Synthesis of robust boundary control for systems governed by semi-discrete differential equations

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Synthesis of robust boundary control for systems governed by semi-discrete differential equations

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Abstract. Boundary control for systems governed by partial differential equations (PDEs) is an important field with many practical and theoretical issues. The topic of boundary control of PDEs has been the subject of a considerable literature since the seminal works of J.-L. Lions in the 90s. In this paper, we consider the boundary control of systems represented by spatial discretizations of PDEs (i.e. semi-discrete equations). We focus on control laws which are sampled and piecewise constant: periodically, at every sampling time, a fixed control amplitude is applied to the system until the next sampling instant. We show that, under some conditions, sampled piecewise-constant boundary control allows to achieve *approximate controllability*: Given a time $T > 0$, the controlled system evolves to a neighborhood of a given final state. The result is illustrated on the boundary control of the semi-discrete version of the heat equation.

1. Introduction

In this paper, we consider finite dimensional systems obtained by spatial discretization of systems governed by partial differential equations (PDEs), i.e, systems governed by *semi-discrete* equations. More specifically, we will focus on the (spatial discretization of the) heat equation, although our method applies in principle more generally. Also, we will always consider *boundary* control, i.e., control applied at the boundary of the structure under consideration.

We will consider the problem of *approximate controllability*: Given a time $T > 0$, the goal is to drive the solution of an evolution problem to a *neighborhood* of a given final state (typically the origin) in T, but not exactly to this final state. In other words, approximate controllability states that the solution of the controlled PDE equation does not reach exactly the equilibrium in T but rather an α-state y such that

$$\|y(T)\|_2 \leq \alpha.$$

Actually, we are concerned by finding an optimal control of minimal L^∞-norm. It is known (see [4]) that such a control exists and is unique. Furthermore, it is of bang-bang form (it takes only two values). This allows us to reformulate the problem as a design problem where the new unknowns are the amplitude of the bang-bang control and the space-time regions where the control takes its two possible values. In other words, we address directly the problem of computing bang-bang controls with minimal amplitude. This problem is of major interest in practice.

We will show that the state space can be decomposed into regions so that a uniform control applies to any point of the same region. In other words, we put in evidence the *robustness* of the synthesized control with respect to small variations of the initial condition.

Plan

In Section 2, we give the semi-discrete model (SDH) of the heat equation obtained by space discretization, then consider the fully discrete model (FDH) obtained by further time discretization. We also state related approximate controllability results. In Section 3, we prove that there are bang-bang controls (with only two values) which are robust with respect to initial conditions. We give preliminary numerical experiments in Section 4, and conclude in Section 5.

2. Discretization

Consider the following one-dimensional linear heat equation (1DH):

$$\begin{cases} y_t - y_{xx} + ay = 0, & 0 < x < 1, \quad 0 < t < T, \\ y(0,t) = 0; y(1,t) = v(t) & 0 < t < T, \\ y(x,0) = y_0(x), & 0 < x < 1, \end{cases}$$

Here, $v \in L^\infty(0,T)$ is the *control* and y is the associated *state*, a is a constant potential. The control is done at the right end of the structure. The temperature is maintained at 0 at the left end.

It is known (see for instance [8]) that, for any $y_0 \in L^2(0,1)$, $T > 0$ and $v \in L^\infty(0,T)$ there is exactly one solution y of (1DH). For a fixed $\alpha > 0$, the *optimal approximate control* problem consists, for any $y_0 \in L^2(0,1)$, in finding a control v of minimal L^∞-norm such that the solution y of the associated equation (1DH) satisfies

$$\|y(T)\|_2 \leq \alpha.$$

It is shown in [4, 2] that the unique solution of this optimization problem is a control of *bang-bang* type, i.e., it takes only two values.

Actually, as in [9] (cf [7]), we will not address the problem at this level, but consider a numerical approximation of bang-bang controls for the heat equation obtained first by spatial discretization (semi-discrete level).

2.1. Space discretization

Given $M \in \mathbb{N}$ we define $h = \frac{1}{M+1} > 0$. We consider the mesh $\{x^j = jh, j = 0, \ldots, M+1\}$ which divides $[0,1]$ into $M+1$ subintervals $I_j = [x^j, x^{j+1}]$, $j = 0, \ldots, M$. Consider the following finite difference approximation of the controlled heat equation:

$$\begin{cases} (y')^j - \frac{1}{h^2}[y^{j+1} + y^{j-1} + (-2 + ah^2)y^j] = 0, & 0 < t < T, \quad j = 1, \ldots, M, \\ y^0(t) = 0; \quad y^{M+1}(t) = v(t) & 0 < t < T, \\ y^j(0) = y_0^j, & j = 1, \ldots, M, \end{cases}$$

which is a coupled system of M linear differential equations. In it the function $y^j(t)$ provides an approximation of $y(x^j, t)$ for all $j = 1, \ldots, M$, y being the solution of the above finite difference approximation heat equation. The conditions $y^0 = 0$ and $y^{M+1} = v$ take account the Dirichlet boundary conditions.

Henceforth, we will suppose that the values of $\alpha > 0, h > 0$ and $T > 0$ are fixed.[1]

[1] This means that the controls v that we will derive, are always implicitly dependent on the values of α, T and h.

We shall use a vector notation to simplify the expressions. In particular, the M-column vector $y(t) = (y^1(t), \ldots, y^M(t))^\top$ will represent the set of unknowns of the system. Likewise, the M-column vector $\overrightarrow{v}(t) = \frac{1}{h^2}(0, \ldots, 0, v(t))^\top$ will represent the control vector. Introducing the $M \times M$ matrix

$$A = \frac{1}{h^2} \begin{pmatrix} (2-ah^2) & -1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & (2-ah^2) \end{pmatrix} \tag{1}$$

the spatial approximation of the heat equation reads now as (SDH):

$$\begin{cases} y'(t) = -Ay(t) + \overrightarrow{v}(t), & 0 < t < T, \\ y^j(0) = y_0^j, & j = 1, \ldots, M, \end{cases}$$

In the following, we will denote the M-vector of initial points $(y_0^1, \ldots, y_0^M) \in \mathbb{R}^M$ by y_0.

We are interested in the following problem (OQ):

For any given $T > 0$, $\alpha > 0$ and initial state $y_0 \in \mathbb{R}^M$, find a control v of minimal L^∞-norm such that:

$$\|y(T)\|_2 \leq \alpha.^2$$

As explained in [9] (cf. [4]), such a control exists and is unique. It can be seen furthermore that this control is of bang-bang type (it takes only two values).

In the following, we will denote such an optimal bang-bang control by: $\overrightarrow{\lambda}(y_0)$. The function $\overrightarrow{\lambda}(y_0)$ goes from $(0, T)$ to $\{-\lambda, \lambda\} \subset \mathbb{R}^2$. Here λ is the amplitude of the piecewise constant control. [3]

The *switching instants* correspond to the elements of $(0, T)$ at which the piecewise constant control function $\overrightarrow{\lambda}(y_0)$ changes its value.

Proposition 1 *Given $T > 0$, $\alpha > 0$, and $y_0 \in \mathbb{R}^M$, the solution y of (SDH) with y_0 as as initial point, controlled by the optimal bang-bang control $\overrightarrow{\lambda}(y_0)$, is such that:*

$$\|y(T)\|_2 \leq \alpha.$$

Furthermore, the number of switching instants is finite.

Proof

(Sketch.) By adapting the method of [9], constructing a control minimizing a relaxation of a functional of the form $J(v) = \frac{1}{2}|v|^2 + \frac{1}{\alpha}\|y(T)\|_2^2$. □

Remark 1 *The above proposition implies that the optimization problem of (SDH) satisfies hypotheses (A1)-(A2) of [1].*

Proposition 2 *If $a \leq 0$, then the real parts of the eigenvalues of matrix $-A$ are negative, hence $\|e^{-AT}\|_2$ converges to 0 as T tends to ∞.*

[2] i.e.: $((y^1(T))^2 + \cdots + (y^M(T))^2)^{\frac{1}{2}} \leq \alpha$.

[3] Recall furthermore that this control is boundary, i.e., applies only at $x = 1$.

Proof
The matrix $-A$ is an $(M \times M)$ tridiagonal matrix with coefficients of the form: α, β, γ with $\beta = -\frac{2}{h^2} + a$, $\alpha = \gamma = \frac{1}{h}^2$. The eigenvalues of this matrix are

$$-\frac{2}{h^2} + a + \frac{2}{h^2} \cos \frac{\pi j}{M+1}, \quad j = 1, \dots, M,$$

with $r = \frac{1}{h^2}$ It follows that the eigenvalues are negative when $a \leq 0$.[4] Hence $\|e^{-AT}\|_2$ converges to 0 as $T \to \infty$. □

In the following, we will assume: $a \leq 0$.
Let us denote by $B(y_0, \alpha)$ the ball in L^2-norm of radius α, centered at y_0. The following result states the robustness of control $\overrightarrow{\lambda}$ with respect to initial conditions.

Proposition 3 *Given $\alpha > 0$, $h > 0$, $T > 0$ and $y_0 \in \mathbb{R}^M$, the solution z of (SDH) with control $\overrightarrow{\lambda}(y_0)$ and initial condition z_0 in $B(y_0, \alpha)$, is such that:*

$$\|z(T)\|_2 \leq (\|e^{-AT}\|_2 + 1)\alpha.$$

Proof
(Sketch). Since the control $\overrightarrow{\lambda}(y_0)$ is piecewise constant (with a finite number of switchings at $0 < t_1 < t_2 < \cdots < t_m < T$), the solution starting from y_0 corresponds to a continuous piecewise exponential function. At switching instants, the value of the solution starting from y_0 (resp. z_0) is thus $y_1 = y_0 e^{-At_1}$, $y_2 = y_1 e^{-A(t_2 - t_1)}$, ..., $y_T = y_m e^{-A(T - t_m)}$ (resp., $z_1 = z_0 e^{-At_1}$, $z_2 = z_1 e^{-A(t_2 - t_1)}$, ..., $z_T = z_m e^{-A(T - t_m)}$). We have then
$\|y_T - z_T\|_2 = \|y_0 - z_0\|_2 \|e^{-AT}\|_2 \leq \alpha \|e^{-AT}\|_2$. It follows $\|z_T\|_2 \leq (\|e^{-AT}\|_2 + 1)\alpha$ since $\|y_T\|_2 \leq \alpha$ by Proposition 1. □

2.2. Time discretization
Given $N \in \mathbb{N}$ we define $\tau = \frac{T}{N+1} > 0$. We now consider the mesh $\{t_j = j\tau, j = 0, \dots, N+1\}$ which divides $[0, T]$ into $N + 1$ subintervals $T_j = [t_j, t_{j+1}]$, $j = 0, \dots, T$.
One can approximate the space X_2 of controls by functions in the subspace $X_{2,N} \subset X_2$ of piecewise constant functions v_τ represented by their values $v_\tau(t_j) = v_j$ at the gridpoints $j\tau$ for $j = 1, \dots, N-1$. This leads to an Euler discretization of the semi-discrete optimization problem (OQ), denoted by $(OQ)_\tau$, defined by:
For any given $T > 0$, $\alpha > 0, h > 0, \tau > 0$ and initial state $y_0 \in \mathbb{R}^M$, find a control v_τ of minimal L^∞-norm such that:

$$\|y(T)\|_2 \leq \alpha.$$

This problem has also a bang-bang control solution denoted by $\overrightarrow{\lambda}_\tau(y_0)$. The switching instants of this control are multiples of τ. The controlled system can be seen as a *discrete-time (or sampled) switched linear system* (see, e.g., [10, 3]).[5] The control can be built by minimizing an appropriate functional J_τ (see [1]). Furthermore, the sequence of time-discrete controls $\overrightarrow{\lambda}_\tau(y_0)$ converges to the continuous-time control $\overrightarrow{\lambda}(y_0)$ as τ tends to zero (see Theorem 4.3, [1]). In particular:

[4] Actually, they are negative as soon as $a < \cos \frac{\pi}{M}$.
[5] Note however that, here, the "switching" component is relevant only to the control part, the matrix A staying invariant.

Proposition 4 *Given $\alpha > 0$, $h > 0$, $T > 0$ and $y_0, z_0 \in \mathbb{R}^m$, if z (resp. z_τ) is the solution of (SDH) with initial condition z_0 under control $\overrightarrow{\lambda}(y_0)$ (resp. under control $\overrightarrow{\lambda}_\tau(y_0)$), then*

$$\|z(T) - z_\tau(T)\|_2 \le \alpha,$$

for sufficiently small τ.

This Proposition holds because assumptions (A1)-(A2) of [1] are satisfied here (see Remark 1).

3. Control Robustness

We are now interested in showing that the discrete-time control $\overrightarrow{\lambda}_\tau(y_0)$ is "robust" with respect to some small variations of the initial condition y_0. In other words, we want to show that, when one applies the discrete-time control $\overrightarrow{\lambda}_\tau(y_0)$ at a starting point z_0 close to y_0 (instead of y_0 exactly), one still reaches a point close to the origin in time T.

Proposition 5 *Given $\alpha > 0$, $h > 0$, $T > 0$ and $y_0 \in \mathbb{R}^M$, the solution z_τ of (SDH) with control $\overrightarrow{\lambda}_\tau(y_0)$ and initial condition z_0 in $B(y_0, \alpha)$ satisfies:*

$$\|z_\tau(T)\|_2 \le (\|e^{-AT}\|_2 + 2)\alpha.$$

for sufficiently small τ.

Proof

Consider a solution z_τ of (SDH) with initial condition z_0 in $B(y_0, \alpha)$ and control $\overrightarrow{\lambda}_\tau(y_0)$. Let z be the solution of (SDH) with initial condition z_0 in $B(y_0, \alpha)$ under *continuous-time* control $\overrightarrow{\lambda}(y_0)$, By Proposition 3, z is such that: $\|z(T)\|_2 \le (\|e^{-AT}\|_2 + 1)\alpha$. Besides, by Proposition 4, we know that $\|z(T) - z_\tau(T)\|_2 \le \alpha$ for sufficiently small τ. Hence
$\|z_\tau(T)\|_2 \le \|z(T) - z_\tau(T)\|_2 + \|z(T)\|_2 \le (\|e^{-AT}\|_2 + 2)\alpha$. □

Given a dense region $R \subset \mathbb{R}^M$, one can applies iteratively Proposition 5 in order to decompose R into a finite number of subregions where discrete-time controls apply uniformly. Formally, we have:

Proposition 6 *Given $\alpha > 0$, $h > 0$, $T > 0$, and a region $R \subset \mathbb{R}^M$, there exist a finite set of points $\{y_1, \ldots, y_m\}$ of $\mathbb{R}^M$ such that, for all sufficiently small time step τ:*

- *$R \subseteq \bigcup_{i=1,\ldots,m} B(y_i, \alpha)$, and*
- *for all $i = 1, \ldots, m$, the solution y of (SDH) with control $\overrightarrow{\lambda}_\tau(y_i)$ and initial condition in $B(y_i, \alpha)$, is such that:*
  $$\|y(T)\|_2 \le (\|e^{-AT}\|_2 + 2)\alpha.$$

If we regard the approximation of the controlled heat equation as a *sampled switched linear system* (in the sense of [3]), then the above result guarantees the existence of a *decomposition* of the state space as studied in [6], as well as the existence of associated bang-bang controls. Actually, if we are primarily interested by the *robustness* of the control at the expense of its *optimality*, then we can practically look for such controls using the tool MINIMATOR described in [6, 5]. In the following section, we give numerical experiments obtained with (an adaptation of) tool MINIMATOR.

4. Numerical Experiments

4.1. Robustness with respect to initial conditions

We now describe first experiments realized with tool MINIMATOR. We consider the heat equation with $a = 0$. We consider a spatial discretization of step $h = 1/3$. This means that $M = 2$, and the interior points of the grid space are at $x = \frac{1}{3}, \frac{2}{3}$. For T, we take: $T = 0.4$. For the time discretization step, we take: $\tau = 0.04$. We are thus looking for bang-bang controls that move an original point y_0 into the neighborhood of O in 10 steps ($T = 10\tau$, $N = 9$). For the initial point, we take $y_0 = (\sin\frac{\pi}{3}, \sin\frac{2\pi}{3})$. [6]

For the amplitude of the bang-bang control, we take $\lambda = 0.1$. Given these parameter values, MINIMATOR explores all the bang-bang controls of length 10, and finds that the control sequence $(-1, -1, -1, +1, -1, -1, -1, -1, +1, -1)$ [7] maps y_0 into the ε-neighborhood of O with $\varepsilon = 0.015$.

Besides, MINIMATOR finds that this bang-bang control maps the whole square box centered at $y = 0$ of side length 0.02 into this ε-neighborhood of O.

The square of side 0.02 centered at y_0 corresponds to the upper right box in Figure 1, and the ε-neighborhood of O to the lower left box.

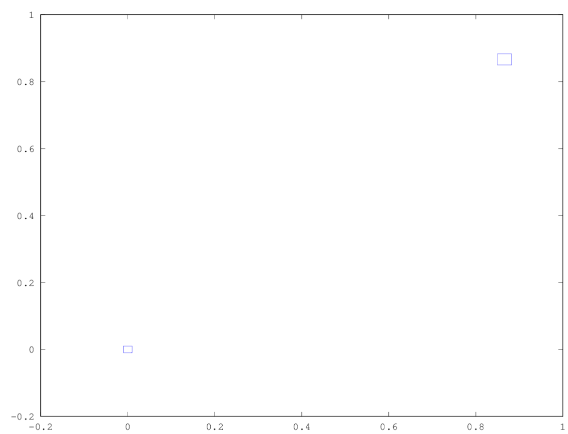

Figure 1. Upper right box: initial region centered at $y_0 = (\sin\frac{\pi}{3}, \sin\frac{2\pi}{3})$; lower left box: overapproximation of the image of the initial region after control $(-1, -1, -1, +1, -1, -1, -1, -1, +1, -1)$

For a slighly different initial point $z_0 = y_0 + 0.01$ (i.e., $z_0 = (\sin\frac{\pi}{3} + 0.01, \sin\frac{2\pi}{3} + 0.01)$), a similar result can be obtained with a bang-bang control of same amplitude, but of control sequence $(-1, -1, -1, -1, +1, -1, -1, -1, +1, -1)$.

More generally, let us consider a rectangular zone $R = [-1, 1] \times [-1, 1]$ in $\mathbb{R}^2$. By iterating the research of square boxes centered in R for which there exists a uniform control sequence of length 10 and amplitude $\lambda = 0.1$ mapping them into the ε-neighborhood of O, on finds the decomposition depicted in Figure 2.

One can see that the square boxes do not cover the whole zone R: this means that, for non covered subregions, control sequences of length greater than 10 are needed (i.e, we need to refine our time discretization by taking smaller τ).

[6] i.e., $y_0^1 = \sin\frac{\pi}{3}, y_0^2 = \sin\frac{2\pi}{3}$, at $t = 0$

[7] In other words, this corresponds to a control with value $-\lambda$ for $t \in [0, 3\tau) \cup [4\tau, 8\tau) \cup [9\tau, 10\tau)$, and value $+\lambda$ for $t \in [3\tau, 4\tau) \cup [8\tau, 9\tau)$.

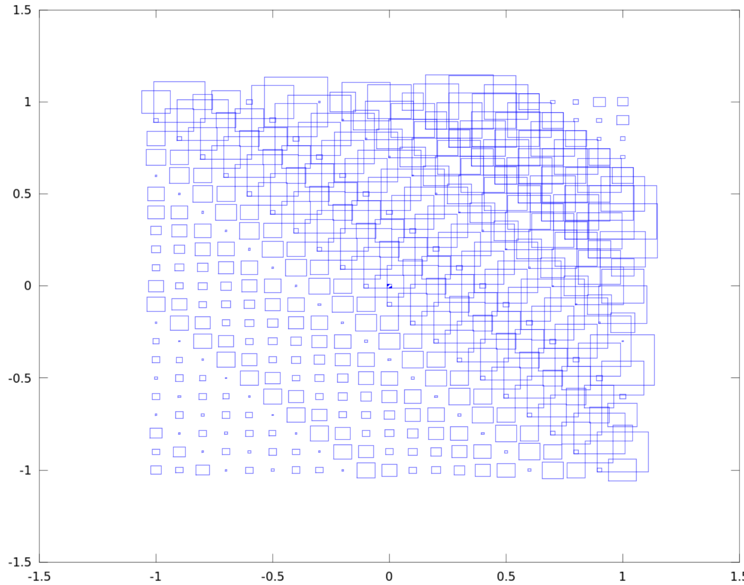

Figure 2. Partial decomposition of $R = [-1,1] \times [-1,1]$ into square boxes with uniform control sequence of length 10 ($\tau = 0.04$, $\lambda = 0.1$, $h = \frac{1}{3}$)

4.2. Robustness with respect to space gridding refinement

We have seen that, for $M = 2$ and $h = \frac{1}{3}$, the discrete-time control $(-1,-1,-1,+1,-1,-1,-1,-1,+1,-1)$ maps $y_0 = (\sin\frac{\pi}{3}, \sin\frac{2\pi}{3})$ into the neighborhood of O in $T = 0.4$. Here, we would like to know if the same control sequence still maps y_0 when one drastically refines the space gridding, and take $M = 99$ and $h = \frac{1}{100}$, and $y_0 = (\sin\frac{\pi}{100}, \sin\frac{2\pi}{100}, \ldots, \sin\frac{99\pi}{100})$. The answer is positive: the same control sequence (corresponding to the same time mesh size $\tau = 0.04$) maps y_0 to the ε-neighborhood of O. A simulation of the control starting at y_0 in time T is given in Figure 3.

This suggests more generally that the boundary control found for a coarse space gridding can be useful for controlling the original system governed by the partial differential heat equation (1DH).

5. Final Remarks

We have considered the semi-discrete heat equation obtained by spatial discretization of the partial difference equation, as well as time discretization of the semi-discrete model. We have shown that, for sufficiently small time discretization steps, there exist time-discrete bang-bang controls which steer the solutions of the semi-discrete equation to the origin, and are *robust* with respect to small perturbations of initial conditions. This guarantees the existence of a state space decomposition into regions with uniform control for all the points of a same region. First experiments performed with tool MINIMATOR confirm the existence of such regions with uniform control. In further work, we will study whether this property still applies to the original heat equation. We will study in particular the uniform convergence of the control sequences as the space discretization step tends to 0.

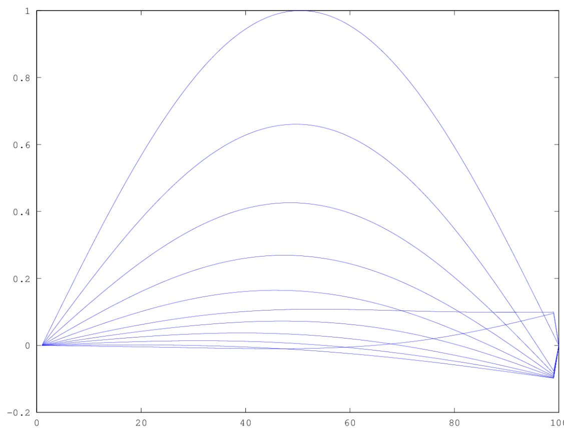

Figure 3. Simulation of the application of control $(-1,-1,-1,+1,-1,-1,-1,-1,+1,-1)$ to $y_0 = \sin\frac{\pi}{x}$ at $t = 0, \tau, \ldots, 10\tau$ ($\tau = 0.04$, $\lambda = 0.1$, $h = \frac{1}{100}$)

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