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On Flatness for 2-dimensional Vector Addition Systems with States

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Abstract. Vector addition systems with states (VASS) are counter automata where (1) counters hold nonnegative integer values, and (2) the allowed operations on counters are increment and decrement. Accelerated symbolic model checkers, like FAST, LASH or TReX, provide generic semi-algorithms to compute reachability sets for VASS (and for other models), but without any termination guarantee. Hopcroft and Pansiot proved that for 2-dim VASS (i.e. VASS with two counters), the reachability set is effectively semilinear. However, they use an ad-hoc algorithm that is specifically designed to analyze 2-dim VASS. In this paper, we show that 2-dim VASS are *flat* (i.e. they "intrinsically" contain no nested loops). We obtain that — forward, backward and binary — reachability sets are effectively semilinear for the class of 2-dim VASS, and that these sets can be computed using generic acceleration techniques.

1 Introduction

Distributed systems have regained much attention recently, especially due to the popularization of the Internet. Ensuring correctness of distributed systems is usually challenging, as these systems may contain subtle errors that are very hard to find. To overcome this difficulty, a *formal verification* approach can be employed: model the system, model the desired property, and algorithmically check that the system satisfies the property.

Petri nets, and equivalently *vector addition systems with states* (VASS), are a widely used formalism to model concurrent distributed systems. Basically, a VASS is a *counter automaton* where (1) counters hold nonnegative integer values, and (2) the allowed operations on counters are *increment* and *decrement*. As the counters are unbounded, VASS are naturally *infinite-state* systems.

Various formalisms have been proposed to model desired properties on systems. In this work, we only consider *safety* properties: these properties (of the original system) may often be expressed by *reachability properties* on the model.

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Reachability properties are algorithmically checkable for *finite-state* systems (and efficient implementations exist). However, the situation is more complex for *infinite-state* systems: the reachability problem is undecidable even for restricted classes of systems, such as Minsky machines [Min67].

Accelerated symbolic model-checking. Verification of reachability properties usually proceeds through an iterative fixpoint computation of the *forward reachability set* post^* (resp. *backward reachability set* pre^*), starting from the initial states (resp. from the error states). When the state space is infinite, finite *symbolic representations* for sets of states are required [HM00]. To help termination of this fixpoint computation, so-called *acceleration* techniques (or *meta-transitions*) are applied [BW94, BGWW97, BH99, BJNT00, FIS03, FL02]. Basically, acceleration consists in computing in one step the effect of iterating a given loop (of the control flow graph). Accelerated symbolic model checkers such as `LASH` [Las], `TReX` [ABS01], and `FAST` [BFLP03] implement this approach.

Accelerated symbolic model-checking is only a *semi*-algorithm: it does not provide any guarantee of termination. Still, this approach is very promising as it behaves well in practice. It would be desirable to determine some classes of systems for which termination is guaranteed. A natural sufficient condition for termination is *flatness* [CJ98]: a system S is called *flat*[3] when we can "extract" from S a finite number of subsystems $S'_0, \ldots, S'_n$ such that (1) each S'_i contains no nested loops, and (2) reachability in S is equivalent to reachability in $\bigcup_i S'_i$. When the system is flat, and if every loop can be accelerated, then (a symbolic representation of) post^* / pre^* is computable. In fact, flatness is also a necessary condition for termination of acceleration-based semi-algorithms. Hence, flatness turns out to be a crucial notion for termination analysis of accelerated symbolic model-checking.

Dedicated algorithms for VASS. Many specialized algorithms have been designed to solve verification problems for various extensions and restrictions of VASS. The reachability problem for VASS has been proved decidable [May84, Kos82]. The reachability sets post^* and pre^* are effectively semilinear for Lossy VASS [BM99]. The class of 2-dimensional VASS (i.e. VASS with only two counters) has received much attention. Hopcroft and Pansiot proved that the reachability sets post^* and pre^* are effectively semilinear for this class [HP79]. It was later shown that post^* / pre^* are still effectively semilinear for various extensions of 2-dim VASS [FS00b, FS00a]. However, these methods suffer from serious drawbacks: (1) they cannot be easily extended or combined, and (2) from an implementation perspective, a dedicated tool would be needed for each specialized algorithm.

Our contribution. Recall that the reachability sets post^* and pre^* have been shown to be effectively semilinear for the class of 2-dim VASS [HP79]. In this

[3] Our notion of flatness is actually more general than in [CJ98]: there, a system is called flat when it contains no nested loops.

paper, we investigate termination of the generic acceleration-based computation of $\text{post}^* / \text{pre}^*$ for this class. The reader familiar with Petri nets will observe that our results also hold for the class of Petri nets having only two unbounded places. Our main result is the following:

Every 2-dim VASS is flat

Since every loop in a VASS can be accelerated [CJ98, FL02], we also obtain the following new results:

i) the binary reachability relation $\mathcal{R}^*$ is effectively semilinear for the class of 2-dim VASS.
ii) the semilinear sets post^*, pre^* and $\mathcal{R}^*$ of any 2-dim VASS can be computed using generic acceleration techniques.

In particular, we get that accelerated symbolic model checkers such as `LASH`, `TReX`, or `FAST`, terminate on 2-dim VASS (if a suitable search strategy is used). From a practical viewpoint, our approach has several benefits: (1) we can apply a generic algorithm, which was designed for a much larger class of systems, with no need for a preliminary syntactic check, and (2) the sets post^*, pre^* and $\mathcal{R}^*$ can be computed using the same algorithm. For instance, we directly obtain that all six VASS examples given in [BLW03] are flat, and hence their binary reachability relation can be computed by means of acceleration.

The effective semilinearity of the binary reachability relation $\mathcal{R}^*$ is also a surprising theoretical result, which can prove useful in pratice. Indeed, we may express many properties on the model as a first order formula over states using (binary) predicates $\mathcal{R}$ and $\mathcal{R}^*$. For instance, we can check relationships between input values and output values for a 2-dim VASS modeling a function. We may also use this computation of $\mathcal{R}^*$ for *parameter synthesis*: for instance, we can compute the set of initial states such that the counters stay bounded, or such that the system terminates. We may also express and check subtle properties on the system's strongly connected components (SCC).

Our results obviously extend to the class of (arbitrary) VASS where only two counters are modified. To guide the analysis of a complex modular VASS, we may choose to replace some flat subsystems (e.g. subsystems manipulating only two counters) by equivalent semilinear meta-transitions. Such a flatness-guided approach would surely help termination of accelerated symbolic model checkers.

Outline. The paper is organized as follows. Section 2 presents vector addition systems with states. We introduce the notion of flatness in Section 3 and we show that the binary reachability relation of any flat VASS is effectively semilinear. As a first step towards flatness, Section 4 investigates the notion of ultimate flatness. Finally, in Section 5, we focus on dimension 2 and we prove our main result: every 2-dim VASS is flat.

For the sake of completeness, all results are proved in detail (except for two easy small proofs), and we also prove a few well known facts. This paper is the long version of our CONCUR 2004 paper.

2 Vector addition systems with states (VASS)

This section is devoted to the presentation of vector addition systems with states. We first give basic definitions and notations that will be used throughout the paper.

2.1 Numbers, vectors, relations

Let $\mathbb{Z}$ (resp. $\mathbb{N}$, $\mathbb{Z}^-$, $\mathbb{Q}$, $\mathbb{Q}_+$) denote the set of *integers* (resp. *nonnegative integers*, *nonpositive integers*, *rational numbers*, *nonnegative rational numbers*). We denote by $\leq$ the *usual total order on* $\mathbb{Q}$. Given $k, l \in \mathbb{N}$, we write $[k \mathinner{..} l]$ (resp. $[k \mathinner{..} \infty[$) for the *interval of integers* $\{i \in \mathbb{N} \;/\; k \leq i \leq l\}$ (resp. $\{i \in \mathbb{N} \;/\; k \leq i\}$). We write $|X|$ the *cardinal* of any finite set X.

Given a set X and $n \in \mathbb{N}$, we write X^n for the set of n-dim *vectors* x of elements in X. For any index $i \in [1 \mathinner{..} n]$, we denote by $\mathsf{x}[i]$ the i^{th} *component* of an n-tuple x.

We now focus on n-dim vectors of (integer or rational) numbers. We write $\mathsf{0}$ for the *all zero vector*: $\mathsf{0}[i] = 0$ for all $i \in [1 \mathinner{..} n]$. We also denote by $\leq$ the *usual partial order on* $\mathbb{Q}^n$, defined by $\mathsf{x} \leq \mathsf{y}$ if for all $i \in [1 \mathinner{..} n]$ we have $\mathsf{x}[i] \leq \mathsf{y}[i]$.

Operations on n-dim vectors are componentwise extensions of their scalar counterpart (e.g. for $\mathsf{x}, \mathsf{x}' \in \mathbb{Q}^n$, $\mathsf{x} + \mathsf{x}'$ is the vector $\mathsf{y} \in \mathbb{Q}^n$ defined by $\mathsf{y}[i] = \mathsf{x}[i] + \mathsf{x}'[i]$ for all $i \in [1 \mathinner{..} n]$). For $\alpha \in \mathbb{Q}$ and $\mathsf{x} \in \mathbb{Q}^n$, $\alpha \mathsf{x}$ is the vector $\mathsf{y} \in \mathbb{Q}^n$ defined by $\mathsf{y}[i] = \alpha \mathsf{x}[i]$ for all $i \in [1 \mathinner{..} n]$.

These operations are classically extended on sets of n-dim vectors (e.g. for $P, P' \subseteq \mathbb{Q}^n$, $P + P' = \{\mathsf{p} + \mathsf{p}' \;/\; \mathsf{p} \in P, \mathsf{p}' \in P'\}$). Moreover, in an operation involving sets of n-dim vectors, we shortly write x for the singleton $\{\mathsf{x}\}$ (e.g. for $P \subseteq \mathbb{Q}^n$ and $\mathsf{x} \in \mathbb{Q}^n$, we write $\mathsf{x} + P$ for $\{\mathsf{x}\} + P$).

A *binary relation* R on some set X is any subset of $X \times X$. We shortly write $x \, R \, x'$ whenever $(x, x') \in R$. Given two binary relations R_1, R_2 on X, the *composed binary relation* $R_1 \cdot R_2$ on X is defined by $x \, (R_1 \cdot R_2) \, x'$ if we have $x \, R_1 \, y$ and $y \, R_2 \, x'$ for some $y \in X$. We denote by R^* the *reflexive and transitive closure* of R. The *identity relation on* X is the binary relation $Id_X = \{(x, x) \;/\; x \in X\}$. In the rest of the paper, we will only consider binary relations, and they will shortly be called *relations*.

2.2 Vector addition systems with states

Definition 2.1. *An n-dim* vector addition system with states *(VASS for short) is a 5-tuple $V = (Q, T, \alpha, \beta, \delta)$ where Q is a finite non empty set of* locations*, T is a finite non empty set of* transitions*, $\alpha : T \to Q$ and $\beta : T \to Q$ are the* source *and* target *mappings, and $\delta : T \to \mathbb{Z}^n$ is a* transition displacement *labeling.*

An n-dim VASS is basically a finite graph whose edges are labeled by n-dim vectors of integers. Each component $i \in [1 \mathinner{..} n]$ corresponds to a counter ranging over $\mathbb{N}$. Operationally, control flows from one location to another along transitions, and counters simultaneously change values by adding the transition's displacement (as long as the counters remain nonnegative).

Formally, let $V = (Q, T, \alpha, \beta, \delta)$ be an n-dim VASS. The *set of configuration* $\mathcal{C}_V$ of V is $Q \times \mathbb{N}^n$, and the semantics of each transition $t \in T$ is given by the *transition reachability relation* $\mathcal{R}_V(t)$ over $\mathcal{C}_V$ defined by:

$$(q, \mathsf{x})\ \mathcal{R}_V(t)\ (q', \mathsf{x}') \quad \text{if} \quad q = \alpha(t),\ q' = \beta(t) \text{ and } \mathsf{x}' = \mathsf{x} + \delta(t)$$

We write T^+ for the set of all *non empty words* $t_0 \cdots t_k$ with $t_i \in T$, and ε denotes the *empty word*. The set $T^+ \cup \{\varepsilon\}$ of all *words* π over T is denoted by T^*. Transition displacements and transition reachability relations are naturally extended to words:

$$\begin{cases} \delta(\varepsilon) = 0 \\ \delta(\pi \cdot t) = \delta(\pi) + \delta(t) \end{cases} \qquad\qquad \begin{cases} \mathcal{R}_V(\varepsilon) = Id_{\mathcal{C}_V} \\ \mathcal{R}_V(\pi \cdot t) = \mathcal{R}_V(\pi) \cdot \mathcal{R}_V(t) \end{cases}$$

A *language* over T is any subset L of T^*. We also extend displacements and reachability relations to languages: $\delta(L) = \{\delta(\pi)\ /\ \pi \in L\}$ and $\mathcal{R}_V(L) = \bigcup_{\pi \in L} \mathcal{R}_V(\pi)$.

Definition 2.2. *Given a VASS $V = (Q, T, \alpha, \beta, \delta)$, the* one-step reachability relation *of V is the relation $\mathcal{R}_V(T)$, shortly written $\mathcal{R}_V$. The* global reachability relation *of V is the relation $\mathcal{R}_V(T^*)$, shortly written $\mathcal{R}_V^*$.*

Remark that the global reachability relation is the reflexive and transitive closure of the one-step reachability relation. The *global reachability relation* of a VASS V is also usually called *the binary reachability relation* of V. A *reachability subrelation* is any relation $R \subseteq \mathcal{R}_V^*$. For the reader familiar with transition systems, the operational semantics of V can be viewed as the infinite-state transition system $(\mathcal{C}_V, \mathcal{R}_V)$.

Consider two locations q and q' in a VASS V. A word $\pi \in T^*$ is called a *path from q to q'* if either (1) $\pi = \varepsilon$ and $q = q'$, or (2) $\pi = t_0 \cdots t_k$ and satisfies: $q = \alpha(t_0)$, $q' = \beta(t_k)$ and $\beta(t_{i-1}) = \alpha(t_i)$ for every $i \in [1\,..\,k]$. A path from q to q is called a *loop on q*, or shortly a *loop*. We denote by $\Pi_V(q, q')$ the set of all paths from q to q' in V. The set $\bigcup_{q,q' \in Q} \Pi_V(q, q')$ of all *paths* in V is written Π_V.

Notation. In the following, we will simply write $\mathcal{R}$ (resp. Π, $\mathcal{C}$) instead of $\mathcal{R}_V$ (resp. Π_V, $\mathcal{C}_V$), when the underlying VASS is unambiguous. We will also sometimes write $\rightarrow$ (resp. $\xrightarrow{\pi}$, $\xrightarrow{L}$, $\xrightarrow{*}$) instead of $\mathcal{R}$ (resp. $\mathcal{R}(\pi)$, $\mathcal{R}(L)$, $\mathcal{R}^*$).

We will later use the following fact, which we leave unproved as it is a well known property of VASS. Recall that a *prefix* of a given word $\pi \in T^*$ is any word σ such that $\pi = \sigma \cdot \sigma'$ for some word σ'.

Fact 1. *For any configurations (q, x) and (q', x') of a VASS V, and for any word $\pi \in T^*$, we have:*

$$(q, \mathsf{x}) \xrightarrow{\pi} (q', \mathsf{x}') \quad \textit{iff} \quad \begin{cases} \pi \in \Pi(q, q') \\ \mathsf{x}' = \mathsf{x} + \delta(\pi) \\ \mathsf{x} + \delta(\sigma) \geq 0 \textit{ for every prefix } \sigma \textit{ of } \pi \end{cases}$$

Observe that for any word $\pi \in T^*$, the relation $\mathcal{R}_V(\pi)$ is non empty iff π is a path.

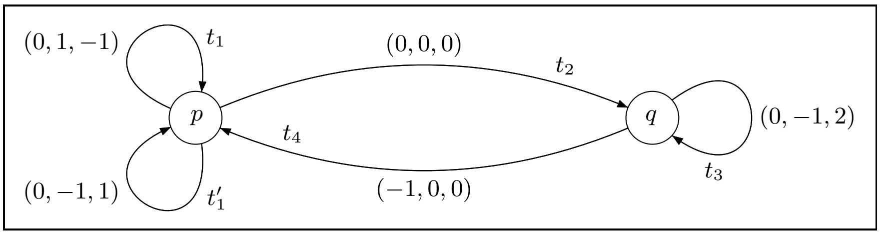

Fig. 1. A 3-dim VASS weakling computing the powers of 2.

Example 2.3. Consider the 3-dim VASS E depicted on Figure 1. This example is a variation of an example in [HP79]. Formally, this VASS is the 5-tuple $E = (\{p, q\}, \{t_1, t'_1, t_2, t_3, t_4\}, \alpha, \beta, \delta)$ where $\alpha(t_1) = \alpha(t'_1) = \alpha(t_2) = \beta(t_1) = \beta(t'_1) = \beta(t_4) = p$, and $\alpha(t_3) = \alpha(t_4) = \beta(t_3) = \beta(t_2) = q$, and δ is defined by: $\delta(t_1) = -\delta(t'_1) = (0, 1, -1)$, $\delta(t_2) = (0, 0, 0)$, $\delta(t_3) = (0, -1, 2)$ and $\delta(t_4) = (-1, 0, 0)$.

Intuitively, the loop t_1 on p transfers the contents of the third counter into the second counter, while the loop t_3 on q transfers twice as much as the contents of the second counter into the third counter. However, the VASS may change location (using transition t_2 or t_4) before the transfer completes (a "zero-test" would be required to ensure that the transfer always completes). Transition t_2 acts as a "silent transition", and transition t_4 decrements the first counter by 1. The loop t'_1 on p has been added to simplify the expression of $\mathcal{R}^*$.

Consider the path $\pi = t_1 t_1 t_2 t_3 t_3 t_4 t_2$. It is readily seen that the reachability subrelation $\mathcal{R}(\pi)$ is precisely the set of pairs $((p, (x+1, y, z+2)), (q, (x, y, z+4)))$ with $(x, y, z) \in \mathbb{N}^3$. This little VASS exhibits a rather complex global reachability relation, since it can be proved[4] that: $(p, (x, y, z)) \xrightarrow{*} (p, (x', y', z'))$ iff $x' \leq x$, $y + z \leq y' + z'$, and $2^{x'}(y' + z') \leq 2^x(y + z)$. □

3 Effective semilinearity of $\mathcal{R}^*$ for flat VASS

An important concept used in this paper is that of *semilinear sets* [GS66]. For any subset $P \subseteq \mathbb{Z}^n$, we denote by P^* the set of all (finite) linear combinations of vectors in P:

$$P^* = \left\{ \textstyle\sum_{i=0}^{k} c_i \, \mathsf{p}_i \;/\; k, c_0, \ldots, c_k \in \mathbb{N} \text{ and } \mathsf{p}_0, \ldots, \mathsf{p}_k \in P \right\}$$

A subset $S \subseteq \mathbb{Z}^n$ is said to be a *linear set* if $S = (\mathsf{x} + P^*)$ for some $\mathsf{x} \in \mathbb{Z}^n$ and for some finite subset $P \subseteq \mathbb{Z}^n$; moreover x is called the *basis* and vectors

[4] This proof is an adpatation of the proof in [HP79], and is left to the reader.

in P are called *periods*. A *semilinear set* is any finite union of linear sets. Let us recall that semilinear sets are precisely the subsets of $\mathbb{Z}^n$ that are definable in Presburger arithmetic $\langle \mathbb{Z}, \leq, + \rangle$ [GS66].

Observe that any finite non empty set Q can be "encoded" using a bijection η from Q to $[1\,..\,|Q|]$. Thus, these semilinearity notions naturally carry[5] over subsets of $Q \times \mathbb{Z}^n$ and over relations on $Q \times \mathbb{Z}^n$.

Definition 3.1. *A* linear path scheme *(LPS for short) for a VASS V is any language $\rho \subseteq \Pi_V$ of the form $\rho = \sigma_0 \theta_1^* \sigma_1 \cdots \theta_k^* \sigma_k$ where $\sigma_0, \theta_1, \sigma_1, \ldots, \theta_k, \sigma_k$ are words. A* semilinear path scheme *(SLPS for short) is any finite union of LPS.*

Remark that a language of the form $\sigma_0 \theta_1^* \sigma_1 \cdots \theta_k^* \sigma_k$, with $\sigma_i, \theta_i \in T^*$, is an LPS iff (1) $\sigma_0 \theta_1 \sigma_1 \cdots \theta_k \sigma_k$ is a path, and (2) θ_i is a loop for every $i \in [1\,..\,k]$.

Definition 3.2. *Given a VASS V, a reachability subrelation $R \subseteq \mathcal{R}_V^*$ is called* flat *if $R \subseteq \mathcal{R}_V(\rho)$ for some SLPS ρ. We say that V is* flat *when $\mathcal{R}_V^*$ is flat.*

The class of flat reachability subrelations is obviously closed under union and under composition.

From a computability viewpoint, any (finitely "encoded") set S is said to be *effectively semilinear* if (1) S is semilinear, and (2) a finite basis-period description (or equivalently a Presburger formula) for S can be computed (from its "encoding"). The following *acceleration theorem* shows that the reachability subrelation "along" any SLPS is an effectively semilinear set. This theorem was proved in [CJ98, FL02] for considerably richer classes of counter automata. We give a simple proof for the simpler case of VASS.

Theorem 3.3 ([CJ98, FL02]). *For any SLPS ρ in a VASS V, the reachability subrelation $\mathcal{R}_V(\rho)$ is effectively semilinear.*

Proof. Let V denote an n-dim VASS. Observe that for any transition t in V, the reachability subrelation $\mathcal{R}(t)$ is effectively semilinear. As the class of effectively semilinear reachability subrelations is closed under union and under composition, it suffices to show that $\mathcal{R}(\theta^*)$ is effectively semilinear for any loop θ. Consider a loop θ on some location q. It is readily seen that:

$$\begin{array}{rcl} \mathcal{R}(\theta^*) & = & Id_{\{q\} \times \mathbb{N}^n} \\ & & \bigcup \ \mathcal{R}(\theta) \\ & & \bigcup \ \mathcal{R}(\theta) \cdot \{((q, \mathsf{x}), (q, \mathsf{x}')) \ / \ \mathsf{x}, \mathsf{x}' \in \mathbb{N}^n \text{ and } \mathsf{x}' \in (\mathsf{x} + \{\delta(\theta)\}^*)\} \cdot \mathcal{R}(\theta) \end{array}$$

Hence we get that $\mathcal{R}(\theta^*)$ is effectively semilinear, which concludes the proof. $\square$

Corollary 3.4. *The global reachability relation $\mathcal{R}_V^*$ of any flat VASS V is effectively semilinear.*

[5] Obviously, the extension of these notions does not depend on the "encoding" η.

Proof. Assume that V is a flat VASS. Since V is flat, there exists an SLPS ρ satisfying $\mathcal{R}_V^* = \mathcal{R}_V(\rho)$. In order to compute such an SLPS, we may enumerate all SLPS ρ, and stop as soon as ρ satifies $Id_{\mathcal{C}_V} \cup \mathcal{R}_V(\rho) \cdot \mathcal{R}_V \subseteq \mathcal{R}_V(\rho)$. All required computations are effective: $\mathcal{R}_V$ is readily seen to be effectively semilinear, semilinear relations are effectively closed by composition, and equality is decidable between semilinear relations. We then apply Theorem 3.3 on ρ. □

Moreover, the semilinear global reachability relation $\mathcal{R}_V^*$ of any flat VASS V can be computed using an existing "accelerated" symbolic model checker such as `LASH` [Las], `TReX` [ABS01], or `FAST` [BFLP03]. In this paper, we prove that every 2-dim VASS is flat, and thus we get that the global reachability relation of any 2-dim VASS is effectively semilinear. This result cannot be extended to dimension 3 as the 3-dim VASS E of Example 2.3 has a non semilinear global reachability relation.

Given an n-dim VASS V and a subset $S \subseteq \mathcal{C}_V$ of configurations, we denote by $\mathrm{post}_V^*(S)$ the set $\{s' \in \mathcal{C}_V \;/\; \exists\, s \in S, s\, \mathcal{R}_V^*\, s'\}$ of *successors* of S, and we denote by $\mathrm{pre}_V^*(S)$ the set $\{s \in \mathcal{C}_V \;/\; \exists\, s' \in S, s\, \mathcal{R}_V^*\, s'\}$ of *predecessors* of S. It is well known that for any 2-dim VASS V, the sets $\mathrm{post}^*(S)$ and $\mathrm{pre}^*(S)$ are effectively semilinear for every semilinear subset S of configurations [HP79]. One may be tempted to think that the semilinearity of $\mathcal{R}_V^*$ is a consequence of this result. The following proposition shows that this is not the case.

Proposition 3.5. *There exists a 3-dim VASS V such that (1) $\mathrm{post}_V^*(S)$ and $\mathrm{pre}_V^*(S)$ are effectively semilinear for every semilinear subset $S \subseteq \mathcal{C}_V$, and (2) the global reachability relation $\mathcal{R}_V^*$ is not semilinear.*

Proof. We first recall some results on *lossy VASS* [BM99]. The only difference beween (regular) VASS and lossy VASS lies in the definition of the *transition reachability relation* $\mathcal{R}_V(t)$. For lossy VASS, this relation on $\mathcal{C}_V$ is defined by:

$$(q, \mathsf{x})\ \mathcal{R}_V(t)\ (q', \mathsf{x}') \quad \text{if} \quad q = \alpha(t),\, q' = \beta(t) \text{ and } \mathsf{x}' \leq \mathsf{x} + \delta(t)$$

As a straightforward consequence of Theorems 10 and 11 of [BM99], we get that:

> *For every lossy VASS V, the sets $\mathrm{post}_V^*(S)$ and $\mathrm{pre}_V^*(S)$ are effectively semilinear for every semilinear subset $S \subseteq \mathcal{C}_V$.*

Now consider the 3-dim VASS E^λ obtained from the VASS E of Example 2.3 by adding six transitions: three loops on p labeled respectively by $(-1,0,0)$, $(0,-1,0)$ and $(0,0,-1)$, and three loops on q labeled with the same displacements. Basically, E^λ corresponds to the VASS E augmented with explicit "losses". Therefore, we may as well interpret E^λ with a "lossy semantics" since this won't change $\mathcal{R}_{E^\lambda}^*$. Thus, we obtain that $\mathrm{post}_{E^\lambda}^*(S)$ and $\mathrm{pre}_{E^\lambda}^*(S)$ are effectively semilinear for every semilinear subset $S \subseteq \mathcal{C}_V$. However, following Example 2.3, it can be proved that: $(p,(x,y,z)) \xrightarrow{*} (p,(x',y',z'))$ iff $x' \leq x$ and $2^{x'}(y'+z') \leq 2^x(y+z)$. Consequently, the global reachability relation $\mathcal{R}_{E^\lambda}^*$ is not semilinear. □

4 Acceleration works better in absence of zigzags

The rest of the paper is devoted to the proof that every 2-dim VASS is flat. We first establish in this section some preliminary results that hold in any dimension. We will restrict our attention to dimension 2 in the next section.

It is well known that the set $\delta(\Pi(q,q'))$ of displacements of all paths between any two locations q and q' is a semilinear set. We now give a stronger version of this result: this set of displacements can actually be "captured" by an SLPS.

Lemma 4.1. *For every pair (q,q') of locations in a VASS V, there exists an SLPS $\rho_{q,q'} \subseteq \Pi(q,q')$ such that $\delta(\rho_{q,q'}) = \delta(\Pi(q,q'))$.*

Proof. A loop is said *reducible* if there exists a decomposition of $\theta = \sigma\theta'\sigma'$ such that $\sigma\sigma' \neq \varepsilon$ and such that $\theta' \neq \varepsilon$ is a loop. A path $\sigma \in \Pi(q,q')$ is said *reducible* if there exists a decomposition of $\sigma = \sigma'\theta\sigma''$ such that $\theta \neq \varepsilon$ is a loop. Remark that for any reducible path σ, there exists a decomposition $\sigma = \sigma'\theta\sigma''$ such that $\theta \neq \varepsilon$ is an irreducibly loop.

Let (q,q') be a pair of locations fixed for the remaining of the proof. A *decomposition* $d = (\sigma_0, \theta_1, \alpha_1, \sigma_1, \dots \theta_k, \alpha_k, \sigma_k)$ of a path $\pi_d = \sigma_0\theta_1^{\alpha_1}\sigma_1 \dots \theta_k^{\alpha_k}\sigma_k$ in $\Pi(q,q')$ is a finite sequence such that σ_i is a path, $\theta_i \neq \varepsilon$ is an irreducibly loop and $\alpha_i \geq 1$. The vector $l(d) = (|\sigma_0| + \dots + |\sigma_k|, k) \in \mathbb{N}^2$ is called the *length* of d.

The decomposition d is said *reducible* if one of the following conditions is satisfied:

- There exists $i \neq i'$ such that $\theta_i = \theta_{i'}$, or
- There exists i such that σ_i is a reducible path.

In fact, in both of these two cases, there exists another decomposition d' such that $\delta(\pi_d) = \delta(\pi_{d'})$ and such that $l(d') <_{lex} l(d)$ where $\leq_{lex}$ is the lexicographic order defined over $\mathbb{N}^2$ by $(x,k) \leq_{lex} (x',k')$ if and only if $(x \leq x')$ or $(x = x'$ and $k \leq k')$. As the length of the new decomposition strictly decrease, for any $\pi \in \Pi(q,q')$, an irreducibly decomposition d' such that $\delta(\pi_{d'}) = \delta(\pi)$, can be deduced from the decomposition $d = (\pi)$.

Let us consider an irreducibly decomposition $d = \sigma_0, \theta_1, \alpha_1, \sigma_1, \dots \theta_k, \alpha_k, \sigma_k$. As σ_i and θ_i are irreducibly, we have $|\sigma_i|, |\theta_i| \leq |Q|$. In particular, $k \leq |T|^{|Q|+1}$ and π_d is in the following SLPS $\rho_{q,q'}$:

$$\rho_{q,q'} = \bigcup_{k \leq |T|^{|Q|+1}} \quad \bigcup_{\substack{q_0,\dots q_{k+1} \in Q \\ q_0 = q,\ q_{k+1} = q'}} \quad \bigcup_{\substack{\sigma_i \in \Pi(q_i, q_{i+1}) \\ |\sigma_i| \leq |Q|}} \quad \bigcup_{\substack{\theta_i \in \Pi(q_i, q_i) \\ |\theta_i| \leq |Q|}} \sigma_0\theta_1^*\sigma_1 \cdots \theta_k^*\sigma_k$$

By construction, we have $\rho_{q,q'} \subseteq \Pi(q,q')$ and $\delta(\rho_{q,q'}) = \delta(\Pi(q,q'))$. □

Given any two locations q and q' in a VASS V, the "counter reachability subrelation" $\{(\mathsf{x},\mathsf{x}') \in (\mathbb{N}^n)^2 \ / \ (q,\mathsf{x}) \xrightarrow{*} (q',\mathsf{x}')\}$ between q and q' is clearly contained in the relation $\{(\mathsf{x},\mathsf{x}') \in (\mathbb{N}^n)^2 \ / \ \mathsf{x}' - \mathsf{x} \in \delta(\Pi(q,q'))\}$. According to the lemma, there exists an SLPS $\rho \subseteq \Pi(q, q')$ such that $\delta(\Pi(q, q')) = \delta(\rho)$. Still, $\mathcal{R}(\rho)$ does not necessarily contain the reachability subrelation between q and q', as shown by the following example.

Example 4.2. Consider again the VASS E of Example 2.3. The set of displacements $\delta(\Pi(p, p))$ is equal to $\delta(\rho)$ where ρ is the SLPS contained in $\Pi(p, p)$ defined by: $\rho = (t_1)^*(t'_1)^* \cup (t_1)^*(t'_1)^* t_2 (t_3)^* t_4 (t_2 t_4)^*$. Note that $\delta(\rho)$ is the semilinear set $\delta(\rho) = {P_1}^* \cup ((-1, 0, 0) + {P_2}^*)$ with $P_1 = \{(0, 1, -1), (0, -1, 1)\}$ and $P_2 = P_1 \cup \{(0, -1, 2), (-1, 0, 0)\}$.

It is readily seen that $\mathcal{R}(\rho)$ satisfies: $(p, (x, y, z)) \xrightarrow{\rho} (p, (x', y', z'))$ iff either (1) $x' = x$ and $y' + z' = y + z$, or (2) $x' < x$ and $y + z \leq y' + z' \leq 2\,(y + z) - y'$. Hence, according to Example 2.3, $\mathcal{R}(\rho)$ does not contain all pairs $((p, \mathsf{x}), (p, \mathsf{x}'))$ such that $(p, \mathsf{x}) \xrightarrow{*} (p, \mathsf{x}')$. □

As a first step towards flatness, we now focus on reachability between configurations that have "big counter values". This leads us to the notion of *ultimate flatness*, but we first need some new notations.

Notation. Consider an n-dim VASS V with a set of locations Q, and let R denote any (binary) relation on $Q \times \mathbb{N}^n$. For any subset $X \subseteq \mathbb{N}^n$, the *restriction of R to X*, written $R|_X$, is the relation $R|_X = R \cap (Q \times X)^2$.

Definition 4.3. *An n-dim VASS V is called* ultimately flat *if the restriction $\mathcal{R}_V^*|_{[c\,..\,\infty[^n}$ is flat for some $c \in \mathbb{N}$.*

Remark 4.4. For any ultimately flat VASS V, there exists $c \in \mathbb{N}$ such that the restriction $\mathcal{R}_V^*|_{[c\,..\,\infty[^n}$ is semilinear. □

In the rest of this section, we give a sufficient condition for ultimate flatness. This will allow us to prove, in the next section, ultimate flatness of every 2-dim VASS. This sufficient condition basically consists in assuming a stronger version of Lemma 4.1 where the considered SLPS $\rho_{q,q'}$ are zigzag-free. In the following, we consider a fixed n-dim VASS $V = (Q, T, \alpha, \beta, \delta)$.

Definition 4.5. *An LPS $\rho = \sigma_0 \theta_1^* \sigma_1 \cdots \theta_k^* \sigma_k$ is said to be* zigzag-free *if for every $i \in [1\,..\,n]$, the k integers $\delta(\theta_1)[i], \ldots, \delta(\theta_k)[i]$ have the same sign. A* zigzag-free SLPS *is any finite union of zigzag-free LPS.*

Intuitively, an LPS ρ is zigzag-free iff the displacements of all loops in ρ "point" in the same hyperquadrant, where by hyperquadrant, we mean a subset of $\mathbb{Z}^n$ of the form $Z_1 \times \cdots \times Z_n$ with $Z_i \in \{\mathbb{N}, \mathbb{Z}^-\}$.

The following lemma shows that the intermediate displacements along any path in a zigzag-free LPS ρ belong to fixed hypercube (that only depends on ρ and π). This result is not very surprising: since all loops in ρ "point" in the same "direction", the intermediate displacements along any path in ρ can not deviate much from this direction.

Lemma 4.6. *Given any zigzag-free LPS ρ, there exists an integer $c \geq 0$ such that for every path $\pi \in \rho$, the displacement $\delta(\sigma)$ of any prefix σ of π satisfies: $\delta(\sigma)[i] \geq \mathrm{Min}\{0, \delta(\pi)[i]\} - c$ for every $i \in [1 \mathinner{..} n]$.*

Proof. The zigzag-free LPS ρ can be written $\rho = \sigma_0 \theta_1^* \sigma_1 \cdots \theta_k^* \sigma_k$. As ρ is a zigzag-free LPS, for any $i \in [1 \mathinner{..} n]$, there exists $\epsilon_i \in \{-1, 1\}$ such that $\epsilon_i \delta(\theta_1)[i]$, ..., $\epsilon_i \delta(\theta_k)[i]$ are in $\mathbb{N}$.

Let $\delta' : T^* \to \mathbb{N}^n$ be the function defined over T by $\delta'(t)[i] = |\delta(t)[i]|$ and naturally extended over T^*. By definition of δ', for any pair $(\sigma, \sigma') \in \Sigma^* \times \Sigma^*$ such that σ is a *sub-word*[6] of σ', we have $|\delta(\sigma)[i]| \leq \delta'(\sigma')[i]$ for any $i \in [1 \mathinner{..} n]$.

Let us prove that $c = \mathrm{Max}_i(\delta'(\sigma_0 \theta_1 \sigma_1 \cdots \theta_k \sigma_k)[i])$ satisfies the proposition.

There exists two type of prefixes σ of a word $\pi = \sigma_0 \theta_1^{\alpha_1} \sigma_1 \cdots \theta_k^{\alpha_k} \sigma_k$, following that it can be written in one of the two following form:

$$\sigma = \begin{cases} \sigma_0 \theta_1^{\alpha_1} \cdots \sigma_{m-1} \theta_m^{\alpha_m} \sigma' & \text{where } \sigma' \text{ is a prefix of } \sigma_m \\ \sigma_0 \theta_1^{\alpha_1} \cdots \sigma_{m-1} \theta_m^{\beta_m} \sigma' & \begin{array}{c}\text{where } \sigma' \text{ is a prefix of } \theta_m \\ \text{and } 0 \leq \beta_m < \alpha_m\end{array} \end{cases}$$

Assume that σ is a prefix of the first form. If $\epsilon_i = 1$, from the equality $\delta(\sigma)[i] = \delta(\sigma_0 \cdots \sigma_m \sigma')[i] + \sum_{j=1}^{m-1} \alpha_j \epsilon_i \delta(\theta_j)[i]$ we deduce that $\delta(\sigma)[i] \geq -c$. If $\epsilon_i = -1$, as σ' is a prefix of σ_m, there exists $\sigma'' \in T^*$ such that $\sigma' \sigma'' = \sigma_m$. We have:

$$\begin{aligned} \delta(\sigma)[i] &= \delta(\sigma_0 \cdots \sigma_{m-1} \sigma')[i] + \sum_{j=1}^{m} \alpha_j \delta(\theta_j)[i] \\ &= \delta(\pi)[i] - \delta(\sigma'' \sigma_{m+1} \cdots \sigma_k) + \sum_{j=m+1}^{k} \alpha_j \epsilon_i \delta(\theta_j)[i] \\ &\geq \delta(\pi)[i] - c \end{aligned}$$

Hence, whatever $\epsilon_i \in \{-1, 1\}$, we have $\delta(\sigma)[i] \geq \mathrm{Min}\{0, \delta(\pi)[i]\} - c$. The same bound can be obtained symmetrically for a prefix of the second form. $\square$

We may now express, in Proposition 4.8, our sufficient condition for ultimate flatness. The proof is based on the following lemma.

Lemma 4.7. *Let q, q' denote two locations, and let $\rho \subseteq \Pi(q, q')$ be any zigzag-free SLPS such that $\delta(\rho) = \delta(\Pi(q, q'))$. There exists $c \in \mathbb{N}$ such that for every $\mathsf{x}, \mathsf{x}' \in [c \mathinner{..} \infty[^n$, if $(q, \mathsf{x}) \xrightarrow{*} (q', \mathsf{x}')$ then $(q, \mathsf{x}) \xrightarrow{\rho} (q', \mathsf{x}')$.*

Proof. Let us consider a zigzag-free SLPS $\rho \subseteq \Pi(q, q')$ such that $\delta(\rho) = \delta(\Pi(q, q'))$. The zigzag-free SLPS ρ can be decomposed into a finite union of zigzag-free LPS $\rho = \bigcup_{m \in M} \rho_m$. For each $m \in M$, we apply Lemma 4.6, and we get an integer $c_m \geq 0$. Let us prove that $c = \max_m(c_m)$ satisfies the proposition. Consider

[6] A word σ is a *sub-word* of σ' if there exists a decomposition of $\sigma' = w_1 \ldots w_k$ and a set $I = \{i_1 < \ldots < i_n\} \subseteq [1 \mathinner{..} k]$ such that $\sigma = w_{i_1} \ldots w_{i_n}$

$x, x' \in [c\,..\,\infty[^n$ such that $(q, x) \xrightarrow{*} (q', x')$. There exists a path $\pi \in \Pi(q, q')$ such that $(q, x) \xrightarrow{\pi} (q', x')$. From $x' - x = \delta(\pi)$ and $\delta(\rho) = \delta(\Pi(q, q'))$, we deduce that there exists $\pi_m \in \rho_m$ such that $x' - x = \delta(\pi_m)$. For any prefix σ of π_m, we have either $x[i] + \delta(\sigma)[i] \geq x[i] - c_m \geq 0$ or $x[i] + \delta(\sigma)[i] \geq x[i] - c_m + \delta(\pi_m)[i] = x'[i] - c_m \geq 0$. So, for any prefix σ of π_m, we have $x + \delta(\sigma) \in \mathbb{N}^n$. As $\pi_m \in \Pi(q, q')$, Fact 1 shows that $(q, x) \xrightarrow{\pi_m} (q', x')$. In particular $(q, x) \xrightarrow{\rho} (q', x')$. □

Proposition 4.8. *Let V be a VASS. Assume that for every pair (q, q') of locations, there exists a zigzag-free SLPS $\rho_{q,q'} \subseteq \Pi(q, q')$ such that $\delta(\rho_{q,q'}) = \delta(\Pi(q, q'))$. Then V is ultimately flat.*

Proof. For each pair (q, q') of locations, we apply Lemma 4.7, and we get that there exists $c_{q,q'} \in \mathbb{N}$ such that for every $\mathsf{x}, \mathsf{x}' \in [c\,..\,\infty[^n$, if $(q, \mathsf{x}) \xrightarrow{*} (q', \mathsf{x}')$ then $(q, \mathsf{x}) \xrightarrow{\rho_{q,q'}} (q', \mathsf{x}')$. We take $\rho = \bigcup_{q,q'} \rho_{q,q'}$, and we choose $c \in \mathbb{N}$ such that $c \geq c_{q,q'}$ for every q, q'. We obtain that $\mathcal{R}_V^*|_{[c\,..\,\infty[^n}$ is contained in $\mathcal{R}_V(\rho)$, and hence V is ultimately flat. □

Observe that all proofs in this section are constructive. From Lemma 4.1, we can compute for each pair (q, q') of locations an SLPS $\rho_{q,q'} \subseteq \Pi_V(q, q')$ such that $\delta(\rho_{q,q'}) = \delta(\Pi_V(q, q'))$. Assume that these SLPS can be effectively "straightened" into zigzag-free SLPS $\rho'_{q,q'} \subseteq \Pi_V(q, q')$ with the same displacements: $\delta(\rho'_{q,q'}) = \delta(\Pi_V(q, q'))$. Then we can compute an integer $c \in \mathbb{N}$ such that $\mathcal{R}_V^*|_{[c\,..\,\infty[^n}$ is contained in $\mathcal{R}_V(\rho)$, where $\rho = \bigcup_{q,q'} \rho_{q,q'}$. Consequently, we can conclude using the acceleration theorem, Theorem 3.3, that $\mathcal{R}_V^*|_{[c\,..\,\infty[^n}$ is effectively semilinear. We will prove in the next section that this "straightening" assumption holds in dimension 2.

5 Flatness of 2-dim VASS

We now have all the necessary background to prove our main result. We first show that every 2-dim VASS is ultimately flat. We then prove that every 1-dim VASS is flat, and we finally prove that every 2-dim VASS is flat.

5.1 Ultimate flatness in dimension 2

In order to prove ultimate flatness of all 2-dim VASS, we will need the following technical proposition.

Proposition 5.1. *For any finite subset P of $\mathbb{Z}^2$, and for any vector $\mathsf{x} \in P$, there exists two finite subsets B', P' of $(\mathsf{x} + P^*) \cap \mathbb{N}^2$ such that:*

$$(\mathsf{x} + P^*) \cap \mathbb{N}^2 = B' + \left(P' \cup (P \cap \mathbb{N}^2)\right)^*$$

Proof. In appendix, section A. □

Of course, this proposition also holds in dimension 1. The following remark shows that the proposition does not hold in dimension 3 (nor in any dimension above 3).

Remark 5.2. Consider the linear set $(\mathsf{x} + P^*)$ with basis $\mathsf{x} = (1, 0, 0)$ and set of periods $P = \{(1, 0, 0), (0, 1, -1), (0, -1, 2)\}$. Observe that $(\mathsf{x} + P^*) \cap \mathbb{N}^3 = (1, 0, 0) + \mathbb{N}^3$. Let B' and P' denote two finite subsets of $(1, 0, 0) + \mathbb{N}^3$. There exists $c \in \mathbb{N}$ such that $B' \subseteq [0 \,..\, c]^3$, and hence $(1, c+1, 0) \notin B' + (P' \cup \{(1, 0, 0)\})^*$. Therefore, there does not exist two finite subsets B', P' of $(\mathsf{x} + P^*) \cap \mathbb{N}^3$ such that $(\mathsf{x} + P^*) \cap \mathbb{N}^3 = B' + \big(P' \cup (P \cap \mathbb{N}^3)\big)^*$. □

We may now prove that every 2-dim VASS is ultimately flat. We first show that any LPS in a 2-dim VASS can be "straightened" into a zigzag-free SLPS with the same displacements.

Lemma 5.3. *For any location q of a 2-dim VASS V and for any LPS $\rho \subseteq \Pi(q, q)$, there exists a zigzag-free SLPS $\rho' \subseteq \Pi(q, q)$ such that $\delta(\rho) \subseteq \delta(\rho')$.*

Proof. An LPS $\rho = \sigma_0 \theta_1^* \sigma_1 \ldots \sigma_{k-1} \theta_k^* \sigma_k$ satisfies $\delta(\rho) = x + P^*$ where $x = \delta(\sigma_0 \ldots \sigma_k)$ and $P = \{\delta(\theta_1), \ldots, \delta(\theta_k)\}$. As $\rho' = (\sigma_0 \ldots \sigma_k)^* \rho$ is an LPS such that $\rho' \subseteq \Pi(q, q)$ and $\delta(\rho) \subseteq \delta(\rho')$, we can assume without loss of generality that $x \in P$.

Let $W \in \{\mathbb{N} \times \mathbb{N}, \mathbb{N} \times \mathbb{Z}^-, \mathbb{Z}^- \times \mathbb{N}, \mathbb{Z}^- \times \mathbb{Z}^-\}$. As $x \in P$, proposition 5.1 shows that there exist two finite subsets of $(x + P^*) \cap W$, written B_W and $P_W = \{p_{1,W}, \ldots, p_{n_W,W}\}$ such that $(x + P^*) \cap W = B_W + (P_W \cup (P \cap W))^*$.

A path $\pi_y \in \Pi(q, q)$ can be associated to any vector $y \in x + P^*$. In fact, there exist $\alpha_1(y)$, ..., $\alpha_k(y)$ in $\mathbb{N}$ such that the following path π_y satisfies $\delta(\pi_y) = y$:

$$\pi_y = \sigma_0 \theta_1^{\alpha_1(y)} \sigma_1 \ldots \theta_k^{\alpha_k(y)} \sigma_k$$

For any $i \in [1 \,..\, k]$, let $\Pi_{W,i}$ be defined by $\Pi_{W,i} = \theta_i^*$ if $\delta(\theta_i) \in W$, and $\Pi_{W,i} = \{\varepsilon\}$ otherwise. By construction, the following SLPS ρ' is zigzag-free and satisfies $\delta(\rho') = \bigcup_W (B_W + (P_W \cup (P \cap W))^*) = \bigcup_W ((x + P^*) \cap W) = x + P^* = \delta(\rho)$.

$$\rho' = \bigcup_W \bigcup_{b \in B_W} \left(\sigma_0 \theta_1^{\alpha_1(b)} \Pi_{W,1} \sigma_1 \ldots \theta_k^{\alpha_k(b)} \Pi_{W,k} \sigma_k \right) \left(\pi_{p_{1,W}}^* \ldots \pi_{p_{n_W,W}}^* \right)$$

□

Proposition 5.4. *Every 2-dim VASS is ultimately flat.*

Proof. According to Proposition 4.8, it is sufficient to prove that for any pair (q, q') of locations, there exists a zigzag-free SLPS $\rho_{q,q'} \subseteq \Pi(q, q')$ such that $\delta(\Pi(q, q')) = \delta(\rho_{q,q'})$. From lemma 4.1 and 5.3, we deduce that for any $q'' \in Q$, there exists a zigzag-free SLPS $\rho_{q'',q''} \subseteq \Pi(q'', q'')$ such that $\delta(\rho_{q'',q''}) =$

$\delta(\Pi(q'', q''))$. Let us consider the following zigzag-free SLPS $\rho_{q,q'} \subseteq \Pi(q, q')$:

$$\rho_{q,q'} = \left(\bigcup_{\substack{\pi \in \Pi(q,q') \\ |\pi| \leq |Q|}} \pi \right) \cup \left(\bigcup_{q'' \in Q} \bigcup_{\substack{\sigma \in \Pi(q,q'') \\ |\sigma| \leq |Q|}} \bigcup_{\substack{\sigma' \in \Pi(q'',q') \\ |\sigma'| \leq |Q|}} \sigma.\rho_{q'',q''}.\sigma' \right)$$

As $\rho_{q,q'} \subseteq \Pi(q, q')$, we just have to prove the inclusion $\delta(\Pi(q, q')) \subseteq \delta(\rho_{q,q'})$ to deduce the equality. A path $\pi \in \Pi(q, q')$ such that $|\pi| \leq |Q|$ satisfies $\pi \in \rho_{q,q'}$. If $|\pi| > |Q|$, there exists $q'' \in Q$ such that π can be decomposed into $\pi = \sigma\theta\sigma'$ where $\sigma \in \Pi(q, q'')$, $\theta \in \Pi(q'', q'')$ and $\sigma' \in \Pi(q'', q')$. As $\delta(\rho_{q'',q''}) = \delta(\Pi(q'', q''))$, there exists $\theta' \in \rho_{q'',q''}$ such that $\delta(\theta) = \delta(\theta')$. From $\delta(\sigma\theta'\sigma') = \delta(\sigma\theta\sigma')$ and $\sigma\theta'\sigma' \in \rho_{q,q'}$, we deduce $\delta(\Pi(q, q')) \subseteq \delta(\rho_{q,q'})$. □

Remark 5.5. There exists a 3-dim VASS that is not ultimately flat. To prove this claim, consider VASS E from Example 2.3. For every $c \in \mathbb{N}$, the restriction $\mathcal{R}^*|_{[c\,..\,\infty[^n}$ is clearly non semilinear. According to Remark 4.4, we conclude that E is not ultimately flat. □

5.2 Flatness and effective semilinearity of $\mathcal{R}^*$ for 1-dim VASS

Let $V = (Q, T, \alpha, \beta, \delta)$ be any 1-dim VASS and let us prove that V is flat. Proposition 5.4 is trivially extended to 1-dim VASS as any 1-dim VASS is "equal" to a 2-dim VASS whose second counter remains unchanged. Therefore, V is ultimately flat, and hence there exists $c \in \mathbb{N}$ such that $\mathcal{R}^*|_{[c\,..\,\infty[}$ is flat. Let $c' = c + \mathrm{Max}_t(|\delta(t)|)$ and let us denote by F and F_∞ the intervals $F = [0\,..\,c']$ and $F_\infty = [c\,..\,\infty[$.

Recall that $\mathcal{R}^*|_{F_\infty}$ is flat. The restriction $\mathcal{R}^*|_F$ is also flat since it is a finite reachability subrelation. As the class of flat reachability subrelations is closed under union and under composition, we just have to prove the following inclusion:

$$\mathcal{R}^* \subseteq (\mathcal{R}^*|_F \cup \mathcal{R}^*|_{F_\infty})^2$$

Assume that $(q, x) \xrightarrow{\pi} (q', x')$ for some path π. If $x, x' \leq c'$ or $x, x' \geq c$ then $((q, x), (q', x')) \in \mathcal{R}^*|_F$ or $((q, x), (q', x')) \in \mathcal{R}^*|_{F_\infty}$, which concludes the proof since $(\mathcal{R}^*|_F \cup \mathcal{R}^*|_{F_\infty})$ contains $Id_{Q\times\mathbb{N}}$.

Now suppose that either (1) $x \leq c$ and $x' > c'$, or (2) $x > c'$ and $x' \leq c$, and consider the case (1) $x \leq c$ and $x' > c'$. Let σ be the longest prefix of π such that $x + \delta(\sigma) \leq c$. As $x + \delta(\pi) = x' > c' \geq c$, the prefix σ can not be equal to π. So the path π can be decomposed into $\pi = \sigma t \sigma'$ with $\sigma, \sigma' \in T^*$ and $t \in T$, and such that $x + \delta(\sigma) \leq c$ and $x + \delta(\sigma t) > c$. We have $(q, x) \xrightarrow{\sigma t} (q'', x'') \xrightarrow{\sigma'} (q', x')$ where $q'' = \beta(t)$ and $x'' = x + \delta(\sigma t)$. Remark that $x'' = x + \delta(\sigma) + \delta(t)$ and hence $x'' \leq c + \delta(t) \leq c'$. From $x, x'' \leq c'$, we deduce that $((q, x), (q'', x'')) \in \mathcal{R}^*|_F$, and as $x'', x' \geq c$, we obtain that $((q'', x''), (q', x')) \in \mathcal{R}^*|_{F_\infty}$. So far, we have proved that $((q,x),(q',x')) \in (\mathcal{R}^*|_F \cup \mathcal{R}^*|_{F_\infty})^2$. Symmetrically, for the case (2) $x > c'$ and $x' \leq c$, we deduce $((q,x),(q',x')) \in (\mathcal{R}^*|_F \cup \mathcal{R}^*|_{F_\infty})^2$.

This concludes the proof that V is flat. We have just proved the following theorem.

Theorem 5.6. *Every* 1*-dim VASS is flat.*

5.3 Flatness and effective semilinearity of $\mathcal{R}^*$ for 2-dim VASS

Let $V = (Q,T,\alpha,\beta,\delta)$ be any 2-dim VASS and let us prove that V is flat. According to Proposition 5.4, V is ultimately flat, and hence there exists $c \in \mathbb{N}$ such that $\mathcal{R}^*|_{[c\,..\,\infty[^2}$ is flat. Let $c' = c + \mathrm{Max}_t\{|\delta(t)[1]|, |\delta(t)[2]|\}$ and let us denote by F and F_∞ the intervals $F = [0\,..\,c']$ and $F_\infty = [c\,..\,\infty[$. The set $\mathbb{N}^2$ is covered by 4 subsets:

$$\mathbb{N}^2 \;=\; (F \times F) \cup (F_\infty \times F) \cup (F \times F_\infty) \cup (F_\infty \times F_\infty)$$

Recall that $\mathcal{R}^*|_{F_\infty \times F_\infty}$ is flat. The restriction $\mathcal{R}^*|_{F \times F}$ is also flat since it is a finite reachability subrelation.

Lemma 5.7. *The reachability subrelations* $(\mathcal{R}|_{F \times \mathbb{N}})^*$ *and* $(\mathcal{R}|_{\mathbb{N} \times F})^*$ *are flat.*

Proof. We only prove that $(\mathcal{R}|_{F \times \mathbb{N}})^*$ is flat (the proof that $(\mathcal{R}|_{\mathbb{N} \times F})^*$ is flat is symmetric). Observe that this reachability subrelation is the reachability relation of a 2-dim VASS whose first counter remains in the finite set F. So the relation $(\mathcal{R}_V|_{F \times \mathbb{N}})^*$ is first shown to be "equal" to the reachability relation of the 1-dim VASS $V' = (Q',T',\alpha',\beta',\delta')$ defined as follows:

$$\begin{cases} Q' = Q \times F \\ T' = \{(t,f) \in T \times F \;/\; f + \delta(t)[1] \in F\} \end{cases} \qquad \begin{cases} \alpha'((t,f)) = (\alpha(t), f) \\ \beta'((t,f)) = (\beta(t), f + \delta(t)[1]) \\ \delta'((t,f)) = \delta(t)[2] \end{cases}$$

Observe that reachability in V and V' are closely related: for every $t \in T$, $q, q' \in Q$, and $(f,y), (f',y') \in F \times \mathbb{N}$, we have:

$$(q,(f,y))\; \mathcal{R}_V(t)\; (q',(f',y')) \quad \text{iff} \quad ((q,f),y)\; \mathcal{R}_{V'}((t,f))\; ((q',f'),y')$$

Let $\varphi : T'^* \to T^*$ denote the letter morphism defined by $\varphi((t,f)) = t$. We deduce from the previous equivalence, that the two following assertions hold for every $\pi' \in T'^*$, $q, q' \in Q$, and $x, y, x', y' \in \mathbb{N}$:

$$\begin{aligned} (q,(x,y))\; \mathcal{R}_V(\varphi(\pi))\; (q',(x',y')) &\quad \text{if} \quad ((q,x),y)\; \mathcal{R}_{V'}(\pi')\; ((q',x'),y') \\ (q,(x,y))\; (\mathcal{R}_V|_{F \times \mathbb{N}})^*\; (q',(x',y')) &\quad \text{iff} \quad ((q,x),y)\; \mathcal{R}^*_{V'}\; ((q',x'),y') \end{aligned}$$

As V' is a 1-dim VASS, Theorem 5.6 shows that there exists a SLPS ρ' for V' such that $\mathcal{R}^*_{V'} \subseteq \mathcal{R}_{V'}(\rho')$. The language $\rho = \varphi(\rho')$ is an SLPS for V. Let us prove that $(\mathcal{R}_V|_{F \times \mathbb{N}})^* \subseteq \mathcal{R}_V(\rho)$.

Consider $((q,(x,y)),(q',(x',y'))) \in (\mathcal{R}_V|_{F\times\mathbb{N}})^*$. Since $(\mathcal{R}_V|_{F\times\mathbb{N}})^*$ is "equal" to $\mathcal{R}^*_{V'}$, we obtain that $((q,x),y)\ \mathcal{R}^*_{V'}\ ((q',x'),y')$. As $\mathcal{R}^*_{V'} \subseteq \mathcal{R}_{V'}(\rho')$, we get that there exists a path $(t_0,f_0)\cdots(t_k,f_k) \in \rho'$ such that the pair $(((q,x),y), ((q',x'),y'))$ belongs $\mathcal{R}_{V'}((t_0,f_0)\cdots(t_k,f_k))$. Recall that $\mathcal{R}_V(\varphi(\pi))$ "contains" $\mathcal{R}_{V'}((t_0,f_0)\cdots(t_k,f_k))$. We deduce that $(q,(x,y))\ \mathcal{R}_V(t_0\cdots t_k)\ (q',(x',y'))$. We have shown that $(\mathcal{R}_V|_{F\times\mathbb{N}})^* \subseteq \mathcal{R}_V(\rho)$, which concludes the proof. $\square$

Let us denote by R_1 the reachability subrelation $R_1 = Id \cup \mathcal{R} \cup (\mathcal{R}|_{\mathbb{N}\times F})^* \cup (\mathcal{R}|_{F\times\mathbb{N}})^*$, where Id denotes the identity relation on $Q \times \mathbb{N}^2$. Recall that we want to prove that V is flat. As the class of flat reachability subrelations is closed under union and under composition, we just have to prove the following "flatness witness" inclusion:

$$\mathcal{R}^* \subseteq R_1 \cdot (Id \cup \mathcal{R}^*|_{F_\infty\times F_\infty}) \cdot R_1 \cdot (Id \cup \mathcal{R}^*|_{F\times F}) \cdot R_1 \cdot (Id \cup \mathcal{R}^*|_{F_\infty\times F_\infty}) \cdot R_1$$

Consider two configurations (q,x) and (q',x'), and a path $\pi \in \Pi(q,q')$, such that $(q,\mathsf{x}) \xrightarrow{\pi} (q',\mathsf{x}')$. An *intermediate vector* for the triple $((q,\mathsf{x}),\pi,(q',\mathsf{x}'))$ is a vector x'' such that $\mathsf{x}'' = \mathsf{x} + \delta(\sigma)$ for some prefix σ of π with $\sigma \notin \{\varepsilon,\pi\}$. Observe that for any such intermediate vector x'', there exists a state $q'' \in Q$ and a decomposition of π into $\pi = \sigma\sigma'$ with $\sigma,\sigma' \neq \varepsilon$, satisfying:

$$(q,\mathsf{x}) \xrightarrow{\sigma} (q'',\mathsf{x}'') \xrightarrow{\sigma'} (q',\mathsf{x}')$$

Let $G = (F \times F) \cup (F_\infty \times F_\infty)$. We first prove the following lemma.

Lemma 5.8. *For any $(q,\mathsf{x}) \xrightarrow{\pi} (q',\mathsf{x}')$ such that there is no intermediate vector in G, we have $((q,\mathsf{x}),(q',\mathsf{x}')) \in R_1$.*

Proof. Assume that $(q,\mathsf{x}) \xrightarrow{\pi} (q',\mathsf{x}')$ is such that there is no intermediate vector in G. Remark that we can assume that $\pi \notin T \cup \{\varepsilon\}$. The intermediate vectors x'' are either in $[c'+1\mathinner{..}\infty[\times[0\mathinner{..}c-1]$ or in $[0\mathinner{..}c-1]\times[c'+1\mathinner{..}\infty[$. Assume by contradiction that there exists both an intermediate vector in $[c'+1\mathinner{..}\infty[\times[0\mathinner{..}c-1]$ and in $[0\mathinner{..}c-1]\times[c'+1\mathinner{..}\infty[$. So there exists $t \in T$ such that either $(q_1,\mathsf{x}_1) \xrightarrow{t} (q_2,\mathsf{x}_2)$ or $(q_2,\mathsf{x}_2) \xrightarrow{t} (q_1,\mathsf{x}_1)$ with $\mathsf{x}_1 \in [c'+1\mathinner{..}\infty[\times[0\mathinner{..}c-1]$ and $\mathsf{x}_2 \in [0\mathinner{..}c-1]\times[c'+1\mathinner{..}\infty[$. Let us consider the case $(q_1,\mathsf{x}_1) \xrightarrow{t} (q_2,\mathsf{x}_2)$. We have $\mathsf{x}_2 = \mathsf{x}_1 + \delta(t)$. From $\mathsf{x}_1[1] \geq c'+1$, we obtain $\mathsf{x}_2[1] \geq c'+1-(c'-c) \geq c+1$ which contradicts $\mathsf{x}_1[1] \leq c-1$. As the case $(q_2,\mathsf{x}_2) \xrightarrow{t} (q_1,\mathsf{x}_1)$ is symmetric, we have proved that we cannot have both an intermediate state in $[c'+1\mathinner{..}\infty[\times[0\mathinner{..}c-1]$ and in $[0\mathinner{..}c-1]\times[c'+1\mathinner{..}\infty[$. By symmetry, we can assume that all the intermediate states are in $[c'+1\mathinner{..}\infty[\times[0\mathinner{..}c-1]$. Let t be the first transition of π. As $\mathsf{x}+\delta(t)$ is an intermediate state, we have $\mathsf{x}+\delta(t) \in [c'+1\mathinner{..}\infty[\times[0\mathinner{..}c-1]$. In particular, $\mathsf{x} \in \mathbb{N}\times[0\mathinner{..}c']$. Symmetrically, by considering the last transition of π, we deduce $\mathsf{x}' \in \mathbb{N}\times[0\mathinner{..}c']$. Therefore, we have proved that $((q,\mathsf{x}),(q',\mathsf{x}')) \in R_1$. $\square$

We may now prove the "flatness witness" inclusion given above. Consider any two configurations (q,x) and (q',x') such that $(q,\mathsf{x}) \xrightarrow{*} (q',\mathsf{x}')$. There exists a path $\pi \in \Pi(q,q')$ such that $(q,\mathsf{x}) \xrightarrow{\pi} (q',\mathsf{x}')$. We are going to prove that there exists a prefix σ of π and a suffix σ' of π such that there is no intermediate vectors of $((q,\mathsf{x}),\sigma,(q_1,\mathsf{x}_1))$ or $((q_2,\mathsf{x}_2),\sigma',(q',\mathsf{x}'))$ in $F \times F$ and such that $((q_1,\mathsf{x}_1),(q_2,\mathsf{x}_1)) \in Id \cup \mathcal{R}^*|_{F\times F}$. If there is no intermediate vector of $(q,\mathsf{x}) \xrightarrow{\pi} (q',\mathsf{x}')$ in $F \times F$, then we can choose $\sigma = \pi$ and $\sigma' = \varepsilon$. So we can assume that there is at least one intermediate state in $F \times F$. Let σ be the least prefix of π such that there is no intermediate vector of $((q,\mathsf{x}),\sigma,(q_1,\mathsf{x}_1))$ in $F\times F$ and $\mathsf{x}_1 \in F$, and let σ' be the least suffix of π such that there is no intermediate vector of $((q_2,\mathsf{x}_2),\sigma',(q',\mathsf{x}'))$ in $F \times F$ and $\mathsf{x}_2 \in F$. Now, just remark that $((q_1,\mathsf{x}_1),(q_2,\mathsf{x}_2)) \in \mathcal{R}^*|_{F\times F}$.

By decomposing in the same way the two paths σ and σ' such that there is no intermediate vector in $F_\infty \times F_\infty$, we have proved that for any path $\pi \in \Pi(q,q')$ and for any $(q,\mathsf{x}) \xrightarrow{\pi} (q',\mathsf{x}')$, there exists $(q,\mathsf{x}) \xrightarrow{\pi_0} (q_1,\mathsf{x}_1)$, $(q'_1,\mathsf{x}'_1) \xrightarrow{\pi_1} (q_2,\mathsf{x}_2)$, $(q'_2,\mathsf{x}'_2) \xrightarrow{\pi_2} (q_3,\mathsf{x}_3)$, $(q'_3,\mathsf{x}'_3) \xrightarrow{\pi_3} (q',\mathsf{x}')$ such that the intermediate vectors are not in G and such that $((q_1,\mathsf{x}_1),(q'_1,\mathsf{x}'_1))$ and $((q_3,\mathsf{x}_3),(q'_3,\mathsf{x}'_3))$ are in $Id \cup \mathcal{R}^*|_{F_\infty\times F_\infty}$ and such that $((q_2,\mathsf{x}_2),(q'_2,\mathsf{x}'_2)) \in Id \cup \mathcal{R}^*|_{F\times F}$. Therefore, we have proved the "flatness witness" inclusion given above.

This concludes the proof that V is flat. We have just proved the following theorem.

Theorem 5.9. *Every 2-dim VASS is flat.*

Corollary 5.10. *The global reachability relation $\mathcal{R}_V^*$ of any 2-dim VASS V is effectively semilinear.*

The generic semi-algorithm implemented in the accelerated symbolic model checker `FAST` is able to compute the reachability set of 40 practical VASS [BFLP03]. Theorem 5.9 shows that this model checker, which was designed to *often* compute the reachability set of practical VASS, also provides a generic algorithm that *always* computes the reachability relation of any 2-dim VASS.

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A Proof of Proposition 5.1

We extensively use in this section the concept of *cones*. For any subset $P \subseteq \mathbb{Z}^n$, we denote by $\widehat{P}$ the *cone* generated by P:

$$\widehat{P} = \left\{ \sum_{i=0}^{k} \gamma_i \, \mathsf{p}_i \ / \ k \in \mathbb{N}, \gamma_0, \ldots, \gamma_k \in \mathbb{Q}_+ \text{ and } \mathsf{p}_0, \ldots, \mathsf{p}_k \in P \right\}$$

The following lemma is borrowed from [HP79]. As we extend it to $\mathbb{Z}^n$ (the lemma only considered subsets of $\mathbb{N}^n$ in [HP79]), we give a complete proof.

First, let us recall that a *minimal element* of a subset $X \subseteq \mathbb{Q}^n$ is any $\mathsf{m} \in X$ such that for every $\mathsf{x} \in X$, if $\mathsf{x} \leq \mathsf{m}$ then $\mathsf{x} = \mathsf{m}$. We denote by $\mathrm{Min}(X)$ the *set of minimal elements* of X. It is well known that any subset of $\mathbb{N}^n$ has finitely many minimal elements [Dic13].

Lemma A.1 ([HP79]). *Let B be a (possibly infinite) subset of $\mathbb{Z}^n$ and let P be a finite subset of $\mathbb{Z}^n$. If $B \subseteq (\mathsf{x} + \widehat{P})$ for some $\mathsf{x} \in \mathbb{Z}^n$ then there exists a finite subset $B' \subseteq B$ such that $(B + P^*) = (B' + P^*)$.*

Proof. The lemma is trivial when P is empty. Assume that $P = \{\mathsf{p}_1, \ldots, \mathsf{p}_k\}$. Let $f : \mathbb{Z}^n \times \mathbb{Q}_+^k \to \mathbb{Q}^n$ be the total function defined by $f(\mathsf{x}, (\gamma_1, \ldots, \gamma_k)) = \mathsf{x} + \sum_{i=1}^{k} \gamma_i \, \mathsf{p}_i$. Observe that $(\mathsf{x} + \widehat{P})$ is equal to $f(\{\mathsf{x}\} \times \mathbb{Q}_+^k)$. Let D denote the image under f of the set $\{\mathsf{x}\} \times \{\gamma \in \mathbb{Q}_+ \ / \ \gamma < 1\}^k$. For every $\mathsf{y} \in (\mathsf{x} + \widehat{P})$, there exists $\mathsf{d} \in D$ and $(c_1, \ldots, c_k) \in \mathbb{N}^k$ such that $\mathsf{y} = \mathsf{d} + \sum_{i=1}^{k} c_i \mathsf{p}_i$. Hence we get that $(\mathsf{x} + \widehat{P}) = f(D \times \mathbb{N}^k)$. Therefore, we come to $(\mathsf{x} + \widehat{P}) \cap \mathbb{Z}^n = f(E \times \mathbb{N}^k)$ where $E = D \cap \mathbb{Z}^n$.

We extend on $E \times \mathbb{N}^k$ the partial ordering $\leq$ on $\mathbb{N}^k$ as follows: for all $(\mathsf{e}, \mathsf{y}), (\mathsf{e}', \mathsf{y}') \in E \times \mathbb{N}^k$, $(\mathsf{e}, \mathsf{y}) \leq (\mathsf{e}', \mathsf{y}')$ if $\mathsf{e} = \mathsf{e}'$ and $\mathsf{y} \leq \mathsf{y}'$. Note that any subset of $E \times \mathbb{N}^k$ has finitely many minimal elements.

Let $M = \mathrm{Min}(\{(\mathsf{e}, \mathsf{c}) \in E \times \mathbb{N}^k \ / \ f(\mathsf{e}, \mathsf{c}) \in B\})$ and define B' as $B' = f(M)$. Observe that $B' \subseteq B$ and that B' is finite. Moreover, since we have $B \subseteq (\mathsf{x} + \widehat{P})$ and $(\mathsf{x} + \widehat{P}) \cap \mathbb{Z}^n = f(E \times \mathbb{N}^k)$, we get that $B \subseteq f(E \times \mathbb{N}^k)$. Hence for every $\mathsf{b} \in B$, there exists $(\mathsf{e}, \mathsf{c}) \in M$ and $\mathsf{c}' \in \mathbb{N}^k$ such that $\mathsf{b} = f(\mathsf{e}, \mathsf{c} + \mathsf{c}') = f(\mathsf{e}, \mathsf{c}) + \sum_{i=1}^{k} \mathsf{c}'[i] \, \mathsf{p}_i$. Consequently, we obtain that $B \subseteq (B' + P^*)$ and hence $(B + P^*) = (B' + P^*)$. □

We now introduce our notations for *dot products* and *angles*, as these notions will be helpful for proofs in dimension 2. We write $\mathsf{x} \cdot \mathsf{y}$ the *dot product* $\mathsf{x} \cdot \mathsf{y} = \mathsf{x}[1] \, \mathsf{y}[1] + \mathsf{x}[2] \, \mathsf{y}[2]$ between any two vectors x and y in $\mathbb{Q}^2$, and we denote by $(\mathsf{x} \angle \mathsf{y})$ the *angle* in $[0, 2\pi[$ between x and y (when x and y are non zero). Formally, $(\cdot \angle \cdot)$ may be defined as the total function from $(\mathbb{Q}^2 \setminus \{\mathsf{0}\})^2$ to the interval $[0, 2\pi[$ of real numbers, defined by $(\mathsf{x} \angle \mathsf{y}) = \arg\left(\frac{\mathsf{y}[1] + i \, \mathsf{y}[2]}{\mathsf{x}[1] + i \, \mathsf{x}[2]}\right)$, where $\arg(z)$ denotes the *complex argument* of any complex number z.

Lemma A.2. *For any two vectors p and p' in $\mathbb{Z}^2 \setminus \{\mathsf{0}\}$, if $(\mathsf{p} \angle \mathsf{p}') < \pi$ then the cone $\widehat{\{\mathsf{p}, \mathsf{p}'\}}$ is equal to $\{\mathsf{0}\} \cup \{\mathsf{x} \in \mathbb{Q}^2 \setminus \{\mathsf{0}\} \ / \ (\mathsf{p} \angle \mathsf{x}) \leq (\mathsf{p} \angle \mathsf{p}')\}$.*

Proof. From a geometric viewpoint, the proof of this lemma is trivial. The formal proof is left to the reader. $\square$

Lemma A.3. *Let* $\mathsf{x} \in \mathbb{Z}^2$ *and let* P *be a finite subset of* $\mathbb{Z}^2$*. If* $(\mathsf{x}+\widehat{P}) \cap \mathbb{Q}_+^2 \not\subseteq \{\mathsf{0}\}$ *then there exists a vector* $\mathsf{m} \in \mathbb{N}^2$ *such that* $\mathsf{m} \in (\mathsf{x} + (P \cup \{\mathsf{x}\})^*) \cup P$ *and* $((\mathsf{x} + \widehat{P}) \cap \mathbb{Q}_+^2) \subseteq \{\widehat{\mathsf{m}, (0,1)}\}$.

Proof. First, observe that we may suppose w.l.o.g. that $\mathsf{x} \in P$. Indeed, if the lemma holds whenever $\mathsf{x} \in P$, then we may replace P with $P' = P \cup \{\mathsf{x}\}$ and apply the lemma on x, P' to infer that the lemma holds for x, P.

We consider two cases, depending on the intersection between $(\mathsf{x} + \widehat{P})$ and $(\mathbb{Q}_+ \times \{0\})$.

- if $(\mathsf{x} + \widehat{P}) \cap (\mathbb{Q}_+ \times \{0\}) \not\subseteq \{\mathsf{0}\}$: then we have $(\gamma, 0) \in (\mathsf{x} + \widehat{P})$ for some $\gamma \in \mathbb{Q}_+$, with $\gamma \neq 0$. Assume that $P = \{\mathsf{p}_1, \ldots, \mathsf{p}_k\}$. Therefore, we have $(\gamma, 0) = \mathsf{x} + \sum_{i=1}^{k} \gamma_i \, \mathsf{p}_i$ for some $(\gamma_1, \ldots, \gamma_k) \in \mathbb{Q}_+^k$. Hence, there exists $d \in \mathbb{N}$, with $d \neq 0$, and $(c_1, \ldots, c_k) \in \mathbb{N}^k$ such that $(d\gamma, 0) = d\mathsf{x} + \sum_{i=1}^{k} c_i \, \mathsf{p}_i$ and $d\gamma \in \mathbb{N}$. Define m as $\mathsf{m} = (d\gamma, 0)$. We get that $\mathsf{m} \in (\mathsf{x} + P^*)$ and $\{\widehat{\mathsf{m}, (0,1)}\} = \mathbb{Q}_+^2$.
- otherwise: we have $(\mathsf{x} + \widehat{P}) \cap (\mathbb{Q}_+ \times \{0\}) \subseteq \{\mathsf{0}\}$. Since $(\mathsf{x} + \widehat{P}) \cap \mathbb{Q}_+^2 \not\subseteq \{\mathsf{0}\}$, we get that $P = P \cup \{\mathsf{x}\}$ intersects $(\mathbb{Z} \times \mathbb{N}) \setminus \{\mathsf{0}\}$ (as otherwise, we would get $\mathsf{y}[2] < 0$ for every $\mathsf{y} \in (\mathsf{x} + \widehat{P})$, a contradiction). Let m be a vector among those in P that are non zero and that have the smallest angle with $(1, 0)$ (i.e. $\mathsf{m} \neq \mathsf{0}$ and for all $\mathsf{y} \in P$, with $\mathsf{y} \neq \mathsf{0}$, we have $((0,1) \angle \mathsf{m}) \leq ((0,1) \angle \mathsf{y})$). Note that $\mathsf{m} \in (\mathbb{Z} \times \mathbb{N}) \setminus \{0\}$ and hence $((0,1) \angle \mathsf{m}) \leq \pi$. Define $\mathsf{m}_\perp$ as $\mathsf{m}_\perp = (-\mathsf{m}[2], \mathsf{m}[1])$ (i.e. $\mathsf{m}_\perp$ is obtained from m by a rotation of $\frac{\pi}{2}$).

  We now show that for every $\mathsf{y} \in P$, we have $\mathsf{y} \cdot \mathsf{m}_\perp \geq 0$. By contradiction, assume that some $\mathsf{y} \in P$ satisfies $\mathsf{y} \cdot \mathsf{m}_\perp < 0$. Note that y necessarily satisfies $\mathsf{y} \neq \mathsf{0}$ and $(\mathsf{m} \angle \mathsf{y}) > \pi$. We prove that $(\mathsf{x} + \widehat{P}) \cap (\mathbb{Q}_+ \times \{0\}) \not\subseteq \{\mathsf{0}\}$ to obtain a contradiction. Since $(\mathsf{x} + \widehat{P})$ is convex and contains a vector in $\mathbb{Q}_+^2 \setminus \{\mathsf{0}\}$, it is actually sufficient to prove that $(\mathsf{x} + \widehat{P})$ intersects $(\mathbb{Q}_+ \setminus \{0\}) \times (-\mathbb{Q}_+)$. Recall that $((0,1) \angle \mathsf{m}) \leq ((0,1) \angle \mathsf{y})$, thus we come to $((0,1) \angle \mathsf{y}) = ((0,1) \angle \mathsf{m}) + (\mathsf{m} \angle \mathsf{y})$. In particular, this implies that $(\mathsf{y} \angle (0,1)) \neq 0$. But we also get that $(\mathsf{y} \angle (0,1)) = (\mathsf{y} \angle \mathsf{m}) - ((0,1) \angle \mathsf{m})$, and hence $(\mathsf{y} \angle (0,1)) \leq (\mathsf{y} \angle \mathsf{m})$. Since $(\mathsf{y} \angle (0,1)) \leq (\mathsf{y} \angle \mathsf{m}) < \pi$, we deduce from Lemma A.2 that $(0,1) \in \{\widehat{\mathsf{m}, \mathsf{y}}\}$. Moreover, we also get that $\mathsf{y}[2] < 0$, as we have $0 < (\mathsf{y} \angle (0,1)) < \pi$. Thus, there exists $\gamma, \chi \in \mathbb{Q}_+$ such that the vector z defined by $\mathsf{z} = \mathsf{x} + \gamma \, \mathsf{y} + \chi \, (1,0)$ satisfies $\mathsf{z}[1] > 0$ and $\mathsf{z}[2] < 0$. As $\mathsf{z} \in (x + \widehat{P})$, we obtain that $(\mathsf{x} + \widehat{P})$ intersects $(\mathbb{Q}_+ \setminus \{0\}) \times (-\mathbb{Q}_+)$, a contradiction.

  We have proved that P is contained in the half plane $H = \{\mathsf{z} \in \mathbb{Q}_+^2 \;/\; \mathsf{z} \cdot \mathsf{m}_\perp \geq 0\}$. Therefore, we get that $(\mathsf{x} + \widehat{P}) \subseteq \widehat{P} \subseteq \widehat{H} = H$. Since $(\mathsf{x} + \widehat{P}) \cap \mathbb{Q}_+^2 \not\subseteq \{\mathsf{0}\}$, we obtain that $H \cap \mathbb{Q}_+^2 \not\subseteq \{\mathsf{0}\}$. Recall that $\mathsf{m}[2] \geq 0$, hence we have either $\mathsf{m} \in \mathbb{N}^2$ or $\mathsf{m} \in (-\mathbb{N}) \times \{0\}$. However, if we had $\mathsf{m} \in (-\mathbb{N}) \times \{0\}$, then we would get that $H \cap \mathbb{Q}_+^2 = \mathbb{Q}_+ \times \{0\}$ and hence $(\mathsf{x} + \widehat{P}) \cap \mathbb{Q}_+^2 = (\mathsf{x} + \widehat{P}) \cap$

$(\mathbb{Q}_+ \times \{0\}) \not\subseteq \{0\}$, a contradiction. Consequently, we have $\mathsf{m} \in \mathbb{N}^2$, and hence $H \cap \mathbb{Q}_+^2 = \{\widehat{\mathsf{m}, (0,1)}\}$. We finally come to $((\mathsf{x} + \widehat{P}) \cap \mathbb{Q}_+^2) \subseteq \{\widehat{\mathsf{m}, (0,1)}\}$.

In both case, we obtain that there exists a vector $\mathsf{m} \in \mathbb{N}^2$ such that $\mathsf{m} \in (\mathsf{x} + (P \cup \{\mathsf{x}\})^*) \cup P$ and $((\mathsf{x} + \widehat{P}) \cap \mathbb{Q}_+^2) \subseteq \{\widehat{\mathsf{m}, (0,1)}\}$. □

Corollary A.4. *Let $\mathsf{x} \in \mathbb{Z}^2$ and let P be a finite subset of $\mathbb{Z}^2$. If $(\mathsf{x}+\widehat{P}) \cap \mathbb{Q}_+^2 \not\subseteq \{0\}$ then there exists a finite subset P' of $\mathbb{N}^2$ such that $P' \subseteq (\mathsf{x} + (P \cup \{\mathsf{x}\})^*) \cup P$ and $((\mathsf{x} + \widehat{P}) \cap \mathbb{Q}_+^2) \subseteq \widehat{P'}$.*

Proof. We apply Lemma A.3 two times: the time first on x, P, and the second time on $\bar{\mathsf{x}}, P'$, with $P' = \{\bar{\mathsf{p}} \ / \ \mathsf{p} \in P\}$, where for any vector $\mathsf{z} \in \mathbb{Z}^2$ we denote by $\bar{\mathsf{z}}$ the vector $\bar{\mathsf{z}} = (\mathsf{z}[2], \mathsf{z}[1])$. We get that there are two vectors $\mathsf{m}, \mathsf{m}' \in \mathbb{N}^2$ such that :

- m and m' are in $(\mathsf{x} + (P \cup \{\mathsf{x}\})^*) \cup P$, and
- $((\mathsf{x} + \widehat{P}) \cap \mathbb{Q}_+^2) \subseteq \{\widehat{\mathsf{m}, (0,1)}\}$ and $((\mathsf{x} + \widehat{P}) \cap \mathbb{Q}_+^2) \subseteq \{\widehat{\mathsf{m}', (1,0)}\}$.

Observe that $\{\widehat{\mathsf{m}, (0,1)}\} \cap \{\widehat{\mathsf{m}', (1,0)}\} = \{\widehat{\mathsf{m}, \mathsf{m}'}\}$. Hence we define P' as $P' = \{\mathsf{m}, \mathsf{m}'\}$, and we conclude that $((\mathsf{x} + \widehat{P}) \cap \mathbb{Q}_+^2) \subseteq \widehat{P'}$. □

We now have all the necessary lemmas to prove Proposition 5.1. It is worth noting that our proof extensively uses properties of dimension 2. Unsurprisingly, Proposition 5.1 does not hold in dimension 3 (see Remark 5.2 page 13).

Proposition 5.1. *For any finite subset P of $\mathbb{Z}^2$, and for any vector $\mathsf{x} \in P$, there exists two finite subsets B', P' of $(\mathsf{x} + P^*) \cap \mathbb{N}^2$ such that:*

$$(\mathsf{x} + P^*) \cap \mathbb{N}^2 = B' + \left(P' \cup (P \cap \mathbb{N}^2)\right)^*$$

Proof. If we have $((\mathsf{x}+P^*) \cap \mathbb{N}^2) \subseteq \{0\}$, then the proposition clearly holds, so we consider the case where $((\mathsf{x}+P^*) \cap \mathbb{N}^2) \not\subseteq \{0\}$. Observe that $(\mathsf{x}+\widehat{P}) \cap \mathbb{Q}_+^2 \not\subseteq \{0\}$, so according to Corollary A.4, there exists a finite subset P' of $\mathbb{N}^2$ such that $P' \subseteq (\mathsf{x} + (P \cup \{\mathsf{x}\})^*) \cup P$ and $((\mathsf{x} + \widehat{P}) \cap \mathbb{Q}_+^2) \subseteq \widehat{P'}$. Therefore, we also have $((\mathsf{x} + P^*) \cap \mathbb{N}^2) \subseteq \widehat{P'}$. We deduce from Lemma A.1 that there exists a finite subset $B' \subseteq ((\mathsf{x}+P^*) \cap \mathbb{N}^2)$ such that $((\mathsf{x}+P^*) \cap \mathbb{N}^2) + P'^* = (B' + P'^*)$. Since $\mathsf{x} \in P$, we get that $P' \subseteq ((\mathsf{x} + P^*) \cup P) \cap \mathbb{N}^2 = ((\mathsf{x} + P^*) \cap \mathbb{N}^2) \cup (P \cap \mathbb{N}^2)$. Thus, we finally obtain that $(\mathsf{x} + P^*) \cap \mathbb{N}^2 = B' + \left(P'' \cup (P \cap \mathbb{N}^2)\right)^*$, where P'' is defined as $P'' = P' \setminus (P \cap \mathbb{N}^2)$, which concludes the proof. □