# Logique & Calculabilité

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**Exercise 8 (interpolation theorem)**
Let $\phi$ and $\psi$ be two formulas such that $\phi \models \psi$. We have shown that there is a formula $\theta$ (called the *interpolant*) such that $\phi \models \theta$ and $\theta \models \psi$, and all propositional variables occurring in $\theta$ occur in both $\phi$ and $\psi$.

1. Apply the method to obtain an interpolant for $p \wedge q \models \phi$ when $q$ does not occur in $\phi$.

2. Compute an interpolant for $\phi \models p \wedge q$ when $q$ does not occur in $\phi$, by reducing the problem to the earlier situation.

**Exercise 15**
A set of formulas $E$ is *independent* if, for any $\phi \in E$, $E \setminus \{\phi\} \not\models \phi$.

1. Show that any finite set of formulas $E$ admits an independent subset $E'$ such that, for any $\phi \in E$, $E' \models \phi$.

2. Show that any countable set of formulas $E$ admits an equivalent independent set of formulas $E'$, *i.e.*, for any $\phi \in E'$, $E \models \phi$ and for any $\psi \in E$, $E' \models \psi$.

3. Show that it is impossible to require additionally that $E' \subseteq E$.

**Exercise 17**
Show that a graph can be colored with $k$ colors iff each of its finite sub-graphs can be colored with $k$ colors.

**Exercise 11,13,14**

1. Show that $\vee$, $\wedge$, $\neg$ are definable using only the connective $\rightarrow$ and the constant $\bot$. We say that the set $\{\rightarrow, \bot\}$ is *functionally complete.*

2. Give a binary connective which, alone, is functionally complete.

3. Show that $\{\leftrightarrow, \neg\}$ is not functionnally complete.

**Exercise 20**
Give a formula which admits two different clausal normal forms.

**Exercise 21**
We aim to estimate the maximal blowup in the size of formulas when putting them in clausal normal form. For $\phi \in \mathcal{F}_0(\mathcal{P})$, let $\tau(\phi)$ be the minimal size of a clausal formal for $\phi$.

1. Find a family of formulas $\phi_n$ such that $|\phi_n|$ grows linearly in $n$ and $\tau(\phi_n)$ grows exponentially in $n$. More precisely, we wish to obtain:

$$\lim_{n \to +\infty} \frac{\tau(\phi_n)}{\sqrt{2^{|\phi_n|}}} > 0$$

2. Show that $\tau(\phi) < |\phi| \times 2^{\frac{|\phi|+3}{2}}$ for any $\phi$.

**Exercise 19'**
Consider a set $E$ of *Horn clauses*, *i.e.*, clauses containing at most one positive literal. We shall view interpretations as sets of predicates variables, instead of functions from predicates to Booleans.

1. Show that models of $E$ are closed under finite intersection.

2. Show that there is a least model of $E$.

3. Given a predicate variable $p$, propose an algorithm for deciding whether $E \vDash p$ or not.