Intuitionistic Logic

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ENS Cachan, L3, 2014–2015

1 Kripke semantics

Definition 1.1 (Kripke structure). A *Kripke structure* is given by:

- a set $\mathcal{W}$ of worlds;
- an order $\leq$ on worlds, often called *accessibility relation*;
- a monotonic mapping $\alpha : \mathcal{W} \to 2^{\mathcal{W}}$.

The monotonicity condition means that $\alpha(P) \subseteq \alpha(Q)$ whenever $P \leq Q$.

When $\mathcal{K}$ is a Kripke structure, we shall denote its set of worlds by $\mathcal{W}(\mathcal{K})$.

Definition 1.2 (Satisfaction). Given a Kripke structure $\mathcal{K}$, a world $w \in \mathcal{W}(\mathcal{K})$ and a formula $\phi \in \mathcal{F}_0(\mathcal{P})$, the *satisfaction* relation is defined by induction on ϕ:

- $\mathcal{K}, w \models P$ iff $P \in \alpha(w)$, for $P \in \mathcal{P}$;
- $\mathcal{K}, w \models \top$ always holds;
- $\mathcal{K}, w \models \bot$ never holds;
- $\mathcal{K}, w \models \phi \wedge \psi$ iff $\mathcal{K}, w \models \phi$ and $\mathcal{K}, w \models \psi$;
- $\mathcal{K}, w \models \phi \vee \psi$ iff $\mathcal{K}, w \models \phi$ or $\mathcal{K}, w \models \psi$;
- $\mathcal{K}, w \models \phi \Rightarrow \psi$ iff for all $w' \geq w$, $\mathcal{K}, w' \models \phi$ implies $\mathcal{K}, w' \models \psi$;
- $\mathcal{K}, w \models \neg\psi$ iff for all $w' \geq w$, $\mathcal{K}, w' \not\models \phi$.

We say that a set of formulas E is satisfied by $w \in \mathcal{W}(\mathcal{K})$ when $\mathcal{K}, w \models \phi$ for all $\phi \in E$. When $\mathcal{K}$ is obvious, we simply omit it and write $w \models \phi$ or $w \models \phi$.

Definition 1.3 (Validity, logical consequence). Let ϕ, ψ be formulas. We define *validity* ($\models \phi$) and *logical consequence* ($\phi \models \psi$) as follows:

- $\models \phi$ when for all $\mathcal{K}$ and all $w \in \mathcal{W}(\mathcal{K})$, $w \models \phi$.
- $\phi \models \psi$ when $\mathcal{K}, w \models \psi$ for all $\mathcal{K}$ and $w \in \mathcal{W}(\mathcal{K})$ such that $\mathcal{K}, w \models \phi$.

When E is a set of formulas, $E \models \phi$ means that $\mathcal{K}, w \models \phi$ for all $\mathcal{K}$ and $w \in \mathcal{W}(\mathcal{K})$ such that $w \models E$.

Remark 1.4. Note that $\neg\phi$ is logically equivalent to $\phi \Rightarrow \bot$. This observation allows us to often ignore negation in the following.

Example 1.5. Consider the validity of a few interesting formulas:

- $\neg\neg\phi \Rightarrow \phi$ and $\phi \Rightarrow \neg\neg\phi$;
- de Morgan laws;
- $\big((\phi \wedge \phi') \vee \psi\big) \Rightarrow \big((\phi \vee \psi) \wedge (\phi' \vee \psi)\big)$ and the converse;
- $(\phi \Rightarrow \psi) \vee (\psi \Rightarrow \phi)$;
- $\phi \vee \neg\phi$.

Proposition 1.6 (Satisfaction is monotonic). $w \models \phi$ and $w \leq w'$ implies $w' \models \phi$.

Proof. By (structural) induction on ϕ. This is obvious for logical constants (their satisfaction does not depend on the world being considered) and propositional variables (because α is assumed to be monotonic). It follows immediately from induction hypotheses for disjunction and conjunction formulas. We consider the case of implication: assuming $w \leq w'$ and $w \models \phi \Rightarrow \psi$, let us show that $w' \models \phi \Rightarrow \psi$. We have to show that $w'' \models \psi$ for all $w'' \geq w'$ such that $w'' \models \phi$. By transitivity of the accessibility relation, we have $w'' \geq w$. By $w \models \phi \Rightarrow \psi$ and $w'' \models \phi$, we conclude $w'' \models \psi$. The case of negation is similar, as observed in Remark 1.4. $\square$

Proposition 1.7. Intuitionistically valid formulas are also classically valid.

Proof. It suffices to observe that any classical interpretation $\mathcal{I} \subseteq \mathcal{P}$ can be seen as Kripke structure $\mathcal{K}_{\mathcal{I}}$ with a single world w_0 such that $\alpha(w_0) = \mathcal{I}$, in such a way that $\mathcal{I} \models \phi$ (in the classical sense) is equivalent to $\mathcal{K}_{\mathcal{I}}, w_0 \models \phi$ (in the intuitionistic sense). $\square$

2 Sequent calculus proof system

A sequent $\Gamma \vdash \phi$ is built from formula ϕ and a multiset of formulas Γ. It should be read as “the conjunction of all formulas in Γ implies ϕ”.

Definition 2.1. The rules of intuitionistic sequent calculus LJ_0 are given in Figure 1. We write $\Gamma \vdash_{\mathrm{LJ}} \phi$ when the sequent $\Gamma \vdash \phi$ admits a derivation in LJ_0.

We briefly motivate the organization of rules in three groups. The *logical group* describes how connectives should be treated. For each connective, there is only rule allowing to introduce a formula with that toplevel connective on the left of a sequent, and one introducing such a formula on the right. The *identity group* contains the only two rules whose application requires to check that two formulas are equal. The *structural group* deals with the multiset structure, allowing to increase of decrease the arity of a formula in the multiset.

Logical group

$$\frac{}{\Gamma, \bot \vdash \phi}\ \bot_L \qquad \frac{}{\vdash \top}\ \bot_R$$

$$\frac{\Gamma, \phi_1, \phi_2 \vdash \psi}{\Gamma, \phi_1 \wedge \phi_2 \vdash \psi}\ \wedge_L \qquad \frac{\Gamma \vdash \phi_1 \quad \Delta \vdash \phi_2}{\Gamma, \Delta \vdash \phi_1 \wedge \phi_2}\ \wedge_R$$

$$\frac{\Gamma, \phi_1 \vdash \psi \quad \Gamma, \phi_2 \vdash \psi}{\Gamma, \phi_1 \vee \phi_2 \vdash \psi}\ \vee_L \qquad \frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2}\ \vee_R$$

$$\frac{\Gamma \vdash \phi_1 \quad \Delta, \phi_2 \vdash \psi}{\Gamma, \Delta, \phi_1 \Rightarrow \phi_2 \vdash \psi}\ \Rightarrow_L \qquad \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \Rightarrow \psi}\ \Rightarrow_R$$

Identity group

$$\frac{}{\phi \vdash \phi}\ \text{axiom} \qquad \frac{\Gamma \vdash \psi \quad \psi, \Delta \vdash \phi}{\Gamma, \Delta \vdash \phi}\ \text{cut}$$

Structural group

$$\frac{\Gamma, \phi, \phi \vdash \psi}{\Gamma, \phi \vdash \psi}\ \text{contraction} \qquad \frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi}\ \text{weakening}$$

Figure 1: Inference rules for LJ_0

3 Soundness

A sequent $\Gamma \vdash \phi$ is said to be valid when $\Gamma \models \phi$, *i.e.*, ϕ is a logical consequence of Γ seen as a set of formulas.

Theorem 3.1. $\Gamma \vdash_{\mathrm{LJ}} \phi$ implies $\Gamma \models \phi$.

Proof. Straightforward (structural) induction on ϕ: for each rule of LJ_0, can show that, if the premises are valid, the conclusion is also valid. □

Corollary 3.2. The sequent $\phi \vee \neg\phi$ is not derivable in LJ_0.

4 Completeness

We shall now establish completeness: any sequent that is valid with respect to Kripke semantics can be derived in LJ_0. To do so, we introduce the universal Kripke structure in which satisfaction is closely related to derivability.

We shall work under the assumption that the set of propositional variables $\mathcal{P}$ is countably infinite. This implies that there exists a bijection $r : \mathcal{F}_0 \to \mathbb{N}$.

Definition 4.1 (Saturated). Given a (possibly infinite) set E of formulas, we write $E \vdash_{\mathrm{LJ}} \phi$ when there is a finite subset $\Gamma \subseteq E$ such that $\Gamma \vdash_{\mathrm{LJ}} \phi$. A set of formulas E is *saturated* if, for any ϕ such that $E \vdash_{\mathrm{LJ}} \phi$, we have $\phi \in E$. Given a set F, the set $F^* = \{\ \phi\ :\ F \vdash_{\mathrm{LJ}} \phi\ \}$ is saturated.

Definition 4.2 (World-set). We say that E is *consistent* if $\bot \notin E$. We say that E has the *disjunction property* if for all $\phi_1 \vee \phi_2 \in E$, there is some $i \in \{1, 2\}$ such that $\phi_i \in E$. We say that Γ is a *world-set* when it is saturated, consistent and has the disjunction property.

Definition 4.3 (Universal Kripke structure). The universal structure $\mathcal{U}$ is defined by: $\mathcal{W}(\mathcal{U}) = \{\ w_E\ :\ E \text{ is a world-set}\ \}$; $w_E \leq w_{E'}$ iff $E \subseteq E'$; $\alpha(w_E) = E \cap \mathcal{P}$.

Lemma 4.4. Let E be a set of formulas, and ϕ a formula such that $E \nvdash_{\mathrm{LJ}} \phi$. There exists a world-set E' such that $E \subseteq E'$ and $E' \nvdash_{\mathrm{LJ}} \phi$.

Proof. We define an increasing sequence $(E_i)_{i\in\mathbb{N}}$ of saturated sets such that for all i, $\phi \notin E_i$. We set $E_0 = E^*$. If E_n enjoys the disjunction property, then $E_{n+1} = E_n$. Otherwise, let $\phi_1 \vee \phi_2$ be the formula in E_n such that $\phi_1 \notin E_n$ and $\phi_2 \notin E_n$, and such that $r(\phi_1 \vee \phi_2)$ is minimal among the formulas having that property. It cannot be that both $E_n \cup \{\phi_1\} \vdash_{\mathrm{LJ}} \phi$ and $E_n \cup \{\phi_2\} \vdash_{\mathrm{LJ}} \phi$, because by rule $\vee_L$ that would contradict $E_n \nvdash_{\mathrm{LJ}} \phi$. Let i be such that $E_n \cup \{\phi_i\} \nvdash_{\mathrm{LJ}} \phi$, and let $E_{n+1} = (E_n \cup \{\phi_i\})^*$.

Let us show that $E' = \bigcup_{i\in\mathbb{N}} E_i$ satisfies the expected conditions. The set is saturated: if for a finite subset $\Gamma \in E'$, we have $\Gamma \vdash_{\mathrm{LJ}} \psi$, then because Γ is finite we have $\Gamma \subseteq E_k$ for some k, and by saturation of E_k we have $\psi \in E_k \subseteq E'$. The same argument shows that $E' \nvdash_{\mathrm{LJ}} \phi$, and thus E' is consistent: if $\bot$ could be derived, ϕ would also be derivable by rule $\bot_L$. It only remains to show that E' enjoys the disjunction property. Let $\phi = \phi_1 \vee \phi_2 \in E'$, there must be some k such that $\phi \in E_k$. By construction, the disjunction property will be restored for that formula in at least $r(\phi)$ steps, thus we have $\phi_1 \in E_{k+r(\phi)}$ or $\phi_2 \in E_{k+r(\phi)}$, and the disjunction property is satisfied for ϕ in E'. $\square$

Lemma 4.5. Let E be a world-set and ϕ a formula. We have $\mathcal{U}, w_E \models \phi$ iff $\phi \in E$.

Proof. We proceed by (structural) induction on the formula.

- Case of $\top$. We always have $w_E \models \top$ and also always have $\top \in E$ by saturation and rule $\top_R$.
- Case of $\bot$. We never have $w_E \models \bot$, and never have $\bot \in E$ for a consistent E.
- Case of P. By definition, $w_E \models P$ iff $P \in \alpha(w_E) = E \cap P$ iff $P \in E$.
- Case of $\phi_1 \wedge \phi_2$.
  - ($\Rightarrow$) From $w_E \models \phi_1 \wedge \phi_2$ we obtain $w_E \models \phi_1$ and $w_E \models \phi_2$. By induction hypotheses we thus have $E \vdash_{\mathrm{LJ}} \phi_1$ and $E \vdash_{\mathrm{LJ}} \phi_2$, and we can conclude by rule $\wedge_R$.
  - ($\Leftarrow$) By assumption we have $E \vdash_{\mathrm{LJ}} \phi_1 \wedge \phi_2$. This allows us to conclude $E \vdash_{\mathrm{LJ}} \phi_i$ for each $i \in \{1, 2\}$, using rules $\wedge_L$, cut and axiom. By induction hypotheses this yields $w_E \models \phi_i$ for each i, which allows us to conclude.
- Case of $\phi_1 \vee \phi_2$.
  - ($\Rightarrow$) As in the previous case, but using rule $\vee_R$ instead of $\wedge_R$.
  - ($\Leftarrow$) If $\phi_1 \vee \phi_2 \in E$, then by the disjunction property of world-sets we have $\phi_i \in E$ for some i. By induction hypothesis this yields $w_E \models \phi_i$ and thus $w_E \models \phi_1 \vee \phi_2$.
- Case of $\phi_1 \Rightarrow \phi_2$.
  - ($\Rightarrow$) By rule $\Rightarrow_R$ it suffices to show $E \cup \{\phi_1\} \vdash_{\mathrm{LJ}} \phi_2$. Assume the contrary. Then by Lemma 4.4 there is some world-set E' such that $E \leq E'$, $\phi_1 \in E'$ and $\phi_2 \notin E'$. By induction hypothesis $w_{E'} \models \phi_1$, but then by our assumption $w_E \models \phi_1 \Rightarrow \phi_2$ we must also have $w_{E'} \models \phi_2$. We then have $\phi_2 \in E'$ by induction hypothesis, which is a contradiction.
  - ($\Leftarrow$) Assuming $E \vdash_{\mathrm{LJ}} \phi_1 \Rightarrow \phi_2$, we show $w_E \models \phi_1 \Rightarrow \phi_2$. We simply follow the definition of satisfaction for an implication. For any $E \leq E'$ such that $w_{E'} \models \phi_1$, we have to establish $w_{E'} \models \phi_2$. By induction hypothesis we have $\phi_1 \in E'$, or in other words $E' \vdash_{\mathrm{LJ}} \phi_1$. Since we also have $E' \vdash_{\mathrm{LJ}} \phi_1 \Rightarrow \phi_2$, we conclude $E' \vdash_{\mathrm{LJ}} \phi_2$ by rules cut, axiom and $\Rightarrow_L$. By induction hypothesis we can finally conclude: $w_{E'} \models \phi_2$.

$\square$

Theorem 4.6. $\Gamma \models \phi$ implies $\Gamma \vdash_{\mathrm{LJ}} \phi$.

Proof. Assume $\Gamma \models \phi$ and $\Gamma \not\vdash_{\mathrm{LJ}} \phi$. By Lemma 4.4 we have some world-set E such that $\Gamma \subseteq E$ and $\phi \notin E$. We obviously have $w_E \models \Gamma$, so by $\Gamma \models \phi$ we also have $w_E \models \phi$. By Lemma 4.5, this implies $\phi \in E$, which is a contradiction. □

5 Atomic axiom

Proposition 5.1. The axiom rule can be restricted to apply only to propositional variables, without loosing completeness.

Proof. By induction on the ϕ, we build a derivation of $\phi \vdash \phi$ in which the axiom is only used on propositional variables. □

6 Cut elimination

The cut rule is admissible. This is shown by means of a cut elimination procedure, which gradually pushes cuts towards the leaves of the derivation tree. We do not detail it here; it will be covered as an exercise.

Cut elimination has many consequences. It is at the basis of the Curry-Howard correspondence between proofs and programs: proofs, equipped with their cut elimination procedure, can naturally be seen as programs equipped with a reduction semantics. Cut elimination also directly implies the consistency of the calculus. We see next another simple but less classical corollary of cut elimination.

Proposition 6.1. The rule $\wedge_L$ is invertible, meaning that if its conclusion is derivable, then so is its premise. From the viewpoint of somebody trying to build a proof of the conclusion, this means that applying the rule will never loose provability.

Proof. Assuming a derivation Π, with a conjunctive hypothesis in its conclusion sequent, we create a new one that ends with $\wedge_L$, then uses cut to "undo" the left conjunction rule and get back to the original sequent, proved by Π:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi_1 \vdash \phi_1} \quad \dfrac{}{\phi_2 \vdash \phi_2}}{\phi_1, \phi_2 \vdash \phi_1 \wedge \phi_2} \quad \dfrac{\Pi}{\Gamma, \phi_1 \wedge \phi_2 \vdash \psi}}{\Gamma, \phi_1, \phi_2 \vdash \psi}\ \mathrm{cut}}{\Gamma, \phi_1 \wedge \phi_2 \vdash \psi}$$

This shows that $\wedge_L$ is invertible and by cut elimination it is also invertible in the cut-free fragment of LJ_0.

This simple argument is not so useful for proof search: we would like to know not only that the application does not loose provability but also that it gets us closer to completing a proof. To get some insight in that respect, we present an argument based on proof transformations, or more precisely rule permutations. As a corollary, we will obtain that the derivation starting with $\wedge_L$ is no higher than the original one — under the (harmless) hypothesis that the axiom is restricted to propositional variables.

Let Π be a derivation of $\Gamma, (\phi_1 \wedge \phi_2)^n \vdash \psi$. We prove by induction on Π that there is a derivation of $\Gamma, \phi_1^n, \phi_2^n \vdash \psi$.

- If the last rule does not apply to one of the $\phi_1 \wedge \phi_2$ formulas, we apply essentially the same rule and conclude by induction hypothesis where needed. For instance, if Π performs a left introduction rule on $\perp$, we do the same on $\Gamma, \phi_1^n, \phi_2^n \vdash \psi$ and conclude immediately. If Π performs a contraction in Γ, we do the same and conclude by induction hypothesis.
- If $\wedge L$ is performed on some occurrence of $\phi_1 \wedge \phi_2$ we conclude by induction hypothesis with $n-1$.
- Otherwise, a structural rule must be applied to one of our $\phi_1 \vee \phi_2$ formulas. If one of our formulas is weakened away, we produce the required derivation by weakening the corresponding subformulas ϕ_1 and ϕ_2. If one of the $\phi_1 \wedge \phi_2$ is contracted then by induction hypothesis we have a derivation of $\Gamma, \phi_1^{n+1}, \phi_2^{n+1} \vdash \psi$, and by applying two contraction rules we obtain the expected result.

□

Obviously, not all rules are invertible. The right rule for disjunction is not invertible. Because of the splitting of the context, the right rule for conjunction is also not invertible.

7 Eliminating structural rules

We give in Figure 2 a new set of rules, forming the system LJ_0^-. Our purpose is to obtain a system in which proof search is as simple as possible. They are similar to the rules of LJ_0, but we did not include the cut and structural rules. The latter removal has forced us to modify the way other rules deal with their contexts. Rules axiom and $\top_R$ do not require an empty context; without this, weakening would not be admissible. Rule $\wedge_R$ and $\Rightarrow_L$ do not split their conclusion context, but copy it to their premises.

Proposition 7.1. LJ_0^- enjoys the subformula property: all formulas occurring in a derivation are subformulas of formulas occuring in the conclusion sequent.

Proof. Obvious, by a simple inspection of the rules. If needed, it may be done by induction over the derivation. □

Proposition 7.2. Weakening is admissible: if $\Gamma \vdash \phi$ is derivable in LJ_0^-, then so is $\Gamma, \psi \vdash \phi$.

Proof. Same as previously. □

Proposition 7.3. Contraction is admissible in LJ_0^-.

Proof. We do not give a complete proof. The idea is to add a contraction rule, then eliminate it by repeatedly applying local proof transformations that push the contraction towards the leaves, where it can be absorbed. The tricky cases are when a contraction is followed by a left introduction rule on the contracted formula. Consider for instance:

$$\dfrac{\dfrac{\dfrac{\Pi}{\Gamma, \phi_1 \wedge \phi_2, \phi_1, \phi_2 \vdash \psi}}{\Gamma, \phi_1 \wedge \phi_2, \phi_1 \wedge \phi_2 \vdash \psi}}{\Gamma, \phi_1 \wedge \phi_2 \vdash \psi}$$

Here we cannot permute the two rules: if we first apply the left introduction rule, the formula $\phi_1 \wedge \phi_2$ won't be available anymore for contraction. Instead we have to show that the hypothesis $\phi_1 \wedge \phi_2$ is not needed in Π, because ϕ_1 and ϕ_2 are already available. Further, we have to eliminate that hypothesis without increasing the height of the proof. This is what we actually proved in the refined argument for Proposition 6.1. Similar arguments are needed for $\vee_L$ and $\Rightarrow_L$. ☐

Proposition 7.4. LJ_0^- is sound and complete with respect to LJ_0.

Proof. Soundness is simple: anything that can be derived in LJ_0^- can be derived in LJ_0, because each rule of LJ_0^- can be simulated by the corresponding LJ_0 rule, plus structural rules.

We now show completeness, *i.e.*,, anything that can be derived in LJ_0 can also be derived in LJ_0^-. We have seen that cut can be eliminated from LJ_0, thus it suffices to show that every other rule can be simulated by rules of LJ_0^-. We have seen that structural rules are admissible. For the logical rules, and axiom, it is simple to obtain the LJ_0 version from the LJ_0^- version plus the admissibility of structural rules. For instance, in the case of $\wedge_R$ we have to show that if $\Gamma \vdash \phi_1$ and $\Delta \vdash \phi_2$ are derivable in LJ_0^-, then so is $\Gamma, \Delta \vdash \phi_1 \wedge \phi_2$. By the admissibility of weakening and contraction we have that $\Gamma, \Delta \vdash \phi_i$ is derivable for both i, and we conclude using the LJ_0^- rule $\wedge_R$. ☐

$$\dfrac{}{\Gamma, \bot \vdash \phi}\ \bot_L \qquad \dfrac{}{\Gamma, \phi \vdash \phi}\ \text{axiom} \qquad \dfrac{}{\Gamma \vdash \top}\ \bot_R$$

$$\dfrac{\Gamma, \phi_1, \phi_2 \vdash \psi}{\Gamma, \phi_1 \wedge \phi_2 \vdash \psi}\ \wedge_L \qquad \dfrac{\Gamma \vdash \phi_1 \quad \Gamma \vdash \phi_2}{\Gamma \vdash \phi_1 \wedge \phi_2}\ \wedge_R$$

$$\dfrac{\Gamma, \phi_1 \vdash \psi \quad \Gamma, \phi_2 \vdash \psi}{\Gamma, \phi_1 \vee \phi_2 \vdash \psi}\ \vee_L \qquad \dfrac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2}\ \vee_R$$

$$\dfrac{\Gamma, \phi_1 \Rightarrow \phi_2 \vdash \phi_1 \quad \Gamma, \phi_2 \vdash \psi}{\Gamma, \phi_1 \Rightarrow \phi_2 \vdash \psi}\ \Rightarrow_L \qquad \dfrac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \Rightarrow \psi}\ \Rightarrow_R$$

Figure 2: A complete intuitionistic sequent calculus without structural rules

Example 7.5. This example shows that, in the system of Figure 2 it is necessary to keep the implication formula in the left premise of the $\Rightarrow_L$ rule:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{A, \neg(A \vee \neg A) \vdash A}\text{ axiom}}{A, \neg(A \vee \neg A) \vdash A \vee \neg A}\vee_R}{A, \neg(A \vee \neg A) \vdash \bot}\Rightarrow_L}{\neg(A \vee \neg A) \vdash \neg A}\Rightarrow_R}{\neg(A \vee \neg A) \vdash A \vee \neg A}\vee_R}{\neg(A \vee \neg A) \vdash \bot}\Rightarrow_L}{\vdash \neg\neg(A \vee \neg A)}\Rightarrow_R$$

Starting from the conclusion sequent (which is valid) we always apply the only possible rule, ruling out applications that would get us back to a previously encountered sequent. The first two rule applications are forced by the structure of the sequent: axiom is not possible, and only the implication can be introduced, first one the right and then on the left. With $\neg(A \vee \neg A) \vdash \neg(A \vee \neg A)$ we could introduce again the implication on the left, but that would cycle, so we apply $\vee_R$. We have to choose a disjunct, but if we chose A we would be forced to do $\Rightarrow_L$ after that, and cycle. Without $\phi_1 \Rightarrow \phi_2$ in the left premise of $\Rightarrow_L$, we would have obtained $\vdash \neg(A \vee \neg A)$ which is not valid, hence not provable.