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Prevision Domains and Convex Powercones*

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Abstract. Two recent semantic families of models for mixed probabilistic and non-deterministic choice over a space X are the convex powercone models, due independently to Mislove, and to Tix, Keimel, and Plotkin, and the continuous prevision model of the author. We show that, up to some minor details, these models are isomorphic whenever X is a continuous, coherent cpo, and whether the particular brand of non-determinism we focus on is demonic, angelic, or chaotic. The construction also exhibits domains of continuous previsions as retracts of well-known continuous cpos, providing simple bases for the various continuous cpos of continuous previsions. This has practical relevance to computing approximations of operations on previsions.

1 Introduction

Continuous lower and upper previsions, and forks, were proposed in [5] as adequate models for mixed non-deterministic (demonic, angelic, and chaotic respectively) and probabilistic choice. At the end of this paper, it was claimed that there was a strong relation between this model and that discovered independently by Mislove [13] and by Tix [16, 17], consisting of convex non-empty subsets of continuous valuations, which are also compact saturated, resp. closed, resp. (compact) lenses—the so-called *convex powercones*. We make this connection more precise, and to show that this "strong relation" is in fact an isomorphism, provided X is a coherent continuous pointed cpo.

Before we go on, let us mention that Keimel and Plotkin [10] solved a very similar problem, under the guise of finding predicate transformers characterizing convex powercones. Keimel and Plotkin's so-called functional representations of predicate transformers map elements of convex powercones over a cpo X to certain continuous functionals very much like our continuous previsions (essentially, up to the replacement of $\mathbb{R}^+$ by $\overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{+\infty\}$). However, Keimel and Plotkin's convex powercones are composed of convex subsets of continuous valuations, and the latter may be unbounded. In practice, convex subsets of continuous *probabilities* (such that the measure of the whole space is 1) or *subprobabilities* (at most 1) seem to fit more tightly our needs in modeling probabilistic choice, and Keimel and Plotkin note that "it would be more natural, from the point of view of computer science applications, to restrict to subprobability valuations, rather than allowing all of them." This is what we do here.

Our isomorphism result is not a consequence of the theorems of Keimel and Plotkin, although it is likely that adapting their proofs (and in fact, making them murkier) would

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give us the desired results. We take the route of [6, Section 11.7], and prove the isomorphism in two steps: first, isomorphism theorems on fairly general classes of topological spaces X, but where convexity has to be replaced by a slightly stronger notion; second, the proof that the strong notions of convexity coincide with ordinary convexity on coherent continuous pointed cpos.

Outline. We quickly go over preliminaries in Section 2, then show our first isomorphism theorem in the demonic case, in Section 3. This relies on the notion of strong convexity, and uses notions of barycenters à la Choquet-Bishop-de Leeuw. Showing that strong convexity reduces to convexity in the case of cpos is the subject of Section 4, and is the only section that relies on theorems by Tix, Keimel, and Plotkin. We deal with the angelic case in Section 5, which we reduce to the demonic case by the so-called *convex-concave duality*. The chaotic case is now ripe for treatment in Section 6. We conclude in Section 7.

Related Work. Clearly [16, 17, 10] and [5] are most relevant. Other relevant material will be cited on the fly.

2 Preliminaries

See [1, 3, 12] for background material on domain theory and topology. A *cpo* X is a partially ordered set (poset) in which every directed set has a least upper bound, or sup. We write $\leq$ its ordering. The Scott topology on a poset has as opens all upward-closed subsets U such that whenever $(z_i)_{i \in I}$ is a directed family having a sup z in U, then some z_i is in U already. The *way-below* relation $\ll$ on a poset is defined by $x \ll y$ iff whenever $(z_i)_{i \in I}$ is a directed family having a sup z with $y \leq z$, then $x \leq z_i$ for some $i \in I$. A poset X is *continuous* iff every $x \in X$ is the directed sup of all elements $y \ll x$. A *basis* of X is then a subset B of X such that every $x \in X$ is the directed sup of all elements $y \in B$ such that $y \ll x$. The Scott topology then has a basis of open sets of the form $\Uparrow y = \{x \in X | y \ll x\}$, $y \in B$.

For any topological space (e.g., cpo) X, let $\langle X \to \mathbb{R}^+\rangle$ be the cpo of all bounded continuous maps from X to $\mathbb{R}^+$. We take $\mathbb{R}^+$ with the Scott topology, whose non-trivial opens are the open intervals $(t, +\infty)$, $t \in \mathbb{R}^+$, and order $\langle X \to \mathbb{R}^+$ pointwise. A *continuous prevision* F on a topological space (e.g., a cpo) X is a Scott-continuous map from $\langle X \to \mathbb{R}^+\rangle$ to $\mathbb{R}^+$ such that $F(af) = aF(f)$ for every $a \in \mathbb{R}^+$ (*positive homogeneity*). A prevision F is *lower* iff $F(h + h') \geq F(h) + F(h')$ for every h, h', *upper* iff $F(h + h') \leq F(h) + F(h')$ for every h, h', *linear* iff $F(h + h') = F(h) + F(h')$, *normalized* iff $F(a + h) = a + F(h)$ for every function h and constant $a \in \mathbb{R}^+$, *subnormalized* iff $F(a + h) \leq a + F(h)$ for every h and constant a. A *fork* is a pair (F^-, F^+) of continuous previsions, where F^- is lower, F^+ is upper, and *Walley's condition* $F^-(h + h') \leq F^-(h) + F^+(h') \leq F^+(h + h')$ holds for every h, h'. A fork is normalized, resp. sub-normalized, whenever both F^- and F^+ are. We shall concentrate on normalized previsions and forks in the sequel.

It was shown in [5] that, among continuous normalized previsions, the lower brand was an adequate model of mixed probabilistic and demonically non-deterministic choice, the upper brand was one of mixed probabilistic and angelically non-deterministic choice, while normalized forks were an adequate model of mixed probabilistic and chaotically non-deterministic choice. It was essentially well-known since Tix [15] that the space of continuous (subnormalized, resp. normalized) *linear* previsions were isomorphic to Jones' space $\mathbf{V}_{\leq 1}(X)$ (resp., $\mathbf{V}_1(X)$) of subprobability (resp. probability) valuations. A *valuation* is a map p from the set $\mathcal{O}(X)$ of all opens of X to $\overline{\mathbb{R}}^+$ that is strict ($p(\emptyset) = 0$), monotone ($U \subseteq V$ implies $p(U) \leq p(V)$), and modular ($p(U \cup V) + p(U \cap V) = p(U) + p(V)$). *Subprobability* valuations (resp., *probability* valuations) are those such that $p(X) \leq 1$ (resp., $p(X) = 1$). The valuation p is *continuous* iff $p(\bigcup_{i \in I} U_i) = \sup_{i \in I} p(U_i)$ for every directed family $(U_i)_{i \in I}$ of opens of X. We shall always equip each cpo, and in fact each poset, X with its Scott topology. However, we shall consider more general topological spaces in the sequel. For every open U, let χ_U map x to 1 if $x \in U$, to 0 otherwise. The isomorphism between the space $\mathbf{P}^{\triangle}_1(X)$ of continuous normalized linear previsions G on X and $\mathbf{V}_1(X)$ maps G to $p = \gamma_{\mathcal{C}}(G)$ defined by $p(U) = G(\chi_U)$ for every open U, and conversely, maps p to $G = \alpha_{\mathcal{C}}(p)$ defined by letting $G(h)$ the Choquet integral of h along p [5]. (The Choquet integral of $h \in \langle X \to \mathbb{R}^+ \rangle$ along p, which we shall write $\oint_{x \in X} h(x) dp$, is defined as the ordinary Riemann integral $\int_0^{+\infty} p(h^{-1}(t, +\infty)) dt$. This is a continuous linear prevision of h and is also linear and Scott-continuous in p, whenever p is a continuous valuation. Note that Choquet integration is also defined when p is merely a so-called *game* [4].)

Let $\bigtriangledown \mathbf{P}_1(X)$ be the space of continuous normalized lower previsions on X, $\mathbf{P}^{\triangle}_1(X)$ that of all continuous normalized linear previsions (with the Scott topology, ordered pointwise), and $\mathbf{P}^{\triangle}_{1\ wk}(X)$ the same space with the *weak topology*, defined as the smallest that contains the subbasic opens $[f > r] = \{G \in \mathbf{P}^{\triangle}_{1\ wk}(X) | G(f) > r\}$, $f \in \langle X \to \mathbb{R}^+ \rangle$, $r \in \mathbb{R}^+$. The Scott topology is always finer than the weak topology. When X is a continuous cpo with a least element, both topologies coincide, i.e., $\mathbf{P}^{\triangle}_{1\ wk}(X) = \mathbf{P}^{\triangle}_1(X)$. This is easily obtained from the coincidence of the two topologies on spaces of valuations, through the isomorphism between continuous valuations and continuous linear previsions above (see [9], who refers to Tix [15, Satz 4.10], who cites Kirch [11, Satz 8.6]; see also [6, Proposition 3.7.12].)

The relation between spaces of previsions and sets of convex subsets of valuations alluded to in the introduction takes the following form, in the demonic case [5, Proposition 4]. Let the *Smyth powerdomain* $\mathcal{Q}(Y)$ of a topological space be the set of all non-empty compact saturated subsets (see below) of Y, ordered by reverse inclusion $\supseteq$. (This is a standard model of demonic non-determinism alone [1].) Then, there is a map $CCoeur_1 : \bigtriangledown \mathbf{P}_1(X) \to \mathcal{Q}(\mathbf{P}^{\triangle}_{1\ wk}(X))$ sending each continuous normalized lower prevision F to its *heart* $CCoeur_1(F) = \{G \in \mathbf{P}^{\triangle}_1(X) | F \leq G\}$. Whenever X is stably compact (see below), the heart is non-empty and compact saturated. (Both properties are non trivial.) Moreover, the heart is *convex*: for any two $G, G' \in CCoeur_1(F)$, for any $\alpha \in [0, 1]$, $\alpha G + (1 - \alpha) G'$ is in $CCoeur_1(F)$ (trivial). Conversely, there is a map $\bigsqcap : \mathcal{Q}(\mathbf{P}^{\triangle}_{1\ wk}(X)) \to \bigtriangledown \mathbf{P}_1(X)$, sending $\mathcal{Q}$ to $\lambda h \in \langle X \to \mathbb{R}^+ \rangle \cdot \min_{G \in \mathcal{Q}} G(h)$. (The min is, indeed, attained.) $CCoeur_1$ and $\bigsqcap$ are Scott-continuous, and form a Galois insertion, i.e., $\bigsqcap \circ CCoeur_1 = \mathrm{id}$ (Rosenmuller's Theorem), and $CCoeur_1 \circ \bigsqcap \supseteq \mathrm{id}$.

A subset Q of X is *compact* iff one can extract a finite subcover from every open cover. It is *saturated* iff it is the intersection of all opens containing it, a.k.a. it is upward-closed in the *specialization quasi-ordering* $\leq$, defined by $x \leq y$ iff every open containing x contains y. A topological space X is *stably compact* (taking Jung's definitions [9]) iff X is T_0 ($\leq$ is an ordering), *well-filtered* (for every filtered family $(Q_i)_{i \in I}$ of compact saturated subsets, for every open U, if $\bigcap_{i \in I} Q_i \subseteq U$ then $Q_i \subseteq U$ already for some $i \in I$), *locally compact* (whenever $x \in U$ with U open, there is a compact saturated subset Q such that $x \in int(Q) \subseteq Q \subseteq U$, where $int(Q)$ denotes the interior of Q), *coherent* (the intersection of any two compact saturated subsets is again so) and *compact*. Every continuous cpo X is well-filtered and locally compact. If additionally X is *pointed*, i.e., has a least element, then X is compact. If finally X is also coherent, then X is stably compact. Stable compactness has a long history, going back to Nachbin (1948; see [9]).

3 Demonic Non-Determinism + Probabilistic Choice

The central question of this paper, in the demonic case, is whether the pair $CCoeur_1 \dashv \bigsqcap$ actually defines an isomorphism between $\bigtriangledown \mathbf{P}_1(X)$ and some suitable subset of $\mathcal{Q}(\mathbf{P}^{\triangle}_{1\ wk}(X))$. One natural candidate is $\mathcal{Q}^{cvx}(\mathbf{P}^{\triangle}_{1\ wk}(X))$, the space of all elements of $\mathcal{Q}(\mathbf{P}^{\triangle}_{1\ wk}(X))$ that are convex—since the heart is always convex.

However, the right notion we need is that of *strong convexity*, defined below. (Connoisseurs will note that the same idea is the root of the classic Choquet-Bishop-de Leeuw extension to the Krein-Milman Theorem.) One first notes that convex sets $\mathcal{Q}$ of linear previsions are those that are stable by taking *finite barycenters*, i.e., such that for any finite set $G_0, G_1, \ldots, G_n$ of $n+1$ elements of $\mathcal{Q}$, for any coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}^+$ with $\sum_{i=0}^n a_i = 1$, $\sum_{i=0}^n a_i G_i$ is again in $\mathcal{Q}$. On any space Y, the *Dirac valuation* δ_y defined so that $\delta_y(V) = 1$ if $y \in V$, 0 otherwise, is a continuous probability valuation, and so are the *simple probability valuations* of the form $\sum_{i=0}^n a_i \delta_{y_i}$, with $a_0, a_1, \ldots, a_n$ as above. The Choquet integral of $h \in \langle Y \to \mathbb{R}^+ \rangle$ along such a simple probability valuation yields $\sum_{i=0}^n a_i h(y_i)$. It follows that we can rewrite the finite barycenter $\sum_{i=0}^n a_i G_i$ as $\lambda h \in \langle X \to \mathbb{R}^+ \rangle \cdot \oint_{G \in \mathbf{P}^{\triangle}_{1\ wk}(X)} G(h) d\mathcal{P}$, where $\mathcal{P}$ is the simple probability valuation $\sum_{i=0}^n a_i \delta_{G_i}$ on $\mathbf{P}^{\triangle}_{1\ wk}(X)$ (an element of $\mathbf{P}^{\triangle}_{1}(\mathbf{P}^{\triangle}_{1\ wk}(X))$!). This allows to define the general notion of *barycenter* $Bary(\mathcal{P})$ of a continuous probability valuation $\mathcal{P} \in \mathbf{P}^{\triangle}_{1}(\mathbf{P}^{\triangle}_{1\ wk}(X))$ as $\lambda h \in \langle X \to \mathbb{R}^+ \rangle \cdot \oint_{G \in \mathbf{P}^{\triangle}_{1\ wk}(X)} G(h) d\mathcal{P} \in \mathbf{P}^{\triangle}_{1}(X)$.

Say that $\mathcal{P}$ is *supported* on $\mathcal{Q} \subseteq \mathbf{P}^{\triangle}_{1\ wk}(X)$ iff $\mathcal{P}(\mathcal{U}) = 1$ for every open set of continuous probability valuations containing $\mathcal{Q}$. This intuitively says that $\mathcal{P}$ bears no mass outside $\mathcal{Q}$. One can check that $\sum_{i=0}^n a_i G_i$ is supported on $\mathcal{Q}$ (where $\mathcal{Q}$ is upward-closed) iff $G_i \in \mathcal{Q}$ for all i such that $a_i \neq 0$.

We then say that $\mathcal{Q}$ is *strongly convex* iff $Bary(\mathcal{P}) \in \mathcal{Q}$ for every $\mathcal{P} \in \mathbf{P}^{\triangle}_{1}(\mathbf{P}^{\triangle}_{1\ wk}(X))$ that is supported on $\mathcal{Q}$. Whenever $\mathcal{Q}$ is strongly convex, this property must hold whenever $\mathcal{P}$ is a simple probability valuation, showing that every strongly convex upward-closed set is convex. The converse fails, as we now show. Let X be $\mathbb{N} \cup \{+\infty\}$, with the obvious ordering and its Scott topology. Let $\mathcal{Q}$ be the set of all linear previsions of the form $\lambda h \cdot \sum_{i=0}^n a_i h(k_i)$, where $a_0, \ldots, a_n$ are as above and $k_i \in \mathbb{N}$. This is the convex hull of the set of linear previsions of the form $\alpha_{\mathcal{C}}(\delta_k)$, $k \in \mathbb{N}$, and is therefore convex. (Recall that $\alpha_{\mathcal{C}}(\delta_k)$ is the image of δ_k through the isomorphism between continuous valuations and continuous linear previsions, and maps h to $h(k)$.) It is clear that $\delta_{+\infty}$ is not in $\mathcal{Q}$. However, $\delta_{+\infty}$ arises as $Bary(\delta_{\delta_{+\infty}})$, and we check that $\delta_{\delta_{+\infty}}$ is supported on $\mathcal{Q}$: any open containing $\mathcal{Q}$ must contain, say, δ_0, hence also $\delta_{+\infty}$, since $\delta_0 \leq \delta_{+\infty}$.

Proposition 1. *Let X be stably compact, and $F \in \bigtriangledown \mathbf{P}_1(X)$. Then $CCoeur_1(F)$ is strongly convex.*

Proof. We first observe that, given any compact saturated subset Q of a space Y, given any continuous probability valuation p on Y that is supported on Q, for any $h \in \langle Y \to \mathbb{R}^+\rangle$: $(*)$ $\oint_{y\in Y} h(y)dp \geq \min_{y\in Q} h(y)$. Let $a = \min_{y\in Q} h(y)$ (which is attained since Q is compact). For all $t < a$, $f^{-1}(t, +\infty)$ contains Q, so $p(f^{-1}(t, +\infty)) = 1$; hence $\oint_{y\in Y} h(y)dp = \int_0^{+\infty} p(f^{-1}(t, +\infty))dt \geq \int_0^a p(f^{-1}(t, +\infty))dt = a$.

For f an arbitrary element of $\langle X \to \mathbb{R}^+\rangle$, apply $(*)$ to the case $Y = \mathbf{P}^{\triangle}_{1\ wk}(X)$, $p = \mathcal{P}$ supported on $\mathcal{Q} = CCoeur_1(F)$, taking $h(G) = G(f)$. (Note that h is continuous, precisely because Y is equipped with the weak topology.) We get $\oint_{G\in\mathbf{P}^{\triangle}_{1\ wk}(X)} G(f)d\mathcal{P} \geq \min_{G\in\mathcal{Q}} G(f) = \bigsqcap \mathcal{Q}(f)$. However, remember that $\bigsqcap \circ CCoeur_1 = \mathrm{id}$, so $\bigsqcap \mathcal{Q} = F$. Also, $\oint_{G\in\mathbf{P}^{\triangle}_{1\ wk}(X)} G(f)d\mathcal{P} = Bary(\mathcal{P})(f)$. So $Bary(\mathcal{P}) \geq F$, i.e., $Bary(\mathcal{P}) \in CCoeur_1(F)$. ☐

The converse direction, that strongly convex non-empty compact saturated subsets of $\mathbf{P}^{\triangle}_{1\ wk}(X)$ arise from some element of $\bigtriangledown \mathbf{P}_1(X)$, relies on the following key Proposition 2. To appreciate it, look at the case when $\mathcal{Q}$ is the upward closure $\uparrow \{G_1, \ldots, G_n\}$ of $\{G_1, \ldots, G_n\}$ in $\mathbf{P}^{\triangle}_{1\ wk}(X)$: up to some details, the proposition states that if for each h, there is an i with $G(h) \geq G_i(h)$ (not necessarily the same i for each h), then there are coefficients $a_0, a_1, \ldots, a_n$ with $\sum_{i=0}^n a_i = 1$ such that $G(h) \geq \sum_{i=0}^n a_i G_i(h)$ for *all* h. (This is similar to a key step in some proofs of the minimax theorem.)

The proof relies on Roth's Sandwich Theorem ([14], [17, Theorem 3.1]), which states that on every ordered cone C, for every positively homogeneous super-additive function $q : C \to \overline{\mathbb{R}}^+$ and every positively homogeneous sub-additive function $p : C \to \overline{\mathbb{R}}^+$ such that $a \leq b$ implies $q(a) \leq p(b)$ (e.g., when $q \leq p$ and either q or p is monotonic), then there is a monotonic linear function $f : C \to \overline{\mathbb{R}}^+$ such that $q \leq f \leq p$. A *cone* is a set C, together with a binary operation $+$ turning it into a commutative monoid and a scalar multiplication $\cdot$ from $\mathbb{R}^+ \times C$ to C, such that $1 \cdot a = a$, $0 \cdot a = 0$, $(rs) \cdot a = r \cdot (s \cdot a)$, $r \cdot (a + b) = r \cdot a + r \cdot b$, and $(r + s) \cdot a = r \cdot a + s \cdot a$. An *ordered cone* is equipped in addition with a partial ordering $\leq$ making $+$ and $\cdot$ monotonic. The function q is *positively homogeneous* iff $q(s \cdot a) = sq(a)$ for all $s \in \mathbb{R}^+$, *super-additive* iff $q(a + b) \geq q(a) + q(b)$, *sub-additive* iff $q(a + b) \leq q(a) + q(b)$, and *linear* iff it has all three properties.

Proposition 2. *Let X be stably compact, $\mathcal{Q}$ be a non-empty compact saturated subset of $\mathbf{P}^{\triangle}_{1\ wk}(X)$, $G \in \mathbf{P}^{\triangle}_{1\ wk}(X)$ such that $\bigsqcap \mathcal{Q} \leq G$. Then there is a continuous probability valuation $\mathcal{P}$ on $\mathbf{P}^{\triangle}_{1\ wk}(X)$ that is supported on $\mathcal{Q}$, and such that $Bary(\mathcal{P}) \leq G$.*

Proof. Consider the ordered cone $C = \langle \mathbf{P}^{\triangle}_{1\ wk}(X) \to \mathbb{R}^+\rangle$, with the obvious $+$ and $\cdot$, and the pointwise ordering. For every $\varphi \in \langle \mathbf{P}^{\triangle}_{1\ wk}(X) \to \mathbb{R}^+\rangle$, let $q(\varphi) =$

$\min_{G \in \mathcal{Q}} \varphi(G)$. This is clearly positively homogeneous, super-additive, and monotonic. In fact, q is even (Scott-)continuous. This is non-obvious. However, recall from [4] that, for any compact saturated subset Q of a space Y, the *unanimity game* $\mathfrak{u}_Q$ (mapping each open U containing Q to 1 and all others to 0), the Choquet integral $\oint_{y \in Y} f(y) d\mathfrak{u}_Q$ equals $\min_{y \in Q} f(y)$—so that $q(\varphi) = \oint_{G \in \mathbf{P}^{\triangle}_{1\ wk}(X)} \varphi(G) d\mathfrak{u}_{\mathcal{Q}}$—and that Choquet integration is Scott-continuous in the integrated function.

Let $p(\varphi) = \inf_{\substack{f \in \langle X \to \mathbb{R}^+ \rangle \\ \forall G' \in \mathcal{Q} \cdot \varphi(G') \leq G'(f)}} G(f)$. (We take this to mean $+\infty$ if the inf is taken over an empty family of values.) It is easy to see that $p(a\varphi) = ap(\varphi)$ for every $a \in \mathbb{R}^+$: the case $a = 0$ works by realizing that $f = 0$ satisfies $\varphi(G') = G'(f)$ for all $G \in \mathcal{Q}$, and then $G(f) = 0$, the case $a \neq 0$ works by substituting f/a for f in the formula for p. Checking that p is super-additive is only slightly harder: $p(\varphi) + p(\varphi')$ equals the inf of $G(f) + G(g) = G(f + g)$ when f and g range over functions such that $\varphi(G') \leq G'(f)$ and $\varphi(G') \leq G'(g)$ for all $G' \in \mathbf{P}^{\triangle}_{1\ wk}(X)$. Since every such G' is linear, they all satisfy $(\varphi + \varphi')(G') \leq G'(f + g)$, so $p(\varphi) + p(\varphi')$ is greater than of equal to the inf of $G(f+g)$ when f and g satisfy the weaker condition $(\varphi+\varphi')(G') \leq G'(f + g)$. This is then greater than or equal to $p(\varphi + \varphi')$.

We now check that $q \leq p$. This is where we use the assumption $\bigsqcap \mathcal{Q} \leq G$. Fix $\varphi \in \langle \mathbf{P}^{\triangle}_{1\ wk}(X) \to \mathbb{R}^+ \rangle$. For all $f \in \langle X \to \mathbb{R}^+ \rangle$ such that $\forall G' \in \mathcal{Q} \cdot \varphi(G') \leq G'(f)$, we have $G(f) \geq \bigsqcap \mathcal{Q}(f) = \min_{G' \in \mathcal{Q}} G'(f) \geq \min_{G' \in \mathcal{Q}} \varphi(G') = q(\varphi)$. Now take infs over f on each side, whence $p(\varphi) \geq q(\varphi)$.

Using Roth's Sandwich Theorem, there is a monotonic linear functional $\mathcal{G}_0$ from $\langle \mathbf{P}^{\triangle}_{1\ wk}(X) \to \mathbb{R}^+ \rangle$ to $\overline{\mathbb{R}}^+$ such that $q \leq \mathcal{G}_0 \leq p$. We claim that $\mathcal{G}_0$ never takes the value $+\infty$. Indeed, for any $\varphi \in \langle \mathbf{P}^{\triangle}_{1\ wk}(X) \to \mathbb{R}^+ \rangle$, letting $a = \sup_{G' \in \mathbf{P}^{\triangle}_{1\ wk}(X)} \varphi(G')$, $\mathcal{G}_0(\varphi) \leq p(\varphi) \leq ap(\chi_{\mathbf{P}^{\triangle}_{1\ wk}(X)}) \geq G(\chi_X) = 1$ because we may take $f = \chi_X$ in the definition of p, as $\varphi(G') \leq a = aG'(\chi_X)$, G' being normalized. So $\mathcal{G}_0$ is a linear prevision. Since $q \leq \mathcal{G}_0$ and $q(\chi_{\mathbf{P}^{\triangle}_{1\ wk}(X)}) = 1$, it follows that $\mathcal{G}_0(\chi_{\mathbf{P}^{\triangle}_{1\ wk}(X)}) = 1$; since $\mathcal{G}_0$ is linear, this is enough to show that $\mathcal{G}_0$ is normalized

But $\mathcal{G}_0$ is not necessarily continuous. We use the machinery, based on the Scott extension formula, developed in [5, Long version, Appendix], and which we recall briefly now. By Claim Q of op.cit., for any stably compact space Y (the result is stated for the slightly more general class of compact, stably core compact spaces), we may define a continuous functional $\mathfrak{r}(F)$ from $\langle Y \to \mathbb{R}^+ \rangle$ to $\overline{\mathbb{R}}^+$ from any functional F from $\langle Y \to \mathbb{R}^+ \rangle$ to $\overline{\mathbb{R}}^+$ by the formula $\mathfrak{r}(F)(f) = \sup_{g \in B, g \ll f} F(g)$, where B is a basis of the continuous poset $\langle Y \to \mathbb{R}^+ \rangle$ (described in Claim K) and $\ll$ is its way-below relation; then $\mathfrak{r}(F)$ is the largest continuous functional below F, and is a continuous normalized linear prevision whenever F is a normalized linear prevision.

In our case, observe that $Y = \mathbf{P}^{\triangle}_{1\ wk}(X)$ is stably compact: the isomorphism $\alpha_{\mathcal{C}}$, $\gamma_{\mathcal{C}}$ between $\mathbf{P}^{\triangle}_{1}(X)$ and $\mathbf{V}_1(X)$ also defines an isomorphism between $\mathbf{P}^{\triangle}_{1\ wk}(X)$ and $\mathbf{V}_{1\ wk}(X)$ (where the latter is defined with the weak topology, whose subbasic opens are $[f > r] = \{p \in \mathbf{V}_1(X) | \oint_{x \in X} f(x) dp > r\}$ are in one to one correspondence to those of $\mathbf{P}^{\triangle}_{1\ wk}(X)$). And $\mathbf{V}_{1\ wk}(X)$ is stably compact as soon as X is, a result due to Jung [9, Theorem 3.2]. So the machinery applies: $\mathcal{G} = \mathfrak{r}(\mathcal{G}_0)$ is a continuous normalized linear prevision, and $\mathcal{G} \leq \mathcal{G}_0$. Moreover, since $\mathcal{G}$ is the largest continuous functional below $\mathcal{G}_0$, and q is continuous, we have $q \leq \mathcal{G} \leq \mathcal{G}_0 \leq p$.

Using the isomorphism $\alpha_\mathcal{C}$, $\gamma_\mathcal{C}$, let $\mathcal{P} = \gamma_\mathcal{C}(\mathcal{P})$: this is a continuous normalized valuation on $\mathbf{P}^\triangle_{1\,wk}(X)$. We claim it is supported on $\mathcal{Q}$. For every open $\mathcal{U}$ containing $\mathcal{Q}$ indeed, $q(\chi_\mathcal{U}) \leq \mathcal{G}(\chi_\mathcal{U})$. But $q(\chi_\mathcal{U}) = \min_{G' \in \mathcal{Q}} \chi_\mathcal{U}(G') = 1$, and $\mathcal{G}(\chi_\mathcal{U}) = \mathcal{P}(\mathcal{U})$. Since $\mathcal{P}(\mathcal{U}) \leq 1$ anyway, $\mathcal{P}(\mathcal{U}) = 1$.

For every $f \in \langle X \to \mathbb{R}^+ \rangle$, let φ be the function mapping G' to $G'(f)$. Note that $\mathcal{G}(\varphi) = \alpha_\mathcal{C}(\mathcal{P})(\varphi) = \oint_{G' \in \mathbf{P}^\triangle_{1\,wk}(X)} \varphi(G') d\mathcal{P} = \oint_{G' \in \mathbf{P}^\triangle_{1\,wk}(X)} G'(f) d\mathcal{P} = Bary(\mathcal{P})(f)$, while $p(\varphi) = \inf_{\substack{g \in \langle X \to \mathbb{R}^+ \rangle \\ \forall G' \in \mathcal{Q} \cdot \varphi(G') \leq G'(g)}} G(g) = \inf_{\substack{g \in \langle X \to \mathbb{R}^+ \rangle \\ \forall G' \in \mathcal{Q} \cdot G'(f) \leq G'(g)}} G(g) \leq G(f)$. So $Bary(\mathcal{P}) \leq G$, as announced. □

Let $Conv(\mathcal{Q})$, the *strong convex closure* of $\mathcal{Q}$, be $\{Bary(\mathcal{P}) | \mathcal{P}$ supported on $\mathcal{Q}\}$. Using Proposition 2, we may characterize the action of $CCoeur_1 \circ \bigsqcap$ by $CCoeur_1(\bigsqcap \mathcal{Q}) = \uparrow Conv(\mathcal{Q})$ for every $\mathcal{Q} \in \mathcal{Q}(\mathbf{P}^\triangle_{1\,wk}(X))$. Recall that $\mathcal{Q} \subseteq CCoeur_1(\bigsqcap \mathcal{Q})$. By Proposition 1, it follows that $Conv(\mathcal{Q}) \subseteq CCoeur_{\leq 1}(\bigsqcap \mathcal{Q})$. Since the heart is upward-closed, $\uparrow Conv(\mathcal{Q}) \subseteq CCoeur_1(\bigsqcap \mathcal{Q})$. Conversely, if $G \in CCoeur_1(\bigsqcap \mathcal{Q})$, i.e., $\bigsqcap \mathcal{Q} \leq G$, then by Proposition 2, there is a continuous probability valuation $\mathcal{P}$ supported on $\mathcal{Q}$, such that $Bary(\mathcal{P}) \leq G$. This means that $Bary(\mathcal{P}) \in Conv(\mathcal{Q})$, so $G \in \uparrow Conv(\mathcal{Q})$.

Theorem 1 (Isomorphism). *Let X be stably compact. Then $CCoeur_1$ and $\bigsqcap$ define an isomorphism between $\bigtriangledown \mathbf{P}_1(X)$ and the space $\mathcal{Q}^{Cvx}(\mathbf{P}^\triangle_{1\,wk}(X))$ of strongly convex non-empty compact saturated subsets of $\mathbf{P}^\triangle_{1\,wk}(X)$, ordered by $\supseteq$.*

Proof. It is enough to realize that for every $\mathcal{Q} \in \mathcal{Q}^{Cvx}(\mathbf{P}^\triangle_{1\,wk}(X))$, $\uparrow Conv(\mathcal{Q}) = \uparrow \mathcal{Q} = \mathcal{Q}$, while $CCoeur_1(\bigsqcap \mathcal{Q}) = \uparrow Conv(\mathcal{Q})$. The identity $\bigsqcap \circ CCoeur_1 = \mathrm{id}$ is already known from [5]. □

4 The Cpo Case

We may refine Theorem 1 and replace strong convexity by the mere notion of convexity, when X is a coherent, continuous and pointed (hence stably compact) cpo. This comes close to the results of Keimel and Plotkin [10], who show that the space of super-additive, positively homogeneous and Scott-continuous functionals from $[X \to \overline{\mathbb{R}}^+]$ to $\overline{\mathbb{R}}^+$, is isomorphic to $\mathcal{Q}^{cvx}(\overline{\mathbf{V}}(X))$, where $[X \to Y]$ denotes the cpo of all continuous maps from X to Y (not just the bounded ones), and $\overline{\mathbf{V}}(X)$ is the set of all valuations (not just the normalized ones, not even those that are bounded, i.e. do not take the value $+\infty$). Note also the use of $\mathcal{Q}^{cvx}$ here instead of $\mathcal{Q}^{Cvx}$. Despite the apparent added generality of the results of [10], they do not seem to entail ours. (Try it!)

The key to our result is to realize that any compact saturated, convex subset of $\mathbf{P}^\triangle_{1\,wk}(X)$ is in fact strongly convex. We need the following variant of [17, Theorem 3.8] first, which is proved in Appendix A. Call *continuous cone* any ordered cone C which is continuous qua poset, and where $+$ and $\cdot$ are Scott-continuous. (This is as the continuous d-cones of [17], except we don't require C to be a cpo.) It is *additive* iff $x_1 \ll y_1$ and $x_2 \ll y_2$ imply $x_1 + x_2 \ll y_1 + y_2$. Call a subset Z of an additive continuous cone *sane* whenever Z is continuous as a sub-partial order of C, and whenever

$x, y \in Z$ are such that $x \ll_Z y$, then $x \ll_C y$, where $\ll_Z$ and $\ll_C$ are the way-below relations of Z and C respectively.

Proposition 3. *Let C be an additive continuous cone, and Z a sane subspace of C. For every convex compact subset K of Z, for every non-empty convex closed subset F of Z disjoint from K, there is $a \in \overline{\mathbb{R}}^+$, $a > 1$, and a continuous linear function $f : C \to \overline{\mathbb{R}}^+$ such that $f(z) > a$ for all $z \in K$ and $f(y) \leq 1$ for every $y \in F$.*

Caveat: The topology of Z is that induced by that of C (whose opens are of the form $V \cap Z$, V open in C), but there is no reason in general that this should coincide with the Scott topology of the ordering $\leq$ on Z. Such pathologies do not arise when Z is a sane subspace of C (see Claim D, Appendix A). For every continuous cpo Y, the space $C = \mathbf{V}(Y)$ of all continuous bounded valuations is an additive continuous cone, and $Z = \mathbf{V}_{\leq 1}(Y)$ is a sane subspace of C (Claim F, Appendix A). However, $\mathbf{V}_1(Y)$ is *not* sane in $\mathbf{V}(Y)$, when Y has a least element $\bot$, as $\delta_\bot \ll_{\mathbf{V}_1(X)} \delta_\bot$ but $\delta_\bot \not\ll_{\mathbf{V}(X)} \delta_\bot$ (the family $r\delta_\bot$, $r < 1$, has the right-hand side as sup, but no element of this family is greater than or equal the left-hand side).

Recall that a saturated subset is one that is the intersection of all opens containing it. Say that a subset A is *linearly saturated* iff A is the intersection of all convex opens containing it. It is tempting to think that any convex saturated subset should be linearly saturated. This is indeed the case for those subsets that are also compact, as the following variant of [17, Corollary 3.13], due to Jung, shows.

Proposition 4. *Let C be a continuous cone, Z be a sane convex subspace of C. Every convex compact saturated subset Q of Z is linearly saturated.*

Proof. We must show that for every $x \in Z \setminus Q$, there is a convex open subset V containing Q but not x. Let $F = \{z \in Z | z \leq x\}$: this is convex, non-empty, and disjoint from Q. Build f and a as in Proposition 3. The open $V = f^{-1}(a, +\infty) \cap Z$ of Z fits the bill. In particular, V is convex because f is linear. ☐

Theorem 2. *Let X be a continuous pointed cpo. Every convex compact saturated subset $\mathcal{Q}$ of $\mathbf{P}^\triangle_1(X) = \mathbf{P}^\triangle_{1\,wk}(X)$ is strongly convex.*

Proof. Fix X, with least element $\bot$. By Edalat's variant of Jones' Theorem [2, Section 3], $Y = \mathbf{V}_1(X)$ is then also a continuous pointed cpo, with least element $\delta_\bot$. Using a trick by Edalat, $X' = X \setminus \{\bot\}$ is a continuous cpo again, and $\mathbf{V}_1(X)$ is isomorphic to $\mathbf{V}_{\leq 1}(X')$. (Send $\nu \in \mathbf{V}_1(X)$ to $\lambda U \in \mathcal{O}(X') \cdot \nu(U)$, and send back $\nu \in \mathbf{V}_{\leq 1}(X')$ to ν' defined as $\nu'(U) = \nu(U)$ if $U \neq X$, i.e., $\bot \notin U$, $\nu'(U) = 1$ otherwise.) Take $C = \mathbf{V}(X')$, $Z = \mathbf{V}_{\leq 1}(X')$. Let $\mathcal{Q}$ be a convex saturated compact of Z, $\mathcal{P}$ an element of $\mathbf{V}_1(Z)$ supported on $\mathcal{Q}$. Since Z is a continuous pointed cpo again, we use again one of Edalat's results [2, Section 3]: every element of $\mathbf{V}_1(Z)$ is the sup of a directed family of simple probability valuations $\mathcal{P}_i$, $i \in I$. Fix an arbitrary convex open $\mathcal{U}$ containing $\mathcal{Q}$. By definition of the Scott topology, some $\mathcal{P}_i$ is in $\mathcal{U}$, from which we deduce easily that the subfamily of those $\mathcal{P}_i$ that are in $\mathcal{U}$ is again directed, with sup $\mathcal{P}$. Let J be the set of indices i such that $\mathcal{P}_i \in \mathcal{U}$. However, since $\mathcal{U}$ is convex and $\mathcal{P}_i$ is simple, the finite barycenter $Bary(\mathcal{P}_i)$ is in $\mathcal{U}$. It is easy to see that the family $(Bary(\mathcal{P}_i))_{i \in J}$ is directed. Its sup is in $\mathcal{U}$, since $\mathcal{U}$ is upward-closed. Since Choquet integration is continuous in the valuation argument, $Bary$ is continuous, so this sup is just $Bary(\mathcal{P})$. We have shown that $Bary(\mathcal{P}) \in \mathcal{U}$ for every convex open $\mathcal{U}$ containing $\mathcal{Q}$. By Proposition 4, $\mathcal{Q}$ is the intersection of all such convex opens, so $Bary(\mathcal{P}) \in \mathcal{Q}$. This shows that every convex compact saturated subset of $Z = \mathbf{V}_{\leq 1}(X')$ is strongly convex. Now note that Z is isomorphic to $\mathbf{V}_1(X)$, which is isomorphic to $\mathbf{P}_1^{\triangle}(X) = \mathbf{P}_{1\ wk}^{\triangle}(X)$. □

Since convex and strong convexity coincide for compact saturated subsets, the following is immediate.

Corollary 1 (Isomorphism). *Let X be a continuous, coherent pointed cpo. $CCoeur_1$ and $\prod$ define an isomorphism between $\bigtriangledown \mathbf{P}_1(X)$ and $\mathcal{Q}^{cvx}(\mathbf{P}_{1\ wk}^{\triangle}(X)) = \mathcal{Q}^{cvx}(\mathbf{P}_1^{\triangle}(X))$ $\cong \mathcal{Q}^{cvx}(\mathbf{V}_1(X))$.*

Note that $CCoeur_1$ and $\prod$ also exhibit $\bigtriangledown \mathbf{P}_1(X)$ as a retract of $\mathcal{Q}(\mathbf{P}_1^{\triangle}(X))$, i.e., they are continuous, and $\prod \circ CCoeur_1 = \mathrm{id}$. ($\prod$ is the *retraction*, and $CCoeur_1$ the associated *section*.) By [2], $\mathbf{P}_1^{\triangle}(X) \cong \mathbf{V}_1(X)$ has a basis of *simple normalized linear previsions*, i.e., previsions of the form $\alpha_{\mathcal{C}}(p)$, with p a simple probability valuation. Concretely, these are previsions of the form $\lambda h \in \langle X \to \mathbb{R}^+\rangle \cdot \sum_{i=0}^n a_i h(x_i)$, where $a_0, a_1, \ldots, a_n \in \mathbb{R}^+$, $\sum_{i=0}^n a_i = 1$. It is well-known that, whenever Y is a continuous cpo, $\mathcal{Q}(Y)$ is also a continuous cpo with basis given by the *finitary compacts* $\uparrow \mathcal{E}$, $\mathcal{E}$ a finite subset of Y [1]. It is also known that any retract Z' of a continuous cpo Z is again a continuous cpo, with basis given by the image of any basis of Z by the retraction. So:

Theorem 3. *For any continuous, coherent pointed cpo X, $\bigtriangledown \mathbf{P}_1(X)$ is a continuous, coherent pointed cpo. A basis is given by previsions of the form $\lambda h \in \langle X \to \mathbb{R}^+\rangle \cdot \min_{i=1}^m \sum_{j=1}^n a_{ij} h(x_j)$, where $a_{ij} \in \mathbb{R}^+$ and $\sum_{j=1}^n a_{ij} = 1$ for each i.*

Proof. The only things that remain to be proved are that $\bigtriangledown \mathbf{P}_1(X)$ has a least element ($\lambda h \cdot h(\bot)$, i.e., $\alpha_{\mathcal{C}}(\delta_\bot)$), and that $\bigtriangledown \mathbf{P}_1(X)$ is coherent. Note that $\mathcal{Q}(\mathbf{P}_1^{\triangle}(X))$ is a bc-domain, i.e., a continuous cpo where any two elements Q_1 and Q_2 having an upper bound have a least upper bound (namely $Q_1 \cap Q_2$, which is non-empty because Q_1 and Q_2 have an upper bound $Q \subseteq Q_1, Q_2$). Every bc-domain is coherent, hence stably compact (see, e.g., [8]), and by a result of Lawson quoted by Jung [9], every retract of a stably compact space is again stably compact. □

The above theorem means that we can always approximate, from below, any continuous normalized lower prevision by one that is computable, using only finitely many $\min$, $+$ and $\cdot$ operations. It is remarkable that we know no proof of this fact that would avoid the relatively daunting constructions above.

5 Angelic Non-Determinism + Probabilistic Choice

Let the *Hoare powerdomain* $\mathcal{H}(Y)$ be the set of all non-empty closed subsets of Y, ordered by inclusion (a standard model of angelic non-determinism alone). This is also a cpo, which is continuous as soon as Y is, and is usually used to model angelic non-determinism. Let $\bigtriangleup \mathbf{P}_1(X)$ be the space of all continuous normalized *upper* previsions on X. Then [5, Proposition 5] there is a map $CPeau_1 : \bigtriangleup \mathbf{P}_1(X) \to \mathcal{H}(\mathbf{P}_{1\ wk}^{\triangle}(X))$, defined by $CPeau_1(F) = \{G \in \mathbf{P}_1^{\triangle}(X) | G \leq F\}$, and a map $\bigsqcup : \mathcal{H}(\mathbf{P}_{1\,wk}^{\triangle}(X)) \to \triangle \mathbf{P}_1(X)$, sending $\mathcal{F}$ to $\lambda h \in \langle X \to \mathbb{R}^+ \rangle \cdot \sup_{G \in \mathcal{F}} G(h)$. When X is stably compact, $\bigsqcup \dashv CPeau_1$ defines what we called a Galois surrection, i.e., $\bigsqcup$ and $CPeau_1$ are monotonic, $\bigsqcup \circ CPeau_1 = \mathrm{id}$, and $CPeau_1 \circ \bigsqcup(\mathcal{F}) \supseteq \mathcal{F}$ for all $\mathcal{F}$. $\bigsqcup$ is continuous, but we do not know whether $CPeau_1$ is continuous in general.

We shall prove that $\bigsqcup$ and $CPeau_1$ define an isomorphism similar to those of the previous sections. The main trick is in using a nice duality between demonic and angelic non-determinism, which we called *convex-concave duality* on games in [4], and which extends to previsions. Very roughly, the idea is to turn any prevision F into the functional $F^\perp = \lambda h \in \langle X \to \mathbb{R}^+ \rangle \cdot -F(-h)$. If F is lower, then $F^\perp$ will be upper, and conversely, moreover $F^{\perp\perp} = F$. Unfortunately, $F(-h)$ is in general ill-defined: First, $-h$ does not take its values in $\mathbb{R}^+$ (easy to repair, see below); second, $-h$ is very far from being continuous from X to $\mathbb{R}^+$: the inverse image of the (Scott-)open $(t, +\infty)$ by $-h$ is $h^{-1}(-\infty, -t)$, of which we know nothing.

To correct the first problem, extend any normalized prevision F on X to a functional $\widehat{F} : \langle X \to \mathbb{R} \rangle \to \mathbb{R}$ by letting $\widehat{F}(f) = F(f + a) - a$, for any $a \geq -\inf_{x \in X} f(x)$. (As before, $\langle X \to \mathbb{R} \rangle$ is the space of all bounded continuous maps from X to $\mathbb{R}$, with the Scott topology of the pointwise ordering.) This is independent of a, because F is normalized. It is easy to see that $\widehat{F}$ is monotonic (if $f \leq f'$ then $\widehat{F}(f) \leq \widehat{F}(f')$), positively homogeneous (if $r \geq 0$ then $\widehat{F}(rf) = r\widehat{F}(f)$), normalized, and lower, resp. upper, resp. linear, resp. Scott-continuous when F is.

Solving the second problem is harder. We will have to approximate functions $-h$ with $h \in \langle X \to \mathbb{R}^+ \rangle$ by functions g not from X, but from the *de Groot dual* X^{d} of X, to $\mathbb{R}^+$. This is defined (when X is stably compact) as X, only with the so-called *cocompact topology*, whose opens are the cocompacts, i.e., subsets of the form $X \setminus Q$, Q compact saturated subset of X. Observe that well-filteredness, coherence, and compactness imply that this is indeed a topology. Then X^{d} is again stably compact, and $X^{\mathsf{dd}} = X$ (see [9] for more background material on this).

A function $f : X \to \mathbb{R}$ is a *step function* if and only if it is of the form $\sum_{i=0}^{n} a_i \chi_{U_i}$, where $X = U_0 \supseteq U_1 \supseteq \ldots \supseteq U_n$ is a sequence of opens, and $a_0 \in \mathbb{R}$, $a_1, \ldots, a_n \in \mathbb{R}^+$. It is well-known (see e.g., [15]) that any element f of $\langle X \to \mathbb{R} \rangle$ is the sup of a directed family of step functions, namely $f_K = a + \frac{1}{2^K} \sum_{k=1}^{\lfloor (b-a)2^K \rfloor} \chi_{f^{-1}(a + \frac{k}{2^K}, +\infty)}$, $K \in \mathbb{N}$, where a is any lower bound for f and b is any upper bound for f.

Definition 1 **($F^\perp$).** *Let X be a stably compact space, F a normalized prevision on X. The* dual *$F^\perp$ of F is the map from $\langle X^{\mathsf{d}} \to \mathbb{R}^+ \rangle$ to $\mathbb{R}^+$ defined by $F^\perp(g) = -\inf_{\substack{f \text{ step function} \\ f \geq -g}} \widehat{F}(f)$.*

We sum up the main properties of this construction. The proof is omitted for lack of space, but can be found in Appendix B.

Theorem 4. *Let X be a stably compact space. For every normalized prevision F on X, $F^\perp$ is a normalized prevision on X^{d}. Moreover: (1) $F^\perp$ is continuous; (2) if F is lower, then $F^\perp$ is upper; (3) if F is upper, then $F^\perp$ is lower; (4) if F is linear, then so is $F^\perp$; (5) if F is continuous, then $F^{\perp\perp} = F$; (6) if $F \leq F'$ then $F'^\perp \leq F^\perp$.*

The main step is to show that, when g is a step function, $F^\perp(g)$ can be defined alternatively as $-\inf_{f \sqsupseteq^{\mathsf{d}} -g} \widehat{F}(f)$, where $\sqsupseteq^{\mathsf{d}}$ is a more constrained relation: $f \sqsupseteq^{\mathsf{d}} -g$ iff one can write f as $-\sum_{i=0}^n a_i \chi_{X \setminus U_i}$ (U_i opens, $\emptyset = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_n$, $a_0 \in \mathbb{R}$, $a_1, \ldots, a_n \in \mathbb{R}^+$), g as $\sum_{i=0}^n a_i \chi_{X \setminus Q_i}$ (Q_i compact saturated subsets, $\emptyset = Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_n$), with the *same* coefficients a_i, and $Q_i \subseteq U_i$ for each i. While the proofs require $\sqsupseteq^d$, we observe that Definition 1 can be simplified by eliminating the recourse to step functions (see Appendix C for a proof).

Lemma 1. *Let X be a stably compact space, and F a normalized continuous prevision on X. For every $g \in \langle X^{\mathsf{d}} \to \mathbb{R} \rangle$, $F^\perp(g) = -\inf_{\substack{f \in \langle X \to \mathbb{R} \rangle \\ f \geq -g}} \widehat{F}(f)$.*

For every continuous game ν in the sense of [4] (in particular, a continuous valuation) on a stably compact space, we may define $\nu^\perp$ as $\gamma_{\mathcal{C}}(F^\perp)$ where $F = \alpha_{\mathcal{C}}(\nu)$. One may check that $\nu^\perp(X \setminus Q) = 1 - \nu^\dagger(Q)$, where $\nu^\dagger(Q) = \inf_{U \supseteq Q} \nu(U)$. (See Claim X, Appendix B.) In the case where ν is a continuous valuation, the $\nu^\dagger$ construction was already studied by Tix [15, Satz 3.4].

Proposition 5. *Let X be stably compact. The map $p \mapsto p^\perp$ defines an isomorphism between $\mathbf{V}_{1\ wk}(X)^{\mathsf{d}}$ and $\mathbf{V}_{1\ wk}(X^{\mathsf{d}})$. The map $F \mapsto F^\perp$ defines an isomorphism between $\mathbf{P}^{\triangle}_{1\ wk}(X)^{\mathsf{d}}$ and $\mathbf{P}^{\triangle}_{1\ wk}(X^{\mathsf{d}})$.*

Proof. The second claim follows from the first through the isomorphism $\alpha_{\mathcal{C}}$, $\gamma_{\mathcal{C}}$. For any compact saturated subset Q of X, for any real r, let $\langle Q < r \rangle = \{p \in \mathbf{V}_1(X) | p^\dagger(Q) < r\}$. By [9, Concluding remarks], the sets $\langle Q < r \rangle$ form a subbasis of the cocompact topology on $\mathbf{V}_{1\ wk}(X)$, provided X is stably compact. (This is stated in terms of sets written $[K \geq r]$, which are the complements of $\langle K < r \rangle$. See [6, Section 6.4] for a proof.) So $p \mapsto p^\perp$ is continuous. Then apply Theorem 4 (6). □

For lower previsions, we used probability valuations $\mathcal{P}$ supported on a compact saturated subsets $\mathcal{Q}$ of $\mathbf{P}^{\triangle}_{1\ wk}(X)$. Upper previsions require us to use *co*supports instead. Say that $\mathcal{P}$ is *co-supported* on $\mathcal{F} \subseteq \mathbf{P}^{\triangle}_{1\ wk}(X)$ iff $\mathcal{P}(\mathcal{U}) = 0$, where $\mathcal{U}$ is the complement of the closure $cl(\mathcal{F})$ of $\mathcal{F}$. It is easy to see that $\mathcal{P}$ is supported on the compact saturated subset $\mathcal{Q}$ of $\mathbf{P}^{\triangle}_{1\ wk}(X)$ iff $\mathcal{P}^\perp$ is co-supported on the closed subset $\mathcal{Q}$ of $\mathbf{P}^{\triangle}_{1\ wk}(X)^{\mathsf{d}}$.

Say that a subset $\mathcal{F}$ of $\mathbf{P}^{\triangle}_{1\ wk}(X)$ is *co-strongly convex* iff $Bary(\mathcal{P}) \in \mathcal{F}$ for every $\mathcal{P} \in \mathbf{V}_1(\mathbf{P}^{\triangle}_{1\ wk}(X))$ that is co-supported on $\mathcal{F}$. When $\mathcal{P}$ is simple, say $\mathcal{P} = \sum_{i=0}^n a_i \delta_{G_i}$, then $\mathcal{P}$ is co-supported on $\mathcal{F}$ (when $\mathcal{F}$ is downward-closed) iff every G_i such that $a_i \neq 0$ is in $\mathcal{F}$; so every co-strongly convex downward-closed set is convex. We shall use this notion when $\mathcal{F}$ is a closed subset of $\mathbf{P}^{\triangle}_{1\ wk}(X)$, in which case $\mathcal{F}$ will always be downward-closed. By an argument similar to that of Proposition 1 (see Appendix D):

Proposition 6. *Let X be stably compact, and $F \in \bigtriangleup \mathbf{P}_1(X)$. Then $CPeau_1(F)$ is co-strongly convex.*

The key argument for the converse is the following proposition, which states how $Bary$ behaves w.r.t. the dualizing operation $\_^\perp$. For any continuous map $f : Y \to Z$, and every $p \in \mathbf{V}_{1\ wk}(Y)$, the *push-forward* continuous valuation $f[p] \in \mathbf{V}_{1\ wk}(Z)$ is that which sends each $V \in \mathcal{O}(Z)$ to $p(f^{-1}(V))$. The change-of-variables formula for Choquet integration states that $\oint_{z\in Z} g(z)df[p] = \oint_{y\in Y} g(f(y))dp$ (an easy consequence of the definition). We use the notation $\_^\perp[\mathcal{P}']$ below (with $\mathcal{P}' = \mathcal{P}^\perp$), where $\_^\perp : \mathbf{P}^{\triangle}_{1\ wk}(X)^{\mathsf{d}} \to \mathbf{P}^{\triangle}_{1\ wk}(X^{\mathsf{d}})$.

Proposition 7. *Let X be stably compact. For any continuous probability valuation $\mathcal{P}$ on $\mathbf{P}^{\triangle}_{1\ wk}(X)$, $(Bary(\_^\perp[\mathcal{P}^\perp]))^\perp = Bary(\mathcal{P})$.*

Proof. Using the change-of-variables formula, $Bary(\_^\perp[\mathcal{P}^\perp]) = \lambda g \in \langle X^{\mathsf{d}} \to \mathbb{R}^+\rangle \cdot \oint_{G'\in\mathbf{P}^{\triangle}_{1\ wk}(X^{\mathsf{d}})} G'(g)d\_^\perp[\mathcal{P}^\perp] = \lambda g \in \langle X^{\mathsf{d}} \to \mathbb{R}^+\rangle \cdot \oint_{G\in\mathbf{P}^{\triangle}_{1\ wk}(X)^{\mathsf{d}}} G^\perp(g)d\mathcal{P}^\perp = \lambda g \in \langle X^{\mathsf{d}} \to \mathbb{R}^+\rangle \cdot -\inf_{\substack{\varphi\in\langle\mathbf{P}^{\triangle}_{1\ wk}(X)\to\mathbb{R}\rangle \\ \varphi\geq\lambda G\in\mathbf{P}^{\triangle}_{1\ wk}(X)^{\mathsf{d}}\cdot -G^\perp(g)}} \oint_{G\in\mathbf{P}^{\triangle}_{1\ wk}(X)} \varphi(G)d\mathcal{P}$ (using the definition of $\mathcal{P}^\perp$ as $\alpha_{\mathcal{C}}(\mathcal{P}^\perp) = \alpha_{\mathcal{C}}(\mathcal{P})^\perp$, and Lemma 1).

For each $g \in \langle X^{\mathsf{d}} \to \mathbb{R}\rangle$, and every constant $a \geq -\inf_{x\in X} g(x)$, then, $\widehat{Bary(\_^\perp[\mathcal{P}^\perp])}(g) = -\inf_{\substack{\varphi\in\langle\mathbf{P}^{\triangle}_{1\ wk}(X)\to\mathbb{R}\rangle \\ \varphi\geq\lambda G\in\mathbf{P}^{\triangle}_{1\ wk}(X)^{\mathsf{d}}\cdot -G^\perp(g+a)}} \oint_{G\in\mathbf{P}^{\triangle}_{1\ wk}(X)} \varphi(G)d\mathcal{P} - a$. So, for all $h \in \langle X \to \mathbb{R}^+\rangle$, with $a \geq \sup_{x\in X} h(x)$, we obtain:

$$(Bary(\_^\perp[\mathcal{P}^\perp]))^\perp(h) = \sup_{\substack{g\in\langle X^{\mathsf{d}}\to\mathbb{R}\rangle \\ g\geq -h}} \inf_{\substack{\varphi\in\langle\mathbf{P}^{\triangle}_{1\ wk}(X)\to\mathbb{R}\rangle \\ \varphi\geq\lambda G\in\mathbf{P}^{\triangle}_{1\ wk}(X)^{\mathsf{d}}\cdot -G^\perp(g+a)}} \oint_{G\in\mathbf{P}^{\triangle}_{1\ wk}(X)} \varphi(G)d\mathcal{P} + a \quad (1)$$

using Lemma 1. For every open U in X, we claim that $(Bary(\_^\perp[\mathcal{P}^\perp]))^\perp(\chi_U) \leq Bary(\mathcal{P})(\chi_U)$. For every $g \in \langle X^{\mathsf{d}} \to \mathbb{R}\rangle$ with $g \geq -\chi_U$, we have $\chi_U \geq -g$, so (since $G^\perp$ is normalized) $G^\perp(g+a) = G^\perp(g) + a = \sup_{f\geq -g} -G(f) + a \geq -G(\chi_U) + a$. For every $\varphi \in \langle\mathbf{P}^{\triangle}_{1\ wk}(X) \to \mathbb{R}\rangle$ with $\varphi \geq \lambda G \in \mathbf{P}^{\triangle}_{1\ wk}(X)^{\mathsf{d}} \cdot G(\chi_U) - a$, therefore, $\varphi \geq \lambda G \in \mathbf{P}^{\triangle}_{1\ wk}(X)^{\mathsf{d}} \cdot -G^\perp(g+a)$. So $\inf_{\substack{\varphi\in\langle\mathbf{P}^{\triangle}_{1\ wk}(X)\to\mathbb{R}\rangle \\ \varphi\geq\lambda G\in\mathbf{P}^{\triangle}_{1\ wk}(X)^{\mathsf{d}}\cdot G(\chi_U)-a}} \oint_{G\in\mathbf{P}^{\triangle}_{1\ wk}(X)} \varphi(G)d\mathcal{P} \geq \inf_{\substack{\varphi\in\langle\mathbf{P}^{\triangle}_{1\ wk}(X)\to\mathbb{R}\rangle \\ \varphi\geq\lambda G\in\mathbf{P}^{\triangle}_{1\ wk}(X)^{\mathsf{d}}\cdot -G^\perp(g+a)}} \oint_{G\in\mathbf{P}^{\triangle}_{1\ wk}(X)} \varphi(G)d\mathcal{P}$. The left-hand side is exactly $Bary(\mathcal{P})(\chi_U) - a$: take $\varphi = \lambda G \in \mathbf{P}^{\triangle}_{1\ wk}(X)^{\mathsf{d}} \cdot G(\chi_U) - a$, and use the fact that $Bary(\mathcal{P})$ is normalized. (The key point is that this φ is indeed in $\langle\mathbf{P}^{\triangle}_{1\ wk}(X) \to \mathbb{R}\rangle$, in particular continuous, as the inverse image of $(t, +\infty)$ is the open $[\chi_U > a + t]$.) So, using (1), $(Bary(\_^\perp[\mathcal{P}^\perp]))^\perp(\chi_U) \leq \sup_{\substack{g\in\langle X^{\mathsf{d}}\to\mathbb{R}\rangle \\ g\geq -\chi_U}} Bary(\mathcal{P})(\chi_U) - a + a = Bary(\mathcal{P})(\chi_U)$.

Conversely, let us show that, for any two opens U and V of X with $V \Subset U$, $Bary(\mathcal{P})(\chi_V) \leq (Bary(\_^\perp[\mathcal{P}^\perp]))^\perp(\chi_U)$. The relation $\Subset$ is the way-below relation on $\mathcal{O}(X)$, ordered by inclusion; on every locally compact space, $U \Subset V$ iff $U \subseteq Q \subseteq V$ for some compact saturated set Q, and $\mathcal{O}(X)$ is then a continuous cpo [3]. Let Q be a compact saturated subset such that $V \subseteq Q \subseteq U$. Then $g = -\chi_Q$ satisfies $g \geq -\chi_U$, so $(Bary(\_^\perp[\mathcal{P}^\perp]))^\perp(\chi_U) \geq \inf_{\substack{\varphi\in\langle\mathbf{P}^{\triangle}_{1\ wk}(X)\to\mathbb{R}\rangle \\ \varphi\geq\lambda G\in\mathbf{P}^{\triangle}_{1\ wk}(X)^{\mathsf{d}}\cdot -G^\perp(a-\chi_Q)}} \oint_{G\in\mathbf{P}^{\triangle}_{1\ wk}(X)} \varphi(G)d\mathcal{P} + a$.
Let us estimate $-G^\perp(a - \chi_Q) = \inf_{\substack{f\in\langle X\to\mathbb{R}^+\rangle \\ f\geq -(a-\chi_Q)}} G(f)$ (using Lemma 1). Whenever $f \geq -(a - \chi_Q)$, we have $f \geq \chi_Q - a \geq \chi_V - a$, so $-G^\perp(a - \chi_Q) \geq \inf_{\substack{f\in\langle X\to\mathbb{R}^+\rangle \\ f\geq\chi_V - a}} G(f) \geq$

$G(\chi_V - a) = G(\chi_V) - a$, since G is normalized. Using (1), $(Bary(\_^\perp[\mathcal{P}^\perp]))^\perp(\chi_U) \geq \inf_{\substack{\varphi \in \langle \mathbf{P}^\triangle_{1\,wk}(X) \to \mathbb{R}\rangle \\ \varphi \geq \lambda G \in \mathbf{P}^\triangle_{1\,wk}(X)^\mathsf{d} \cdot G(\chi_V) - a}} \oint_{G \in \mathbf{P}^\triangle_{1\,wk}(X)} \varphi(G) d\mathcal{P} + a = \oint_{G \in \mathbf{P}^\triangle_{1\,wk}(X)} G(\chi_V) d\mathcal{P} - a + a = Bary(\mathcal{P})(\chi_V)$.

Let $p = \gamma_\mathcal{C}(Bary(\mathcal{P}))$, $p' = \gamma_\mathcal{C}((Bary(\_^\perp[\mathcal{P}^\perp]))^\perp)$. We have shown that $p'(U) \leq p(U)$ for every open U of X, and that $p'(U) \geq p(V)$ whenever $V \Subset U$. Since $\mathcal{O}(X)$ is a continuous cpo, and p is continuous, $p(U) = \sup_{V \Subset U} p(V) \leq p'(U)$. So $p = p'$. Since $\alpha_\mathcal{C}, \gamma_\mathcal{C}$ form an isomorphism, $Bary(\mathcal{P}) = (Bary(\_^\perp[\mathcal{P}^\perp]))^\perp$. □

Let $Conv^*(\mathcal{F})$, the *co-strong convex closure* of $\mathcal{F}$, be $\{Bary(\mathcal{P}) | \mathcal{P}$ co-supported on $\mathcal{F}\}$, and $\downarrow$ be the downward-closure operator. Similarly to Section 3, we show that $CPeau_1(\bigsqcup \mathcal{F}) = \downarrow Conv^*(\mathcal{F})$ for every $\mathcal{F} \in \mathcal{H}(\mathbf{P}^\triangle_{1\,wk}(X))$. It is easy to see that (on stably compact X), for any normalized continuous upper prevision F on X, $CPeau_1(F)^\perp = CCoeur_1(F^\perp)$; this is because $\_^\perp$ is antitone (Theorem 4 (6)). For every non-empty closed subset $\mathcal{F}$ of $\mathbf{P}^\triangle_{1\,wk}(X)$, $CPeau_1(\bigsqcup \mathcal{F})^\perp = CCoeur_1((\bigsqcup \mathcal{F})^\perp) = CCoeur_1 (\bigsqcap \mathcal{F}^\perp)$ (because $\_^\perp$ is antitone) $= \uparrow Conv(\mathcal{F}^\perp)$ (see Section 3). Apply $\_^\perp$, using Theorem 4 (5) to get $CPeau_1(\bigsqcup \mathcal{F}) = \uparrow Conv(\mathcal{F}^\perp)^\perp = \downarrow Conv(\mathcal{F}^\perp)^\perp = \downarrow Conv^*(\mathcal{F})$. The latter equality is a straightforward exercise, using Proposition 7, and the easily proved facts that, for any continuous function $f : Y \to Z$, for any continuous probability valuation p on Y: (a) if p is supported on some subset A of X, then $f[p]$ is supported on the direct image $f(A)$; (b) if $f[p]$ is co-supported on some closed subset F of Y, then p is co-supported on $f^{-1}(F)$. (See Appendix E.)

Theorem 5 (Isomorphism). *Let X be stably compact. Then $CPeau_1$ and $\bigsqcup$ define an isomorphism between $\triangle \mathbf{P}_1(X)$ and the space $\mathcal{H}^{Cvx^*}(\mathbf{P}^\triangle_{1\,wk}(X))$ of co-strongly convex non-empty closed subsets of $\mathbf{P}^\triangle_{1\,wk}(X)$, ordered by $\subseteq$.*

The case where X is a cpo is much simpler than for the demonic case (Section 4).

Lemma 2. *Let X be a continuous pointed cpo. Every convex closed subset of $\mathbf{P}^\triangle_1(X)$ is co-strongly convex.*

Proof. Let $Z = \mathbf{P}^\triangle_1(X)$, $\mathcal{F}$ a convex closed subset of Z, and $\mathcal{P}$ a continuous probability valuation on Z, co-supported on $\mathcal{F}$: $\mathcal{P}(Z \setminus \mathcal{F}) = 0$. Since $Z \cong \mathbf{V}_1(X)$ is a continuous pointed cpo, $\mathbf{V}_1(Z)$ is one, too, with a basis of simple probability valuations [2]. So write $\mathcal{P}$ as the sup of a directed family $(\mathcal{P}_i)_{i \in I}$, with $\mathcal{P}_i \leq \mathcal{P}$. In particular, $\mathcal{P}_i(Z \setminus \mathcal{F}) = 0$, so $\mathcal{P}_i$ is co-supported on $\mathcal{F}$. Write $\mathcal{P}_i$ as $\sum_{j=1}^n a_j \delta_{G_j}$, where each G_j is in $\mathcal{F}$, and $a_1 + \ldots + a_n = 1$. $\mathcal{F}$ is convex so $Bary(\mathcal{P}_i) = \sum_{j=1}^n a_j G_j$ is in $\mathcal{F}$. Now $Bary$ is continuous, so $Bary(\mathcal{P}) = \sup_{i \in I} Bary(\mathcal{P}_i)$. As $\mathcal{F}$ is Scott-closed, $Bary(\mathcal{P}) \in \mathcal{F}$. □

Writing $\mathcal{H}^{cvx}(Y)$ the subset of $\mathcal{H}(Y)$ consisting of convex subsets, it follows:

Corollary 2 (Isomorphism). *Let X be a continuous, coherent pointed cpo. $CPeau_1$ and $\bigsqcup$ define an isomorphism between $\triangle \mathbf{P}_1(X)$ and $\mathcal{H}^{cvx}(\mathbf{P}^\triangle_{1\,wk}(X)) = \mathcal{H}^{cvx}(\mathbf{P}^\triangle_1(X)) \cong \mathcal{H}^{cvx}(\mathbf{V}_1(X))$.*

One may also show the following. The omitted proof can be found in Appendix F. This depends crucially on the fact that $CPeau_1(\bigsqcup \mathcal{F}) = \downarrow Conv^*(\mathcal{F})$.

Proposition 8. *Let X be a continuous, coherent, pointed cpo. Then $CPeau_1$ is a continuous map from $\bigtriangleup \mathbf{P}_1(X)$ to $\mathcal{H}(\mathbf{P}^{\triangle}_{1\,wk}(X))$.*

Recall that, in this case, $Y = \mathbf{P}^{\triangle}_{1\,wk}(X) = \mathbf{P}^{\triangle}_1(X)$ has a basis of simple normalized linear previsions. For any continuous cpo Y, $\mathcal{H}(Y)$ is continuous cpo too, with basis given by the *finitary closed subsets* $\downarrow \mathcal{E}$, $\mathcal{E}$ a finite subset of Y [1]. As for Theorem 3, we can therefore conclude:

Theorem 6. *For any continuous, coherent pointed cpo X, $\bigtriangleup \mathbf{P}_1(X)$ is a continuous, coherent pointed cpo. A basis is given by previsions of the form $\lambda h \in \langle X \to \mathbb{R}^+\rangle \cdot \max_{i=1}^m \sum_{j=1}^n a_{ij} h(x_j)$, where $a_{ij} \in \mathbb{R}^+$ and $\sum_{j=1}^n a_{ij} = 1$ for each i.*

So we can also approximate from below any continuous normalized upper prevision by one that is computable, using only finitely $\max$, $+$, and $\cdot$ operations.

6 Chaotic Non-Determinism + Probabilistic Choice

In chaotic non-determinism we replace $\mathcal{Q}(Y)$ or $\mathcal{H}(Y)$ by the *Plotkin powerdomain* $\mathcal{P}\ell(Y)$. This is the set of all *lenses* L, which are non-empty intersections of a compact saturated subset Q of Y and a closed subset F of Y. A canonical way of writing L as $Q \cap F$ is then to take $Q = \uparrow L$, $F = cl(L)$. We order $\mathcal{P}\ell(Y)$ by the topological Egli-Milner ordering $\sqsubseteq_{\mathrm{EM}}$ defined by $L \sqsubseteq_{\mathrm{EM}} L'$ iff $\uparrow L \supseteq \uparrow L'$ and $cl(L) \subseteq cl(L')$. If Y is a continuous, coherent pointed cpo, then $\mathcal{P}\ell(Y)$ is one, too. (See [1, Section 6.2.3].) Among all lenses, call *strong* those that obey the stronger property $F = \downarrow L$.

Let $\mathbf{F}_1(X)$ be the space of all normalized forks on X. Then [5, Proposition 6] there is a map $CCorps_1 : \mathbf{F}_1(X) \to \mathcal{P}\ell(\mathbf{P}^{\triangle}_{1\,wk}(X))$, defined by $CCorps_1(F^-, F^+) = CCoeur_1(F^-) \cap CPeau_1(F^+)$. Moreover, $CCoeur_1(F^-) = \uparrow CCorps_1(F)$ and $CPeau_1(F^+) = \downarrow CCorps_1(F)$. (So $CCorps_1(F^-, F^+)$ is a strong lens.) It is clear from our results on $CCoeur_1$ and $CPeau_1$ that $\bigsqcap\bigsqcup \circ CCorps_1 = \mathrm{id}$; the map $\bigsqcap\bigsqcup$: $\mathcal{P}\ell(\mathbf{P}^{\triangle}_{1\,wk}(X)) \to \mathbf{F}_1(X)$ is defined by $\bigsqcap\bigsqcup \mathcal{L} = (\bigsqcap \mathcal{Q}, \bigsqcup \mathcal{C})$, where $\mathcal{Q} = \uparrow \mathcal{L}$ and $\mathcal{C} = cl(\mathcal{L})$. The following is an immediate consequence of the results of previous sections:

Proposition 9. *A subset $\mathcal{A}$ of $\mathbf{P}^{\triangle}_{1\,wk}(X)$ is* bi-strongly convex *iff for every continuous probability valuation $\mathcal{P}$ supported on $\mathcal{A}$, $Bary(\mathcal{P})$ is in $\uparrow \mathcal{A}$, and for every continuous probability valuation $\mathcal{P}$ co-supported on $\mathcal{A}$, $Bary(\mathcal{P})$ is in $\downarrow \mathcal{A}$.*

Let X be stably compact. For every lens $\mathcal{L}$ on $\mathbf{P}^{\triangle}_{1\,wk}(X)$, $CCorps_1(\bigsqcap\bigsqcup\mathcal{L}) = \uparrow Conv(\mathcal{L}) \cap \downarrow Conv^*(\mathcal{L})$. $CCorps_1$ and $\bigsqcap\bigsqcup$ define an isomorphism between $\mathbf{F}_1(X)$ and the space $\mathcal{P}L^{biCvx}(\mathbf{P}^{\triangle}_{1\,wk}(X))$ of all strong bi-strongly convex lenses on $\mathbf{P}^{\triangle}_{1\,wk}(X)$.

Let now X be a continuous, coherent pointed cpo. It is an easy consequence of Theorem 2 and Lemma 2 that any strong lens $\mathcal{L}$ on $\mathbf{P}^{\triangle}_{1\,wk}(X)$ is bi-strongly convex. Also, every lens is in fact strong [1, Lemma 6.2.20]. Write $\mathcal{P}\ell^{cvx}(Y)$ the subspace of convex lenses in $\mathcal{P}\ell(X)$:

Theorem 7. *Let X be a continuous, coherent pointed cpo. $CCorps_1$ and $\bigsqcap\bigsqcup$ define an isomorphism between $\mathbf{F}_1(X)$ and $\mathcal{P}\ell^{cvx}(\mathbf{P}^{\triangle}_{1\,wk}(X)) = \mathcal{P}\ell^{cvx}(\mathbf{P}^{\triangle}_1(X)) \cong \mathcal{P}\ell^{cvx}(\mathbf{V}_1(X))$.*

7 Conclusion

We have solved the problem of relating domains of continuous previsions and forks à la [5], and convex powercones à la [17]: in standard cases, they are isomorphic. This question was raised under this form at the end of [5], while Keimel and Plotkin [10] show similar results for (unbounded) valuations instead of normalized valuations. They justify working on unbounded valuations because "the mathematics seems to be more natural if we take all the valuations, since one can then work with notions of linearity rather than convexity". I hope to have convinced the reader that the mathematics of the normalized case, once generalized to the topological case, is both beautiful and deep. Note in particular that the notion of (generalized) barycenters $Bary(\mathcal{P})$, and above all the convex-concave duality work naturally at the topological level, not directly on cpos.

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A Proof of Proposition 3

We establish a few auxiliary claims, similar to those found in [17].

Claim A. *Let C be an ordered cone, Z a subset of C, U a convex upward-closed subset of Z, F a convex subset of Z, and assume that U and F are non-empty and disjoint. Then there is a continuous linear function $f : C \to \overline{\mathbb{R}}^+$ such that $f(x) \leq 1$ for every $x \in F$, and $f(y) \geq 1$ for every $y \in U$.*

This is analogous to the well-known Separation Theorem in normed vector spaces. This was proved in the special case $Z = C$ in [17].

Proof. Let $\downarrow F$ be the downward-closure of F in C, and $\uparrow U$ the upward-closure of U in C. Then $\uparrow U$ and $\downarrow F$ are disjoint, otherwise for some $z \in C$, $x \in F$, $y \in U$ we would get $y \leq z \leq x$, hence $x \in U$.

Let $q(x) = \sup_{\lambda>0/\lambda\cdot x\in\uparrow U} 1/\lambda$, where the sup is taken to be 0 if no such λ exists. First, $q(0) = 0$. Indeed, there is no $\lambda > 0$ such that $\lambda \cdot 0 \in \uparrow U$, else 0 would belong to $\uparrow U$, hence to U. Since every element of Z (and indeed, of any cone) is greater than or equal to 0, we would have $U = Z$, contradicting the fact that F is non-empty and disjoint from U. For any $a > 0$, $q(a \cdot x) = \sup_{\lambda>0/\lambda\cdot(a\cdot x)\in\uparrow U} 1/\lambda = \sup_{\lambda'>0/\lambda'\cdot x\in\uparrow U} 1/(\lambda'/a)$ (take $\lambda' = \lambda a$) $= aq(x)$. So q is positively homogeneous.

Let us show that q is super-additive. First, observe that $\uparrow U$ is convex: if $a, b \in \uparrow U$, there are $a', b' \in U$ with $a' \leq a$, $b' \leq b$; for every $r \in [0, 1]$, $r \cdot a' + (1-r) \cdot b' \in U$ since U is convex, and $r \cdot a' + (1-r) \cdot b' \leq r \cdot a + (1-r) \cdot b$ since addition and scalar product are monotonic. Let $x, x' \in C$. Then:

$$q(x) + q(x') = \sup_{\substack{\lambda,\lambda'>0\\ \lambda\cdot x\in\uparrow U\\ \lambda'\cdot x'\in\uparrow U}} \left(\frac{1}{\lambda} + \frac{1}{\lambda'}\right) \leq \sup_{\substack{\lambda,\lambda'>0\\ \lambda'/(\lambda+\lambda')\cdot(\lambda\cdot x)+\lambda/(\lambda+\lambda')\cdot(\lambda'\cdot x')\in\uparrow U}} \frac{\lambda+\lambda'}{\lambda\lambda'}$$

since $\uparrow U$ is convex. So $q(x) + q(x') \leq \sup_{\substack{\lambda,\lambda'>0\\ \lambda\lambda'/(\lambda+\lambda')\cdot(x+x')\in\uparrow U}} (\lambda+\lambda')/(\lambda\lambda') \leq q(x + x')$. Therefore q is super-additive.

Let then $p(x) = \inf_{\lambda>0/\lambda\cdot x\in\downarrow F} 1/\lambda$. In case no such λ exists, we take this to denote $+\infty$. For every $\lambda > 0$, $\lambda \cdot 0 = 0$ is in $\downarrow F$, since 0 is the least element of C and F is non-empty. So $p(0) = 0$. By a similar argument as above, p is sub-additive.

Furthermore, for all $a, b \in C$ with $a \leq b$, for every $\lambda > 0$ such that $\lambda \cdot a \in \uparrow U$ and $\lambda' > 0$ such that $\lambda' \cdot b \in \downarrow F$, necessarily $\lambda > \lambda'$. Otherwise, $\lambda \leq \lambda'$, so $\lambda \cdot a \leq \lambda' \cdot a \leq \lambda' \cdot b$. Since $\lambda \cdot a \in \uparrow U$, $\lambda' \cdot b$ would also be in $\uparrow U$. Since it is also in $\downarrow F$, this would contradict the fact that $\uparrow U$ and $\downarrow F$ are disjoint. So, indeed $\lambda > \lambda'$. In particular, $1/\lambda < 1/\lambda'$. Taking sups when λ varies, and infs when λ' varies, we obtain $q(a) \leq p(b)$, for any $a, b \in C$.

We can now apply Roth's Sandwich Theorem: there is a monotonic linear function $f : C \to \overline{\mathbb{R}}^+$ with $q \leq f \leq p$. For every $x \in F$, $p(x) \leq 1$ since $\lambda = 1$ satisfies $\lambda \cdot x \in \downarrow F$, so $f(x) \leq 1$. For every $x \in U$, similarly, $q(x) \geq 1$, so $f(x) \geq 1$. □

We adapt Claim A to the case of additive continuous cones, showing that we can now take f to be continuous (Claim E below). This is a variant of [17, Theorem 3.4].

As above, the only change we make is to consider a subspace Z of a cone C, instead of forcing Z to be the whole cone. We shall use this typically when $C = \mathbf{V}(X)$ (the space of bounded continuous valuations on X) and $Z = \mathbf{V}_{\leq 1}(X)$, or the isomorphic situation $C = \mathbf{P}^{\triangle}(X)$ and $Z = \mathbf{P}^{\triangle}_{\leq 1}(X)$. First, on any continuous cone C (which is a continuous poset), we may form the largest continuous function $\mathfrak{r}(h)$ below h from any monotonic $h : C \to \overline{\mathbb{R}}^+$ using Scott's extension formula $\mathfrak{r}(h)(y) = \sup_{x/x \ll y} h(x)$. This is well-known for continuous cpos; the proof is the same on continuous posets, using e.g. results one can find in Mislove [12, Section 4.2].

Claim B. *In every continuous cone C, if $x \ll y_1 + y_2$, then there are $x_1, x_2 \in C$ with $x \leq x_1 + x_2$, $x_1 \ll y_1$ and $x_2 \ll y_2$.*
Proof. Write y_1 as the sup of a directed family $(y_{1i})_{i \in I}$, $y_{1i} \ll y_1$, and y_2 as the sup of a directed family $(y_{2i})_{i \in I}$, $y_{2i} \ll y_2$. Then (y_1, y_2) is the sup of the directed family $(y_{1i}, y_{2j})_{i,j \in I}$. Since $+$ is continuous, $y_1 + y_2$ is the sup of the directed family $(y_{1i} + y_{2j})_{i,j \in I}$. Since $x \ll y_1 + y_2$, there are $i, j \in I$ such that $x \leq y_{1i} + y_{2j}$. Take $x_1 = y_{1i}$, $x_2 = y_{2j}$. ☐

Claim C. *Let C be an additive continuous cone. Pour every monotonic linear function $h : C \to \overline{\mathbb{R}}^+$, $\mathfrak{r}(h)$ is a continuous linear function from C to $\overline{\mathbb{R}}^+$.*
Proof. Let $f = \mathfrak{r}(h)$. First show that $f(r \cdot x) = rf(x)$. When $r = 0$, $f(r \cdot x) = \sup_{y \ll r \cdot x} h(y) = h(0) = 0$, since $0 \cdot x = 0$ (in any cone), and $y \ll 0$ iff $y = 0$ (since 0 is the least element in the cone C). When $r > 0$, $f(r \cdot x) = \sup_{y \ll r \cdot x} h(y) \geq \sup_{y' \ll x} h(r \cdot y')$. We use the fact that $y' \ll x$ implies $r \cdot y' \ll r \cdot x$, a consequence of the fact that the map $a \mapsto r \cdot a$ is an order isomorphism. Since $h(r \cdot y') = rh(y')$, $f(r \cdot x) \geq rf(x)$. As this holds for all $r > 0$ and all $x \in C$, it follows $f(1/r \cdot (r \cdot x)) \geq 1/rf(r \cdot x)$, i.e., $f(r \cdot x) \leq rf(x)$. So $f(r \cdot x) = rf(x)$, and f is positively homogeneous. Let us show that f is additive:

$$\begin{aligned} f(y_1) + f(y_2) &= \sup_{x_1 \ll y_1} h(x_1) + \sup_{x_2 \ll y_2} h(x_2) \\ &= \sup_{\substack{x_1 \ll y_1 \\ x_2 \ll y_2}} h(x_1 + x_2) \quad \text{since } h \text{ is linear} \\ &\leq \sup_{x \ll y_1 + y_2} h(x) = f(y_1 + y_2) \end{aligned}$$

since whenever $x_1 \ll y_1$, $x_2 \ll y_2$, we obtain $x \ll y_1 + y_2$ by choosing $x = x_1 + x_2$. (This is where we use that C is additive.) Conversely,

$$f(y_1 + y_2) = \sup_{x \ll y_1 + y_2} h(x) \leq \sup_{x_1, x_2 \in C / x_1 \ll y_1, x_2 \ll y_2} h(x_1 + x_2)$$

Indeed, by Claim B, for every $x \ll y_1 + y_2$, there are $x_1, x_2 \in C$ with $x \leq x_1 + x_2$, $x_1 \ll y_1$, and $x_2 \ll y_2$; then $h(x) \leq h(x_1 + x_2)$, since h is monotonic. But the quantity above is just $f(y_1) + f(y_2)$. ☐

Claim D. *Let C be an additive continuous cone, Z a subset of C, and assume that Z is continuous as a sub-partial order of C. Let $\ll_C$ the way-below relation on C,*

$\ll_Z$ *that on* Z*, and assume that whenever* $x, y \in Z$ *are such that* $x \ll_Z y$*, then* $x \ll_C y$*. In this case, we say that* Z *is a* sane *subspace of* C*.*
Then the topology of Z*, induced by that of* C*, is its Scott topology. Any open* U *of* Z *can be written* $V \cap Z$*, where* V *is the open subset of* C *defined as* $\bigcup_{x\in U} \Uparrow_C x$*, where* $\Uparrow_C x = \{y \in C | x \ll_C y\}$*.*
Proof. Let U be any open in Z. Let V be $\bigcup_{x\in U} \Uparrow_C x$. V is Scott-open in C, since C is a continuous poset. We claim that $U = V \cap Z$. For every $y \in V \cap Z$, there is an $x \in U$ with $x \ll_C y$, in particular $x \leq y$. Since U is upward-closed in Z, $y \in U$. Conversely, for every $y \in U$, since Z is continuous, y is the sup of all $x \in Z$ such that $x \ll_Z y$. Note that $U = V' \cap Z$ for some open subset V' of C, by definition. So $x \in V'$ for some $x \in Z$ with $x \ll_Z y$. In particular, $x \in V' \cap Z = U$. Since both x and y are in Z, and $x \ll_Z y$, it follows $x \ll_C y$.
It follows that every open of Z is Scott-open: for any directed family $(z_i)_{i\in I}$ in Z whose sup z is in U, z is in V, so some z_i is in V, hence in U. Conversely, since Z is continuous, for every Scott-open subset U of Z, for every $z \in U$, there is an $x \ll_Z z$ with $x \in U$. In particular, $x \ll_C z$, so $z \in V = \bigcup_{x\in U} \Uparrow_C x$, so $z \in V \cap Z$. But for any $z \in V \cap Z$, by definition there is an $x \in U$ with $x \ll_C z$, hence $x \leq z$, whence $z \in U$. So $U = V \cap Z$ is open in Z. So every Scott-open is open, too. ☐

<u>Claim E.</u> *Let* C *be an additive continuous cone, and* Z *a sane subspace of* C*. Let* U *be a convex open subset of* Z*,* F *a convex subset of* Z*, and assume* U *and* F *non-empty and disjoint. Then there is a continuous linear function* $f : C \to \overline{\mathbb{R}}^+$ *such that* $f(x) \leq 1$ *for every* $x \in F$*, and* $f(y) > 1$ *for every* $y \in U$*.*
Proof. Let V be as in Claim D. Note that V is disjoint from F: otherwise there would be an element $y \in F$ and $x \in U$ with $x \ll_C y$, hence $x \leq y$, which would imply $y \in U$. Note also that V is convex: for any $r \in [0, 1]$, $y, y' \in V$, there are $x, x' \in U$ such that $x \ll_C y$, $x' \ll_C y'$. We have already seen that, then, $r \cdot x \ll_C r \cdot y$ and $(1 - r) \cdot x' \ll_C (1 - r) \cdot y'$. Since C is additive, $r \cdot x + (1-r) \cdot x' \ll_C r \cdot y + (1-r) \cdot y'$. Since U is convex, $r \cdot x + (1-r) \cdot x' \in U$, so $r \cdot y + (1 - r) \cdot y' \in V$.
By Claim A, there is a monotonic linear function $h : C \to \overline{\mathbb{R}}^+$ such that $h(x) \leq 1$ for every $x \in F$ and $h(x) \geq 1$ for every $x \in V$. Let $f = \mathfrak{r}(h)$. By Claim C, f is continuous and linear. Since $f \leq h$, in particular $f(x) \leq 1$ for every $x \in F$. It remains to show that $f(y) > 1$ for every $y \in U$. We show that $f(y) > 1$ for every $y \in V$. Since C is continuous, y is the sup of the directed family of all $x \ll y$. Since V is Scott-open, one of these x is in V. This x is also the sup of all $r \cdot x$, $r < 1$, since scalar product is continuous. So $r \cdot x \in V$ for some $r < 1$. Then $h(r \cdot x) \geq 1$, so $h(x) \geq 1/r$, and $f(y) \geq h(x) \geq 1/r > 1$. ☐
<u>*Claim F.*</u> *Let* X *be a continuous cpo. Then* $\mathbf{V}(X)$ *is an additive continuous cone, and* $\mathbf{V}_{\leq 1}(X)$ *is a sane subspace of* $\mathbf{V}(X)$*.*
Proof. $C = \mathbf{V}(X)$ is clearly an ordered cone, and $+$ and $\cdot$ are Scott-continuous. We also know from Jones [7] that $\mathbf{V}(X)$ is a continuous poset, with a basis composed of *simple valuations* $\sum_{i=1}^m a_i \delta_{x_i}$, with $x_1, \ldots, x_m \in X$ and $a_1, \ldots, a_m \in \mathbb{R}^+$. Moreover, the Splitting Lemma for $\ll_C$ (Lemma 4.13 of [7]) states that $\sum_{i=1}^m a_i \delta_{x_i} \ll_C \sum_{j=1}^n b_j \delta_{y_j}$ (where the b_j's are non-zero) iff there is a matrix $(t_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$ of

coefficients in $\mathbb{R}^+$ such that $\sum_{j=1}^n t_{ij} = a_i$ for each i, $\sum_{i=1}^m t_{ij} < b_j$ for each j, and the only entries t_{ij} that are non-zero are such that $x_i \ll y_j$.

It follows that C is additive. Indeed, assume $\nu_1 \ll_C \nu'_1$ and $\nu_2 \ll_C \nu'_2$. By the interpolation property, valid on every continuous poset (if $x \ll y$ then $x \ll z \ll y$ for some z in the basis, see [12, Lemma 4.16]; repeating this, there are two element z, z' in the basis with $x \ll z \ll z' \ll y$), there are simple valuations $\nu''_1, \nu'''_1, \nu''_2, \nu'''_2$ with $\nu_1 \ll_C \nu''_1 \ll_C \nu'''_1 \ll_C \nu'_1$ and $\nu_2 \ll_C \nu''_2 \ll_C \nu'''_2 \ll_C \nu'_2$. Summing the respect matrices of coefficients obtained from the relations $\nu''_1 \ll_C \nu'''_1$ and $\nu''_2 \ll_C \nu'''_1$ by the Splitting Lemma for $\ll_C$, we obtain $\nu''_1 + \nu''_2 \ll_C \nu'''_1 + \nu'''_2$. Now $\nu_1 + \nu_2 \leq \nu''_1 + \nu''_2 \ll_C \nu'''_1 + \nu'''_2 \leq \nu'_1 + \nu'_2$, from which $\nu_1 + \nu_2 \ll_C \nu'_1 + \nu'_2$ obtains.

$Z = \mathbf{V}_{\leq 1}(X)$ is a continuous cpo by Jones' results, and the Splitting Lemma for $\ll_Z$ is characterized exactly as $\ll_C$, so $\ll_Z = \ll_C$: Z is a sane subspace of C. ☐

The case $Z = \mathbf{V}_1(X)$, $C = \mathbf{V}(X)$ is slightly different. While $\mathbf{V}_1(X)$ is a continuous cpo with a basis of simple probability valuations by [2, Section 3] whenever X is a continuous pointed cpo, it turns out that $\delta_\perp \ll_Z \delta_\perp$ (since $\delta_\perp$ is the least element of $\mathbf{V}_1(X)$), but $\delta_\perp \not\ll_C \delta_\perp$ (since $(r\delta_\perp)_{r\in[0,1)}$ has $\delta_\perp$ as sup, but no element of the family is above $\delta_\perp$). However, $X \setminus \{\perp\}$ is again a continuous cpo, and $\mathbf{V}_1(X)$ is isomorphic to $\mathbf{V}_{\leq 1}(X \setminus \{\perp\})$, through the map sending $\nu \in \mathbf{V}_1(X)$ to $\lambda U \in \mathcal{O}(X \setminus \{\perp\}) \cdot \nu(U)$, and conversely, the map sending $\nu \in \mathbf{V}_{\leq 1}(X \setminus \{\perp\})$ to the valuation sending each open U of X to $\nu(U)$ if $\perp \notin U$, and to 1 otherwise (i.e., when $U = X$). It follows:

Claim G. *Let X be a continuous cpo with a least element $\perp$. Then $\mathbf{V}(X \setminus \{\perp\})$ is an additive continuous cone, and $\mathbf{V}_1(X)$ is a sane subspace of $\mathbf{V}(X \setminus \{\perp\})$.*

A topological space Z is *locally convex* iff every point has a neighborhood basis of convex open subsets, i.e., for every $z \in Z$ and every open U containing z, there is a *convex* open V such that $x \in V \subseteq U$. The following is a slight variant of Proposition 2.5 of [17], due to Jimmie Lawson.

Claim H. *Let C be a continuous cone. Every convex topological subspace Z of C is locally convex.*

Proof. Fix $x \in Z$, and an open U of Z containing x. We may write U as $U' \cap Z$, where U' is open in C. For every $z \in U'$, since C is continuous, there is an element of U' way-below z, call it $f(z)$. Let $V' = \bigcup_{n\in\mathbb{N}} \Uparrow_C f^n(x)$. V' is Scott-open, and contains x. Using an auxiliary induction on n, it is easy to show that $V' \subseteq U'$. So $V = V' \cap Z$ is an open of Z that contains x and contained in U. Now note that $V' = \bigcup_{n\in\mathbb{N}} \uparrow f^n(x)$: the $\subseteq$ direction comes from $\Uparrow f^n(x) \subseteq \uparrow f^n(x)$, the $\supseteq$ direction from $\uparrow f^n(x) \subseteq \Uparrow f^{n+1}(x)$. It follows that V is convex: for every $y, z \in V$ and $r \in [0,1]$, there is $n \in \mathbb{N}$ such that $y, z \in \uparrow f^n(x)$, i.e., $f^n(x) \leq y, z$, so $f^n(z) \leq r \cdot y + (1-r) \cdot z$; i.e., $r \cdot y + (1-r) \cdot z$ is in V'; it is also in Z since Z is convex, so it is in V. ☐

We now recall Lemma 3.7 of [17]. Let $\mathbf{1}$ be the element $(1, \ldots, 1)$ of $\overline{\mathbb{R}}^{+n}$ ($n \in \mathbb{N}$). Let K be a compact convex subset of $\overline{\mathbb{R}}^{+n}$, disjoint from $\downarrow \mathbf{1}$. Then there is a continuous linear function $h : \overline{\mathbb{R}}^{+n} \to \overline{\mathbb{R}}^{+}$, and $a \in \mathbb{R}^+$ with $a > 1$, such that $h(\mathbf{1}) \leq 1$ and $h(\boldsymbol{x}) > a$ for every $\boldsymbol{x} \in K$. We use this to show:

Proposition 3. *Let C be an additive continuous cone, and Z a sane subspace of C. For every convex compact subset K of Z, for every non-empty convex closed subset F of Z disjoint from K, there is $a \in \overline{\mathbb{R}}^{+}$, $a > 1$, and a continuous linear function $f : C \to \overline{\mathbb{R}}^{+}$ such that $f(z) > a$ for all $z \in K$ and $f(y) \leq 1$ for every $y \in F$.*

Proof. Let x be some fixed element of K. By Claim H, Z is locally convex. Since K is disjoint from F, x is in the open $Z \setminus F$, so there is a convex open U such that $x \in U \subseteq Z \setminus F$. In particular, U is a non-empty convex open subset of Z, disjoint from F. Using Claim E, there is a continuous linear function $g_x : C \to \overline{\mathbb{R}}^{+}$ such that $g_x(z) \leq 1$ for every $z \in F$, and $g_x(y) > 1$ for every $y \in U$. Let then $U_x = g_x^{-1}(1, +\infty)$, and $V_x = U_x \cap Z$. Since g_x is continuous, U_x is open in C, so V_x is open in Z. Moreover, U_x is convex, because g_x is linear. Since Z is convex, V_x is, too. Finally, since $x \in U$, $g_x(x) > 1$, i.e., $x \in U_x$. In particular, since $x \in K \subseteq Z$, x belongs to V_x.

For every $x \in K$, $x \in V_x$, and $(V_x)_{x \in K}$ is an open cover of K. Since K is compact, we can extract a finite subcover, say $V_{x_1}, \ldots, V_{x_n}$. Let $g : C \to \overline{\mathbb{R}}^{+n}$ that sends z to the tuple $(g_{x_1}(z), \ldots, g_{x_n}(z))$, a continuous linear function. Also, $g_x(z) \leq 1$ for every $z \in F$, so the image $g(F)$ of F by g is contained in $\downarrow \mathbf{1}$.

Let K' be the image of K by g. Note that K is compact in Z, hence also in C: for every open cover $(U_i)_{i \in I}$ of K in C, $(U_i \cap Z)_{i \in I}$ is an open cover of K in Z; extract a finite subcover $(U_i \cap Z)_{i \in J}$ (J finite contained in I) in Z; then $(U_i)_{i \in J}$ is a finite subcover of K in C. Since g is continuous, K' est compact. Since g is linear and K is convex, K' is also convex. Finally, K' is disjoint from $\downarrow \mathbf{1}$, otherwise there would be a $z \in K$ with $g(z) \leq \mathbf{1}$. But $V_{x_1}, \ldots, V_{x_n}$ is a cover of K, so there would be an i, $1 \leq i \leq n$, with $z \in V_{x_i}$, i.e., $g_{x_i}(z) > 1$, contradiction.

By [17, Lemma 3.7], there is a continuous linear function $h : \overline{\mathbb{R}}^{+n} \to \overline{\mathbb{R}}^{+}$, and $a \in \mathbb{R}^+$ with $a > 1$, such that $h(\mathbf{1}) \leq 1$ and $h(\boldsymbol{x}) > a$ for every $\boldsymbol{x} \in K'$. Let finally $f = h \circ g$. ☐

B Proof of Theorem 4

Claim I. *Let X be stably compact. For every normalized prevision F on X, $F^\perp$ is a normalized prevision on X^{d}.*

Proof. First $F^\perp(0) = -\inf_{\substack{f \text{ step function} \\ f \geq 0}} \widehat{F}(f) = 0$ since $\widehat{F}$ is monotonic, the inf is attained at $f = 0$, and $\widehat{F}(0) = 0$. For every $\alpha > 0$, $F^\perp(\alpha g) = -\inf_{\substack{f \text{ step} \\ f + \alpha g \geq 0}} \widehat{F}(f)$ $= -\inf_{\substack{f' \text{ step} \\ f' + g \geq 0}} \widehat{F}(\alpha f')$, letting $f' = 1/\alpha f$. Since $\widehat{F}(\alpha f') = \alpha \widehat{F}(f')$, it follows $F^\perp(\alpha g) = \alpha F^\perp(g)$. So $F^\perp$ is positively homogeneous.

It is clear that $F^\perp$ is monotonic. Finally, for every $a \in \mathbb{R}^+$, $F^\perp(g) = F^\perp(a + g - a) = -\inf_{\substack{f \text{ step} \\ f + a + g - a \geq 0}} \widehat{F}(f) = -\inf_{\substack{f' \text{ step} \\ f' + g + a \geq 0}} \widehat{F}(f' + a)$ (taking $f' = f - a$)

$$= -\inf_{\substack{f' \text{ step} \\ f'+g+a\geq 0}} [a + \widehat{F}(f')] = -a + F^\perp(g+a), \text{ so } F^\perp(g+a) = a + F^\perp(g),$$

showing that $F^\perp$ is normalized. □

To establish the other properties of $F^\perp$, we need to replace the quantification over all $f \geq -g$ (equivalently, $g \geq -f$) in the definition of $F^\perp$ by a stronger condition $g \sqsupseteq -f$.

For any $f \in \langle X \to \mathbb{R}\rangle$, $g \in \langle X^{\mathsf{d}} \to \mathbb{R}\rangle$, write $g \sqsupseteq -f$ iff one can write f and g as step functions:

$$f = \sum_{i=0}^{n} a_i \chi_{U_i} \quad g = -\sum_{i=0}^{n} a_i \chi_{Q_i}$$

with the same coefficients $a_0 \in \mathbb{R}$, $a_1, \ldots, a_n \in \mathbb{R}^+$, $X = U_0 \supseteq U_1 \supseteq \ldots \supseteq U_n$ is a non-increasing sequence of opens, $X = Q_0 \supseteq Q_1 \supseteq \ldots \supseteq Q_n$ is a non-increasing sequence of compact saturated subsets, and $Q_i \subseteq U_i$ for every i, $0 \leq i \leq n$. Note that $g \sqsupseteq -f$ entails $g \geq -f$.

<u>Claim J.</u> *Let X be stably compact. For any step function f from X to $\mathbb{R}$, for any $g \in \langle X^{\mathsf{d}} \to \mathbb{R}\rangle$ such that $g \geq -f$, there is a step function $h : X^{\mathsf{d}} \to \mathbb{R}$ with $g \geq h \sqsupseteq -f$.*

Proof. Write $f = \sum_{i=0}^{n} a_i \chi_{U_i}$, where $X = U_0 \supseteq U_1 \supseteq \ldots \supseteq U_n$, $a_0 \in \mathbb{R}$, and $a_1, \ldots, a_n \in \mathbb{R}^+$ are non-zero. We look for h of the form $-\sum_{i=0}^{n} a_i \chi_{Q_i}$, with $Q_i \subseteq U_i$ for each i, $Q_n \subseteq \ldots \subseteq Q_1 \subseteq Q_0 = X$, and $g \geq h$. For each i, $1 \leq i \leq n$, $g^{-1}(-a_0 - a_1 - \ldots - a_i, +\infty)$ is open in X^{d}: write it as $X \setminus Q_i$, where Q_i is compact saturated (in X). Let $h = (-\sum_{i=0}^{n} a_i)\, \chi_X + \sum_{i=1}^{n} a_i \chi_{X\setminus Q_i}$. Now:

- h is of the right form: since $\chi_{X\setminus Q_i} = 1 - \chi_{Q_i}$, $h = -a_0\chi_X - \sum_{i=1}^{n} a_i \chi_{Q_i}$;
- For every i, $0 \leq i \leq n$, $Q_i \subseteq U_i$. This is clear for $i = 0$, since $Q_0 = X = U_0$. Let $i \geq 1$. If $x \in Q_i$, then $x \notin X \setminus Q_i$, so $g(x) \leq -a_0 - a_1 - \ldots - a_i$. Since $f(x) + g(x) \geq 0$, $f(x) \geq a_0 + a_1 + \ldots + a_i$. If x was not in U_i, it would not be in $U_{i+1}, \ldots, U_n$ either, so $f(x)$ would be at most $a_0 + a_1 + \ldots + a_{i-1}$. So $x \in U_i$.
- $Q_n \subseteq \ldots \subseteq Q_1 \subseteq Q_0 = X$. Indeed, when i increases from 1 to n, the interval $(-a_0 - a_1 - \ldots - a_i, +\infty)$ increases, so $X \setminus Q_i$ increases, i.e. Q_i decreases in the inclusion ordering $\subseteq$.
- It follows that h is a step function from X^{d} to $\mathbb{R}$, since it can be written as:

$$\left(-\sum_{i=0}^{n} a_i\right) \chi_X + a_n \chi_{X\setminus Q_n} + a_{n-1}\chi_{X\setminus Q_{n-1}} + \ldots + a_1 \chi_{X \setminus Q_1}$$

  where $X \supseteq X \setminus Q_n \supseteq \ldots \supseteq X \setminus Q_1$, and X and the $X \setminus Q_i$ are open in X^{d}.
- Finally, $h \leq g$. Indeed, for every $x \in X$, let k be the integer with $1 \leq k \leq n$ such that $x \notin X \setminus Q_{k-1}$ but $x \in X \setminus Q_k$. Then $h(x) = (-\sum_{i=0}^{n} a_i) + \sum_{i=k}^{n} a_i = -\sum_{i=0}^{k-1} a_i$, and $g(x) \geq -\sum_{i=0}^{k-1} a_i$, since $x \in X \setminus Q_k$, so $g(x) > -\sum_{i=0}^{k} a_i$. □

The corresponding notion obtained by exchanging X and X^{d} yields: $f \sqsupseteq^{\mathsf{d}} -g$ iff one can write f and g as step functions:

$$g = \sum_{i=0}^{n} a_i \chi_{X \setminus Q_i} \quad f = -\sum_{i=0}^{n} a_i \chi_{X \setminus U_i}$$

with the same coefficients $a_0 \in \mathbb{R}$, $a_1, \ldots, a_n \in \mathbb{R}^+$, $X = X \setminus Q_0 \supseteq X \setminus Q_1 \supseteq \ldots \supseteq X \setminus Q_n$ is a non-increasing sequence of opens in X^{d}, $X = X \setminus U_0 \supseteq X \setminus U_1 \supseteq \ldots \supseteq X \setminus U_n$ is a non-increasing sequence of compact saturated subsets of X^{d}, and $X \setminus U_i \subseteq X \setminus Q_i$ for every i, $0 \leq i \leq n$.

Equivalently, $f \sqsupseteq^{\mathsf{d}} -g$ iff:

$$f = -\left(\sum_{i=0}^{n} a_i\right) \chi_X + \sum_{i=1}^{n} a_i \chi_{U_i} \quad g = \left(\sum_{i=0}^{n} a_i\right) \chi_X - \sum_{i=1}^{n} a_i \chi_{Q_i}$$

and $Q_i \subseteq U_i$ for every i, $1 \leq i \leq n$.

By letting $U'_0 = X$, $U'_i = U_{n+1-i}$ ($1 \leq i \leq n$), $a'_0 = -\sum_{i=0}^{n} a_i$, $a'_i = a_{n+1-i}$ ($1 \leq i \leq n$), and similarly $Q'_0 = X$, $Q'_i = Q_{n+1-i}$ ($1 \leq i \leq n$), then $f \sqsupseteq^{\mathsf{d}} -g$ iff one can write:

$$f = \sum_{i=0}^{n} a'_i \chi_{U'_i} \quad g = -\sum_{i=0}^{n} a'_i \chi_{Q'_i}$$

with the same coefficients $a'_0 \in \mathbb{R}$, $a'_1, \ldots, a'_n \in \mathbb{R}^+$, $X = U'_0 \supseteq U'_1 \supseteq \ldots \supseteq U'_n$ is a non-increasing sequence of opens, $X = Q'_0 \supseteq Q'_1 \supseteq \ldots \supseteq Q'_n$ is a non-increasing sequence of compact saturated subsets, and $Q'_i \subseteq U'_i$ for every i, $0 \leq i \leq n$. So:

Claim K. *$g \sqsupseteq -f$ iff $f \sqsupseteq^{\mathsf{d}} -g$.*

Claim L. *Let X be stably compact. For every step function g from X^{d} to $\mathbb{R}$, for every $f \in \langle X \to \mathbb{R} \rangle$ such that $f \geq -g$, there is a step function h from X to $\mathbb{R}$ such that $f \geq h \sqsupseteq^{\mathsf{d}} -g$.*

Proof. By exchanging X and X^{d} in Claim J, using Claim K. ☐

Claim M. *Let X be stably compact, F a normalized prevision on X. For every step function g from X^{d} to $\mathbb{R}^+$, $F^{\perp}(g) = -\inf_{f \sqsupseteq^{\mathsf{d}} -g} \widehat{F}(f)$.*

Proof. For every $f \sqsupseteq^{\mathsf{d}} -g$, $f \geq -g$, so $\inf_{f \sqsupseteq^{\mathsf{d}} -g} \widehat{F}(f) \geq \inf_{f \geq -g} \widehat{F}(f)$. Conversely, if $f \geq -g$, by Claim L, there is an $h \sqsupseteq^{\mathsf{d}} -g$ with $h \leq f$, hence with $\widehat{F}(h) \leq \widehat{F}(f)$. So $\inf_{f \geq -g} \widehat{F}(f) \geq \inf_{h \sqsupseteq^{\mathsf{d}} -g} \widehat{F}(h)$. Therefore $\inf_{f \sqsupseteq^{\mathsf{d}} -g} \widehat{F}(f) = \inf_{f \geq -g} \widehat{F}(f) = F^{\perp}(g)$. ☐

By [5, Claim J, Appendix, Long version], the way-below relation $\ll$ on $\langle X \to \mathbb{R}^+ \rangle$ is characterized as follows. Let $f = \sum_{i=1}^{n} a_i \chi_{U_i}$ a step function from X to $\mathbb{R}^+$ (resp. $[0, 1]$), $U_1 \supseteq \ldots \supseteq U_n$, $a_1, \ldots, a_n \in \mathbb{R}^+ \setminus \{0\}$. Let g a continuous function from X to $\mathbb{R}^+$ (resp. $[0, 1]$). Then $f \ll g$ iff for every i, $1 \leq i \leq n$, $U_i \Subset g^{-1}(\sum_{j=1}^{i} a_i, +\infty)$. The relation $\Subset$ is the way-below relation on opens ordered by inclusion; on every locally compact space, $U \Subset V$ iff $U \subseteq Q \subseteq V$ for some compact saturated set Q [3].

Claim N. *Let X be stably compact. For every step function f from X to $\mathbb{R}^+$, if $g \sqsupseteq -f$, then there is a step function f' from X to $\mathbb{R}^+$ such that $g \sqsupseteq -f'$ and $r.f' \ll f$ for all r, $0 < r < 1$.*
Proof. Write:

$$f = \sum_{i=0}^{n} a_i \chi_{U_i} \quad g = -\sum_{i=0}^{n} a_i \chi_{Q_i}$$

where $X = U_0 \supseteq U_1 \supseteq \ldots \supseteq U_n$, $X = Q_0 \supseteq Q_1 \supseteq \ldots \supseteq Q_n$, and $Q_i \subseteq U_i$ for all i. Since X is locally compact, for each i, there is a compact saturated subset Q'_i with $Q_i \subseteq int(Q'_i) \subseteq Q'_i \subseteq U_i$. (This is a classical argument: for each $x \in Q_i$, find a compact saturated subset Q_x with $x \in int(Q_x) \subseteq Q_x \subseteq U_i$; the collection of all such $int(Q_x)$ covers Q_i, so extract a finite subcover; the union of the corresponding finitely many Q_x is the desired Q'_i.)
Up to the possible replacement of Q'_i by $Q'_i \cup Q'_{i+1} \cup \ldots \cup Q'_n$, we may assume that $X = Q'_0 \supseteq Q'_1 \supseteq \ldots \supseteq Q'_n$. Let $f' = \sum_{i=0}^{n} a_i \chi_{int(Q'_i)}$. Clearly, $g \sqsupseteq -f'$, since $Q_i \subseteq int(Q'_i)$.
It only remains to show $r.f' \ll f$. Since f takes its values in $\mathbb{R}^+$, we may assume $a_0 = 0$, up to a shift of indices i in the summation. We may also assume $a_i > 0$ for all i, $1 \leq i \leq n$. Then $int(Q'_i) \Subset U_i$ since $int(Q'_i) \subseteq Q'_i \subseteq U_i$, and $U_i \subseteq f^{-1}(r.\sum_{j=1}^{i} a_i, +\infty)$; indeed, for every $x \in U_i$, $f(x) \geq \sum_{j=1}^{i} a_i > r.\sum_{j=1}^{i} a_i$. So $int(Q'_i) \Subset f^{-1}(r.\sum_{j=1}^{i} a_i, +\infty)$, from which we conclude $r.f' \ll f$. □
By passing to the dual, we get:
Claim O. *Let X be stably compact. For every step function g from X^{d} to $\mathbb{R}^+$, if $f \sqsupseteq^{\mathsf{d}} -g$, then there is a step function g' from X^{d} to $\mathbb{R}^+$ such that $f \sqsupseteq^{\mathsf{d}} -g'$ and $r.g' \ll g$ for all r, $0 < r < 1$.*
Claim P. *Let X be stably compact, F a normalized prevision on X. For every step function g from X^{d} to $\mathbb{R}^+$, $F^\perp(g) = \sup_{\substack{g'' \text{ step from } X^{\mathsf{d}} \text{ to } \mathbb{R}^+ \\ g'' \ll g}} F^\perp(g'')$.*

Proof. If $g' \ll g$, then $g' \leq g$, so $F^\perp(g') \leq F^\perp(g)$ by Claim I. So the right-hand side is less than or equal to the left-hand side. Conversely, by Claim M, for every $\epsilon > 0$, there is an $f \sqsupseteq^{\mathsf{d}} -g$ such that $F^\perp(g) \leq -\widehat{F}(f) + \epsilon$. By Claim O, there is a step function g' from X^{d} to $\mathbb{R}^+$ such that $f \sqsupseteq^{\mathsf{d}} -g'$ and $r.g' \ll g$ for every r, $0 < r < 1$. By Claim M, $-\widehat{F}(f) \leq F^\perp(g') = 1/rF^\perp(r.g')$. Then $g'' = r.g'$ is a step function from X^{d} to $\mathbb{R}^+$, $g'' \ll g$, and $F^\perp(g) \leq 1/rF^\perp(g'') + \epsilon$. Since $\epsilon > 0$ and r, $0 < r < 1$, are arbitrary, we obtain $F^\perp(g) \leq \sup_{g'' \ll g} F^\perp(g'')$ where g'' ranges over all step functions from X^{d} to $\mathbb{R}^+$. □

By [5, Claim K, Appendix, Long version], whenever X is core compact (in particular, when X is stably compact), $\langle X \to \mathbb{R}^+ \rangle$ is a continuous poset, with basis B consisting of all step functions of the form $1/2^K \sum_{k=1}^{N} \chi_{U_k}$, $K, N \in \mathbb{N}$, where $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_{2^K}$ are opens of X. Dually, let B^{d} the set of all step functions of the form $1/2^K \sum_{k=1}^{N} \chi_{X \setminus Q_k}$, $K, N \in \mathbb{N}$, and where $Q_1 \subseteq Q_2 \subseteq \ldots \subseteq Q_{2^K}$ are compact saturated subsets of X. We may then improve Claim P as follows.

Claim Q. *Let X be stably compact. For every step function g from X^{d} to $\mathbb{R}^+$, $F^\perp(g) = \sup_{g'' \in B^{\mathsf{d}}, g'' \ll g} F^\perp(g'')$.*

Proof. The right-hand side is less than or equal to the left-hand side, by Claim P. Conversely,

$$\begin{aligned}
F^\perp(g) &= \sup_{\substack{g' \text{ step from } X^{\mathsf{d}} \text{ to } \mathbb{R}^+ \\ g' \ll g}} F^\perp(g') && \text{par Claim P} \\
&= \sup_{\substack{g' \text{ step from } X^{\mathsf{d}} \text{ to } \mathbb{R}^+ \\ \exists g'' \in B^{\mathsf{d}} \cdot g' \ll g'' \ll g}} F^\perp(g') && \text{by the interpolation property on } \langle X^{\mathsf{d}} \to \mathbb{R}^+ \rangle \\
&\leq \sup_{g'' \in B^{\mathsf{d}}, g'' \ll g} F^\perp(g'')
\end{aligned}$$

since $g' \ll g''$ implies $g' \leq g''$. So $F^\perp(g') \leq F^\perp(g'')$. ☐

Claim R. *Let X be stably compact, F a normalized prevision on X. For every $g \in \langle X^{\mathsf{d}} \to \mathbb{R}^+ \rangle$, $F^\perp(g) = \sup_{\substack{g' \text{ step from } X^{\mathsf{d}} \text{ to } \mathbb{R} \\ g' \leq g}} F^\perp(g')$*

Proof. The right-hand side is $\sup_{g' \leq g} F^\perp(g') = \sup_{g' \leq g} \sup_{f \geq -g'} (-\widehat{F}(f))$, where g' ranges over the step functions from X^{d} to $\mathbb{R}$, and f those from X to $\mathbb{R}$. If $g' \leq g$ and $f \geq -g'$ then $f \geq -g$, so $\sup_{g' \leq g} F^\perp(g') \leq F^\perp(g)$. Conversely, if $f \geq -g$, namely $g \geq -f$, by Claim J there is a step function g' from X^{d} to $\mathbb{R}$ such that $g \geq g' \sqsupseteq -f$; in particular, $g' \leq g$, and $f \geq -g'$, so $F^\perp(g) = \sup_{f \geq -g} (-\widehat{F}(f)) \leq \sup_{g' \leq g} \sup_{f \geq -g'} (-\widehat{F}(f)) = \sup_{g' \leq g} F^\perp(g')$. Equality follows. ☐

Claim S. *Let X be stably compact, F a normalized prevision on X. Then $F^\perp$ is a continuous prevision on X^{d}. More precisely, $F^\perp(g) = \sup_{g'' \in B^{\mathsf{d}}, g'' \ll g} F^\perp(g'')$ for every $g \in \langle X^{\mathsf{d}} \to \mathbb{R}^+ \rangle$.*

Proof. Let $g \in \langle X^{\mathsf{d}} \to \mathbb{R}^+ \rangle$. By Claim R, $F^\perp(g)$ is the sup of $F^\perp(g')$ when g' ranges over all step functions such that $g' \leq g$. By Claim Q, $F^\perp(g')$ is the sup of $F^\perp(g'')$ when $g'' \in B^{\mathsf{d}}$, $g'' \ll g'$. However $g'' \ll g'$ and $g' \leq g$ entail $g'' \ll g$, so the right-hand side is larger than or equal to the left-hand side. The converse inequality is clear. Finally, the function that maps g to $\sup_{g'' \in B^{\mathsf{d}}, g'' \ll g} F^\perp(g'')$ is continuous: this is Scott's extension formula. ☐

Our next step is to show that $F^\perp$ is lower whenever F is upper and conversely. This requires quite some effort, which is needed to study the compatibility between $\sqsupseteq^{\mathsf{d}}$ and function addition. We first recall Claim N of [5, Claim N, Appendix, Long version], which we shall use several times: Let $f' = 1/2^K \sum_{i=1}^{N} \chi_{A_i}$, $g' = 1/2^K \sum_{j=1}^{N'} \chi_{B_j}$ two functions from X to $\mathbb{R}$, with $X \supseteq A_1 \supseteq A_2 \supseteq \ldots \supseteq A_N$ and $X \supseteq B_1 \supseteq B_2 \supseteq \ldots \supseteq B_{N'}$. By extension, let $A_0 = B_0 = X$, and $A_i = \emptyset$ for every $i > N$, $B_j = \emptyset$ for every $j > N'$. Then $f' + g' = 1/2^K \sum_{k=1}^{N+N'} \chi_{W_k}$, where $W_k = \bigcup_{\substack{i \in \mathbb{N}, j \in \mathbb{N} \\ i+j=k}} (A_i \cap B_j)$.

Claim T. *Let X be stably compact. Fix $h, h' \in B^{\mathsf{d}}$, and assume $f \sqsupseteq^{\mathsf{d}} -h$, $f' \sqsupseteq^{\mathsf{d}} -h'$. Then $f + f' \sqsupseteq^{\mathsf{d}} -(h + h')$.*

Proof. Since h and h' are in B^{d}, let us write:

$$h = \frac{1}{2^K}\sum_{i=1}^{N}\chi_{X\setminus Q_i} \quad h' = \frac{1}{2^K}\sum_{j=1}^{N'}\chi_{X\setminus Q'_j}$$

where $Q_1 \subseteq \ldots \subseteq Q_N$, $Q'_1 \subseteq \ldots \subseteq Q'_{N'}$ consist of compact saturated subsets of X. Clearly we may take the same K in both without losing generality. By definition of $\sqsupseteq^{\mathsf{d}}$, we can then write f and f' as:

$$f = -\frac{1}{2^K}\sum_{i=1}^{N}\chi_{X\setminus U_i} \quad f' = -\frac{1}{2^K}\sum_{j=1}^{N'}\chi_{X\setminus U'_j}$$

with $U_1 \subseteq \ldots \subseteq U_N$, $U'_1 \subseteq \ldots \subseteq U'_{N'}$ consisting of opens, and where $Q_i \subseteq U_i$ ($1 \leq i \leq N$), $Q'_j \subseteq U'_j$ ($1 \leq j \leq N'$).
By [5, Claim N, Appendix, Long version], and using the convention that $X \setminus Q_0 = X \setminus Q'_0 = X$ (i.e., $Q_0 = Q'_0 = \emptyset$), as well as $X \setminus Q_i = \emptyset$ for every $i > N$ (i.e., $Q_i = X$) and $X \setminus Q'_j = \emptyset$ for all $j > N'$ (i.e., $Q'_j = X$),

$$h + h' = \frac{1}{2^K}\sum_{k=1}^{N+N'}\chi_{X\setminus Q''_k}$$

where $X \setminus Q''_k = \bigcup_{i+j=k}(X \setminus Q_i) \cap (X \setminus Q'_j)$, that is, $Q''_k = \bigcap_{i+j=k}(Q_i \cup Q'_j)$. Still using the same Claim N of op.cit., letting $A_i = X \setminus U_i$, $B_j = X \setminus V_j$:

$$f + f' = -\frac{1}{2^K}\sum_{k=1}^{N+N'}\chi_{W_k}$$

where $W_k = \bigcup_{i+j=k}(A_i \cap B_j)$. Again we take the convention that $A_0 = B_0 = X$, that $A_i = \emptyset$ for $i > N$, and $B_j = \emptyset$ for $j > N'$. Letting $U_0 = V_0 = \emptyset$ and $U_i = X$ for $i > N$, $V_j = X$ for $j > N'$, then $A_i = X \setminus U_i$ for all i, and $B_j = X \setminus V_j$ for all j. Let $U''_k = X \setminus W_k$. Then:

$$f + f' = -\frac{1}{2^K}\sum_{k=1}^{N+N'}\chi_{X\setminus U''_k}$$

and $U''_k = \bigcap_{i+j=k}(U_i \cup V_j)$. It follows that $Q''_k \subseteq U''_k$ for every $k \geq 1$. So $f + f' \sqsupseteq^{\mathsf{d}} -(h + h')$. ☐

Claim U. *Let X be stably compact. Fix $h, h' \in B^{\mathsf{d}}$, and let f'' be a step function from X to $\mathbb{R}$ such that $f'' \sqsupseteq^{\mathsf{d}} -(h + h')$. Then there are two step functions f, f' from X to $\mathbb{R}$ such that $f \sqsupseteq^{\mathsf{d}} -h$, $f' \sqsupseteq^{\mathsf{d}} -h'$, and $f + f' \leq f''$.*
Proof. Write:

$$h = \frac{1}{2^K}\sum_{i=1}^{N}\chi_{X\setminus Q_i} \quad h' = \frac{1}{2^K}\sum_{j=1}^{N'}\chi_{X\setminus Q'_j}$$

with $Q_1 \subseteq \ldots \subseteq Q_N$ and $Q'_1 \subseteq \ldots \subseteq Q'_{N'}$. Let $h'' = h + h'$. By [5, Claim N, Appendix, Long version],

$$h'' = \frac{1}{2^K} \sum_{k=1}^{N+N'} \chi_{X \setminus Q''_k}$$

where $X \setminus Q''_k = \bigcup_{\substack{i\in\mathbb{N}, j\in\mathbb{N} \\ i+j=k}} ((X \setminus Q_i) \cap (X \setminus Q'_j))$, i.e. $Q''_k = \bigcap_{\substack{i\in\mathbb{N}, j\in\mathbb{N} \\ i+j=k}} (Q_i \cup Q'_j)$. We have (again) taken the convention that $X \setminus Q_0 = X \setminus Q'_0 = X$ (i.e., $Q_0 = Q'_0 = \emptyset$), $X \setminus Q_i = \emptyset$ for all $i > N$ (i.e., $Q_i = X$), and $X \setminus Q'_j = \emptyset$ for all $j > N'$ (i.e., $Q'_j = X$).
Observe that $Q''_1 \subseteq \ldots \subseteq Q''_{N+N'} \subseteq \ldots$. Indeed, for all $k \geq 1$,

$$\begin{aligned}
Q''_k &= \bigcap_{i+j=k} (Q_i \cup Q'_j) \\
&\subseteq \bigcap_{i+j=k} (Q_{i+1} \cup Q'_j) \cap \bigcap_{i+j=k} (Q_i \cap Q'_{j+1}) \quad \text{since } Q_i \subseteq Q_{i+1} \text{ and } Q'_j \subseteq Q'_{j+1} \\
&= \bigcap_{i+j=k+1} (Q_i \cup Q'_j) = Q''_{k+1}
\end{aligned}$$

Since $f'' \sqsupseteq^{\mathsf{d}} -h''$, f'' is of the form:

$$f'' = -\sum_{k=1}^{N''} \chi_{X \setminus W_k} \tag{2}$$

where $N'' = N + N'$, $W_1 \subseteq \ldots\ldots W_{N''}$, and $Q''_k \subseteq W_k$ for all k, $1 \leq k \leq N''$. Let $W_0 = \emptyset$, and $W_k = X$ for all $k > N''$. Since Q''_k is a finite non-empty intersection of compact saturated subsets $Q_i \cup Q'_j$, there is a family of opens W_{ij}, $i, j \in \mathbb{N}$, $i + j \geq 1$, such that:

$$Q_i \cup Q'_j \subseteq W_{ij} \quad \bigcap_{i+j=k} W_{ij} \subseteq W_k$$

for all $i, j \in \mathbb{N}$, $i + j \geq 1$ on the left, and for all $k \geq 1$ on the right. This is obtained by repeated application of Lemma 3.3 of [15], which states that, in a stably (locally) compact space, for any two compact saturated subsets Q_1, Q_2 and for any open W containing $Q_1 \cap Q_2$, there are two open subsets V_1 and V_2 such that $Q_1 \subseteq V_1$, $Q_2 \subseteq V_2$, and $V_1 \cap V_2 \subseteq W$.
Without loss of generality, we may assume that the family W_{ij} is monotone, in the sense that whenever $i \leq i'$ and $j \leq j'$, then $W_{ij} \subseteq W_{i'j'}$. Otherwise, replace W_{ij} with $W'_{ij} = \bigcap_{\substack{i' \geq i \\ j' \geq j}} W_{ij}$, a trivially monotone family. (This is a well-defined open set, since for $i > N$, $j > N'$, $X = Q_i \cup Q'_j \subseteq W_{ij}$, so that $W_{ij} = X$ for i, j large enough: in particular, W'_{ij} is in fact a finite intersection of opens. Next, $Q_i \cup Q'_j \subseteq Q_{i'} \cup Q'_{j'} \subseteq W_{i'j'}$ for all $i' \geq i$, $j' \geq j$, so $Q_i \cup Q'_j \subseteq W'_{ij}$. Finally, $\bigcap_{i+j=k} W'_{ij} \subseteq \bigcap_{i+j=k} W_{ij} \subseteq W_k$, since clearly $W'_{ij} \subseteq W_{ij}$.)

Define finally $W_{00} = \emptyset$. The family W_{ij}, $i, j \in \mathbb{N}$, consists of opens such that $Q_i \cup Q'_j \subseteq W_{ij}$, $\bigcap_{i+j=k} W_{ij} \subseteq W_k$, and $(W_{ij})_{i,j\in\mathbb{N}}$ is monotone. Also, since $Q_i \cup Q'_j \subseteq W_{ij}$, $Q_i = X$ whenever $i > N$, and $Q'_j = X$ whenever $j > N'$, then $W_{ij} = X$ whenever $i > N$ or $j > N'$.

Define U_i as $\bigcap_{j\in\mathbb{N}} W_{ij}$ for all $i \in \mathbb{N}$, and V_j as $\bigcap_{i\in\mathbb{N}} W_{ij}$. Since $(W_{ij})_{i,j\in\mathbb{N}}$ is monotone, the families $(U_i)_{i\in\mathbb{N}}$ and $(V_j)_{j\in\mathbb{N}}$ are also monotone (i.e., $i \leq i'$ implies $U_i \subseteq U_{i'}$, and $j \leq j'$ implies $V_j \subseteq V_{j'}$). Let $f = -1/2^K \sum_{i=1}^{N} \chi_{X\setminus U_i}$, $f' = -1/2^K \sum_{j=1}^{N'} \chi_{X\setminus V_j}$.

Note that $Q_i \subseteq U_i$ for all i, $1 \leq i \leq N$. Indeed, since $Q_i \cup Q'_j \subseteq W_{ij}$ for all i and j, $Q_i \subseteq W_{ij}$, so $Q_i \subseteq \bigcap_{j\in\mathbb{N}} W_{ij} = U_i$. It follows that $f \sqsupseteq^{\mathsf{d}} -h$. Similarly, $Q'_j \subseteq V_j$ for all j, $1 \leq j \leq N'$, so $f' \sqsupseteq^{\mathsf{d}} -h'$.

It remains to show that $f + f' \leq f''$. Write f, f', and f'' under the standard form for step functions:

$$f = -\frac{N}{2^K}\chi_X + \frac{1}{2^K}\sum_{i=1}^{N} \chi_{U_{N+1-i}}$$

$$f' = -\frac{N'}{2^K}\chi_X + \frac{1}{2^K}\sum_{j=1}^{N'} \chi_{V_{N'+1-j}}$$

$$f'' = -\frac{N''}{2^K}\chi_X + \frac{1}{2^K}\sum_{k=1}^{N''} \chi_{W_{N''+1-k}}$$

To this end, we have re-indexed the sums on i, j, k, so that the sequences $(U_{N+1-i})_{i=1}^{N}$, $(V_{N'+1-j})_{j=1}^{N'}$, and $(W_{N''+1-k})_{k=1}^{N''}$ are non-increasing. Using [5, Claim N, Appendix, Long version], we have:

$$\sum_{i=1}^{N} \chi_{U'_i} + \sum_{j=1}^{N'} \chi_{V'_j} = \sum_{k=1}^{N''} \chi_{W'_k}$$

where $W'_k = \bigcup_{\substack{i,j\in\mathbb{N}\\ i+j=k\\ i\leq N, j\leq N'}} (U'_i \cap V'_j)$. Again, we take the convention that $U'_0 = V'_0 = X$. (Taking $i \leq N$, $j \leq N'$ in the definition W'_k allows us to not assume that $U'_i = \emptyset$ for $i > N$ and $V'_j = \emptyset$ for $j > N'$.) Let $U'_i = U_{N+1-i}$, $0 \leq i \leq N$. Note that $U'_0 = X$, since $U'_0 = U_{N+1} = \bigcap_{j\in\mathbb{N}} W_{(N+1)j} = X$, as $W_{ij} = X$ whenever $i > N$. Similarly, $V'_0 = X$. So:

$$\sum_{i=1}^{N} \chi_{U_{N+1-i}} + \sum_{j=1}^{N'} \chi_{V_{N'+1-j}} = \sum_{k=1}^{N''} \chi_{W'_k}$$

and $W'_k = \bigcup_{\substack{i,j\in\mathbb{N}\\ i+j=k\\ i\le N, j\le N'}} (U'_i \cap V'_j) = \bigcup_{\substack{i,j\in\mathbb{N}\\ i+j=k\\ i\le N, j\le N'}} (U_{N+1-i} \cap V_{N'+1-j})$. However if $i + j = k$, $i \le N$, and $j \le N'$,

$$\begin{aligned}
U_{N+1-i} \cap V_{N'+1-j} &= \bigcap_{j'\in\mathbb{N}} W_{(N+1-i)j'} \cap \bigcap_{i'\in\mathbb{N}} W_{i'(N'+1-j)} \\
&\subseteq \bigcap_{j'=0}^{N'-j} W_{(N+1-i)j'} \cap \bigcap_{i'=0}^{N-i} W_{i'(N'+1-j)} \\
&\subseteq \bigcap_{j'=0}^{N'-j} W_{(N+N'+1-k-j')j'} \cap \bigcap_{i'=0}^{N-i} W_{i'(N+N'+1-k-i')} \\
&\quad \text{since } (W_{i''j''})_{i'',j''\in\mathbb{N}} \text{ is monotone,} \\
&\quad \text{since } N+1-i \le N+N'+1-k-j' \text{ whenever } i+j=k, j' \le N'-j \\
&\quad \text{and since } N'+1-j \le N+N'+1-k-i' \text{ whenever } i+j=k, i' \le N-i \\
&= \bigcap_{i'=N+1-i}^{N+N'+1-k} W_{i'(N+N'+1-k-i')} \cap \bigcap_{i'=0}^{N-i} W_{i'(N+N'+1-k-i')} \\
&\quad \text{by re-indexing the first union using } i' = N+N'+1-k-j', \\
&\quad \text{knowing that } -k+j=-i \\
&= \bigcap_{i'=0}^{N+N'+1-k} W_{i'(N+N'+1-k-i')} = \bigcap_{\substack{i',j'\in\mathbb{N}\\ i'+j'=N+N'+1-k}} W_{i'j'} \\
&\subseteq W_{N+N'+1-k} = W_{N''+1-k}
\end{aligned}$$

So $W'_k \subseteq W_{N''+1-k}$ for every k, $1 \le k \le N''$. So $\sum_{i=1}^{N} \chi_{U_{N+1-i}} + \sum_{j=1}^{N'} \chi_{V_{N'+1-j}} \le \sum_{k=1}^{N''} \chi_{W_{N''+1-k}}$, therefore:

$$\begin{aligned}
f + f' &= -\frac{N+N'}{2^K}\chi_X + \frac{1}{2^K}\left(\sum_{i=1}^{N} \chi_{U_{N+1-i}} + \sum_{j=1}^{N'} \chi_{V_{N'+1-j}}\right) \\
&\le -\frac{N''}{2^K} + \frac{1}{2^K}\sum_{k=1}^{N''} \chi_{W_{N''+1-k}}
\end{aligned}$$

and this is nothing else but f''. □

<u>Claim V.</u> *Let X be stably compact, F a normalized prevision on X. If F is lower, then $F^\perp$ is upper. If F is upper, then $F^\perp$ is lower.*

Proof. Let F be lower, i.e., super-additive. Fix $h, h' \in B^{\mathsf{d}}$:

$$\begin{aligned}
F^\perp(h) + F^\perp(h') &= -\inf_{f \sqsupseteq^{\mathsf{d}} -h} \widehat{F}(f) - \inf_{f' \sqsupseteq^{\mathsf{d}} -h'} \widehat{F}(f') \quad \text{by Claim M} \\
&= -\inf_{\substack{f \sqsupseteq^{\mathsf{d}} -h \\ f' \sqsupseteq^{\mathsf{d}} -h'}} \widehat{F}(f) + \widehat{F}(f') \\
&\geq -\inf_{\substack{f \sqsupseteq^{\mathsf{d}} -h \\ f' \sqsupseteq^{\mathsf{d}} -h'}} \widehat{F}(f + f') \\
&\quad \text{since } F \text{ is lower, hence also } \widehat{F} \\
&= -\inf_{f'' \sqsupseteq^{\mathsf{d}} -(h+h')} \widehat{F}(f'')
\end{aligned}$$

Indeed, if $f \sqsupseteq^{\mathsf{d}} -h$ and $f' \sqsupseteq^{\mathsf{d}} -h'$, then $f'' = f + f'$ satisfies $f'' \sqsupseteq^{\mathsf{d}} -(h+h')$ by Claim T, so $\inf_{\substack{f \sqsupseteq^{\mathsf{d}} -h \\ f' \sqsupseteq^{\mathsf{d}} -h'}} \widehat{F}(f + f') \geq \inf_{f'' \sqsupseteq^{\mathsf{d}} -(h+h')} \widehat{F}(f'')$. Conversely, if $f'' \sqsupseteq^{\mathsf{d}} -(h + h')$ then one can find $f \sqsupseteq^{\mathsf{d}} -h$ and $f' \sqsupseteq^{\mathsf{d}} -h'$ such that $f + f' \leq f''$ by Claim U, so $\inf_{f'' \sqsupseteq^{\mathsf{d}} -(h+h')} \widehat{F}(f'') \geq \inf_{\substack{f \sqsupseteq^{\mathsf{d}} -h \\ f' \sqsupseteq^{\mathsf{d}} -h'}} \widehat{F}(f + f')$.

By Claim F again, it follows that $F^\perp(h) + F^\perp(h') \geq F^\perp(h + h')$.

In the general case, let $g, g' \in \langle X^{\mathsf{d}} \to \mathbb{R}^+ \rangle$:

$$\begin{aligned}
F^\perp(g) + F^\perp(g') &= \sup_{h \in B^{\mathsf{d}}, h \ll g} F^\perp(h) + \sup_{h' \in B^{\mathsf{d}}, h' \ll g'} F^\perp(h') \quad \text{by Claim S} \\
&= \sup_{\substack{h, h' \in B^{\mathsf{d}} \\ h \ll g, h' \ll g'}} F^\perp(h) + F^\perp(h') \\
&\geq \sup_{\substack{h, h' \in B^{\mathsf{d}} \\ h \ll g, h' \ll g'}} F^\perp(h + h') \\
&= \sup_{h'' \in B^{\mathsf{d}}, h'' \ll g + g'} F^\perp(h'')
\end{aligned}$$

since the cone $\langle X \to \mathbb{R}^+ \rangle$ is additive (a consequence of Claims M and O of [5, Appendix, Long version]). But the above is $F^\perp(g + g')$ by Claim S. So $F^\perp$ is upper.

That $F^\perp$ is lower whenever F is lower proceeds along similar lines. □

That $F \mapsto F^\perp$ is a duality is our last claim.

<u>Claim W.</u> *Let X be stably compact. For every continuous normalized prevision F on X, $F^{\perp\perp} = F$.*

Proof. Let f be an arbitrary step function from X to $\mathbb{R}^+$. Fix $a \geq \sup_{x \in X} f(x)$. Then:

$$\begin{aligned} F^{\perp\perp}(f) &= a - \inf_{g \geq -f} F^{\perp}(g+a) \\ &= a - \inf_{g \geq -f} (- \inf_{h \geq -g-a} \widehat{F}(h)) \\ &= a + \sup_{g \geq -f} \inf_{h \geq -g-a} \widehat{F}(h) \\ &= \sup_{g \geq -f} \inf_{h \geq -g-a} \widehat{F}(h+a) \quad \text{since } F \text{ is normalized} \\ &= \sup_{g \geq -f} \inf_{h \geq -g} \widehat{F}(h) \end{aligned}$$

where g and h range over all step functions from X^{d}, resp. X, to $\mathbb{R}$. For every step function $g \geq -f$, $\inf_{h \geq -g} \widehat{F}(h) \leq \widehat{F}(f)$, since we may take $h = f \geq -g$. (Recall that f is a step function.) So $F^{\perp\perp}(f) \leq F(f)$.
The converse is harder. Write f as $\sum_{i=1}^n a_i \chi_{U_i}$, where $U_1 \supseteq \ldots \supseteq U_n$ consists of opens in X, and $a_1, \ldots, a_n \in \mathbb{R}^+$. Take the convention that $U_0 = X$, $U_{n+1} = \emptyset$.
For each i, $1 \leq i \leq n$, for all $x \in U_i$, since X is locally compact, there is a compact saturated subset Q_x with $x \in int(Q)_x \subseteq Q_x \subseteq U_i$. So $U_i = \bigcup_{x \in U_i} int(Q)_x$. It follows that U_i is also the union of the directed family of the sets of the form $\bigcup_{x \in E_i} int(Q)_x$, where E_i ranges over all finite subsets of U_i. Since $\bigcup_{x \in E_i} int(Q)_x \subseteq int(\bigcup_{x \in E_i} Q_x) \subseteq U_i$, U_i is also the directed union of all $int(\bigcup_{x \in E_i} Q_x)$, when E_i ranges over the finite subsets of U_i. Note that $\bigcup_{x \in E_i} Q_x$ is a compact saturated subset of U_i. So U_i is also the union of the directed family of the interiors of all compact saturated subsets Q_i contained in U_i.
It follows that f is the sup of the directed family of the functions $\sum_{i=1}^n a_i \chi_{int(Q_i)}$, where $Q_1, \ldots, Q_n$ range over all compact saturated subsets of $U_1, \ldots, U_n$ respectively, with $Q_1 \supseteq Q_2 \supseteq \ldots \supseteq Q_n$. This is shown by induction on n. This is clear for $n = 0$. When $n \geq 1$, $\sum_{i=2}^n a_i \chi_{U_i}$ is the sup of the directed family of all $\sum_{i=2}^n a_i \chi_{int(Q_i)}$, where $Q_2, \ldots, Q_n$ range over the compact saturated subsets of $U_2, \ldots, U_n$ respectively, and $Q_2 \supseteq \ldots \supseteq Q_n$, by induction hypothesis. Now recall that U_1 is the sup of the directed family of the interiors of all compact saturated subsets Q_1 of U_1, so $\chi_{U_1} = \sup_{Q_1 \subseteq U_1} \chi_{int(Q_1)}$. For every compact saturated subset $Q_1 \subseteq U_1$, there is another one (namely $Q_1 \cup Q_2$) which also contains Q_2. So $\chi_{U_1} = \sup_{\substack{Q_1 \text{ saturated compact} \\ Q_2 \subseteq Q_1 \subseteq U_1}} \chi_{int(Q_1)}$, and we conclude.
For every $\epsilon > 0$, therefore, there are n compact saturated subsets $Q_1 \supseteq Q_2 \supseteq \ldots \supseteq Q_n$ of $U_1, \ldots, U_n$ respectively, such that $f - \epsilon \leq \sum_{i=1}^n a_i \chi_{int(Q_i)} \leq f$. Let $g = -\sum_{i=1}^n a_i \chi_{Q_i}$: g is continuous from X^{d} to $\mathbb{R}$, since $-\chi_{Q_i}$ est continuous from X^{d} to $\mathbb{R}$ for each $i \in I$. Moreover, g only takes finitely many values, so g is a step function from X^{d} to $\mathbb{R}$.
For every step function h from X to $\mathbb{R}$ such that $h \geq -g$, i.e., such that $h \geq \sum_{i=1}^n a_i \chi_{Q_i}$, we obtain $h \geq \sum_{i=1}^n a_i \chi_{int(Q_i)} \geq f - \epsilon$. So $\inf_{h \geq -g} \widehat{F}(h) \geq \widehat{F}(f - \epsilon)$ (since $\widehat{F}$ is monotonic) $= \widehat{F}(f) - \epsilon$ (since $\widehat{F}$ is normalized) $= F(f) - \epsilon$.
By varying g, we get $F^{\perp\perp}(f) = \sup_{g \geq -f} \inf_{h \geq -g} F(h) \geq F(f) - \epsilon$. Since ϵ is arbitrary, $F^{\perp\perp}(f) \geq F(f)$. Since $F^{\perp\perp}(f) \leq F(f)$, $F^{\perp\perp}(f) = F(f)$.

In the general case where f is no longer a step function, $F^{\perp\perp}(f) = \sup_{f' \in B, f' \ll g} F^{\perp\perp}(f')$ by Claim S. We have just seen that $F^{\perp\perp}(f') = F''(f')$ for every $f' \in B$. We conclude since F is continuous, and B is a basis of the continuous poset $\langle X \to \mathbb{R}^+\rangle$. □

Theorem 4. *Let X be a stably compact space. For every normalized prevision F on X, $F^\perp$ is a normalized prevision on X^{d}. Moreover: (1) $F^\perp$ is continuous; (2) if F is lower, then $F^\perp$ is upper; (3) if F is upper, then $F^\perp$ is lower; (4) if F is linear, then so is $F^\perp$; (5) if F is continuous, then $F^{\perp\perp} = F$; (6) if $F \leq F'$ then $F'^\perp \leq F^\perp$.*

Proof. Continuity (1) is by Claim S. (2) and (3) are by Claim V. (4) is a trivial consequence of (2) and (3). That $F^{\perp\perp} = F$ (5) whenever F is continuous is by Claim W. (6) is obvious. □

Claim X. *Let X be stably compact, and ν a continuous game on X, then $\nu^\perp(X \setminus Q) = 1 - \nu^\dagger(Q)$, where $\nu^\dagger(Q) = \inf_{U \supseteq Q} \nu(U)$.*
Proof. By definition, $\nu^\perp(X \setminus Q)$ is $F^\perp(\chi_{X \setminus Q})$, where $F = \alpha_{\mathcal{C}}(\nu)$. Observe that $f \sqsupseteq^{\mathsf{d}} -\chi_{X\setminus Q}$ iff f is of the form $-\chi_{X \setminus U}$ for some open U containing Q; also, that $-\chi_{X\setminus U} = -1 + \chi_U$. By Claim M, $F^\perp(\chi_{X\setminus Q}) = -\inf_{f \sqsupseteq^{\mathsf{d}} -\chi_{X\setminus Q}} \widehat{F}(f) = -\inf_{U \supseteq Q} \widehat{F}(-1 + \chi_U) = 1 - \inf_{U \supseteq Q} F(\chi_U) = 1 - \inf_{U\supseteq Q} \nu(U)$. □

C Proof of Lemma 1

Lemma 1. *Let X be a stably compact space, and F a normalized continuous prevision on X. For every $g \in \langle X^{\mathsf{d}} \to \mathbb{R}\rangle$, $F^\perp(g) = -\inf_{\substack{f \in \langle X \to \mathbb{R}\rangle \\ f \geq -g}} \widehat{F}(f)$.*

Proof. First show that $F^\perp(g) \geq -\widehat{F}(f)$ for every $f \in \langle X \to \mathbb{R}\rangle$ with $f \geq -g$. Apply the construction of a step function f_K approximating f to g instead, on X^{d}, getting g_K ($K \in \mathbb{N}$): $g_K = a + \frac{1}{2^K}\sum_{k=1}^{\lfloor (b-a)2^K \rfloor} \chi_{g^{-1}(a + \frac{k}{2^K}, +\infty)}$, where a is a lower bound and b an upper bound for g. It is easy to see that $g - \frac{1}{2^K} \leq g_K \leq g$. Let $f' = f + \frac{1}{2^K}$, so that $f' \geq -g_K$. By Claim L, there is a step function h from X to $\mathbb{R}$ such that $f' \geq h \sqsupseteq^{\mathsf{d}} -g_K$. In particular $h \geq -g_K$, and h is a step function. So by definition $F^\perp(g_K) \geq -\widehat{F}(h) \geq -\widehat{F}(f') = -\widehat{F}(f) - \frac{1}{2^K}$, since $\widehat{F}$ is normalized. Since $g \geq g_K$, $F^\perp(g) \geq F^\perp(g_K) \geq -\widehat{F}(f) - \frac{1}{2^K}$. Since K is arbitrary, $F^\perp(g) \geq -\widehat{F}(f)$. Since $f \in \langle X \to \mathbb{R}\rangle$ is arbitrary with $f \geq -g$, $F^\perp(g) \geq -\inf_{\substack{f \in \langle X \to \mathbb{R}\rangle \\ f \geq -g}} \widehat{F}(f)$.

The converse inequality is obvious, since every step function is continuous and bounded. □

D Proof of Proposition 6

Proposition 6. *Let X be stably compact, and $F \in \triangle\mathbf{P}_1(X)$. Then $CPeau_1(F)$ is co-strongly convex.*

Proof. We first observe that, given any closed subset F of a space Y, given any continuous probability valuation p on Y that is co-supported on F, for any $h \in \langle Y \to \mathbb{R}^+ \rangle$: $(*)$ $\oint_{y \in Y} h(y)dp \leq \sup_{y \in F} h(y)$. Let $a = \sup_{y \in F} h(y)$. For all $t \geq a$, $f^{-1}(t, +\infty)$ does not intersect F, otherwise there would be an $x \in F$ such that $f(x) > t \geq a \geq f(x)$, a contradiction. So $f^{-1}(t, +\infty) \subseteq X \setminus F$ for every $t \geq a$; hence $\oint_{y \in Y} h(y)dp = \int_0^{+\infty} p(f^{-1}(t, +\infty))dt = \int_0^a p(f^{-1}(t, +\infty))dt \leq a$.

Let $\mathcal{F} = CPeau_1(F)$, $\mathcal{P}$ a continuous probability valuation on $\mathbf{P}^{\triangle}_{1\ wk}(X)$ that is co-supported on $\mathcal{F}$. Fix $f \in \langle X \to \mathbb{R}^+ \rangle$, and let $\varphi(G) = G(f)$. Note that φ is continuous, by the definition of the weak topology. It is bounded since $\varphi(G) = G(f) \leq G(\sup_{x \in X} f(x).\chi_X) = \sup_{x \in X} f(x)$, because G is normalized. By $(*)$, $\oint_{G \in Y} \varphi(G)d\mathcal{P} \leq \sup_{G \in \mathcal{F}} \varphi(G)$ for every $\varphi \in \langle \mathbf{P}^{\triangle}_{1\ wk}(X) \to \mathbb{R}^+ \rangle$. In other words, $Bary(\mathcal{P})(f) \leq \bigsqcup \mathcal{F}(f)$. Since f is arbitrary, $Bary(\mathcal{P}) \leq \bigsqcup \mathcal{F}$, i.e. $Bary(\mathcal{P}) \in CPeau_1(\bigsqcup \mathcal{F})$. But recall that $\bigsqcup \mathcal{F} = \bigsqcup CPeau_1(F) = F$. $\Box$

E Proof of $Conv(\mathcal{F}^{\perp})^{\perp} = Conv^*(\mathcal{F})$

We show that: (a) if p is supported on some subset A of X, then $f[p]$ is supported on the direct image $f(A)$. Indeed, for any open V containing $f(A)$, $A \subseteq f^{-1}(V)$, so $f[p](V) = p(f^{-1}(V)) = 1$.

Next, we show: (b) if $f[p]$ is co-supported on some closed subset F of Y, then p is co-supported on $f^{-1}(F)$. Indeed, let U be the complement of [the closure of] $f^{-1}(F)$. U equals $f^{-1}(V)$, where V is the complement of [the closure of] F. So $p(U) = p(f^{-1}(V)) = f[p](V) = 0$.

We finally show that, whenever X is stably compact, then for every closed subset $\mathcal{F}$ of $Y = \mathbf{P}^{\triangle}_{1\ wk}(X)$, $Conv^*(\mathcal{F}) = Conv(\mathcal{F}^{\perp})^{\perp}$. To be precise, note that $\mathcal{F}^{\perp}$ is the image under $\_^{\perp}$ of $\mathcal{F}$, i.e., the set $\{p^{\perp} | p \in \mathcal{F}\}$.

Start by showing $Conv^*(\mathcal{F}) \subseteq Conv(\mathcal{F}^{\perp})^{\perp}$. Let p an arbitrary element of $Conv^*(\mathcal{F})$. So there is a continuous probability valuation $\mathcal{P}$, co-supported on $\mathcal{F}$, such that $p = Bary(\mathcal{P})$. By Proposition 7, $(Bary(\_^{\perp}[\mathcal{P}^{\perp}]))^{\perp}$. Since $\mathcal{P}$ is co-supported on $\mathcal{F}$, $\mathcal{P}^{\perp}$ is supported on $\mathcal{F}$. Using (a), $\_^{\perp}[\mathcal{P}^{\perp}]$ is supported on $\mathcal{F}^{\perp}$. So $Bary(\_^{\perp}[\mathcal{P}^{\perp}]) \in Conv(\mathcal{F}^{\perp})$, whence $p \in Conv(\mathcal{F}^{\perp})^{\perp}$.

The converse inclusion $Conv(\mathcal{F}^{\perp})^{\perp} \subseteq Conv^*(\mathcal{F})$ is shown as follows. Let p an arbitrary element of $Conv(\mathcal{F}^{\perp})^{\perp}$, and write $p^{\perp} = Bary(\mathcal{P})$, where $\mathcal{P}$ is a continuous probability valuation that is supported on $\mathcal{F}^{\perp}$. Then $\mathcal{P}^{\perp}$ is co-supported on $\mathcal{F}^{\perp}$. By Theorem 4 (5), $\mathcal{P}^{\perp} = \_^{\perp}[\_^{\perp}[\mathcal{P}^{\perp}]]$. Using (b), $\_^{\perp}[\mathcal{P}^{\perp}]$ is co-supported on $\mathcal{F}$. Since $p^{\perp} = Bary(\mathcal{P})$, we obtain $p = Bary(\_^{\perp}[\mathcal{P}^{\perp}])$ by applying Proposition 7 and Theorem 4 (5). So $p \in Conv^*(\mathcal{F})$. $\Box$

F Proof of Proposition 8

Proposition 8. *Let X be a continuous, coherent, pointed cpo. Then $CPeau_1$ is a continuous map from $\bigtriangleup \mathbf{P}_1(X)$ to $\mathcal{H}(\mathbf{P}^{\triangle}_{1\ wk}(X))$.*

Proof. Recall that whenever Y is a continuous cpo, then so is $\mathcal{H}(Y)$, and the sup of the directed family $(C_i)_{i\in I}$ in $\mathcal{H}(Y)$ is $cl(\bigcup_{i\in I} C_i)$, where cl denotes topological closure. Moreover, for any two finite subsets E and E' of Y, the closed subsets $\downarrow E$ and $\downarrow E'$ are such that $\downarrow E \ll \downarrow E'$ iff $E \subseteq \twoheaddownarrow E'$, where $\twoheaddownarrow E'$ denotes the set of all elements of Y way-below some element of E' [1].

Let $(F_i)_{i\in I}$ a directed family of normalized continuous upper previsions on X. We must show that $CPeau_1(\sup_{i\in I} F_i) = cl(\bigcup_{i\in I} CPeau_1(F_i))$. Then inclusion $cl(\bigcup_{i\in I} CPeau_1(F_i)) \subseteq CPeau_1(\sup_{i\in I} F_i)$ is clear, since $CPeau_1$ is monotonic.

Conversely, for every $G \in CPeau_1(\sup_{i\in I} F_i)$, i.e., $G \leq \sup_{i\in I} F_i$, let us show that $G \in cl(\bigcup_{i\in I} CPeau_1(F_i))$. Before we embark on the actual proof, let us remark that the following seemingly obvious argument is wrong: write G as the sup of a directed family of simple normalized linear previsions G_j, $j \in J$, with $G_j \ll G$; since $G \leq \sup_{i\in I} F_i$ we must have $G_j \leq F_i$ for some i, so $G_j \in cl(\bigcup_{i\in I} CPeau_1(F_i))$; since $cl(\bigcup_{i\in I} CPeau_1(F_i))$ is Scott-closed, $G = \sup_{j\in J} G_j$ is in $cl(\bigcup_{i\in I} CPeau_1(F_i))$. This argument is wrong because, while it is true that $G_j \ll G$ *in* $\mathbf{P}_1^{\triangle}(X)$, we have $G \leq \sup_{i\in I} F_i$ *in* $\triangle\mathbf{P}_1(X)$: the spaces do not match, and we actually cannot conclude.

Instead, note that $G \leq \sup_{i\in I} F_i$, and $F_i = \bigsqcup CPeau_1(F_i)$, so $G \leq \bigsqcup \bigcup_{i\in I} CPeau_1(F_i)$. In particular, $G \leq \bigsqcup \mathcal{F}$, where $\mathcal{F}$ is the closed subset $cl(\bigcup_{i\in I} CPeau_1(F_i))$ of $\mathbf{P}_{1\,wk}^{\triangle}(X)$. This means that $G \in CPeau_1(\bigsqcup \mathcal{F})$, so $G \in \downarrow Conv^*(\mathcal{F})$ (see remark before Theorem 5). In other words, there is a continuous probability valuation $\mathcal{P}$ on $\mathbf{P}_{1\,wk}^{\triangle}(X)$, co-supported on $\mathcal{F}$, such that $G \leq Bary(\mathcal{P})$.

Since X is continuous and pointed, $Y = \mathbf{P}_1^{\triangle}(X) = \mathbf{P}_{1\,wk}^{\triangle}(X)$ is a continuous pointed cpo. So $\mathbf{P}_{1\,wk}^{\triangle}(Y) = \mathbf{P}_1^{\triangle}(Y)$ is also a continuous cpo, and a basis consists of the simple normalized linear previsions. Let therefore $\mathcal{P}_j$, $j \in J$, a directed family of simple normalized linear previsions way-below $\mathcal{P}$ in $\mathbf{P}_1^{\triangle}(Y)$, whose sup is $\mathcal{P}$.

Let us show that for every $j \in J$, there is an $i \in I$ such that $\mathcal{P}_j$ is co-supported on $CPeau_1(F_i)$. Write $\mathcal{P}_j = a_0\delta_\perp + \sum_{k=1}^n a_k\delta_{G_k}$, where a_k is non-zero for every $k \geq 1$, and each G_k ($k \geq 1$) is different from the least element $\perp$ of Y. Since $\mathcal{P}_j \ll \mathcal{P}$, using the interpolation property, there is a simple normalized linear prevision $\mathcal{P}' = b_0\delta_\perp + \sum_{\ell=1}^p b_\ell\delta_{G'_\ell}$ (each b_ℓ being non-zero, $\ell \geq 1$)) such that $\mathcal{P}_j \ll \mathcal{P}' \ll \mathcal{P}$. Since $\mathcal{P}' \leq \mathcal{P}$ and $\mathcal{P}$ is co-supported on $\mathcal{F}$, $\mathcal{P}'(Y \setminus \mathcal{F}) \leq \mathcal{P}(Y \setminus \mathcal{F}) = 0$, so $\mathcal{P}'$ is also co-supported on $\mathcal{F}$. It follows that each G'_ℓ, $\ell \geq 1$, is in $\mathcal{F}$. Since $\mathcal{P}_j \ll \mathcal{P}'$, by Edalat's version of the Splitting Lemma on spaces of continuous probability valuations [2], there is a matrix $(t_{k\ell})_{\substack{1\leq k\leq n\\1\leq \ell\leq p}}$ of coefficients in $\mathbb{R}^+$ such that:

$$\sum_{1\leq \ell\leq p} t_{k\ell} = a_k \quad \text{for every } k, 1 \leq k \leq n$$

$$\sum_{1\leq k\leq n} t_{ij} < b_\ell \qquad \text{for every } \ell, 1 \leq \ell \leq p$$

and such that $t_{ij} \neq 0$ implies $G_k \ll G'_\ell$. Note that $\mathcal{P}_j = a_0\delta_\perp + \sum_{k=1}^n a_k\delta_{G_k} = a_0\delta_\perp + \sum_{k=1}^n \sum_{\ell=1}^p t_{k\ell}\delta_{G_k} = a_0\delta_\perp + \sum_{\substack{1\leq k\leq n\\1\leq \ell\leq p\\G_k \ll G'_\ell}} t_{k\ell}\delta_{G_k}$. The only coefficients a_k that are non-zero, with $k \geq 2$, must then be those for which there is an ℓ, $1 \leq \ell \leq p$, such that $G_k \ll G'_\ell$. Since $a_k \neq 0$ for every $k \geq 1$, and $G'_\ell \in \mathcal{F}$, we must have $G_k \in \twoheaddownarrow\mathcal{F}$ for every k, $1 \leq k \leq n$. Also, since $\bot$ is the least element of Y, $\bot \ll \bot$, so $\bot \in \twoheaddownarrow\mathcal{F}$. It follows that $\{\bot, G_1, \ldots, G_n\} \subseteq \twoheaddownarrow\mathcal{F}$, that is, $\downarrow \{\bot, G_1, \ldots, G_n\} \ll \mathcal{F}$ in $\mathcal{H}(Y)$. Recall that $\mathcal{F} = cl(\bigcup_{i \in I} CPeau_1(F_i))$ is the sup of the directed family $(\mathcal{F}_i)_{i \in I}$ in $\mathcal{H}(Y)$, so there is an $i \in I$ such that $\downarrow \{\bot, G_1, \ldots, G_n\} \subseteq CPeau_1(F_i)$. In particular, $\bot, G_1, \ldots, G_n \in CPeau_1(F_i)$. Hence $\mathcal{P}_j = a_0\delta_\bot + \sum_{k=1}^n a_k \delta_{G_k}$ is co-supported on $CPeau_1(F_i)$.

We have shown that, for every $j \in J$, there is an $i_j = i \in I$ such that $\mathcal{P}_j$ is co-supported on $CPeau_{\leq 1}(F_{i_j})$. Then $Bary(\mathcal{P}_j) \in CPeau_1(F_{i_j})$, since $CPeau_1(F_{i_j})$ is convex. In particular, $Bary(\mathcal{P}_j) \in cl(\bigcup_{i \in I} CPeau_1(F_i))$. Since $Bary$ is continuous and $cl(\bigcup_{i \in I} CPeau_1(F_i))$ is Scott-closed, $Bary(\mathcal{P})$, the sup of all $Bary(\mathcal{P}_j)$, is also in $cl(\bigcup_{i \in I} CPeau_1(F_i))$. Recall that $G \leq Bary(\mathcal{P})$. Since $cl(\bigcup_{i \in I} CPeau_1(F_i))$ is downward-closed, $G \in cl(\bigcup_{i \in I} CPeau_1(F_i))$. $\square$