \sigma \models \psi$;
- $\mathcal{K}, w, \sigma \models \phi \vee \psi$ iff $\mathcal{K}, w, \sigma \models \phi$ or $\mathcal{K}, w, \sigma \models \psi$;
- $\mathcal{K}, w, \sigma \models \phi \Rightarrow \psi$ iff for all $w' \geq w$, $\mathcal{K}, w', \sigma \models \phi$ implies $\mathcal{K}, w', \sigma \models \psi$;
- $\mathcal{K}, w, \sigma \models \neg\psi$ iff for all $w' \geq w$, $\mathcal{K}, w', \sigma \not\models \psi$;
- $\mathcal{K}, w, \sigma \models \exists x.\psi$ iff there exists $v \in \mathcal{D}_w$ such that $\mathcal{K}, w, \sigma + \{x \mapsto v\} \models \psi$;
- $\mathcal{K}, w, \sigma \models \forall x.\psi$ iff for all $w' \geq w$ and $v \in \mathcal{D}_{w'}$, $\mathcal{K}, w', \sigma + \{x \mapsto v\} \models \psi$.

We say that a set of formulas E is satisfied by $w \in \mathcal{W}(\mathcal{K})$ when $\mathcal{K}, w \models \phi$ for all $\phi \in E$. When $\mathcal{K}$ is obvious, we simply omit it and write $w \models \phi$ or $w \models \phi$.

Note that the previous definition is only valid because domains are monotonic. Specifically, this is used (in four cases) to be able to consider the assignment $\sigma : \mathsf{fv}(\phi) \to \mathcal{D}_w$ as an assignment of type $\mathsf{fv}(\psi) \to \mathcal{D}_{w'}$ for $w \leq w'$.

Definition 1.3 (Validity, logical consequence)**.** Let ϕ, ψ be formulas. We define *validity* ($\models \phi$) and *logical consequence* ($\phi \models \psi$) as follows:

- $\models \phi$ when for all $\mathcal{K}$ and all $w \in \mathcal{W}(\mathcal{K})$, $w \models \phi$.
- $\phi \models \psi$ when $\mathcal{K}, w \models \psi$ for all $\mathcal{K}$ and $w \in \mathcal{W}(\mathcal{K})$ such that $\mathcal{K}, w \models \phi$.

When E is a set of formulas, $E \models \phi$ means that $\mathcal{K}, w \models \phi$ for all $\mathcal{K}$ and $w \in \mathcal{W}(\mathcal{K})$ such that $w \models E$.

Remark 1.4. Note that $\neg\phi$ is logically equivalent to $\phi \Rightarrow \bot$. This observation allows us to often ignore negation in the following.

Exercise 1.5. Consider the validity of a few interesting formulas:

- $\neg\neg\phi \Rightarrow \phi$ and $\phi \Rightarrow \neg\neg\phi$;
- de Morgan laws;
- $\big((\phi \wedge \phi') \vee \psi\big) \Rightarrow \big((\phi \vee \psi) \wedge (\phi' \vee \psi)\big)$ and the converse;
- $(\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi)$;
- $\phi \vee \neg\phi$;
- $\exists x.b(x) \Rightarrow \forall y.b(y)$.

Proposition 1.6 (Satisfaction is monotonic)**.**
For all $\mathcal{K}$, w and σ, we have that $\mathcal{K}, w, \sigma \models \phi$ and $w \leq w'$ implies $\mathcal{K}, w', \sigma \models \phi$.

Proof. By (structural) induction on ϕ. This is obvious for logical constants (their satisfaction does not depend on the world being considered) and propositional variables (because α is assumed to be monotonic). It follows immediately from induction hypotheses for disjunction and conjunction formulas. We consider the case of implication: assuming $w \leq w'$ and $w \models \phi \Rightarrow \psi$, let us show that $w' \models \phi \Rightarrow \psi$. We have to show that $w'' \models \psi$ for all $w'' \geq w'$ such that $w'' \models \phi$. By transitivity of the accessibility relation, we have $w'' \geq w$. By $w \models \phi \Rightarrow \psi$ and $w'' \models \phi$, we conclude $w'' \models \psi$. The case of negation is similar, as observed in Remark 1.4. The case of quantifiers is left as an exercise. □

Proposition 1.7. Intuitionistically valid formulas are also classically valid.

Proof. It suffices to observe that any classical interpretation $\mathcal{I}$. can be seen as a Kripke structure $\mathcal{K}_\mathcal{I}$ with a single world w_0, in such a way that $\mathcal{I} \models \phi$ (in the classical sense) is equivalent to $\mathcal{K}_\mathcal{I}, w_0 \models \phi$ (in the intuitionistic sense). □

2 Sequent calculus proof system

A sequent $\Gamma \vdash \phi$ is built from formula ϕ and a multiset of formulas Γ. It should be read as "the conjunction of all formulas in Γ implies ϕ".

Definition 2.1. The rules of intuitionistic sequent calculus LJ_1 are given in Figure 1. We write $\Gamma \vdash_{\mathrm{LJ1}} \phi$ when the sequent $\Gamma \vdash \phi$ admits a derivation in LJ_1.

We briefly motivate the organization of rules in three groups. The *logical group* describes how connectives should be treated. For each connective, there is only rule allowing to introduce a formula with that toplevel connective on the left of a sequent, and one introducing such a formula on the right. The *identity group* contains the only two rules whose application requires to check that two formulas are equal. The *structural group* deals with the multiset structure, allowing to increase of decrease the arity of a formula in the multiset.

Exercise 2.2. Show that LJ_1 and NJ_1 are equivalent.

3 Soundness

We say that a sequent $\Gamma \vdash \phi$ is valid (written $\Gamma \models \phi$) when:

- for all $\mathcal{K}$, $w \in \mathcal{W}$ and $\sigma : \mathsf{fv}(\Gamma, \Delta) \to \mathcal{D}_w$,
- if $\mathcal{K}, w, \sigma \models \psi$ for all $\psi \in \Gamma$, then $\mathcal{K}, w, \sigma \models \phi$.

Note that $\vdash \phi$ is valid iff ϕ is valid.

Theorem 3.1. $\Gamma \vdash_{\mathrm{LJ1}} \phi$ implies $\Gamma \models \phi$.

Proof. Straightforward (structural) induction on ϕ: for each rule of LJ_1, can show that, if the premises are valid, the conclusion is also valid. □

Corollary 3.2. The sequent $\phi \vee \neg\phi$ is not derivable in LJ_1.

Logical group

$$\frac{}{\Gamma, \bot \vdash \phi}\ \bot_L \qquad \frac{}{\vdash \top}\ \bot_R$$

$$\frac{\Gamma, \phi_1, \phi_2 \vdash \psi}{\Gamma, \phi_1 \wedge \phi_2 \vdash \psi}\ \wedge_L \qquad \frac{\Gamma \vdash \phi_1 \quad \Delta \vdash \phi_2}{\Gamma, \Delta \vdash \phi_1 \wedge \phi_2}\ \wedge_R$$

$$\frac{\Gamma, \phi_1 \vdash \psi \quad \Gamma, \phi_2 \vdash \psi}{\Gamma, \phi_1 \vee \phi_2 \vdash \psi}\ \vee_L \qquad \frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2}\ \vee_R$$

$$\frac{\Gamma \vdash \phi_1 \quad \Delta, \phi_2 \vdash \psi}{\Gamma, \Delta, \phi_1 \Rightarrow \phi_2 \vdash \psi}\ \Rightarrow_L \qquad \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \Rightarrow \psi}\ \Rightarrow_R$$

$$\frac{\Gamma, \phi[t/x] \vdash \psi}{\Gamma, \forall x.\phi \vdash \psi}\ \forall_L \qquad \frac{\Gamma \vdash \phi}{\Gamma \vdash \forall x.\phi}\ \forall_R\ (x \notin \mathsf{fv}(\Gamma, \psi))$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma, \exists x.\phi \vdash \psi}\ \exists_L\ (x \notin \mathsf{fv}(\Gamma)) \qquad \frac{\Gamma \vdash \phi[t/x]}{\Gamma \vdash \exists x.\phi}\ \exists_R$$

Identity group

$$\frac{}{\phi \vdash \phi}\ \text{axiom} \qquad \frac{\Gamma \vdash \psi \quad \psi, \Delta \vdash \phi}{\Gamma, \Delta \vdash \phi}\ \text{cut}$$

Structural group

$$\frac{\Gamma, \phi, \phi \vdash \psi}{\Gamma, \phi \vdash \psi}\ \text{contraction} \qquad \frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi}\ \text{weakening}$$

Figure 1: Inference rules for LJ_1

4 Completeness

This section only deals with the propositional case, even though the result holds in first-order logic as well. In that case, $\alpha(w)$ is simply seen as a subset of $\mathcal{P}$. The derivation system LJ_0 is obtained from LJ_1 by dropping the rules associated to quantifiers.

We shall now establish completeness: any sequent that is valid with respect to Kripke semantics can be derived in LJ_0. To do so, we introduce the universal Kripke structure in which satisfaction is closely related to derivability.

We shall work under the assumption that the set of propositional variables $\mathcal{P}$ is countably infinite. This implies that there exists a bijection $r : \mathcal{F}_0 \to \mathbb{N}$.

Definition 4.1 (Saturated)**.** Given a (possibly infinite) set E of formulas, we write $E \vdash_{\mathrm{LJ}} \phi$ when there is a finite subset $\Gamma \subseteq E$ such that $\Gamma \vdash_{\mathrm{LJ}} \phi$. A set of formulas E is *saturated* if, for any ϕ such that $E \vdash_{\mathrm{LJ}} \phi$, we have $\phi \in E$. Given a set F, the set $F^* = \{ \phi \ : \ F \vdash_{\mathrm{LJ}} \phi \}$ is saturated.

Definition 4.2 (World-set)**.** We say that E is *consistent* if $\bot \notin E$. We say that E has the *disjunction property* if for all $\phi_1 \vee \phi_2 \in E$, there is some $i \in \{1, 2\}$ such that $\phi_i \in E$. We say that Γ is a *world-set* when it is saturated, consistent and has the disjunction property.

Definition 4.3 (Universal Kripke structure)**.** The universal structure $\mathcal{U}$ is defined by: $\mathcal{W}(\mathcal{U}) = \{ w_E \ : \ E \text{ is a world-set} \}$; $w_E \leq w_{E'}$ iff $E \subseteq E'$; $\alpha(w_E) = E \cap \mathcal{P}$.

Lemma 4.4. Let E be a set of formulas, and ϕ a formula such that $E \nvdash_{\mathrm{LJ}} \phi$. There exists a world-set E' such that $E \subseteq E'$ and $E' \nvdash_{\mathrm{LJ}} \phi$.

Proof. We define an increasing sequence $(E_i)_{i \in \mathbb{N}}$ of saturated sets such that for all i, $\phi \notin E_i$. We set $E_0 = E^*$. If E_n enjoys the disjunction property, then $E_{n+1} = E_n$. Otherwise, let $\phi_1 \vee \phi_2$ be the formula in E_n such that $\phi_1 \notin E_n$ and $\phi_2 \notin E_n$, and such that $r(\phi_1 \vee \phi_2)$ is minimal among the formulas having that property. It cannot be that both $E_n \cup \{\phi_1\} \vdash_{\mathrm{LJ}} \phi$ and $E_n \cup \{\phi_2\} \vdash_{\mathrm{LJ}} \phi$, because by rule $\vee_L$ that would contradict $E_n \nvdash_{\mathrm{LJ}} \phi$. Let i be such that $E_n \cup \{\phi_i\} \nvdash_{\mathrm{LJ}} \phi$, and let $E_{n+1} = (E_n \cup \{\phi_i\})^*$.

Let us show that $E' = \bigcup_{i \in \mathbb{N}} E_i$ satisfies the expected conditions. The set is saturated: if for a finite subset $\Gamma \in E'$, we have $\Gamma \vdash_{\mathrm{LJ}} \psi$, then because Γ is finite we have $\Gamma \subseteq E_k$ for some k, and by saturation of E_k we have $\psi \in E_k \subseteq E'$. The same argument shows that $E' \nvdash_{\mathrm{LJ}} \phi$, and thus E' is consistent: if $\bot$ could be derived, ϕ would also be derivable by rule $\bot_L$. It only remains to show that E' enjoys the disjunction property. Let $\phi = \phi_1 \vee \phi_2 \in E'$, there must be some k such that $\phi \in E_k$. By construction, the disjunction property will be restored for that formula in at least $r(\phi)$ steps, thus we have $\phi_1 \in E_{k+r(\phi)}$ or $\phi_2 \in E_{k+r(\phi)}$, and the disjunction property is satisfied for ϕ in E'. □

Lemma 4.5. Let E be a world-set and ϕ a formula. We have $\mathcal{U}, w_E \models \phi$ iff $\phi \in E$.

Proof. We proceed by (structural) induction on the formula.

- Case of $\top$. We always have $w_E \models \top$ and also always have $\top \in E$ by saturation and rule $\top_R$.

- Case of $\bot$. We never have $w_E \models \bot$, and never have $\bot \in E$ for a consistent E.
- Case of P. By definition, $w_E \models P$ iff $P \in \alpha(w_E) = E \cap P$ iff $P \in E$.
- Case of $\phi_1 \wedge \phi_2$.
  - ($\Rightarrow$) From $w_E \models \phi_1 \wedge \phi_2$ we obtain $w_E \models \phi_1$ and $w_E \models \phi_2$. By induction hypotheses we thus have $E \vdash_{\mathrm{LJ}} \phi_1$ and $E \vdash_{\mathrm{LJ}} \phi_2$, and we can conclude by rule $\wedge_R$.
  - ($\Leftarrow$) By assumption we have $E \vdash_{\mathrm{LJ}} \phi_1 \wedge \phi_2$. This allows us to conclude $E \vdash_{\mathrm{LJ}} \phi_i$ for each $i \in \{1, 2\}$, using rules $\wedge_L$, cut and axiom. By induction hypotheses this yields $w_E \models \phi_i$ for each i, which allows us to conclude.
- Case of $\phi_1 \vee \phi_2$.
  - ($\Rightarrow$) As in the previous case, but using rule $\vee_R$ instead of $\wedge_R$.
  - ($\Leftarrow$) If $\phi_1 \vee \phi_2 \in E$, then by the disjunction property of world-sets we have $\phi_i \in E$ for some i. By induction hypothesis this yields $w_E \models \phi_i$ and thus $w_E \models \phi_1 \vee \phi_2$.
- Case of $\phi_1 \Rightarrow \phi_2$.
  - ($\Rightarrow$) By rule $\Rightarrow_R$ it suffices to show $E \cup \{\phi_1\} \vdash_{\mathrm{LJ}} \phi_2$. Assume the contrary. Then by Lemma 4.4 there is some world-set E' such that $E \leq E'$, $\phi_1 \in E'$ and $\phi_2 \notin E'$. By induction hypothesis $w_{E'} \models \phi_1$, but then by our assumption $w_E \models \phi_1 \Rightarrow \phi_2$ we must also have $w_{E'} \models \phi_2$. We then have $\phi_2 \in E'$ by induction hypothesis, which is a contradiction.
  - ($\Leftarrow$) Assuming $E \vdash_{\mathrm{LJ}} \phi_1 \Rightarrow \phi_2$, we show $w_E \models \phi_1 \Rightarrow \phi_2$. We simply follow the definition of satisfaction for an implication. For any $E \leq E'$ such that $w_{E'} \models \phi_1$, we have to establish $w_{E'} \models \phi_2$. By induction hypothesis we have $\phi_1 \in E'$, or in other words $E' \vdash_{\mathrm{LJ}} \phi_1$. Since we also have $E' \vdash_{\mathrm{LJ}} \phi_1 \Rightarrow \phi_2$, we conclude $E' \vdash_{\mathrm{LJ}} \phi_2$ by rules cut, axiom and $\Rightarrow_L$. By induction hypothesis we can finally conclude: $w_{E'} \models \phi_2$.

$\square$

Theorem 4.6. $\Gamma \models \phi$ implies $\Gamma \vdash_{\mathrm{LJ}} \phi$.

Proof. Assume $\Gamma \models \phi$ and $\Gamma \nvdash_{\mathrm{LJ}} \phi$. By Lemma 4.4 we have some world-set E such that $\Gamma \subseteq E$ and $\phi \notin E$. We obviously have $w_E \models \Gamma$, so by $\Gamma \models \phi$ we also have $w_E \models \phi$. By Lemma 4.5, this implies $\phi \in E$, which is a contradiction. $\square$