Adding support for induction in Dedukti

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Context. Lambdapi is a new proof assistant based on a logical framework called the $\lambda\Pi$-calculus modulo rewriting, which is an extension of the simply-typed λ-calculus (the basis of functional programming languages like OCaml or Haskell) with dependent types (e.g. vectors and matrices of some given dimension) and an equivalence relation on types generated by user-defined rewrite rules [3]. Thanks to rewriting, Lambdapi allows the formalization of proofs that cannot be done in other proof assistants (e.g. simplicial sets of infinite dimensions).

However, there is currently no user support for doing inductive proofs. Currently, the user must define induction principles by hand. This can quickly become heavy when there are many constructors, and difficult when one considers complex inductive types.

To take a simple example, from the following definition:

```
inductive N: TYPE ≔ // natural numbers
| 0: N
| s: N ⇒ N
```

we would like the following code to be internally generated:

```
constant symbol N: TYPE
constant symbol 0: N
constant symbol s: N ⇒ N
symbol ind_N: ∀p,π(p0) ⇒ (∀n,π(pn) ⇒ π(p(sn))) ⇒ ∀n,π(pn)
rule ind_N p p0 ps 0     → p0
and  ind_N p p0 ps (s n) → ps n (ind_N p p0 ps n)
```

where `p:N⇒Prop` is a predicate on `N` (`Prop` is the type of propositions) and `π:Prop⇒TYPE` maps propositions to types.

Goal. The goal of the internship is to automate the definition of the induction principle associated to an inductive type [11]. As a follow-up, one could consider how to handle co-inductive types [5] and co-recursion [1] in Dedukti as well, that is, terms representing infinite objects like streams .

Workplan. One can start by considering the case of a simple first-order inductive type like N, and then consider increasingly more complex classes of inductive types:

- polymorphic inductive types

```
inductive L: Set ⇒ TYPE ≔ // lists
| nil{α}: Lα
| cons{α}: τα ⇒ Lα ⇒ Lα
```

  where Set is the type of sets and τ:Set⇒TYPE maps sets to types.

- dependent inductive types

```
inductive V: Set ⇒ N ⇒ TYPE ≔ // vectors
| nil{α}: Vα0
| cons{α}: τα ⇒ ∀n,Vαn ⇒ Vα(sn)
```

- mutually defined inductive types

```
inductive tree: TYPE ≔ // trees with finite branching
| leaf: tree
| node: forest ⇒ tree
and forest: TYPE ≔
| empty: forest
| cons: tree ⇒ forest ⇒ forest
```

- strictly positive inductive types

```
inductive O: TYPE ≔ // ordinals
| 0: O
| s: O ⇒ O
| sup: (N ⇒ O) ⇒ O
```

- positive inductive types [4]

```
inductive Rou: TYPE ≔ // continuations
| over: Rou
| next: ((Rou ⇒ LN) ⇒ LN) ⇒ Rou
```

- nested inductive types [10]

```
inductive Bush: Set ⇒ TYPE ≔
| nil{α}: Bushα
| cons{α}: τα ⇒ Bush(Bushα) ⇒ Bushα
```

- inductive predicates

```
inductive ≤: N ⇒ N ⇒ Prop ≔
| 0≤ : ∀x,π(0 ≤ x)
| s≤ : ∀x y,π(x ≤ y) ⇒ π(s x ≤ s y)
```

- inductive-inductive types [9]

```
inductive SortL: TYPE ≔ // sorted lists
| nil:SortL
| cons: ∀y l (p:Le y l), SortL
and Le: N ⇒ SortL ⇒ TYPE ≔
| Le_nil: ∀x,Le x nil
| Le_cons: ∀x y l (p:Le y l),
  x ≤ y ⇒ Le x l ⇒ Le x (cons y l p)
```

- inductive-recursive definitions [6]

```
inductive UniqL: TYPE ≔ // lists with unique elements
| nil: UniqL
| cons: ∀x l, x ∉ l ⇒ UniqL
and ∉: N ⇒ UniqL ⇒ TYPE
modulo x ∉ nil          → True
       x ∉ cons y l _ → x ≠ y ∧ x ∉ l
```

- the combination of both [7]

- quotient types [8]

```
inductive Bag: TYPE ≔ // multisets
| nil: Bag
| cons: N ⇒ Bag ⇒ Bag
modulo
| swap: ∀x y b, cons x y b = cons y x b
```

where `=` is Leibniz's polymorphic equality (`x=y` iff `x` and `y` satisfies the same propositions).

- higher dimensional inductive types [12]

```
inductive S2: TYPE ≔ // homotopic sphere
| base: S2
modulo
| surf: eq (eq base base) refl refl
```

where `eq` is Leibniz equality and `refl` the canonical reflexivity proof.

- co-inductive types [1]

```
coinductive stream : TYPE // streams of natural numbers
| head: stream ⇒ N
| tail: stream ⇒ stream

constant symbol zeros : stream // stream of 0
head zeros → 0
tail zeros → zeros
```

An interesting working example is to define in Lambdapi the set of well-typed terms of λΠ itself [2], the simplest dependent type theory, and compare it with its definition in Agda.

Requirements. Some familiarity with a functional programming language with pattern-matching.

References

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