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# Continuous Capacities on Continuous State Spaces*

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**Abstract.** We propose axiomatizing some stochastic games, in a continuous state space setting, using continuous belief functions, resp. plausibilities, instead of measures. Then, stochastic games are just variations on continuous Markov chains. We argue that drawing at random along a belief function is the same as letting the probabilistic player $\mathsf{P}$ play first, then letting the non-deterministic player $\mathsf{C}$ play demonically. The same holds for an angelic $\mathsf{C}$, using plausibilities instead. We then define a simple modal logic, and characterize simulation in terms of formulae of this logic. Finally, we show that (discounted) payoffs are defined and unique, where in the demonic case, $\mathsf{P}$ maximizes payoff, while $\mathsf{C}$ minimizes it.

## 1 Introduction

The theory of Markov chains is well established. Such a system evolves from state $x \in X$ by drawing the next state $y$ in the state space $X$ according to some probability distribution $\sigma(x)$. One may enrich this model to take into account decisions made by a *player* $\mathsf{P}$, which can take actions $\ell$ in some set $L$. In state $x \in X$, $\mathsf{P}$ chooses an action $\ell \in L$, and draws the next state $y$ according to a probability distribution $\sigma_\ell(x)$ depending on $\ell \in L$: these are *labeled Markov processes* (LMPs) [8]. Adding rewards $r_\ell(x)$ on taking action $\ell$ from state $x$ yields *Markov decision processes* [10]. The main topic there is to evaluate strategies that maximize the expected payoff, possibly discounted.

These notions have been generalized in many directions. Consider stochastic games, where there is not one but several players, with different goals. In security protocols, notably, it is meaningful to assume that the honest agents collectively define a player $\mathsf{P}$ as above, who may play probabilistically, and that attackers define a second player $\mathsf{C}$, who plays *non-deterministically*. Instead of drawing the next state at random, $\mathsf{C}$ deliberately chooses its next state, typically to minimize $\mathsf{P}$'s expected payoff or to maximize the probability that a bad state is reached—this is *demonic* non-determinism.

A nice idea of F. Laviolette and J. Desharnais (private comm., 2003), which we develop, is that the theory of these games could be simplified by relaxing the requirements of Markov chains: if $\nu = \sigma_\ell(x)$ is not required to be a measure, but the additivity requirement is relaxed to sub-additivity (i.e., $\nu(A)+\nu(B) \leq \nu(A \cup B)$ for disjoint measurable sets $A, B$), then such "preprobabilities" include both ordinary probabilities and the following funny-looking *unanimity game* $\mathfrak{u}_A$, which represents the demonic non-deterministic choice of an element from the set $A$: the preprobability $\mathfrak{u}_A(B)$ of drawing

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an element in $B$ is 1 if $A \subseteq B$, 0 otherwise. The intuition is as follows. Assume that, starting from state $x$, you would like the next state $y$ to be in $B$. A demonic adversary $\mathsf{C}$ will then strive to pick $y$ outside $B$. Now if $\mathsf{C}$'s moves are given by $\delta_\ell(x) = \mathfrak{u}_A$, then either $A \not\subseteq B$, then it is $\mathsf{C}$'s interest to pick $y$ from $A \setminus B$, so that the preprobability that $y$ be in $B$ is 0; or $A \subseteq B$, then $\mathsf{C}$ is forced to play $y \in B$, and the preprobability is 1.

However, sub-additive set functions are not quite the right notion; and second (which does not detract from F. Laviolette and J. Desharnais' great intuition), the right notions had been invented by economists in the 1950s under the name of "cooperative game with transferable utility" [21] and by statisticians in the 1960s under the names of belief functions and plausibilities, while capacities and Choquet integration are even more ancient [4]. A nice survey is [12]. These notions are well-known in discrete state spaces. Our generalization to topological spaces is new, and non-trivial. The spaces we consider include finite spaces as well as infinite ones such as $\mathbb{R}^n$, but also cpos and domains.

**Outline.** We introduce necessary mathematical notions in Section 2. We then develop the theory of continuous games, and continuous belief functions in particular in Section 3, showing in a precise sense how the latter model both probabilistic and demonic non-deterministic choice. We then recall the Choquet integral in Section 4, and show how taking averages reflects the fact that $\mathsf{C}$ aims at minimizing $\mathsf{P}$'s gains. We briefly touch the dual notion of plausibilities (angelic non-determinism) in passing. Finally, we define ludic transition systems, the analogue of Markov chains, except using continuous games, in Section 5, and define a notion of simulation topologies. We show that the coarsest simulation topology is exactly that defined by a simple modal logic, à la Larsen-Skou [18]. This illustrates how continuous games allow us to think of certain stochastic games as really being just LMPs, only with a relaxed notion of probability.

This work is a summary of most of Chapters 1-9 of [13], in which all proofs, and many more results can be found.

**Related Work.** Many models of Markov chains or processes, or stochastic games are discrete or even finite-state. Desharnais *et al.* [8] consider LMPs over *analytic* spaces, a class of topological spaces that includes not only finite spaces but also spaces such as $\mathbb{R}^n$. They show an extension of Larsen and Skou's Theorem [18]: two states are probabilistically bisimilar iff they satisfy the same formulae of the logic whose formulae are $F ::= \top | F \wedge F | [\ell]_{>r} F$, where $[\ell]_{>r} F$ is true at those states $x$ where the probability $\sigma_\ell(x)(\llbracket F \rrbracket_\sigma)$ of going to some state satisfying $F$ by doing action $\ell$ is greater than $r$. This is extended to any measurable space through *event bisimulations* in [5].

Mixing probabilistic (player $\mathsf{P}$) and non-deterministic ($\mathsf{C}$) behavior has also received some attention. This is notably at the heart of the *probabilistic I/O automata* of Segala and Lynch [24]. The latter can be seen as labeled Markov processes with discrete probability distributions $\sigma_\ell(x)$ (i.e., linear combinations of Dirac masses), where the set $L$ of actions is partitioned into internal (hidden) actions and external actions. While $\mathsf{P}$ controls the latter, the former represent non-deterministic transitions, i.e., under the control of $\mathsf{C}$. Our model of stochastic games is closer to the strictly alternating variant of probabilistic automata, where at each state, a non-deterministic choice is made among several distributions, then the next state is drawn at random according to the chosen distribution. I.e., $\mathsf{C}$ plays, then $\mathsf{P}$, and there is no intermediate state where $\mathsf{C}$ would have played but not $\mathsf{P}$. This is similar to the model by Mislove *et al.* [20], who consider state spaces that are continuous cpos. In our model, this is the other way around: in each state, $\mathsf{P}$ draws at random a possible choice set for $\mathsf{C}$, who then picks non-deterministically from it. Additionally, our model accommodates state spaces that are discrete, or continuous cpos, or topological spaces such as $\mathbb{R}^n$, without any change to be made. Mislove *et al.* [20] consider a model where non-determinism is chaotic, i.e., based on a variant of Plotkin's powerdomain. We concentrate on demonic non-determinism, which is based on the Smyth powerdomain instead. For angelic non-determinism, see [13, chapitre 6], and [13, chapitre 7] for chaotic non-determinism.

Bisimulations have been studied in the above models. There are many variants on probabilistic automata [25, 15, 22]. Mislove *et al.* [20] show that (bi)simulation in their model is characterized by a logic similar to [8], with an added disjunction operator. Our result is similar, for a smaller logic, with one less modality. Segala and Turrini [26] compare various notions of bisimulations in these contexts.

We have already mentioned cooperative games and belief functions. See the abundant literature [6, 7, 27, 12, 23, 2]. We view belief functions as generalized probabilities; the competing view as a basis for a theory of evidence is incompatible [14].

An obvious approach to studying probabilistic phenomena is to turn to measure theory and measurable spaces, see e.g. [3]. However, we hope to demonstrate that the theory of cooperative games in the case of infinite state spaces $X$ is considerably more comfortable when $X$ is a topological space, and we only measure opens instead of Borel subsets. This is in line with the theory of continuous valuations [16], which has had considerable success in semantics.

We use Choquet integration to integrate along capacities $\nu$ [4]. This is exactly the notion that Tix [28] used more recently, too, and coincides with the Jones integral [16] for integration along continuous valuations. Finally, we should also not that V. Danos and M. Escardo have also come up (private comm.) with a notion of integration that generalizes Choquet integration, at least when integrating with respect to a convex game.

## 2 Preliminaries

Our state spaces $X$ are topological spaces. We assume the reader to be familiar with (point-set) topology, in particular topology of $T_0$ but not necessarily Hausdorff spaces. See [11, 1, 19] for background. Let $\mathring{A}$ denote the interior of $A$, $cl(A)$ its closure.

The *Scott topology* on a poset $X$, with ordering $\leq$, has as opens the upward-closed subsets $U$ (i.e., $x \in U$ and $x \leq y$ imply $y \in U$) such that for every directed family $(x_i)_{i \in I}$ having a least upper bound $\sup_{i \in I} x_i$ inside $U$, some $x_i$ is already in $U$. The *way-below* relation $\ll$ is defined by $x \ll y$ iff for any directed family $(z_i)_{i \in I}$ with a least upper bound $z$ such that $y \leq z$, then $x \leq z_i$ for some $i \in I$. A poset is *continuous* iff $\twoheaddownarrow y = \{x \in X | x \ll y\}$ is directed, and has $x$ as least upper bound. Then every open $U$ can be written $\bigcup_{x \in U} \twoheaduparrow x$, where $\twoheaduparrow x = \{y \in X | x \ll y\}$.

Every topological space $X$ has a specialization quasi-ordering $\leq$, defined by: $x \leq y$ iff every open that contains $x$ contains $y$. $X$ is $T_0$ iff $\leq$ is a (partial) ordering. That of the Scott topology of a quasi-ordering $\leq$ is $\leq$ itself. A subset $A \subseteq X$ is *saturated* if and only if $A$ is the intersection of all opens that contain it; alternatively, iff $A$ is upward-closed in $\leq$. Every open is upward-closed. Let $\uparrow A$ denote the upward-closure of $A$ under a quasi-ordering $\leq$, $\downarrow A$ its downward-closure. A $T_0$ space is *sober* iff every irreducible closed subset is the closure $cl\{x\} = \downarrow x$ of a (unique) point $x$. The Hofmann-Mislove Theorem implies that every sober space is *well-filtered* [17]: given any filtered family of saturated compacts $(Q_i)_{i \in I}$ in $X$, and any open $U$, $\bigcap_{i \in I} Q_i \subseteq U$ iff $Q_i \subseteq U$ for some $i \in I$. In particular, $\bigcap_{i \in I} Q_i$ is saturated compact. $X$ is *locally compact* iff whenever $x \in U$ ($U$ open) there is a saturated compact $Q$ such that $x \in Q \subseteq Q \subseteq U$. Every continuous cpo is sober and locally compact in its Scott topology. We shall consider the space $\mathbb{R}$ of all reals with the Scott topology of its natural ordering $\leq$. Its opens are $\emptyset$, $\mathbb{R}$, and the intervals $(t, +\infty)$, $t \in \mathbb{R}$. $\mathbb{R}$ is a stably locally compact, continuous cpo. Because we equip $\mathbb{R}$ with the Scott topology, our *continuous* functions $f : X \to \mathbb{R}$ are those which are usually called *lower semi-continuous* in the mathematical literature.

We call *capacity* on $X$ any function $\nu$ from $\mathcal{O}(X)$, the set of all opens of $X$, to $\mathbb{R}^+$, such that $\nu(\emptyset) = 0$ (a.k.a., a *set function*.) A *game* $\nu$ is a *monotonic capacity*, i.e., $U \subseteq V$ implies $\nu(U) \leq \nu(V)$[1]. A *valuation* is a *modular* game $\nu$, i.e., one such that $\nu(U \cup V) + \nu(U \cap V) = \nu(U) \cap \nu(V)$ for every opens $U, V$. A game is *continuous* iff $\nu(\bigcup_{i \in I} U_i) = \sup_{i \in I} \nu(U_i)$ for every directed family $(U_i)_{i \in I}$ of opens. Continuous valuations have a convenient theory that fits topology well [16, 17].

The *Dirac valuation* $\delta_x$ at $x \in X$ is the continuous valuation mapping each open $U$ to 1 if $x \in U$, to 0 otherwise. (Note that $\delta_x = \mathfrak{u}_{\{x\}}$, by the way.) A finite linear combination $\sum_{i=1}^n a_i \delta_{x_i}$, $a_i \in \mathbb{R}^+$, is a *simple valuation*. All simple valuations are continuous. Conversely, Jones' Theorem [16, Theorem 5.2] states that, if $X$ is a continuous cpo, then every continuous valuation $\nu$ is the least upper bound $\sup_{i \in I} \nu_i$ of a directed family $(\nu_i)_{i \in I}$ of simple valuations way-below $\nu$. Continuous valuations are canonically ordered by $\nu \leq \nu'$ iff $\nu(U) \leq \nu'(U)$ for every open $U$ of $X$.

## 3 Continuous Games, and Belief Functions

Defining the "preprobabilities" alluded to in the introduction is best done by strengthening super-additivity. A game $\nu$ on $X$ on $X$ is *convex* iff $\nu(U \cup V) + \nu(U \cap V) \geq \nu(U) \cap \nu(V)$ for every opens $U, V$. It is *concave* if the opposite inequality holds. Convex games are one of the cornerstones of economic theory. E.g., Shapley's celebrated Theorem states that (on a finite space) every convex game has a core, which implies the existence of economic equilibria [12, 21]. However, this notion has only been studied on finite spaces (an implicit assumption in [12], notably). Finite, and more generally discrete spaces will be the particular case where $X$ is equipped with the discrete topology, so one may see our topological approach as a generalization of previous approaches.

Clearly, $\mathfrak{u}_A$ is convex. It is in fact more. A game $\nu$ is *totally convex*[2] iff:

$$\nu\left(\bigcup_{i=1}^n U_i\right) \geq \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} U_i\right) \quad (1)$$

for every finite family $(U_i)_{i=1}^n$ of opens ($n \geq 1$), where $|I|$ denotes the cardinality of $I$.

[1] The name "game" is unfortunate, as there is no obvious relationship between this and games as they are usually handled in computer science, in particular with stochastic games. The notion stems from (cooperative) games in economics, where $X$ is the set of players, not of states.

[2] The standard name, when $X$ is discrete, i.e., when $U_i$ is an arbitrary subset of $X$, is "totally monotonic". We changed the name so as to name total concavity the dual of total monotonicity.

A *belief function* is a totally convex game. (The dual notion of *total concavity* is obtained by replacing $\bigcup$ by $\bigcap$ and conversely in (1), and turning $\geq$ into $\leq$. A *plausibility* is a totally concave game.) One checks that $\mathfrak{u}_A$ is totally convex. If $\geq$ is replaced by $=$ in (1), then we retrieve the familiar inclusion-exclusion principle from statistics. In particular any (continuous) valuation is a (continuous) belief function. Clearly, any belief function is a convex game. The converses of both statements fail: On $X = \{1,2,3\}$ with the discrete topology, $\mathfrak{u}_{\{1,2\}}$ is a belief function but not a valuation, and $\frac{1}{2}(\mathfrak{u}_{\{1,2\}} + \mathfrak{u}_{\{1,3\}} + \mathfrak{u}_{\{2,3\}} - \mathfrak{u}_{\{1,2,3\}})$ is a convex game but not a belief function.

When $X$ is finite, it is well-known [12] that every capacity $\nu$ can be written $\sum_{A\neq\emptyset, A\subseteq X} \alpha_A \mathfrak{u}_A$ for some coefficients $\alpha_A \in \mathbb{R}$, in a unique way. Also, $\nu$ is a belief function iff all coefficients are non-negative. An interpretation of this formula is that $\nu$ is essentially a probabilistic choice of some non-empty subset $A$, with probability $\alpha_A$, from which $\mathsf{C}$ can choose an element $y \in A$ non-deterministically.

Our first result is to show that this result transfers, in some form, to the general topological case. Let $\mathcal{Q}(X)$ be the *Smyth powerdomain* of $X$, i.e., the space of all non-empty compact saturated subsets $Q$ of $X$, ordered by reverse inclusion $\supseteq$. $\mathcal{Q}(X)$ is equipped with its Scott topology, and is known to provide an adequate model of demonic non-determinism in semantics [1]. When $X$ is well-filtered and locally compact, $\mathcal{Q}(X)$ is a continuous cpo. Its Scott topology is generated by the basic open sets $\Box U = \{Q \in \mathcal{Q}(X) | Q \subseteq U\}$, $U$ open in $X$.

The relevance of $\mathcal{Q}(X)$ here can be obtained by realizing that a finite linear combination $\sum_{i=1}^n a_i \mathfrak{u}_{A_i}$ with positive coefficients is a *continuous* belief function iff every subset $A_i$ is compact (Proposition 3, see Appendix); and that $\mathfrak{u}_{A_i} = \mathfrak{u}_{\uparrow A_i}$. Any such linear combination that is continuous is therefore of the form $\sum_{i=1}^n a_i \mathfrak{u}_{Q_i}$, with $Q_i \in \mathcal{Q}(X)$. We call such belief functions *simple*. Returning to the interpretation above, this can be intuitively seen as a probabilistic choice of some set $Q_i$ with probability $a_i$, from which $\mathsf{C}$ will choose $y \in Q_i$; additionally, $Q_i$ is an element of $\mathcal{Q}(X)$, the traditional domain for *demonic* non-determinism.

So any simple belief function $\nu$ can be matched with a (simple) valuation $\nu^* = \sum_{i=1}^n a_i \delta_{Q_i}$ on $\mathcal{Q}(X)$. Note that $\nu^*(\Box U) = \nu(U)$ for every open $U$ of $X$. This is exactly the sense in which continuous belief functions are essentially continuous valuations on the space $\mathcal{Q}(X)$ of non-deterministic choices (see the Appendix for proof):

**Theorem 1.** *For any continuous valuation $P$ on $\mathcal{Q}(X)$, the capacity $\nu$ defined by $\nu(U) = P(\Box U)$ is a continuous belief function on $X$.*

*Conversely, let $X$ be a well-filtered and locally compact space. For every continuous belief function $\nu$ on $X$ there is a unique continuous valuation $\nu^*$ on $\mathcal{Q}(X)$ such that $\nu(U) = \nu^*(\Box U)$ for every open $U$ of $X$.*

Next, we show that this bijection is actually an isomorphism, i.e., it also preserves order and therefore the Scott topology. To this end, define the ordering $\leq$ on all capacities, not just valuations, by $\nu \leq \nu'$ iff $\nu(U) \leq \nu'(U)$ for every open $U$ of $X$. We start by characterizing it in the manner of Jones' splitting lemma. This [16, Theorem 4.10] states that $\sum_{i=1}^m a_i \delta_{x_i} \leq \sum_{j=1}^n b_j \delta_{y_j}$ iff there is matrix $(t_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$ of coefficients in $\mathbb{R}^+$ such that $\sum_{j=1}^{n} t_{ij} = a_i$ for each $i$, $\sum_{i=1}^{m} t_{ij} \leq b_j$ for each $j$, and whenever $t_{ij} \neq 0$ then $x_i \leq y_j$. (Jones proves it for cpos, but it works on any topological space.) We show:

**Lemma 1 (Splitting Lemma).** $\sum_{i=1}^{m} a_i \mathfrak{u}_{Q_i} \leq \sum_{j=1}^{n} b_j \mathfrak{u}_{Q'_j}$ *iff there is matrix* $(t_{ij})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$ *of coefficients in* $\mathbb{R}^+$ *such that* $\sum_{j=1}^{n} t_{ij} = a_i$ *for each* $i$, $\sum_{i=1}^{m} t_{ij} \leq b_j$ *for each* $j$, *and whenever* $t_{ij} \neq 0$ *then* $Q_i \leq Q'_j$.

It follows that: (A) for any two simple belief functions $\nu$, $\nu'$ on $X$, $\nu \leq \nu'$ iff $\nu^* \leq \nu'^*$, since the two are equivalent to the existence of a matrix $(t_{ij})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$ satisfying the same conditions. This can be extended to all continuous belief functions, see below. Let $\mathbf{Cd}_{\leq 1}(X)$ be the space of continuous belief functions $\nu$ on $X$ with $\nu(X) \leq 1$, ordered by $\leq$. Let $\mathbf{V}_{\leq 1}(X)$ the subspace of continuous valuations. We have:

**Theorem 2.** *Let $X$ be well-filtered and locally compact. Every continuous belief function $\nu$ on $X$ is the least upper bound of a directed family of simple belief functions $\nu_i$ way-below $\nu$.* $\mathbf{Cd}_{\leq 1}(X)$ *is a continuous cpo.*

It follows that continuous belief functions are really the same thing as (sub-)probabilities over the set of demonic choice sets $Q \in \mathcal{Q}(X)$.

**Theorem 3.** *Let $X$ be well-filtered and locally compact. The function $\nu \mapsto \nu^*$ defines an order-isomorphism from* $\mathbf{Cd}_{\leq 1}(X)$ *to* $\mathbf{V}_{\leq 1}(\mathcal{Q}(X))$.

As a side note, (up to the $\leq 1$ subscript) $\mathbf{V}_{\leq 1}(\mathcal{Q}(X))$ is exactly the space into which Edalat [9] embeds a space of measures on $X$. The above Theorem states that the space of objects for which we can do this is exactly $\mathbf{Cd}_{\leq 1}(X)$.

Dually, we may mix probabilistic choice with angelic non-determinism. Space does not permit us to describe this in detail, see [13, chapitre 6]. The point is that the space $\mathbf{Pb}_{\leq 1}(X)$ of continuous plausibilities is order-isomorphic to $\mathbf{V}_{\leq 1}(\mathcal{H}_u(X))$, whenever $X$ is stably locally compact, where the (topological) *Hoare powerdomain* $\mathcal{H}_u(X)$ of $X$ is the set of non-empty closed subsets of $X$, with the upper topology of the inclusion ordering, generated by the subbasic sets $\Diamond U = \{F \in \mathcal{H}(X) | F \cap U \neq \emptyset\}$, $U$ open in $X$. The argument goes through a nice notion of convex-concave duality, which intuitively exchanges good (concave) and evil (convex). The case of chaotic non-determinism is more complex, see [13, chapitre 7].

## 4 Choquet Integration

We introduce the standard notion of integration along games $\nu$. This is mostly well-known [12], and adapting to the topological case is easy, so we omit proofs [13, chapitre 4].

Let $\nu$ be a game on $X$, and $f$ be continuous from $X$ to $\mathbb{R}$. Recall that we equip $\mathbb{R}$ with its Scott topology, so that $f$ is really what is known otherwise as *lower semi-continuous*. Assume $f$ bounded, too, i.e., $\inf_{x\in X} f(x) > -\infty$, $\sup_{x\in X} f(x) < +\infty$. The *Choquet integral* of $f$ along $\nu$ is:

$$\oint^{C}_{x\in X} f(x) d\nu = \int_{0}^{+\infty} \nu(f^{-1}(t, +\infty)) dt + \int_{-\infty}^{0} [\nu(f^{-1}(t, +\infty)) - \nu(X)] dt \quad (2)$$

where both integrals on the right are improper Riemann integrals. This is well-defined, since $f^{-1}(t, +\infty)$ is open for every $t \in \mathbb{R}$ by assumption, and $\nu$ measures opens. Also, since $f$ is bounded, the improper integrals above really are ordinary Riemann integrals over some closed intervals. The function $t \mapsto \nu(f^{-1}(t, +\infty))$ is decreasing, and every decreasing (even non-continuous, in the usual sense) function is Riemann-integrable, therefore the definition makes sense.

An alternate definition consists in observing that any *step function* $\sum_{i=0}^{n} a_i \chi_{U_i}$, where $a_0 \in \mathbb{R}$, $a_1, \ldots, a_n \in \mathbb{R}^+$, $X = U_0 \supseteq U_1 \supseteq \ldots \supseteq U_n$ is a decreasing sequence of opens, and $\chi_U$ denotes the indicator function of $U$ ($\chi_U(x) = 1$ if $x \in X$, $\chi_U(x) = 0$ otherwise) is continuous, and of integral along $\nu$ equal to $\sum_{i=0}^{n} a_i \nu(U_i)$—for *any* game $\nu$. It is well-known that every bounded continuous function $f$ can be written as the least upper bound of a sequence of step functions $f_K = a + \frac{1}{2^K} \sum_{k=1}^{\lfloor (b-a)2^K \rfloor} \chi_{f^{-1}(a+\frac{k}{2^K}, +\infty)}(x)$, $K \in \mathbb{N}$, where $a = \inf_{x \in X} f(x)$, $b = \sup_{x \in X} f(x)$. Then the integral of $f$ along $\nu$ is the least upper bound of the increasing sequence of the integrals of $f_K$ along $\nu$.

The main properties of Choquet integration are as follows. First, the integral is increasing in its function argument: if $f \leq g$ then the integral of $f$ along $\nu$ is less than or equal to that of $g$ along $\nu$. If $\nu$ is continuous, then integration is also Scott-continuous in its function argument. The integral is also monotonic and Scott-continuous in the game $\nu$, provided the function we integrate takes only non-negative values, or provided $\nu$ is *normalized*, i.e., $\nu(X) = 1$. Integration is linear in the game, too, so integrating along $\sum_{i=1}^{n} a_i \nu_i$ is the same as taking the integrals along each $\nu_i$, and computing the obvious linear combination. However, Choquet integration is *not* linear in the function integrated, unless the game $\nu$ is a valuation. Still, it is *positively homogeneous*: integrating $\alpha f$ for $\alpha \in \mathbb{R}^+$ yields $\alpha$ times the integral of $f$. It is additive on *comonotonic* functions $f, g : X \to \mathbb{R}$ (i.e., there is no pair $x, x' \in X$ such that $f(x) < f(x')$ and $g(x) > g(x')$). It is super-additive (the integral of $f + g$ is at least that of $f$ plus that of $g$) when $\nu$ is convex, in particular when $\nu$ is a belief function, and sub-additive when $\nu$ is concave. See [12] for the finite case, [13, chapitre 4] for the topological case.

One of the most interesting things is that integrating with respect to a unanimity game consists in taking minima. This suggests that unanimity games indeed model some *demonic* form of non-determinism. Imagine $f(x)$ is the amount of money you gain by going to state $x$. The following says that taking the average amount of money with respect to a demonic adversary $\mathsf{C}$ will give you back the least amount possible.

**Proposition 1.** *For any continuous $f : X \to \mathbb{R}^+$,*

$$\oint_{x \in X} f(x) d\mathfrak{u}_A = \inf_{x \in A} f(x)$$

*Moreover, if $A$ is compact, then the inf is attained: this equals $\min_{x \in A} f(x)$.*

Since Choquet integration is linear in the game, the integral of $f$ along a simple belief function $\sum_{i=1}^{n} a_i \mathfrak{u}_{Q_i}$ yields $\sum_{i=1}^{n} a_i \min_{x \in Q_i} f(x)$: this is the expected min-value of $f$ obtained by drawing $Q_i$ at random with probability $a_i$ ($\mathsf{P}$ plays) then letting $\mathsf{C}$ non-deterministically move to the state $x \in Q_i$ that minimizes the gain. We can generalize this to non-discrete probabilities over $\mathcal{Q}(X)$ by using the $\nu \mapsto \nu^*$ isomorphism:

**Theorem 4.** *For any bounded continuous function $f : X \to \mathbb{R}$, let $f_*$ be the function from $\mathcal{Q}(X)$ to $\mathbb{R}$ defined by $f_*(Q) = \min_{x \in Q} f(x)$. Say that a capacity $\nu$ is* linearly extensible from below *if and only if there is continuous valuation $P$ on $\mathcal{Q}(X)$ with:*

$$\oint_{x \in X} f(x) d\nu = \oint_{Q \in \mathcal{Q}(X)} f_*(Q) dP \tag{3}$$

*for every bounded continuous $f$. If $X$ is well-filtered and locally compact, then the capacities that are linearly extensible from below are exactly the continuous belief functions, and $P$ must be $\nu^*$ in (3).*

It follows in particular that whenever $\nu$ is the least upper bound of a directed family $(\nu_i)_{i \in I}$ of simple belief functions $\nu_i$, then integrating $f : X \to \mathbb{R}$ with respect to $\nu$ can be computed by taking least upper bounds of linear combinations of mins. Therefore the Choquet integral along continuous belief functions coincides with Edalat's *lower R-integral* [9], which was only defined for measures.

This can be dualized to the case of plausibilities $\nu$, assuming $X$ stably locally compact [13, théorème 6.3.17]. Then we talk about capacities that are linearly extensible from *above*. There is an isomorphism $\nu \mapsto \nu_*$ such that $\nu_*(\Diamond U) = \nu(U)$ for all $U$, and integrating $f$ along $\nu$ amounts to integrating $f^*$ along $\nu_*$, where for every $F \in \mathcal{H}_u(X)$, $f^*(F) = \sup_{x \in F} f(x)$. (I.e., $\complement$ now *maximizes* our gain.) Then the Choquet integral along continuous plausibilities coincides with Edalat's *upper R-integral* [9].

# 5 Ludic Transition Systems, Logic, Simulation, Rewards

Let $\mathbf{J}_{\leq 1}(X)$ be the space of all continuous games $\nu$ on $X$ with $\nu(X) \leq 1$. This is equipped with its Scott topology. It will be practical to consider another topology. The *weak topology* on a subspace $Y$ of $\mathbf{J}_{\leq 1}(X)$ is the topology generated by the subbasic open sets $[U > r] = \{\nu \in Y | \nu(U) > r\}$, $U$ open in $X$, $r \in \mathbb{R}$. It is in general coarser than the Scott topology, and coincides with it when $Y = \mathbf{V}_{\leq 1}(X)$ and $X$ is a continuous cpo [28, Satz 4.10]. One can show that the weak topology is exactly the coarsest that makes continuous all functionals mapping $\nu \in Y$ to the integral of $f$ along $\nu$, for all $f : X \to \mathbb{R}^+$ bounded continuous. (See [13, section 4.5] for details.)

By analogy with Markov kernels and LMPs, define a *ludic transition system* as a family $\sigma = (\sigma_\ell)_{\ell \in L}$, where $L$ is a given set of *actions*, and each $\sigma_\ell$ is a continuous map from the state space $X$ to $\mathbf{J}_{\leq 1\ wk}(X)$. (See [13, chapitres 8, 9] for missing details.) The main change is that, as announced in the introduction, we replace probability distributions by continuous games. One may object that LMPs are defined as *measurable*, not continuous, so that this definition overly restricts the class of transition systems we are considering. This is certainly a shortcoming of our theory. However, the mathematics are considerably cleaner when assuming continuity. Moreover, the weak topology is so weak that, for example, it only restrains $\sigma_\ell$ so that $x \mapsto \sigma_\ell(x)(U)$ is continuous as a function from $X$ to $\mathbb{R}^+$, equipped with its *Scott* topology; this certainly allows it to have jumps. Finally, one may argue, following Edalat [9], that any second countable locally compact Hausdorff space $X$ can be embedded as a set of maximal elements of a continuous cpo (namely $\mathcal{Q}(X)$; other choices are possible) so that any measure on $X$ extends to a continuous valuation on $\mathcal{Q}(X)$. This provides a theory of approximation of integration on $X$ through domain theory. One may hope a similar phenomenon will apply to games—for some notion of games yet to be defined on Borel subsets, not opens.

**Logic.** Following [8, 5], define the logic $\mathcal{L}^{\top\wedge\vee}_{\text{open}}$ by the grammar shown right, where $\ell \in L$, $r \in \mathbb{Q} \cap [0,1]$ in the last line. Compared to [8, 5], we only have one extra disjunction operator.

| | | |
|---|---|---|
| $F ::=$ | $\top$ | true |
| $\mid$ | $F \wedge F$ | conjunction (and) |
| $\mid$ | $F \vee F$ | disjunction (or) |
| $\mid$ | $[\ell]_{>r} F$ | modality |

Let $[\![F]\!]_\sigma$ be the set of states $x \in X$ where $F$ holds: $[\![\top]\!]_\sigma = X$, $[\![F_1 \wedge F_2]\!]_\sigma = [\![F_1]\!]_\sigma \wedge [\![F_2]\!]_\sigma$, $[\![F_1 \vee F_2]\!]_\sigma = [\![F_1]\!]_\sigma \vee [\![F_2]\!]_\sigma$, and $[\![[\ell]_{>r}F]\!]_\sigma = \delta_\ell^{-1}[[\![F]\!]_\sigma > r]$ is the set of states $x$ such that the preprobability $\delta_\ell([\![F]\!]_\sigma)$ that the next state $y$ will satisfy $F$ on firing an $\ell$ action is strictly greater than $r$. Note that this is well-defined, precisely because $\delta_\ell$ is continuous from $X$ to a space of games with the weak topology. Also, it is easy to see that $[\![F]\!]_\sigma$ is always open.

**Simulation.** Now define *simulation* in the spirit of event bisimulation [5] (we shall see below why we do not call it *bi*simulation). For any topology $\mathcal{O}$ on $X$ coarser than that of $X$, let $X : \mathcal{O}$ be $X$ equipped with the topology $\mathcal{O}$. A *simulation topology* for $\sigma$ is a topology $\mathcal{O}$ on $X$, coarser than that of $X$, such that $\delta_\ell$ is continuous from $X : \mathcal{O}$ to $\mathbf{J}_{\leq 1\ wk}(X : \mathcal{O})$, i.e., $\delta_\ell^{-1}[U > r] \in \mathcal{O}$ for each $U \in \mathcal{O}$ and each $r \in \mathbb{R}$. One non-explanation for this definition is to state that this is exactly event bisimulation [5], only replacing $\sigma$-algebras by topologies. A better explanation is to revert back to Larsen and Skou's original definition of probabilistic bisimulation in terms of an algebra of *tests* (in slightly more abstract form). A (bi)simulation should not be thought as an arbitrary equivalence relation, rather as one generated from a collection $Tst$ of tests, which are subsets $A$ of $X$: $x \in X$ passes the test iff $x \in A$, it fails it otherwise. Two elements are equivalent iff they pass the same tests. Now in a continuous setting it only makes sense that the tests be open: any open $U$ defines a continuous predicate $\chi_U$ from $X$ to the Sierpiński space $\mathbb{S} = \{0,1\}$ (with the Scott topology of $0 \leq 1$), and conversely. Let $\mathcal{O}_{Tst}$ be the topology generated by the tests $Tst$. It is sensible to require that $\delta_\ell^{-1}[U > r]$ be a test, too, at least when $U$ is a finite union of finite intersections of tests (for the general case, appeal to the fact that $\delta_\ell(x)$ is continuous, and that any open can be approximated by such a finite union): one can indeed test whether $x \in \delta_\ell^{-1}[U > r]$ by firing transitions according to the preprobability $\delta_\ell(x)$, and test (e.g., by sampling, knowing that if $\delta_\ell(x)$ is a belief function for example, then we are actually playing also against a demonic adversary $\mathsf{C}$) whether our chances of getting to a state $y \in U$ exceed $r$. And this is essentially how we defined simulation topologies.

Every simulation topology $\mathcal{O}$ defines a specialization quasi-ordering $\preceq_\mathcal{O}$, which is the analogue of the standard notion of simulation here. (Note that in the case of event bisimulation, i.e., taking $\sigma$-algebras instead of topologies, $\preceq_\mathcal{O}$ would be an equivalence relation—because $\sigma$-algebras are closed under complements—justifying the fact that event bisimulation really is a bisimulation, while our notion is a simulation.) Write $\equiv_\mathcal{O} = \preceq_\mathcal{O} \cap \succeq_\mathcal{O}$ the equivalence associated with simulation $\preceq_\mathcal{O}$. Clearly, there is a coarsest (largest) simulation topology $\mathfrak{O}_\sigma$. The following is then easy:

**Theorem 5.** *Let $\mathcal{O}$ be a simulation topology for $\sigma$ on $X$. For any $F \in \mathcal{L}_{open}$, $[\![F]\!]_\sigma \in \mathcal{O}$. In particular [Soundness], if $x \in [\![F]\!]_\sigma$ and $x \preceq_\mathcal{O} y$ then $y \in [\![F]\!]_\sigma$. Conversely [Completeness], the coarsest simulation topology $\mathfrak{O}_\sigma$ is exactly that generated by the opens $[\![F]\!]_\sigma$, $F \in \mathcal{L}^{\top\wedge\vee}_{open}$.*

This can be used, as is standard in the theory of Markov chains, to *lump* states. Given a topology $\mathcal{O}$, let $X/\mathcal{O}$ be the quotient space $X/{\equiv_{\mathcal{O}}}$, equipped with the finest topology such that $q_{\mathcal{O}} : X : \mathcal{O} \to X/\mathcal{O}$ is continuous. Let the *direct image* $f[\nu]$ of a game $\nu$ on $X$ by a continuous map $f : X \to Y$ be $f[\nu](V) = \nu(f^{-1}(V))$. Taking direct images preserves monotonicity, modularity, (total) convexity, (total) concavity, and continuity.

**Proposition 2.** *Let $\mathcal{O}$ be a simulation topology for $\sigma$. The function $\sigma_\ell/\mathcal{O}$ mapping $q_{\mathcal{O}}(x)$ to $q_{\mathcal{O}}[\sigma_\ell(x)]$ is well defined and continuous from $X/\mathcal{O}$ to $\mathbf{J}_{\leq 1\ wk}(X/\mathcal{O})$ for every $\ell \in L$. The family $\sigma/\mathcal{O} = (\sigma_\ell/\mathcal{O})_{\ell \in L}$ is then a ludic transition system on $X/\mathcal{O}$, which we call the* lumped *ludic transition system.*

*For any $F \in \mathcal{L}^{\top\wedge\vee}_{open}$ and $x \in X$, $x$ and $q_{\mathcal{O}}(x)$ satisfy the same formulae: $q_{\mathcal{O}}(\llbracket F \rrbracket_\sigma) = \llbracket F \rrbracket_{\sigma/\mathcal{O}}$, and $\llbracket F \rrbracket_\sigma = q_{\mathcal{O}}^{-1}(\llbracket F \rrbracket_{\sigma/\mathcal{O}})$, in particular, $x \in \llbracket F \rrbracket_\sigma$ iff $q_{\mathcal{O}}(x) \in \llbracket F \rrbracket_{\sigma/\mathcal{O}}$.*

**Rewards and payoffs.** A classical problem on Markov decision processes is to evaluate average payoffs. Since LMPs and ludic transition systems are so similar, we can do exactly the same. Imagine P plays according to a finite-state program $\Pi$, i.e., an automaton with *internal states* $q, q'$ and transitions $q \stackrel{\ell}{\longrightarrow} q'$. Let $r_{q \stackrel{\ell}{\longrightarrow} q'} : X \to \mathbb{R}$ be a family of bounded continuous *reward* functions: we may think that $r_{q \stackrel{\ell}{\longrightarrow} q'}(x)$ is the amount of money P gains if she fires her internal transition $q \stackrel{\ell}{\longrightarrow} q'$, drawing the next state $y$ at random along $\sigma_\ell(x)$. Let $\gamma_{q \stackrel{\ell}{\longrightarrow} q'} \in (0, 1]$ be a family of so-called *discounts*. Define the average payoff, starting from state $x$ when P is in its internal state $q$, by:

$$V_q(x) = \sup_{\ell, q'/q \stackrel{\ell}{\longrightarrow} q'} \left[ r_{q \stackrel{\ell}{\longrightarrow} q'}(x) + \gamma_{q \stackrel{\ell}{\longrightarrow} q'} \oint^{\mathcal{C}}_{y \in X} V_{q'}(y) d\sigma_\ell(x) \right] \tag{4}$$

This formula would be standard if $\sigma_\ell(x)$ were a probability distribution. What is less standard is what (4) means when $\sigma_\ell(x)$ is a game. E.g., when $\sigma_\ell(x)$ is a simple belief function $\sum_{i=1}^{n_\ell} a_{i\ell x} \mathfrak{u}_{Q_{i\ell x}}$, then:

$$V_q(x) = \sup_{\ell, q'/q \stackrel{\ell}{\longrightarrow} q'} \left[ r_{q \stackrel{\ell}{\longrightarrow} q'}(x) + \gamma_{q \stackrel{\ell}{\longrightarrow} q'} \sum_{i=1}^{n_\ell} a_{i\ell x} \min_{y \in Q_{i\ell x}} V_{q'}(y) \right] \tag{5}$$

where we see that P has control over the visible transitions $\ell$, and tries to maximize his payoff (sup), while C will minimize it, and some averaging is taking place in-between. The equation (4) does not always have a solution in the family of all $V_q$s. But there are two cases where it has, similar to those encountered in Markov decision processes.

**Theorem 6.** *Assume $\sigma$ is* standard, *i.e., $\sigma_\ell(X)$ is always either $0$ or $1$, and the set $\{x \in X | \sigma_\ell(x) = 0\}$ of* deadlock states *is open; or that $r_{q \stackrel{\ell}{\longrightarrow} q'}(x) \geq 0$ for all $q$, $\ell$, $q'$, $x \in X$. Assume also that there are $a, b \in \mathbb{R}$ with $a \leq r_{q \stackrel{\ell}{\longrightarrow} q'}(x), \gamma_{q \stackrel{\ell}{\longrightarrow} q'} \leq b$ for all $q$, $\ell$, $q'$, $x \in X$. Then (4) has a unique solution in any of the following two cases:*
*[Finite Horizon] If all paths in $\Pi$ have bounded length.*
*[Discount] If there is a constant $\gamma \in (0, 1)$ such that $\gamma_{q \stackrel{\ell}{\longrightarrow} q'} \leq \gamma$ for every $q$, $\ell$, $q'$.*

When $\sigma_\ell$ is a simple belief function, Equation (5) is then a Bellman-type equation that can be solved by dynamic programming techniques. Then observe that any continuous belief function is the directed lub of simple belief functions by Theorem 2, under mild assumptions. This offers a canonical way to approximate the average payoff $V_q$.

## References

1. Samson Abramsky and Achim Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3, pages 1–168. Oxford University Press, 1994.
2. Gilbert W. Bassett Jr., Roger Koenker, and Gregory Kordas. Pessimistic portfolio allocation and Choquet expected utility. Available from `http://www.econ.uiuc.edu/~roger/research/risk/choquet.pdf`, January 2004.
3. Stefano Cattani, Roberto Segala, Marta Z. Kwiatkowska, and Gethin Norman. Stochastic transition systems for continuous state spaces and non-determinism. In V. Sassone, editor, *Proc. 8th Int. Conf. Foundations of Software Science and Computation Structures (FoSSaCS 2005)*. Springer-Verlag LNCS 3441, 125–139.
4. Gaston Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953–54.
5. Vincent Danos, Josée Desharnais, François Laviolette, and Prakash Panangaden. Bisimulation and cocongruence for probabilistic systems. *Information and Computation*, 2005. Special issue for selected papers from CMCS04, 22 pages. A paraître.
6. A. P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Annals of Mathematical Statistics*, 38:325–339, 1967.
7. A. P. Dempster. A generalization of Bayesian inference. *Journal of the Royal Statistical Society*, B 30:205–247, 1968.
8. Josée Desharnais, Abbas Edalat, and Prakash Panangaden. Bisimulation for labelled Markov processes. *Information and Computation*, 2002. To appear.
9. Abbas Edalat. Domain theory and integration. *Theoretical Computer Science*, 151:163–193, 1995.
10. Eugene A. Feinberg and Adam Schwartz, editors. *Handbook of Markov Decision Processes, Methods and Applications*. Kluwer, 2002. 565 pages.
11. Gerhard Gierz, Karl Heinrich Hofmann, Klaus Keimel, Jimmie D. Lawson, Michael Mislove, and Dana S. Scott. *A Compendium of Continuous Lattices*. Springer Verlag, 1980.
12. Itzhak Gilboa and David Schmeidler. Additive representation of non-additive measures and the Choquet integral. Discussion Papers 985, Northwestern University, Center for Mathematical Studies in Economics and Management Science, 1992.
13. Jean Goubault-Larrecq. Une introduction aux capacités, aux jeux et aux prévisions. `http://www.lsv.ens-cachan.fr/~goubault/ProNobis/pp.pdf`, January 2007. 516 pages.
14. Joseph Y. Halpern and Ronald Fagin. Two views of belief: Belief as generalized probability and belief as evidence. *Artificial Intelligence*, 54:275–317, 1992.
15. Hans A. Hansson and Bengt Jonsson. A calculus for communicating systems with time and probabilities. In *Proc. 11th IEEE Real-time Systems Symp.*, pages 278–287, Silver Spring, MD, 1990. IEEE Computer Society Press.
16. Claire Jones. *Probabilistic Non-Determinism*. PhD thesis, University of Edinburgh, 1990. Technical Report ECS-LFCS-90-105.
17. Achim Jung. Stably compact spaces and the probabilistic powerspace construction. In J. Desharnais and P. Panangaden, editors, *Domain-theoretic Methods in Probabilistic Processes*, volume 87 of *Electronic Lecture Notes in Computer Science*. Elsevier, 2004. 15pp.
18. Kim G. Larsen and A. Skou. Bisimulation through probabilistic testing. *Information and Computation*, 94:1–28, 1991.
19. Michael Mislove. Topology, domain theory and theoretical computer science. *Topology and Its Applications*, 89:3–59, 1998.
20. Michael Mislove, Joël Ouaknine, and James Worrell. Axioms for probability and nondeterminism. *Electronic Notes in Theoretical Computer Science*, 91(3):7–28, 2003. Proc. 10th Int. Workshop on Expressiveness in Concurrency (EXPRESS'03).

21. Martin J. Osborne and Ariel Rubinstein. *A Course in Game Theory*. MIT Press, 1994.
22. A. Philippou, I. Lee, and O. Sokolsky. Weak bisimulation for probabilistic processes. In C. Palamidessi, editor, *Proc. CONCUR 2000*, pages 334–349. Springer-Verlag LNCS 1877, 2000.
23. David Schmeidler. Subjective probability and expected utility without additivity. *Econometrica*, 57:571–587, 1989.
24. Roberto Segala. *Modeling and Verification of Randomized Distributed Real-Time Systems*. MIT Press, Cambridge, MA, 1996.
25. Roberto Segala and Nancy Lynch. Probabilistic simulations for probabilistic processes. *Nordic Journal of Computing*, 2(2):250–273, 1995.
26. Roberto Segala and Andrea Turrini. Comparative analysis of bisimulation relations on alternating and non-alternating probabilistic models. In *2nd Int. Conf. Quantitative Evaluaton of Systems (QEST 2005)*, pages 44–53, Torino, Italy, September 2005. IEEE Computer Society Press.
27. G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, NJ, USA, 1976.
28. Regina Tix. *Stetige Bewertungen auf topologischen Räumen*. Diplomarbeit, TH Darmstadt, June 1995.

## A Proofs of Theorems

**Proposition 3.** *The belief function $\nu = \sum_{i=1}^{n} a_i \mathfrak{u}_{A_i}$, where $a_i \in \mathbb{R}^+ \setminus \{0\}$ for every $i$, $1 \leq i \leq n$, is continuous iff every $A_i$ is compact.*

*Proof.* If $A_i$ is compact, then for every directed family $(U_j)_{j \in J}$ of opens, $\mathfrak{u}_{A_i}(\bigcup_{j \in J} U_j) = 1$ iff $A_i \subseteq \bigcup_{j \in J} U_j$, iff $A_i \subseteq U_j$ for some $j \in J$ by compactness, iff $\mathfrak{u}_{A_i}(U_j) = 1$ for some $j \in J$, iff $\sup_{j \in J} \mathfrak{u}_{A_i}(U_j) = 1$. Since $\mathfrak{u}_{A_i}$ only takes values 0 and 1, $\mathfrak{u}_{A_i}$ is continuous. Then addition and multiplication by a non-negative constant are Scott-continuous in $\mathbb{R}^+$, so $\nu$ is continuous.

Conversely, let $(U_j)_{j \in J}$ be a directed family of opens such that $A_i \subseteq \bigcup_{j \in J} U_j$. W.l.o.g., assume that $A_i$ is upward-closed. This is harmless, as $\mathfrak{u}_{\uparrow A_i} = \mathfrak{u}_{A_i}$. Also assume the $A_i$s are pairwise distinct. Since $A_i$ is upward-closed, it is saturated. Hence whenever $A_{i'} \not\subseteq A_i$, we can find an open subset $V_{ii'}$ among those opens containing $A_i$ that does not contain $A_{i'}$. Let $V_i = \bigcap_{i' / A_{i'} \not\subseteq A_i} V_{ii'}$. $V_i$ is open, contains $A_i$, and does not contain any $A_{i'}$ such that $A_{i'} \subseteq A_i$. The family $(U_j \cap V_i)_{j \in J}$ is then a directed family of opens whose union contains $A_i$, but no $A_{i'}$ with $A_{i'} \not\subseteq A_i$. So $\nu\left(\bigcup_{j \in J} (U_j \cap V_i)\right) = \sum_{i' / A_{i'} \subseteq A_i} a_{i'}$. Since $\nu$ is continuous, the latter is $\sup_{j \in J} \nu(U_j \cap V_i)$. Since $\nu$ only takes finitely many values, there must be some $j \in J$ such that $\nu(U_j \cap V_i) = \sum_{i' / A_{i'} \subseteq A_i} a_{i'}$. Now again $U_j \cap V_i$ can only contain those $A_{i'}$ that are contained in $A_i$, since this is the case of $V_i$. Since $\nu(U_j \cap V_i) = \sum_{i' / A_{i'} \subseteq A_i} a_{i'}$ and every $a_{i'}$ is (strictly) positive, $U_j \cap V_i$ must contain them all, hence contains $A_i$. This implies that $U_j$ contains $A_i$, so $A_i$ is compact. □

**Theorem 1.** *For any continuous valuation $P$ on $\mathcal{Q}(X)$, the capacity $\nu$ defined by $\nu(U) = P(\Box U)$ is a continuous belief function on $X$.*

*Conversely, let $X$ be a well-filtered and locally compact space. For every continuous belief function $\nu$ on $X$ there is a unique continuous valuation $\nu^*$ on $\mathcal{Q}(X)$ such that $\nu(U) = \nu^*(\Box U)$ for every open $U$ of $X$.*

*Proof.* We first prove that $\nu(U) = P(\Box U)$ defines a continuous belief function. It is clear that $\nu$ is a game. Let us show that $\nu$ is continuous. Let $(U_i)_{i \in I}$ be a directed family of opens. Because any element $Q \in \mathcal{Q}(X)$ is compact, $\Box \bigcup_{i \in I} U_i = \bigcup_{i \in I} \Box U_i$. Then $\nu\left(\bigcup_{i \in I} U_i\right) = P\left(\Box \bigcup_{i \in I} U_i\right) = P\left(\bigcup_{i \in I} \Box U_i\right) = \bigcup_{i \in I} P(\Box U_i) = \bigcup_{i \in I} \nu(U_i)$. Let us finally show that $\nu$ is totally convex. We observe that, given any finite family $(U_i)_{i=1}^{n}$ of opens, $\Box \bigcup_{i=1}^{n} U_i \supseteq \bigcup_{i=1}^{n} \Box U_i$. So:

$$\nu\left(\bigcup_{i=1}^{n} U_i\right) = P\left(\Box \bigcup_{i=1}^{n} U_i\right) \geq P\left(\bigcup_{i=1}^{n} \Box U_i\right) \quad \text{since } P \text{ is monotonic}$$

$$= \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} P\left(\bigcap_{i \in I} \Box U_i\right)$$

by the inclusion-exclusion principle

$$= \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} P\left(\Box \bigcap_{i \in I} U_i\right) = \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} U_i\right)$$

The converse direction is harder. Assume $\nu$ is a continuous belief function on $X$.

We first establish a useful lemma on open subsets of $\mathcal{Q}(X)$, then a combinatorial lemma, finally a topological lemma.

Claim A. *For every pair of finite families $(U_i)_{i=1}^n$ and $(V_j)_{j=1}^m$ of opens in $X$, $\bigcup_{i=1}^n \Box U_i \subseteq \bigcup_{j=1}^m \Box V_j$ iff $(U_i)_{i=1}^n$ is smaller than or equal to $(V_j)_{j=1}^m$ in the* Hoare ordering $\subseteq^\flat$*, defined by $(U_i)_{i=1}^n \subseteq^\flat (V_j)_{j=1}^m$ iff for every $i$, $1 \leq i \leq n$, there exists $j$, $1 \leq j \leq m$, such that $U_i \subseteq V_j$.*

*Proof.* The if direction is obvious. Conversely, assume $\bigcup_{i=1}^n \Box U_i \subseteq \bigcup_{j=1}^m \Box V_j$. The idea of the proof is to take compacts contained in a given $U_i$, but arbitrarily big. First notice that any open is a directed union of saturated compacts. In fact, $U$ is the directed union of sets of the form $\uparrow E$, $E$ finite contained in $U$, and each is clearly compact. This allows us to pick a directed family $(Q_{ik})_{k\in K}$ of saturated compacts such that $U_i = \bigcup_{k\in K} Q_{ik}$, and such that: $(*)$ for every $k, k' \in K$, there is $k'' \in K$ such that $Q_{ik} \cup Q_{ik'} = Q_{ik''}$. Since $Q_{ik} \subseteq U_i$, $Q_{ik} \in \Box U_i$, so for some $j$, $1 \leq j \leq n$, we obtain $Q_{ik} \in \Box V_j$.

In general, $j$ depends on $k$; we shall show that we can choose $j$ so that is does not. For each $j$, $1 \leq j \leq m$, let $K_j = \{k \in K | Q_{ik} \subseteq V_j\}$. The argument above shows that $K = \bigcup_{j=1}^m K_j$.

Let $J$ be the subset of all $j \in \{1, \ldots, m\}$ such that $K_j$ is non-empty. $J$ itself is non-empty, since $K \neq \emptyset$ and $K = \bigcup_{j=1}^m K_j$. Now observe that there must be a $j \in J$ such that $K_j$ is *cofinal* in $K$, i.e., that for every $k \in K$, there is $k' \in K_j$ such that $Q_{ik} \subseteq Q_{ik'}$. Otherwise for every $j \in J$, there would be a $k_j \in K$ such that $Q_{ik_j} \not\subseteq Q_{ik'}$ for every $k' \in K_j$. By $(*)$, and since $J$ is finite, there is $k'' \in K$ such that $Q_{ik''} = \bigcup_{j\in J} Q_{ik_j}$. Since $K = \bigcup_{j=1}^m K_j$, there is a $j$ such that $k'' \in K_j$; in particular $K_j \neq \emptyset$, so $j \in J$. By construction, $Q_{ik_j} \not\subseteq Q_{ik'}$ for every $k' \in K_j$. Take $k' = k''$, so $Q_{ik_j} \not\subseteq Q_{ik''} = \bigcup_{j\in J} Q_{ik_j}$, a contradiction.

So let $j \in J$ be such that $K_j$ is cofinal in $K$. So $\bigcup_{k'\in K_j} Q_{ik'} = \bigcup_{k\in K} Q_{ik} = U_i$. By definition of $K_j$, for every $k' \in K_j$, $Q_{ik'} \subseteq V_j$. So $U_i \subseteq V_j$. $\Box$

Claim B.*Let $m, n \in \mathbb{N}$. For every $K \subseteq \{1, \ldots, n\} \times \{1, \ldots, m\}$, define $\pi_1 K = \{i | \exists j \cdot (i, j) \in K\}$, and $\pi_2 K = \{j | \exists j \cdot (i, j) \in K\}$. Then for any non-empty subsets $I$ of $\{1, \ldots, n\}$ and $J$ of $\{1, \ldots, m\}$,*

$$\sum_{\substack{K\subseteq\{1,\ldots,n\}\times\{1,\ldots,m\}\\ \pi_1K=I,\pi_2K=J}} (-1)^{|K|+1} = (-1)^{|I|+|J|} \tag{6}$$

*Proof.* By induction on the cardinality $|J| \geq 1$ of $J$. If $|J| = 1$, there is a unique $K$ such that $\pi_1 K = I$ and $\pi_2 K = J$, namely $I \times J$. Then $|K| = |I|$, and the result is clear. If $|J| > 1$, write $J$ as the disjoint union of some one element set $\{*\}$ and of $J' = J \setminus \{*\}$. Choosing $K$ (see Figure 1) such that $\pi_1 K = I$ and $\pi_2 K = J$ means picking the set $I'$ of indices $i$ such that $(i, *)$ is in $K$ (an arbitrary non-empty subset of $\{1, \ldots, n\}$), the set $I''$ of indices $i$ such that there is a $j \in J'$ with $(i, j) \in K$ (an arbitrary subset of $\{1, \ldots, n\}$ such that $I' \cup I'' = I$), and finally a subset $K'$ of $\{1, \ldots, n\} \times \{1, \ldots, m\}$ such that $\pi_1 K' = I''$ and $\pi_2 K' = J'$; the latter must

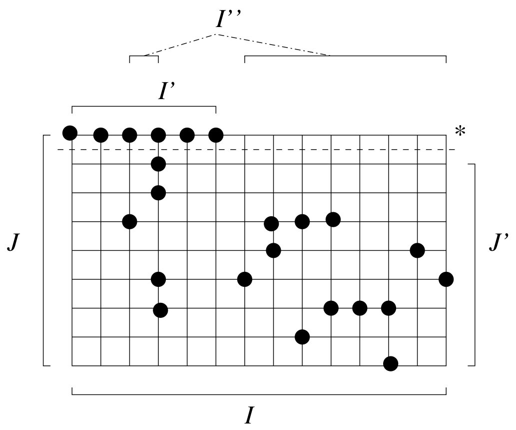


**Fig. 1.** Choosing $K$ so that $\pi_1 K = I$, $\pi_2 K = J$

be $K' = \{(i,j) \in K | j \neq *\}$. On Figure 1, $K$ is represented as the set of fat black dots, $K'$ is the set of dots above the dashed line.
So:

$$\begin{aligned}\sum_{\substack{K\subseteq\{1,\ldots,n\}\times\{1,\ldots,m\}\\ \pi_1K=I,\pi_2K=J}} (-1)^{|K|+1} &= \sum_{\substack{I'\subseteq\{1,\ldots,n\}\\ I'\neq\emptyset}} \sum_{\substack{I''\subseteq\{1,\ldots,n\}\\ I'\cup I''=I}} \sum_{\substack{K'\subseteq\{1,\ldots,n\}\times\{1,\ldots,m\}\\ \pi_1K'=I'',\pi_2K'=J'}} (-1)^{|I'|+|K'|+1} \\ &= \sum_{\substack{I'\subseteq\{1,\ldots,n\}\\ I'\neq\emptyset}} \sum_{\substack{I''\subseteq\{1,\ldots,n\}\\ I'\cup I''=I}} (-1)^{|I'|} \sum_{\substack{K'\subseteq\{1,\ldots,n\}\times\{1,\ldots,m\}\\ \pi_1K'=I'',\pi_2K'=J'}} (-1)^{|K'|+1}\end{aligned}$$

If $I'' = \emptyset$, the sum $\sum_{\substack{K'\subseteq\{1,\ldots,n\}\times\{1,\ldots,m\}\\ \pi_1K'=I'',\pi_2K'=J'}} (-1)^{|K'|+1}$ is 0. Otherwise, it is $(-1)^{|I''|+|J'|}$ by induction hypothesis. So

$$\sum_{\substack{K\subseteq\{1,\ldots,n\}\times\{1,\ldots,m\}\\ \pi_1K=I,\pi_2K=J}} (-1)^{|K|+1} = \sum_{\substack{I'\subseteq\{1,\ldots,n\}\\ I'\neq\emptyset}} \sum_{\substack{I''\subseteq\{1,\ldots,n\}\\ I''\neq\emptyset, I'\cup I''=I}} (-1)^{|I'|+|I''|+|J'|}$$

However, when $I' \neq \emptyset$, we can match those $I''$ such that $I' \cup I'' = I$ in pairs: fix $i' \in I'$, consider those $I''$ which contain $i'$, and those which do not. Then the sum of $(-1)^{|I'|+|I''|+|J'|}$, over all $I'' \subseteq \{1,\ldots,n\}$ such that $I' \cup I'' = I$ (including the empty $I''$) is 0. If $I' \neq I$, the empty set cannot be one of these $I''$, so $\sum_{\substack{I''\subseteq\{1,\ldots,n\}\\ I''\neq\emptyset, I'\cup I''=I}} (-1)^{|I'|+|I''|+|J'|}$ is 0, too. If $I' = I$, $\sum_{\substack{I''\subseteq\{1,\ldots,n\}\\ I''\neq\emptyset, I'\cup I''=I}} (-1)^{|I'|+|I''|+|J'|}$ is equal to $-(-1)^{|I'|+|J'|} = (-1)^{|I|+|J'|+1}$. So $\sum_{\substack{K\subseteq\{1,\ldots,n\}\times\{1,\ldots,m\}\\ \pi_1K=I,\pi_2K=J}} (-1)^{|K|+1} = (-1)^{|I|+|J'|+1} = (-1)^{|I|+|J|}$. □

Claim C.*Let $X$ be locally compact, $Q$ a saturated compact contained in some open $U$ of $X$. Then there is a saturated compact $Q_1$ such that $Q \subseteq \mathring{Q}_1 \subseteq Q_1 \subseteq U$.*

*Proof.* This is well-known. For each $x \in Q$, let $Q_x$ a saturated compact such that $x \in \mathring{Q}_x \subseteq Q_x \subseteq U$. The open cover $(\mathring{Q}_x)_{x \in Q}$ of $Q$ contains a finite subcover $(\mathring{Q}_x)_{x \in E}$, $E$ a finite subset of $Q$. Then $Q_1 = \bigcup_{x \in E} Q_x$ fits the bill. $\Box$

Claim A can be used to show that if $\nu^*$ exists such that $\nu(U) = \nu^*(\Box U)$ for every open $U$ of $X$, there it is unique. Since $\Box U$ are basic opens of the Scott topology of $\mathcal{Q}(X)$, every open $\mathcal{U}$ of $\mathcal{Q}(X)$ is a union $\bigcup_{i \in I} \Box U_i$, where $(U_i)_{i \in I}$ is some family of opens in $X$. We can also assume $I$ to be non-empty. For every finite non-empty subset $J$ of $I$, write $\Box U_J$ for the finite union $\bigcup_{i \in J} \Box U_i$. Then the family of all $U_J$s is directed, and:

$$\nu^*(\mathcal{U}) = \sup_{J \text{ finite}, J \subseteq I} \nu^* \left( \bigcup_{i \in J} \Box U_i \right) \tag{7}$$

By the inclusion-exclusion principle, the latter rewrites as a linear combination of quantities of the form $\nu^* \left( \Box \bigcap_{k \in K} U_k \right) = \nu(\bigcap_{k \in K} U_k)$, $K \subseteq J$. So $\nu^*$ is uniquely determined.

Let us now show that $\nu^*$ exists. This will take 12 steps. Start by defining a function $P$ that maps every finite family $(U_i)_{i=1}^n$ of opens of $X$ to:

$$P((U_i)_{i=1}^n) = \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu \left( \bigcap_{i \in I} U_i \right)$$

This is intended to be $\nu^* \left( \bigcup_{i=1}^n \Box U_i \right)$.

Step 1. We note that for any extra open $U_{n+1}$,

$$P((U_i)_{i=1}^{n+1}) \geq P((U_i)_{i=1}^n) \tag{8}$$

with equality if $U_{n+1} \subseteq U_i$ for some $i \in \{1, \ldots, n\}$. Indeed,

$$\begin{aligned}
P((U_i)_{i=1}^{n+1}) &= \sum_{I \subseteq \{1,\ldots,n+1\}, I \neq \emptyset} (-1)^{|I|+1} \nu \left( \bigcap_{i \in I} U_i \right) \\
&= \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu \left( \bigcap_{i \in I} U_i \right) + \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \nu \left( \bigcap_{i \in I} U_i \cap U_{n+1} \right) \\
&\qquad \text{splitting those non-empty subsets of } \{1, \ldots, n+1\} \\
&\qquad \text{into those that do not contain } n+1 \text{, and those of the form } I \cup \{n+1\} \\
&= P((U_i)_{i=1}^n) + \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \nu \left( \bigcap_{i \in I} U_i \cap U_{n+1} \right)
\end{aligned}$$

The last sum can be rewritten into:

$$\begin{aligned}\sum_{I\subseteq\{1,\dots,n\}}(-1)^{|I|}\nu\left(\bigcap_{i\in I}U_i\cap U_{n+1}\right) &= \nu\left(U_{n+1}\right)-\sum_{I\subseteq\{1,\dots,n\},I\neq\emptyset}(-1)^{|I|+1}\nu\left(\bigcap_{i\in I}(U_i\cap U_{n+1})\right)\\ &\geq \nu\left(U_{n+1}\right)-\nu\left(\bigcup_{i=1}^{n}(U_i\cap U_{n+1}\right) \quad \text{by total convexity}\\ &= \nu\left(U_{n+1}\right)-\nu\left(U_{n+1}\cap\bigcup_{i=1}^{n}U_i\right)\geq 0 \quad \text{by monotonicity}\end{aligned}$$

and this shows (8).

Furthermore, if for some $i_0$ we have $U_{n+1}\subseteq U_{i_0}$, we can instead rewrite the same sum as:

$$\begin{aligned}\sum_{I\subseteq\{1,\dots,n\}}(-1)^{|I|}\nu\left(\bigcap_{i\in I}U_i\cap U_{n+1}\right) &= \sum_{I\subseteq\{1,\dots,n\},i_0\notin I}(-1)^{|I|+1}\nu\left(\bigcap_{i\in I}U_i\cap U_{i_0}\cap U_{n+1}\right)\\ &+\sum_{I\subseteq\{1,\dots,n\},i_0\notin I}(-1)^{|I|}\nu\left(\bigcap_{i\in I}U_i\cap U_{n+1}\right)\end{aligned}$$

by splitting the second sum into those subsets of $\{1,\dots,n\}$ which contain $i_0$, that is, of the form $I\cup\{i_0\}$, $I\subseteq\{1,\dots,n\}$, $i_0\notin I$, and those which do not contain $i_0$. Since $U_{i_0}\cap U_{n+1}=U_{n+1}$ by assumption, the corresponding terms in the two sums cancel each other out. We therefore obtain $P((U_i)_{i=1}^{n+1})=P((U_i)_{i=1}^{n})$ in this case.

Step 2. It follows that for any permutation $\sigma$ if $\{1,\dots,n\}$, $P((U_i)_{i=1}^{n})=P(U_{\sigma(i)}{}_{i=1}^{n})$.

Step 3. Then whenever $(U_i)_{i=1}^{n}\subseteq^\flat (V_j)_{j=1}^{m}$, we must have $P((U_i)_{i=1}^{n})\leq P((V_j)_{j=1}^{m})$. Indeed, one can get $(V_j)_{j=1}^{m}$ from $(U_i)_{i=1}^{n}$ by first adding the $V_j$s one by one, which makes the $P$ value increase, and then removing the $U_i$s one by one; since each removed $U_i$ is contained in some $V_j$, this preserves the $P$ value.

Step 4. It follows that $P((U_i)_{i=1}^{n})$ only depends on $\bigcup_{i=1}^{n}\Box U_i$, not on the $U_i$s per se. Indeed, if $\bigcup_{i=1}^{n}\Box U_i=\bigcup_{j=1}^{m}\Box V_j$, by Claim A we must have $(U_i)_{i=1}^{n}\subseteq^\flat (V_j)_{j=1}^{m}$ and $(V_j)_{j=1}^{m}\subseteq^\flat (U_i)_{i=1}^{n}$, so $P((U_i)_{i=1}^{n})=P((V_j)_{j=1}^{m})$.

We may therefore legitimately write $P\left(\bigcup_{i=1}^{n}\Box U_i\right)$ instead of $P((U_i)_{i=1}^{n})$.

Step 5. A second consequence of Step 3 is that $P$ is monotonic, in the sense that $\bigcup_{i=1}^{n}\Box U_i\subseteq\bigcup_{j=1}^{m}\Box V_j$ implies $P\left(\bigcup_{i=1}^{n}\Box U_i\right)\leq P\left(\bigcup_{j=1}^{m}\Box V_j\right)$.

Step 6. Clearly, $P(\emptyset)=P\left(\bigcup_{i\in\emptyset}U_i\right)=0$.

Step 7. We show that $P$ is modular, i.e., that:

$$\begin{aligned}&P\left(\bigcup_{i=1}^{n}\Box U_i\cup\bigcup_{j=n+1}^{n+m}\Box U_j\right)+P\left(\bigcup_{i=1}^{n}\Box U_i\cap\bigcup_{j=n+1}^{n+m}\Box U_j\right)\\ &=P\left(\bigcup_{i=1}^{n}\Box U_i\right)+P\left(\bigcup_{j=n+1}^{n+m}\Box U_j\right)\end{aligned} \tag{9}$$

To this end, compute:

$$P\left(\bigcup_{i=1}^{n}\Box U_i \cap \bigcup_{j=n+1}^{n+m}\Box U_j\right) = P\left(\bigcup_{\substack{i\in\{1,\ldots,n\}\\ j\in\{1,\ldots,m\}}} (\Box U_i \cap \Box U_{j+n})\right)$$
$$= P\left(\bigcup_{\substack{i\in\{1,\ldots,n\}\\ j\in\{1,\ldots,m\}}} \Box (U_i \cap U_{j+n})\right)$$

since $\Box(U\cap V) = \Box U \cap \Box V$ for every $U, V$. By the definition of $P$:

$$P\left(\bigcup_{i=1}^{n}\Box U_i \cap \bigcup_{j=n+1}^{n+m}\Box U_j\right) = \sum_{\substack{K\subseteq\{1,\ldots,n\}\times\{1,\ldots,m\}\\ K\neq\emptyset}} (-1)^{|K|+1}\nu\left(\bigcap_{(i,j)\in K}(U_i\cap U_{j+n})\right)$$

Clearly, $\bigcap_{(i,j)\in K}(U_i\cap U_{j+n})$ only depends on $\pi_1 K$ and $\pi_2 K$, not on $K$: this is the same as $\bigcap_{i\in\pi_1 K} U_i \cap \bigcap_{j\in\pi_2 K} U_j$. So we can split the sum into one on the possible values $I$ and $J$ of $\pi_1 K$ and $\pi_2 K$ respectively:

$$\begin{aligned}
&P\left(\bigcup_{i=1}^{n}\Box U_i \cap \bigcup_{j=n+1}^{n+m}\Box U_j\right)\\
&= \sum_{\substack{I\subseteq\{1,\ldots,n\},I\neq\emptyset\\ J\subseteq\{1,\ldots,m\},J\neq\emptyset}} \sum_{\substack{K\subseteq\{1,\ldots,n\}\times\{1,\ldots,m\}\\ \pi_1K=I,\pi_2K=J}} (-1)^{|K|+1}\nu\left(\bigcap_{i\in I}U_i \cap \bigcap_{j\in J}U_{j+n}\right)\\
&= \sum_{\substack{I\subseteq\{1,\ldots,n\},I\neq\emptyset\\ J\subseteq\{1,\ldots,m\},J\neq\emptyset}} \left(\sum_{\substack{K\subseteq\{1,\ldots,n\}\times\{1,\ldots,m\}\\ \pi_1K=I,\pi_2K=J}} (-1)^{|K|+1}\right)\nu\left(\bigcap_{i\in I}U_i \cap \bigcap_{j\in J}U_{j+n}\right)\\
&= \sum_{\substack{I\subseteq\{1,\ldots,n\},I\neq\emptyset\\ J\subseteq\{1,\ldots,m\},J\neq\emptyset}} (-1)^{|I|+|J|}\nu\left(\bigcap_{i\in I}U_i \cap \bigcap_{j\in J}U_{j+n}\right) \qquad (10)
\end{aligned}$$

by Claim B. On the other hand, we may split those non-empty sets $L \subseteq \{1,\ldots,n+m\}$ into those of the form $I \subseteq \{1,\ldots,n\}$, those of the form $J \subseteq \{n+1,\ldots,m\}$, and those that are unions of sets of the first two forms:

$$
\begin{aligned}
P\left(\bigcup_{i=1}^{n} \Box U_i \cup \bigcup_{j=n+1}^{n+m} \Box U_j\right) &= \sum_{L\subseteq\{1,\dots,n+m\}, L\neq\emptyset} (-1)^{|L|+1}\nu\left(\bigcap_{\ell\in L} U_\ell\right) \\
&= \sum_{I\subseteq\{1,\dots,n\}, I\neq\emptyset} (-1)^{|I|+1}\nu\left(\bigcap_{i\in I} U_i\right) \\
&+ \sum_{J\subseteq\{1,\dots,m\}, J\neq\emptyset} (-1)^{|J|+1}\nu\left(\bigcap_{j\in J} U_{j+n}\right) \\
&+ \sum_{\substack{I\subseteq\{1,\dots,n\}, I\neq\emptyset \\ J\subseteq\{1,\dots,m\}, J\neq\emptyset}} (-1)^{|I|+|J|+1}\nu\left(\bigcap_{i\in I} U_i \cap \bigcap_{j\in J} U_{j+n}\right) \\
&= P\left(\bigcup_{i=1}^{n} \Box U_i\right) + P\left(\bigcup_{j=1}^{m} \Box U_{j+n}\right) - P\left(\bigcup_{i=1}^{n} \Box U_i \cap \bigcup_{j=n+1}^{n+m} \Box U_j\right)
\end{aligned}
$$

using (10). This proves that $P$ is modular.

Step 8. Let:

$$
\nu^*(\mathcal{U}) = \sup_{\mathcal{J} \text{ finite } \subseteq \mathcal{U}} P\left(\bigcup_{Q\in\mathcal{J}} \Box \mathring{Q}\right)
$$

The point is that $\mathcal{Q}(X)$, as a continuous cpo, has its Scott-topology generated by the subsets $\Uparrow Q$, $Q \in \mathcal{Q}(X)$, where $\Uparrow Q = \{Q' | Q \ll Q'\}$, and $\ll$ is the way-below relation on $\mathcal{Q}(X)$: $\mathcal{U} = \bigcup_{Q\in\mathcal{U}} \Uparrow Q$. It is well-known that $Q \ll Q'$ iff $Q' \subseteq \mathring{Q}$, so that $\Uparrow Q = \Box\mathring{Q}$. The formula above is then dictated by (7). Note that this is a well-defined real: $P\left(\bigcup_{Q\in\mathcal{J}} \Box\mathring{Q}\right) \leq P\left(\Box\bigcup_{Q\in\mathcal{J}} \mathring{Q}\right) = \nu\left(\bigcup_{Q\in\mathcal{J}} \mathring{Q}\right) \leq \nu(X)$, so $\nu^*(\mathcal{U})$ is well-defined as a least upper bound of reals bounded above by $\nu(X)$.

By Step 6, $\nu^*(\emptyset) = P(\emptyset) = 0$, so $\nu^*$ is a capacity. Then $\nu^*$ is monotonic by construction.

Step 9. We claim that $\nu^*$ is continuous. This is almost by construction. Let $(\mathcal{U}_i)_{i\in I}$ any directed family of opens of $\mathcal{Q}(X)$. For every finite set $\mathcal{J}$, $\mathcal{J} \subseteq \bigcup_{i\in I} \mathcal{U}_i$ iff there is a

$i \in I$ with $\mathcal{J} \subseteq \mathcal{U}_i$, by directedness. Then:

$$\begin{aligned}\nu^*\left(\bigcup_{i\in I}\mathcal{U}_i\right) &= \sup_{\mathcal{J} \text{ finite } \subseteq \bigcup_{i\in I}\mathcal{U}_i} P\left(\bigcup_{Q\in\mathcal{J}} \Box\overset{\circ}{Q}\right) \\ &= \sup_{\mathcal{J} \text{ finite } / \exists i\in I\cdot \mathcal{J}\subseteq\mathcal{U}_i} P\left(\bigcup_{Q\in\mathcal{J}} \Box\overset{\circ}{Q}\right) \\ &= \sup_{i\in I}\sup_{\mathcal{J}\subseteq\mathcal{U}_i} P\left(\bigcup_{Q\in\mathcal{J}} \Box\overset{\circ}{Q}\right) = \sup_{i\in I}\nu^*(\mathcal{U}_i)\end{aligned}$$

Step 10. For every finite subset $\mathcal{J}$ of $\mathcal{Q}(X)$,

$$\nu^*\left(\bigcup_{Q\in\mathcal{J}} \Box\overset{\circ}{Q}\right) = P\left(\bigcup_{Q\in\mathcal{J}} \Box\overset{\circ}{Q}\right)$$

The $\geq$ direction is by definition. Conversely, let $\mathcal{U} = \bigcup_{Q\in\mathcal{J}} \Box\overset{\circ}{Q}$, and consider any finite subset $\mathcal{J}'$ of $\mathcal{U}$. For every $Q' \in \mathcal{J}'$, $Q'$ is in $\mathcal{U}$, so there is $Q \in \mathcal{J}$ such that $Q' \in \Box\overset{\circ}{Q}$, in particular $\overset{\circ}{Q'} \subseteq \overset{\circ}{Q}$. By Claim A, $\bigcup_{Q'\in\mathcal{J}'} \Box\overset{\circ}{Q'} \subseteq \bigcup_{Q\in\mathcal{J}} \Box\overset{\circ}{Q}$. By Step 5,

$$P\left(\bigcup_{Q'\in\mathcal{J}'} \Box\overset{\circ}{Q'}\right) \leq P\left(\bigcup_{Q\in\mathcal{J}} \Box\overset{\circ}{Q}\right)$$

Now take the least upper bound of each size over all finite subsets $\mathcal{J}'$ of $\mathcal{U}$.

Step 11. Let us show that $\nu^*$ is modular. We have $\mathcal{U}\cup\mathcal{U}' = \bigcup_{Q\in\mathcal{U}\cup\mathcal{U}'} \Box\overset{\circ}{Q}$, so $\mathcal{U}\cup\mathcal{U}'$ is the directed union of all $\bigcup_{Q\in\mathcal{J}\cup\mathcal{J}'} \Box\overset{\circ}{Q} = \bigcup_{Q\in\mathcal{J}} \Box\overset{\circ}{Q} \cup \bigcup_{Q'\in\mathcal{J}'} \Box\overset{\circ}{Q'}$, when $\mathcal{J}$ ranges over all finite subsets of $\mathcal{U}$ and $\mathcal{J}'$ over all finite subsets of $\mathcal{U}'$. So:

$$\begin{aligned}\nu^*(\mathcal{U}\cup\mathcal{U}') &= \sup_{\substack{\mathcal{J} \text{ finite } \subseteq\mathcal{U} \\ \mathcal{J}' \text{ finite } \subseteq\mathcal{U}'}} P\left(\bigcup_{Q\in\mathcal{J}} \Box\overset{\circ}{Q} \cup \bigcup_{Q'\in\mathcal{J}'} \Box\overset{\circ}{Q'}\right) \\ &= \sup_{\substack{\mathcal{J} \text{ finite } \subseteq\mathcal{U} \\ \mathcal{J}' \text{ finite } \subseteq\mathcal{U}'}} \left[P\left(\bigcup_{Q\in\mathcal{J}} \Box\overset{\circ}{Q}\right) + P\left(\bigcup_{Q'\in\mathcal{J}'} \Box\overset{\circ}{Q'}\right) - P\left(\bigcup_{Q\in\mathcal{J}} \Box\overset{\circ}{Q} \cap \bigcup_{Q'\in\mathcal{J}'} \Box\overset{\circ}{Q'}\right)\right] \quad (11)\end{aligned}$$

by Step 7. The case of intersections is subtler (at least if we don't want to assume $X$ coherent).

$$\nu^*(\mathcal{U}\cap\mathcal{U}') = \sup_{\mathcal{J}'' \text{ finite } \subseteq\mathcal{U}\cap\mathcal{U}'} P\left(\bigcup_{Q''\in\mathcal{J}''} \Box\overset{\circ}{Q''}\right)$$

Now every finite subset $\mathcal{J}''$ of $\mathcal{U} \cap \mathcal{U}'$ is both a finite subset $\mathcal{J}$ of $\mathcal{U}$ and a finite subset $\mathcal{J}'$ of $\mathcal{U}'$, so:

$$\bigcup_{Q'' \in \mathcal{J}''} \Box \mathring{Q}'' = \bigcup_{Q \in \mathcal{J}} \Box \mathring{Q} \cap \bigcup_{Q' \in \mathcal{J}'} \Box \mathring{Q}'$$

Therefore:

$$\nu^*(\mathcal{U} \cap \mathcal{U}') \leq \sup_{\substack{\mathcal{J} \text{ finite } \subseteq \mathcal{U} \\ \mathcal{J}' \text{ finite } \subseteq \mathcal{U}'}} P\left(\bigcup_{Q \in \mathcal{J}} \Box \mathring{Q} \cap \bigcup_{Q' \in \mathcal{J}'} \Box \mathring{Q}'\right) \tag{12}$$

Let us show the converse inequality. If $\mathcal{J}$ is a finite subset of $\mathcal{U}$ and $\mathcal{J}'$ a finite subset of $\mathcal{U}'$, every $Q'' \in \bigcup_{Q \in \mathcal{J}} \Box \mathring{Q} \cap \bigcup_{Q' \in \mathcal{J}'} \Box \mathring{Q}'$ satisfies $Q'' \subseteq \mathring{Q}$ and $Q'' \subseteq \mathring{Q}'$ for some $Q \in \mathcal{J}$, and some $Q' \in \mathcal{J}'$. Since $X$ is locally compact, there is a saturated compact $Q_1$ such that $Q'' \subseteq \mathring{Q}_1 \subseteq Q_1 \subseteq \mathring{Q} \cap \mathring{Q}'$ by Claim C. Since $Q_1 \subseteq \mathring{Q}$ and $Q \in \mathcal{U}$, $Q_1 \in \mathcal{U}$ (every open is upward-closed); similarly, $Q_1 \in \mathcal{U}'$, so $Q_1 \in \mathcal{U} \cap \mathcal{U}'$. As $Q''$ is arbitrary,

$$\begin{aligned}\bigcup_{Q \in \mathcal{J}} \Box \mathring{Q} \cap \bigcup_{Q' \in \mathcal{J}'} \Box \mathring{Q}' &\subseteq \bigcup_{Q_1 \in \mathcal{U} \cap \mathcal{U}'} \Box \mathring{Q}_1 \\ &= \bigcup_{\mathcal{J}'' \text{ finite } \subseteq \mathcal{U} \cap \mathcal{U}'} \bigcup_{Q_1 \in \mathcal{J}''} \Box \mathring{Q}_1\end{aligned}$$

By Step 8, $\nu^*$ is monotonic and by Step 9, $\nu^*$ is continuous, so:

$$\begin{aligned}\nu^*\left(\bigcup_{Q \in \mathcal{J}} \Box \mathring{Q} \cap \bigcup_{Q' \in \mathcal{J}'} \Box \mathring{Q}'\right) &\leq \nu^*\left(\bigcup_{\mathcal{J}'' \text{ finite } \subseteq \mathcal{U} \cap \mathcal{U}'} \bigcup_{Q_1 \in \mathcal{J}''} \Box \mathring{Q}_1\right) \\ &= \sup_{\mathcal{J}'' \text{ finite } \subseteq \mathcal{U} \cap \mathcal{U}'} \nu^*\left(\bigcup_{Q_1 \in \mathcal{J}''} \Box \mathring{Q}_1\right)\end{aligned}$$

By Step 10,

$$\begin{aligned}P\left(\bigcup_{Q \in \mathcal{J}} \Box \mathring{Q} \cap \bigcup_{Q' \in \mathcal{J}'} \Box \mathring{Q}'\right) &\leq \sup_{\mathcal{J}'' \text{ finite } \subseteq \mathcal{U} \cap \mathcal{U}'} P\left(\bigcup_{Q_1 \in \mathcal{J}''} \Box \mathring{Q}_1\right) \\ &= \nu^*\left(\mathcal{U} \cap \mathcal{U}'\right)\end{aligned}$$

Taking least upper bounds, we get:

$$\sup_{\substack{\mathcal{J} \text{ finite } \subseteq \mathcal{U} \\ \mathcal{J}' \text{ finite } \subseteq \mathcal{U}'}} P\left(\bigcup_{Q \in \mathcal{J}} \Box \mathring{Q} \cap \bigcup_{Q' \in \mathcal{J}'} \Box \mathring{Q}'\right) \leq \nu^*(\mathcal{U} \cap \mathcal{U}')$$

Combining this with (12), we obtain:

$$\nu^*(\mathcal{U} \cap \mathcal{U}') = \sup_{\substack{\mathcal{J} \text{ finite } \subseteq \mathcal{U} \\ \mathcal{J}' \text{ finite } \subseteq \mathcal{U}'}} P\left(\bigcup_{Q \in \mathcal{J}} \Box \overset{\circ}{Q} \cap \bigcup_{Q' \in \mathcal{J}'} \Box \overset{\circ}{Q'}\right) \tag{13}$$

Now add (13) with (11), and recall that $+ : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is Scott-continuous:

$$\begin{aligned}\nu^*(\mathcal{U} \cup \mathcal{U}') + \nu^*(\mathcal{U} \cap \mathcal{U}') &= \sup_{\substack{\mathcal{J} \text{ finite } \subseteq \mathcal{U} \\ \mathcal{J}' \text{ finite } \subseteq \mathcal{U}'}} \left[P\left(\bigcup_{Q \in \mathcal{J}} \Box \overset{\circ}{Q}\right) + P\left(\bigcup_{Q' \in \mathcal{J}'} \Box \overset{\circ}{Q'}\right)\right] \\ &= \nu^*(\mathcal{U}) + \nu^*(\mathcal{U}')\end{aligned}$$

So $\nu^*$ is a continuous valuation.

Step 12. Let us finally show that for every open $U$ of $X$, $\nu^*(\Box U) = \nu(U)$. By definition,

$$\begin{aligned}\nu^*(\Box U) &= \sup_{\mathcal{J} \text{ finite } \subseteq \Box U} P\left(\bigcup_{Q \in \mathcal{J}} \Box \overset{\circ}{Q}\right) \\ &\geq \sup_{Q \text{ saturated compact } \subseteq U} P(\Box \overset{\circ}{Q}) = \sup_{Q \text{ saturated compact } \subseteq U} \nu(\overset{\circ}{Q})\end{aligned}$$

Since $X$ is locally compact, and using Claim C, $U$ is the directed union of $\overset{\circ}{Q}$, when $Q$ ranges over the saturated compacts contained in $U$. Since $\nu$ is continuous, $\sup_{Q \subseteq U} \nu(\overset{\circ}{Q})$ equals $\nu(U)$. So $\nu^*(\Box U) \geq \nu(U)$.

Conversely, for every finite subset $\mathcal{J}$ of $\Box U$, $\bigcup_{Q \in \mathcal{J}} \Box \overset{\circ}{Q} \subseteq \bigcup_{Q \in \Box U} \Box \overset{\circ}{Q} \subseteq \Box U$, so $\nu^*(\Box U) \leq P(\Box U) = \nu(U)$. Therefore $\nu^*(\Box U) = \nu(U)$. □

**Lemma 1.** $\sum_{i=1}^m a_i \mathfrak{u}_{Q_i} \leq \sum_{j=1}^n b_j \mathfrak{u}_{Q'_j}$ *iff there is matrix* $(t_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ *of coefficients in* $\mathbb{R}^+$ *such that* $\sum_{j=1}^n t_{ij} = a_i$ *for each* $i$, $\sum_{i=1}^m t_{ij} \leq b_j$ *for each* $j$, *and whenever* $t_{ij} \neq 0$ *then* $Q_i \leq Q'_j$.

*Proof.* By the max-flow min-cut theorem, as in [16]. We first prove two claims. Note that, since the ordering of $\mathcal{Q}(X)$ is *reverse* inclusion, a subset $J$ of $\mathcal{Q}(X)$ upward-closed if and only if $Q \in J$ and $Q \supseteq Q''$ implies $Q'' \in J$. Claim A states that we can always separate a given finite family of saturated compacts by finitely many opens.

Claim D. *Let $A$ be a finite subset of $\mathcal{Q}(X)$. There is family of opens $(U_Q)_{Q \in A}$ such that, for any upward-closed subset $J$ of $A$, $J = \{Q' \in A | Q' \subseteq \bigcup_{Q \in J} U_Q\}$.*
*Proof.* For each pair of saturated compacts $Q, Q' \in A$ such that $Q \not\supseteq Q'$, there is an open $U_{Q,Q'}$ containing $Q$ but not $Q'$. Indeed, if every open containing $Q$ contained $Q'$, then $\bigcap_{U/Q \subseteq U} U$ would contain $Q'$. But since $Q$ is saturated, $\bigcap_{U/Q \subseteq U} U = Q$. Since $A$ is finite, for every $Q \in A$, $\bigcap_{Q' \in A/Q \not\supseteq Q'} U_{Q,Q'}$ is open, let us name it $U_Q$. We observe that for every $Q'' \in A$, $Q'' \subseteq U_Q$ iff $Q \supseteq Q''$. Indeed, if $Q'' \subseteq U_Q$,

then $Q'' \subseteq U_{Q,Q'}$ for every $Q' \in A$ such that $Q \not\supseteq Q'$. In particular, for $Q' = Q''$, if $Q \not\supseteq Q''$, then $Q'' \subseteq U_{Q,Q''}$, a contradiction. So $Q \supseteq Q''$. Conversely, if $Q \supseteq Q''$, then for every $Q' \in A$ such that $Q \not\supseteq Q'$, in particular, $Q'' \subseteq U_{Q,Q'}$, since $U_{Q,Q'}$ contains $Q$.
For every upward-closed subset $J$ of $A$, therefore, $Q'' \in J$ iff $Q'' \subseteq \bigcup_{Q \in J} U_Q$. Indeed, if $Q'' \in J$, then $Q'' \subseteq U_{Q''}$, since $Q'' \subseteq U_{Q'',Q'}$ for every $Q'$. Conversely, if $Q'' \subseteq \bigcup_{Q \in J} U_Q$, there is $Q \in J$ such that $Q'' \subseteq U_Q$, so such that $Q \supseteq Q''$. Since $J$ is upward-closed in $A$, $Q'' \in J$. $\square$

Claim E. *For every simple belief functions $\sum_{Q \in A} a_Q \mathfrak{u}_Q$ and $\sum_{Q \in A} b_Q \mathfrak{u}_Q$, where $A$ is a finite subset of $\mathfrak{Q}(X)$, the inequality $\sum_{Q \in A} a_Q \mathfrak{u}_Q \leq \sum_{Q \in A} b_Q \mathfrak{u}_Q$ holds iff for every upward-closed subset $J$ of $A$, $\sum_{Q \in J} a_Q \leq \sum_{Q \in J} b_Q$.*
*Proof.* Assume $\sum_{Q \in J} a_Q \leq \sum_{Q \in J} b_Q$ for every upward-closed subset $J$ of $A$. Let $U$ be any open. Let $J = \{Q \in A | Q \subseteq U\}$. $J$ is upward-closed in $A$, and $(\sum_{Q \in A} a_Q \mathfrak{u}_Q)(U) = \sum_{Q \in J} a_Q$, $(\sum_{Q \in A} b_Q \mathfrak{u}_Q)(U) = \sum_{Q \in J} b_Q$, therefore $\sum_{Q \in A} a_Q \mathfrak{u}_Q \leq \sum_{Q \in A} b_Q \mathfrak{u}_Q$
Conversely, assume $\sum_{Q \in A} a_Q \mathfrak{u}_Q \leq \sum_{Q \in A} b_Q \mathfrak{u}_Q$, and let $J$ an upward-closed subset of $A$. Consider the family $(U_Q)_{Q \in A}$ of Claim D. By construction, we obtain $(\sum_{Q \in A} a_Q \mathfrak{u}_Q)(\bigcup_{Q \in J} U_Q) = \sum_{Q \in J} a_Q$ and $(\sum_{Q \in A} b_Q \mathfrak{u}_Q) = \sum_{Q \in J} b_Q$, whence the claim. $\square$

Now on to the proof of Lemma 1. If the matrix $(t_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ exists as in the Lemma, then:

$$\begin{aligned}
\sum_{i=1}^{m} a_i \mathfrak{u}_{Q_i} &= \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij} \mathfrak{u}_{Q_i} = \sum_{j=1}^{n} \sum_{i=1}^{m} t_{ij} \mathfrak{u}_{Q_i} \\
&\leq \sum_{j=1}^{n} \sum_{i=1}^{m} t_{ij} \mathfrak{u}_{Q'_j} \quad \text{since if } t_{ij} \neq 0, \mathfrak{u}_{Q_i} \leq \mathfrak{u}_{Q'_j} \\
&\leq \sum_{j=1}^{n} b_j \mathfrak{u}_{Q'_j}
\end{aligned}$$

Conversely, assume $\sum_{i=1}^{m} a_i \mathfrak{u}_{Q_i} \leq \sum_{j=1}^{n} b_j \mathfrak{u}_{Q'_j}$. Let $A$ be the set of all $Q_i$, $1 \leq i \leq m$, and all $Q'_j$, $1 \leq j \leq n$. We may write $\sum_{i=1}^{m} a_i \mathfrak{u}_{Q_i}$ as $\sum_{Q \in A} a_Q \mathfrak{u}_Q$, where $a_Q$ is $a_{Q_i}$ if $Q = Q_i$ for some $i$, $1 \leq i \leq m$, and 0 otherwise; similarly we can write $\sum_{j=1}^{n} b_j \mathfrak{u}_{Q'_j}$ as $\sum_{Q \in A} b_Q \mathfrak{u}_Q$, where $b_Q$ is $b_{Q'_j}$ if $Q = Q'_j$ for some $j$, $1 \leq j \leq n$, and 0 otherwise.

So let us assume $\sum_{Q \in A} a_Q \mathfrak{u}_Q \leq \sum_{Q \in A} b_Q \mathfrak{u}_Q$. Build the network $G$ with vertices $s$ (the source), $t$ (the sink), and $2|A|$ vertices $[Q]$ and $\langle Q \rangle$, two for each $Q \in A$. All these vertices are distinct. There is an edge from $s$ to $[Q]$ for each $Q \in A$, with capacity[3] $a_Q$; one from $[Q]$ to $\langle Q' \rangle$ for every $Q, Q' \in A$ with $Q \supseteq Q'$, with large enough capacity $M$, say $M > \max(\sum_{Q \in A} a_Q, \sum_{Q \in A} b_Q)$; and one from $\langle Q \rangle$ to $t$ for each $Q \in A$, of capacity $b_Q$.

[3] A capacity here is just a real number labeling an edge, meant to bound the flow through the edge. It has nothing to do with games and capacities.

If there is a flow $f$ whose value along the edge from $s$ to $[Q]$ is $a_Q$ for every $Q \in A$, then its value will be $\sum_{Q\in A} a_Q$, and this value will be maximal. Let us show that the value of every cut through the network is at least $\sum_{Q\in A} a_Q$. (A cut $c$ through $G$ is a set of vertices containing $s$ but not $t$. Its value is the sum of capacities or edges from a vertex in $c$ to one outside $c$.) The max-flow min-cut Theorem states that the values of the minimum cut equals that of the maximum flow, so the value of the minimum cut will be exactly $\sum_{Q\in A} a_Q$.

Let then $c$ be a cut through $G$. Let $(*)$ the following assumption: there are $[Q] \in c$, $\langle Q'\rangle \notin c$, with an edge from $[Q]$ to $\langle Q'\rangle$. In this case, $Q \supseteq Q'$, and the edge's capacity is $M$, so the cut's value is at least $M \geq \sum_{Q\in A} a_Q$, and we are done. If on the other hand $(*)$ is false, the only edges from $v \in c$ to $v' \notin c$ are those from $s$ to $[Q]$ or from $\langle Q\rangle$ to $t$ for some $Q \in A$. The value of $c$ is $\sum_{[Q]\notin c} a_Q + \sum_{\langle Q\rangle\in c} b_Q$. Since $(*)$ fails, $\langle Q\rangle$ is in $c$ whenever $[Q]$ is. By removing from $c$ those vertices $\langle Q\rangle$ such that $[Q]$ is not in $c$, we get a cut with possibly smaller value, where $(*)$ still fails, and where $[Q] \in c$ iff $\langle Q\rangle \in c$. Let $J$ be the set of all $Q \in A$ such that $[Q]$ is in $c$. The value of the $c$ is then $\sum_{Q\notin J} a_Q + \sum_{Q\in J} b_Q$. $J$ is upward-closed in $A$: if $Q \in J$ (i.e., $[Q] \in c$) and $Q \supseteq Q'$, the fact that $(*)$ fails implies that $\langle Q'\rangle$, hence also $[Q']$ is in $c$, so $Q' \in J$. By Claim E, $\sum_{Q\in J} a_Q \leq \sum_{Q\in J} b_Q$, so the value of $c$ is at least $\sum_{Q\in A} a_Q$.

So there is a maximum flow $f$, of value $\sum_{Q\in A} a_Q$. Necessarily the flow from $s$ to $[Q]$ must be exactly $a_Q$ for every $Q \in A$. Let $T_{QQ'}$ be the value of the flow from $[Q]$ to $[Q']$ when $Q \supseteq Q'$, $T_{QQ'} = 0$ otherwise. We notice that:

$$\sum_{Q'\in A} T_{QQ'} = a_Q \tag{14}$$

for every $Q \in A$. Indeed, $a_Q$ is the incoming value of the flow $f$ at vertex $[Q]$, and (as a flow) this must coincide with its outgoing value $\sum_{Q'\in A} T_{QQ'}$. Next we notice that:

$$\sum_{Q\in A} T_{QQ'} \leq b_{Q'} \tag{15}$$

for every $Q' \in A$. Indeed, $b_{Q'}$ is larger than or equal to the outgoing value of $f$ at vertex $\langle Q'\rangle$, and the latter is exactly the sum $\sum_{Q\in A} T_{QQ'}$.

Observe that, for every $i$, $1 \leq i \leq m$, $T_{Q_iQ'} = 0$ if $Q'$ is not one of $Q'_j$, $1 \leq j \leq n$. Indeed, $T_{Q_iQ'}$ is a non-negative real, and by (15) their sum when $i$ varies is at most $b_{Q'}$, which is zero.

Similarly, for every $j$, $1 \leq j \leq n$, $T_{QQ'_j} = 0$ when $Q$ is not among $Q_i$, $1 \leq i \leq m$. This is because the sum of $T_{QQ'_j}$ when $j$ varies is at most $a_Q = 0$, by (14).

The equation (15) with $Q = Q_i$ then becomes $\sum_{j=1}^{n} T_{Q_iQ'_j} = a_{Q_i} = a_i$, and the inequality (15) with $Q' = Q'_j$ becomes $\sum_{i=1}^{m} T_{Q_iQ'_j} \leq b_{Q'_j} = b_j$, whence the claim.

□

**Theorem 2.** *Let $X$ be well-filtered and locally compact. Every continuous belief function $\nu$ on $X$ is the least upper bound of a directed family of simple belief functions $\nu_i$ way-below $\nu$. $\mathbf{Cd}_{\leq 1}(X)$ is a continuous cpo.*

*Proof.* $\mathcal{Q}(X)$ is a continuous cpo, hence the space $\mathbf{V}_{\leq 1}(\mathcal{Q}(X))$ of continuous valuations $\nu$ with $\nu(X) \leq 1$ is a continuous cpo, with a basis of simple valuations by Jones' Theorem [16, Theorem 5.2]. In particular, given any continuous belief function $\nu$ on $X$, we may write $\nu^*$ as $\sup_{i \in I} P_i$ for some directed family $(P_i)_{i \in I}$ of simple valuations on $\mathcal{Q}(X)$. Define $\nu_i(U) = P_i(\Box U)$, then $P_i = \nu_i^*$. By (A) above, $(\nu_i)_{i \in I}$ is directed in $\mathbf{Cd}_{\leq 1}(X)$, and has therefore a least upper bound. It is easy to check that for each open $U$ of $X$, $\nu(U) = \nu^*(\Box U) = \sup_{i \in I} \nu_i^*(\Box U) = \sup_{i \in I} \nu_i(U)$. The fact that we can take $\nu_i$ to be way-below proceeds by showing that each simple belief function $\nu_i$ is itself the least upper bound of a directed family of simple belief functions way-below it. This is proved by observing a few facts first.

Claim F. *Let $X$ be well-filtered. Let $(Q_i)_{i \in I}$ be any filtered family of non-empty saturated compacts. Then $\mathfrak{u}_{\bigcap_{i \in I} Q_i} = \sup_{i \in I} \mathfrak{u}_{Q_i}$.*
*Proof.* For every open $U$ of $X$, $\mathfrak{u}_{\bigcap_{i \in I} Q_i}(U) = 1$ iff $\bigcap_{i \in I} Q_i \subseteq U$, iff $Q_i \subseteq U$ for some $i$, since $X$ is well-filtered. But this is equivalent to $\sup_{i \in I} \mathfrak{u}_{Q_i}(U) = 1$. □

Claim G. *Let $X$ be well-filtered and locally compact. Assume $a_i < b_i$ and $Q'_i \subseteq \overset{\circ}{Q}_i$ for every $i$, $1 \leq i \leq n$, then*

$$\sum_{i=1}^{n} a_i \mathfrak{u}_{Q_i} \ll \sum_{i=1}^{n} b_i \mathfrak{u}_{Q'_i}$$

*Proof.* Let $(\nu_j)_{j \in I}$ be any directed family of continuous belief functions such that $\sum_{i=1}^{n} b_i \mathfrak{u}_{Q'_i} \leq \sup_{j \in I} \nu_j$. Let $J$ be a non-empty upward-closed subset of $A = \{Q_i | 1 \leq i \leq n\}$. Let $U_J$ be the open $\bigcup_{Q \in J} \overset{\circ}{Q}$. We have $(\sum_{i=1}^{n} b_i \mathfrak{u}_{Q'_i})(U_J) \leq \sup_{j \in I} \nu_j(U_J)$. Since $Q'_i \subseteq U_J$ for every $i$ such that $Q_i \in J$, the left-hand side is at least $\sum_{i/Q_i \in J} b_i$. Since $a_i < b_i$ for every $i$ and $J$ is non-empty, $\sum_{i/Q_i \in J} a_i < \sup_{j \in I} \nu_j(U_J)$. So there is $j \in I$ such that $\sum_{i/Q_i \in J} a_i \leq \nu_j(U_J)$.
The latter inequality also clearly holds when $J = \emptyset$. Moreover, since $(\nu_j)_{j \in I}$ is directed and there are only finitely many upward-closed subsets $J$ of $A$, there is a $j \in I$ such that $\sum_{i/Q_i \in J} a_i \leq \nu_j(U_J)$, whatever $J$.
We now show that $\sum_{i=1}^{n} a_i \mathfrak{u}_{Q_i} \leq \nu_j$. For every open $U$, let $J = \{Q_i | 1 \leq i \leq n, Q_i \subseteq U\}$, so that $(\sum_{i=1}^{n} a_i \mathfrak{u}_{Q_i})(U) = \sum_{i/Q_i \in J} a_i$, which is less than or equal to $\nu_j(U_J)$ by the computation above. For every $x \in U_J$, there is $Q_i \in J$ such that $x \in \overset{\circ}{Q}_i$, so $x \in U$, i.e., $U_J \subseteq U$. So $\nu_j(U_J) \leq \nu_j(U)$. Finally, $\sum_{i=1}^{n} a_i \mathfrak{u}_{Q_i} \leq \nu_j$, and we are done. □

Claim H. *Let $X$ be well-filtered and locally compact, and $\nu = \sum_{j=1}^{n} b_j \mathfrak{u}_{Q'_j}$ a simple belief function on $X$. Then $\nu$ is the least upper bound of the directed family of those simple belief functions of the form $\sum_{j=1}^{n} r b_j \mathfrak{u}_{Q''_j}$, where $0 < r < 1$ and $Q'_j \subseteq \overset{\circ}{Q''_j}$ for every $j$, $1 \leq j \leq n$.*
*Proof.* Consider those continuous belief functions of the form $\sum_{j=1}^{n} r b_j \mathfrak{u}_{Q''_j}$, where $0 < r < 1$ and $Q'_j \subseteq \overset{\circ}{Q''_j}$ for every $j$, $1 \leq j \leq n$, and show first that this family is directed.

It is non-empty: for every $j$, since $Q'_j$ is contained in $X$, which is open, there is a saturated compact $Q''_j$ such that $Q'_j \subseteq \overset{\circ}{Q''_j} \subseteq Q''_j \subseteq X$, by Claim C.
Then, let $\sum_{j=1}^n r_1 b_j \mathfrak{u}_{Q''_{1j}}$ and $\sum_{j=1}^n r_2 b_j \mathfrak{u}_{Q''_{2j}}$ be two belief functions of the form above, namely $0 < r_1, r_2 < 1$ and $Q'_j \subseteq \overset{\circ}{Q''_{1j}} \cap \overset{\circ}{Q''_{2j}}$ for every $j$. Since $X$ is locally compact, by Claim C there is a saturated compact $Q''_j$ such that $Q'_j \subseteq \overset{\circ}{Q''_j} \subseteq Q''_j \subseteq \overset{\circ}{Q''_{1j}} \cap \overset{\circ}{Q''_{2j}}$, for each $j$. Let $r = \max(r_1, r_2)$, then $\sum_{j=1}^n r b_j \mathfrak{u}_{Q''_j}$ is again of the form above, since $0 < r < 1$ and $Q'_j \subseteq \overset{\circ}{Q''_j}$. Moreover, since $Q''_j \subseteq Q''_{1j}$ and $r \geq r_1$, $\sum_{j=1}^n r b_j \mathfrak{u}_{Q''_j} \geq \sum_{j=1}^n r_1 b_j \mathfrak{u}_{Q''_{1j}}$; similarly, $\sum_{j=1}^n r b_j \mathfrak{u}_{Q''_j} \geq \sum_{j=1}^n r_2 b_j \mathfrak{u}_{Q''_{2j}}$.
Let us show that the least upper bound of the family of all $\sum_{j=1}^n r b_j \mathfrak{u}_{Q''_j}$, $0 < r < 1$, $Q'_j \subseteq \overset{\circ}{Q''_j}$ for every $j$, is $\sum_{j=1}^n b_j \mathfrak{u}_{Q'_j}$. Since $\mathcal{Q}(X)$ is a continuous cpo, $Q'_j$ is the intersection of the filtered family of all saturated compacts $Q''_j$ whose interior contains $Q'_j$. By Claim F, the least upper bound of $\mathfrak{u}_{Q''_j}$ when $\overset{\circ}{Q''_j} \supseteq Q'_j$ is $\mathfrak{u}_{Q'_j}$. So the least upper bound of all $r b_j \mathfrak{u}_{Q''_j}$, when $0 < r < 1$ and $Q''_j$ ranges over the saturated compacts whose interior contains $Q'_j$ is $b_j \mathfrak{u}_{Q'_j}$. We conclude since addition is Scott-continuous.. □

We know that $\nu$ is the least upper bound of a directed family $(\nu_i)_{i \in I}$ of simple belief functions. By Claim H, each $\nu_i$ is the least upper bound of a directed family $(\nu_{ik})_{k \in K_i}$ of simple belief functions, which are all way-below $\nu_i$ by Claim G, in particular way-below $\nu$. Clearly $\nu = \sup_{i \in I, k \in K_i} \nu_{ik}$. Moreover the family of all $\nu_{ik}$s is directed. It is clearly non-empty, and whenever $\nu_{ik}$ and $\nu_{i'k'}$ are two elements of this family, $\nu_{ik} \ll \nu_i$ and $\nu_{i'k'} \ll \nu_{i'}$. We know that for some $i'' \in I$, $\nu_i, \nu_{i'} \leq \nu_{i''}$. So $\nu_{ik}, \nu_{i'k'} \ll \nu_{i''}$. Since $\nu_{i''}$ is the least upper bound of all $\nu_{i''k''}$, $k'' \in K_{i''}$, by definition of $\ll$, and since $\nu_{ik} \ll \nu_{i''}$, there is $k_1$ such that $\nu_{ik} \leq \nu_{i''k_1}$. Similarly, for some $k_2$ we have $\nu_{i'k'} \leq \nu_{i''k_2}$. Since $(\nu_{i''k''})_{k'' \in K_{i''}}$ is directed, there is $k_3 \in K_{i''}$ such that $\nu_{i''k_1}, \nu_{i''k_2} \leq \nu_{i''k_3}$, so $\nu_{ik}, \nu_{i'k'} \leq \nu_{i''k_3}$. □

**Theorem 3.** *Let $X$ be well-filtered and locally compact. The function $\nu \mapsto \nu^*$ defines an order-isomorphism from $\mathbf{Cd}_{\leq 1}(X)$ to $\mathbf{V}_{\leq 1}(\mathcal{Q}(X))$.*

*Proof.* First, $\nu^* \leq {\nu'}^*$ implies $\nu \leq \nu'$. Indeed, for every open $U$, $\nu(U) = \nu^*(\Box U) \leq {\nu'}^*(\Box U) = \nu'(U)$. Conversely, assume $\nu \leq \nu'$, we show that $\nu^* \leq {\nu'}^*$. Write $\nu$ as the least upper bound of a directed family $(\nu_i)_{i \in I}$ of simple belief functions way-below $\nu$, and similarly for $\nu'$ and $(\nu'_j)_{j \in J}$. For each $i \in I$, since $\nu_i \ll \nu \leq \nu' = \sup_{j \in J} \nu'_j$, there is $j \in J$ such that $\nu_i \leq \nu'_j$. By (A) $\nu_i^* \leq {\nu'_j}^*$. This entails $\sup_{i \in I} \nu_i^* \leq \sup_{j \in J} {\nu'_j}^*$. For each open $U$ of $X$, $\sup_{i \in I} \nu_i^*(\Box U) = \sup_{i \in I} \nu_i(U) = \nu(U) = \nu^*(U)$, so the left-hand side is $\nu^*$, and similarly the right-hand side is ${\nu'}^*$. □

**Proposition 1.** *For any continuous $f : X \to \mathbb{R}^+$,*

$$\oint_{x \in X} f(x) d\mathfrak{u}_A = \inf_{x \in A} f(x)$$

*Moreover, if $A$ is compact, then the inf is attained: this equals $\min_{x \in A} f(x)$.*

*Proof.* Using the definition,

$$\oint_{x\in X} f(x)d\mathfrak{u}_A = \int_0^{+\infty} \mathfrak{u}_A(f^{-1}(t,+\infty))dt + \int_{-\infty}^{0} [\mathfrak{u}_A(f^{-1}(t,+\infty)) - \mathfrak{u}_(X)]dt$$
$$= \int_{t\geq 0/A\subseteq f^{-1}(t,+\infty)} dt + \int_{t<0/A\not\subseteq f^{-1}(t,+\infty)} -dt$$

But $A \subseteq f^{-1}(t,+\infty)$ iff for every $x \in A$, $f(x) > t$, i.e., $\inf_{x\in A} f(x) \geq t$ or $\inf_{x\in X} f(x) > t$. If $\inf_{x\in A} f(x) \geq 0$, then the right integral is zero and the left one is $\int_0^{\inf_{x\in A} f(x)} dt = \inf_{x\in A} f(x)$. If $\inf_{x\in A} f(x) < 0$, the converse happens, and the right integral is $\inf_{x\in A} f(x)$.

If $A$ is compact, then the image $f(A)$ of $A$ by $f$ is compact, and so is $\uparrow f(A)$. The saturated compacts of $\mathbb{R}$ are the subsets of the form $[t,+\infty)$, whence $\uparrow f(A)$ has a least element $t = \inf_{x\in X} f(x)$. There must be $x \in A$ such that $f(x) \leq t$, whence $f(x) = t$. □

**Theorem 4.** *For any bounded continuous function $f : X \to \mathbb{R}$, let $f_*$ be the function from $\mathcal{Q}(X)$ to $\mathbb{R}$ defined by $f_*(Q) = \min_{x\in Q} f(x)$. Say that a capacity $\nu$ is* linearly extensible from below *if and only if there is continuous valuation $P$ on $\mathcal{Q}(X)$ with:*

$$\oint_{x\in X} f(x)d\nu = \oint_{Q\in\mathcal{Q}(X)} f_*(Q)dP \qquad (3)$$

*for every bounded continuous $f$. If $X$ is well-filtered and locally compact, then the capacities that are linearly extensible from below are exactly the continuous belief functions, and $P$ must be $\nu^*$ in (3).*

*Proof.* Note first that, since $Q$ is compact, $f$ reaches its minimum on $Q$, so the notation $\min_{x\in Q} f(x)$ makes sense.

Assume (3) holds. Take $f = \chi_U$ for any open $U$ of $X$ in (3). For any $Q$, $f_*(Q)$ equals 1 iff $Q \subseteq U$, 0 otherwise, so $f_* = \chi_{\Box U}$. It follows that (3) implies $\nu(U) = P(\Box U)$. So $\nu$ is a continuous belief function by Theorem 1, and $P = \nu^*$.

Conversely, if $\nu$ is a continuous belief function and $P = \nu^*$, then:

$$\oint_{Q\in\mathcal{Q}(X)} f_*(Q)dP = \int_0^{+\infty} \nu^*(f_*^{-1}(t,+\infty))dt + \int_{-\infty}^{0} [\nu^*(f_*^{-1}(t,+\infty)) - \nu^*(\mathcal{Q}(X))]dt$$

Now observe that $\nu^*(f_*^{-1}(t,+\infty)) = \nu^*\{Q \in \mathcal{Q}(X) | \min_{x\in Q} f(x) > t\} = \nu^*\{Q \in \mathcal{Q}(X) | Q \subseteq f^{-1}(t,+\infty)\}$ (using the fact, which is crucial here, that the minimum $\min_{x\in Q} f(x)$ is attained) $= \nu^*(\Box f^{-1}(t,+\infty)) = \nu(f^{-1}(t,+\infty))$. On the other hand $\nu^*(\mathcal{Q}(X)) = \nu^*(\Box X) = \nu(X)$, so:

$$\oint_{Q\in\mathcal{Q}(X)} f_*(Q)dP = \int_0^{+\infty} \nu(f^{-1}(t,+\infty))dt + \int_{-\infty}^{0} [\nu(f^{-1}(t,+\infty)) - \nu(X)]dt$$
$$= \oint_{x\in X} f(x)d\nu$$

□

**Theorem 5.** *Let $\mathcal{O}$ be a simulation topology for $\sigma$ on $X$. For any $F \in \mathcal{L}_{open}$, $[\![F]\!]_\sigma \in \mathcal{O}$. In particular [Soundness], if $x \in [\![F]\!]_\sigma$ and $x \preceq_{\mathcal{O}} y$ then $y \in [\![F]\!]_\sigma$. Conversely [Completeness], the coarsest simulation topology $\mathfrak{O}_\sigma$ is exactly that generated by the opens $[\![F]\!]_\sigma$, $F \in \mathcal{L}^{\top\wedge\vee}_{open}$.*

*Proof.* The fact that $[\![F]\!]_\sigma$ is in $\mathcal{O}$ is an easy structural induction on $F$. Conversely, let $\mathcal{O}$ the topology generated by opens of the form $[\![F]\!]_\sigma$, $F \in \mathcal{L}^{\top\wedge\vee}_{\text{open}}$. We claim that $\mathcal{O}$ is a simulation topology, i.e., that $\mathcal{O}$ is finer than $\mathfrak{O}_\sigma$. Since, by the first part of the Theorem, every open $[\![F]\!]_\sigma$ is in $\mathfrak{O}_\sigma$, $\mathcal{O}$ is also coarser than $\mathfrak{O}_\sigma$. So $\mathcal{O}$ and $\mathfrak{O}_\sigma$ will coincide.

So let us show that $\mathcal{O}$ is a simulation topology. First, by the first part of the Theorem, $\mathcal{O}$ is coarser than the topology of $X$. We now note that for every open $U$ of $\mathcal{O}$ that happens to be of the form $[\![F]\!]_\sigma$, then ${\sigma_\ell}^{-1}_{|\mathcal{O}}[U > r] = \{x \in X | \sigma_\ell(x)([\![F]\!]_\sigma) > r\} = [\![[\ell]_{>r} F]\!]_\sigma$ is again in $\mathcal{O}$, for every $r$ in $\mathbb{Q} \cap [0, 1]$. This must again hold for any real $r \in [0, 1]$, since ${\sigma_\ell}^{-1}_{|\mathcal{O}}[U > r] = \bigcup_{r' \in \mathbb{Q}, r' > r} {\sigma_\ell}^{-1}_{|\mathcal{O}}[U > r']$.

It remains to show that ${\sigma_\ell}^{-1}_{|\mathcal{O}}[U > r]$ is in $\mathcal{O}$ for every $U \in \mathcal{O}$, not just for those $U$ of the form $[\![F]\!]_\sigma$. Then the function $\sigma_\ell$ from $X : \mathcal{O}$ to $\mathbf{J}_{\leq 1\ wk}(X : \mathcal{O})$ will be continuous, and we will be done.

By definition $U \in \mathcal{O}$ is a union of finite intersections of opens of the form $[\![F]\!]_\sigma$, hence of opens that are either empty or directed unions of finite non-empty unions of finite intersections of opens of the form $[\![F]\!]_\sigma$. If $U$ is empty, then ${\sigma_\ell}^{-1}_{|\mathcal{O}}[U > r]$ is empty if $r \geq 0$, is the whole of $X$ if $r < 0$, and therefore in $\mathcal{O}$ in any case.

Otherwise, we note that any finite intersection of opens $[\![F_i]\!]_\sigma$, $1 \leq i \leq m$, is of the form $[\![F_1 \wedge \ldots \wedge F_m]\!]_\sigma$ ($m \geq 1$), or of the form $[\![\top]\!]_\sigma$ if $m = 0$. Every non-empty finite union of opens $[\![F_i]\!]_\sigma$, $1 \leq i \leq m$ ($m \geq 1$), is of the form $[\![F_1 \vee \ldots \vee F_m]\!]_\sigma$. So $U$ is a directed union of sets of the form $[\![F]\!]_\sigma$, say $U = \bigcup_{i \in I} [\![F_i]\!]_\sigma$. Then ${\sigma_\ell}^{-1}_{|\mathcal{O}}[U > r] = \{x \in X | \sigma_\ell(x)(\bigcup_{i \in I} [\![F_i]\!]_\sigma) > r\} = \{x \in X | \sup_{i \in I} \sigma_\ell(x)([\![F_i]\!]_\sigma > r\} = \bigcup_{i \in I} \{x \in X | \sigma_\ell(x)([\![F_i]\!]_\sigma > r\}$. The latter is a union of opens of $\mathcal{O}$, and is therefore in $\mathcal{O}$. □

**Proposition 2.** *Let $\mathcal{O}$ be a simulation topology for $\sigma$. The function $\sigma_\ell/\mathcal{O}$ mapping $q_{\mathcal{O}}(x)$ to $q_{\mathcal{O}}[\sigma_\ell(x)]$ is well defined and continuous from $X/\mathcal{O}$ to $\mathbf{J}_{\leq 1\ wk}(X/\mathcal{O})$ for every $\ell \in L$. The family $\sigma/\mathcal{O} = (\sigma_\ell/\mathcal{O})_{\ell \in L}$ is then a ludic transition system on $X/\mathcal{O}$, which we call the* lumped *ludic transition system.*

*For any $F \in \mathcal{L}^{\top\wedge\vee}_{open}$ and $x \in X$, $x$ and $q_{\mathcal{O}}(x)$ satisfy the same formulae: $q_{\mathcal{O}}([\![F]\!]_\sigma) = [\![F]\!]_{\sigma/\mathcal{O}}$, and $[\![F]\!]_\sigma = q_{\mathcal{O}}^{-1}([\![F]\!]_{\sigma/\mathcal{O}})$, in particular, $x \in [\![F]\!]_\sigma$ iff $q_{\mathcal{O}}(x) \in [\![F]\!]_{\sigma/\mathcal{O}}$.*

*Proof.* Again, we start with a few lemmas. Note that $X/\mathcal{O}$ is not the usual quotient $X/{\equiv_{\mathcal{O}}}$, despite the similarity;

Claim I. *For every $U \in \mathcal{O}$, $q_{\mathcal{O}}^{-1}(q_{\mathcal{O}}(U)) = U$.*
*Proof.* First, $q_{\mathcal{O}}^{-1}(q_{\mathcal{O}}(U))$ contains $U$. Conversely, for every $x \in q_{\mathcal{O}}^{-1}(q_{\mathcal{O}}(U))$, there is a $y \in U$ such that $q_{\mathcal{O}}(x) = q_{\mathcal{O}}(y)$, i.e., such that $x \equiv_{\mathcal{O}} y$. By definition of $\equiv_{\mathcal{O}}$, since $y \in U$, and $U \in \mathcal{O}$, we obtain $x \in U$. So $q_{\mathcal{O}}^{-1}(q_{\mathcal{O}}(U)) \subseteq U$. □

Claim J. *The opens of $X/\mathcal{O}$ are the images $q_{\mathcal{O}}(U)$ of opens $U \in \mathcal{O}$.*
*Proof.* Let $U'$ be any open of $X/\mathcal{O}$. By definition, $q_{\mathcal{O}}^{-1}(U')$ is an open of $X : \mathcal{O}$, and is therefore an element $U$ of $\mathcal{O}$. Since $q_{\mathcal{O}}$ is onto, $q_{\mathcal{O}}(U) = U'$, so $U'$ is of the

required form. Conversely, the opens of $X/\mathcal{O}$ are the subsets $U'$ of $X/\mathcal{O}$ such that $q_{\mathcal{O}}^{-1}(U')$ is in $\mathcal{O}$. For every $U \in \mathcal{O}$, consider $U' = q_{\mathcal{O}}(U)$. Then $q_{\mathcal{O}}^{-1}(U') = U$ by Claim I. So $q_{\mathcal{O}}(U)$ is an open of $X/\mathcal{O}$. $\square$

Claim K. *Let $\mathcal{O}$ be a simulation topology for $\sigma$. For every $x_1, x_2 \in X$ such that $x_1 \equiv_{\mathcal{O}} x_2$, we have $q_{\mathcal{O}}[\sigma_\ell(x_1)] = q_{\mathcal{O}}[\sigma_\ell(x_2)]$ for every $\ell \in L$.*

*Proof.* For every $r \in \mathbb{R}$, for every open of $X/\mathcal{O}$—which is necessarily of the form $q_{\mathcal{O}}(U)$ for some $U \in \mathcal{O}$ by Claim J—, if $q_{\mathcal{O}}[\sigma_\ell(x_1)](q_{\mathcal{O}}(U)) > r$, then by definition $\sigma_\ell(x_1)(q_{\mathcal{O}}^{-1}(q_{\mathcal{O}}(U))) > r$, so $\sigma_\ell(x_1)(U) > r$ by Claim I. So $x_1$ is in the open $\sigma_\ell^{-1}[U > r]$, which is in $\mathcal{O}$ since $\mathcal{O}$ a simulation topology for $\sigma$. Since $x_1 \equiv_{\mathcal{O}} x_2$, $x_2$ is in the same open. By the same argument as above, $q_{\mathcal{O}}[\sigma_\ell(x_2)](q_{\mathcal{O}}(U)) > r$. Since this holds for any $r \in \mathbb{R}$, $q_{\mathcal{O}}[\sigma_\ell(x_2)](q_{\mathcal{O}}(U)) \geq q_{\mathcal{O}}[\sigma_\ell(x_1)](q_{\mathcal{O}}(U))$. By the same argument, exchanging the roles of $x_1$ and $x_2$, $q_{\mathcal{O}}[\sigma_\ell(x_2)](q_{\mathcal{O}}(U)) \leq q_{\mathcal{O}}[\sigma_\ell(x_1)](q_{\mathcal{O}}(U))$. So $q_{\mathcal{O}}[\sigma_\ell(x_2)](q_{\mathcal{O}}(U)) = q_{\mathcal{O}}[\sigma_\ell(x_1)](q_{\mathcal{O}}(U))$. We conclude since $q_{\mathcal{O}}(U)$ is arbitrary. $\square$

By Claim K, $\sigma_\ell/\mathcal{O}$ is well-defined, since $q_{\mathcal{O}}[\sigma_\ell(x)]$ does not depend on $x$, but only on $q_{\mathcal{O}}(x)$.

Let us show that it is continuous. For each open $q_{\mathcal{O}}(U)$ of $X/\mathcal{O}$ (using Claim J), where $U \in \mathcal{O}$,

$$\begin{aligned}
\sigma_\ell/\mathcal{O}^{-1}[q_{\mathcal{O}}(U) > r] &= \{q_{\mathcal{O}}(x) | q_{\mathcal{O}}[\sigma_\ell(x)](q_{\mathcal{O}}(U)) > r\} \\
&= \{q_{\mathcal{O}}(x) | \sigma_\ell(x)(q_{\mathcal{O}}^{-1}(q_{\mathcal{O}}(U))) > r\} \\
&= \{q_{\mathcal{O}}(x) | \sigma_\ell(x)(U) > r\} \qquad \text{by Claim I} \\
&= q_{\mathcal{O}}(\sigma_\ell^{-1}[U > r])
\end{aligned}$$

is an open of $X/\mathcal{O}$, since $\sigma_\ell$ is continuous from $X : \mathcal{O}$ to $\mathbf{J}_{\leq 1\ wk}(X : \mathcal{O})$, and by Claim J.

Let us show that $x \in X$, $x$ and $q_{\mathcal{O}}(x)$ satisfy the same formulae. We show the second statement, that $[\![F]\!]_\sigma = q_{\mathcal{O}}^{-1}([\![F]\!]_{\sigma/\mathcal{O}})$, by structural induction on $F$. The only interesting case is when $F$ is of the form $[\ell]_{>r}F'$:

$$\begin{aligned}
q_{\mathcal{O}}^{-1}([\![F]\!]_{\sigma/\mathcal{O}}) &= q_{\mathcal{O}}^{-1}(\sigma_\ell/\mathcal{O}^{-1}[[\![F']\!]_{q_{\mathcal{O}}[\sigma]} > r]) \\
&= q_{\mathcal{O}}^{-1}\{q_{\mathcal{O}}(x) \mid q_{\mathcal{O}}[\sigma_\ell(x)]([\![F']\!]_{q_{\mathcal{O}}[\sigma]}) > r\} \\
&= q_{\mathcal{O}}^{-1}(q_{\mathcal{O}}(\sigma_\ell^{-1}[q_{\mathcal{O}}^{-1}([\![F']\!]_{q_{\mathcal{O}}[\sigma]}) > r])) \\
&= q_{\mathcal{O}}^{-1}(q_{\mathcal{O}}(\sigma_\ell^{-1}[[\![F']\!]_\sigma > r])) \qquad \text{by induction hypothesis} \\
&= q_{\mathcal{O}}^{-1}(q_{\mathcal{O}}([\![F]\!]_\sigma)) \\
&= [\![F]\!]_\sigma \qquad \text{by Claim I and the Soundness part of Theorem 5}
\end{aligned}$$

Now we know that $[\![F]\!]_\sigma = q_{\mathcal{O}}^{-1}([\![F]\!]_{\sigma/\mathcal{O}})$. The other claim follows: $q_{\mathcal{O}}([\![F]\!]_\sigma) = q_{\mathcal{O}}(q_{\mathcal{O}}^{-1}([\![F]\!]_{\sigma/\mathcal{O}})) = [\![F]\!]_{\sigma/\mathcal{O}}$, since $q_{\mathcal{O}}$ is onto. $\square$

**Theorem 6.** *Assume $\sigma$ is* standard, *i.e., $\sigma_\ell(X)$ is always either* 0 *or* 1, *and the set* $\{x \in X | \sigma_\ell(x) = 0\}$ *of* deadlock states *is open; or that $r_{q \xrightarrow{\ell} q'}(x) \geq 0$ for all $q$, $\ell$, $q'$,*

*$x \in X$. Assume also that there are $a, b \in \mathbb{R}$ with $a \leq r_{q \xrightarrow{\ell} q'}(x), \gamma_{q \xrightarrow{\ell} q'} \leq b$ for all $q$, $\ell$, $q'$, $x \in X$. Then (4) has a unique solution in any of the following two cases:*
*[Finite Horizon] If all paths in $\Pi$ have bounded length.*
*[Discount] If there is a constant $\gamma \in (0, 1)$ such that $\gamma_{q \xrightarrow{\ell} q'} \leq \gamma$ for every $q$, $\ell$, $q'$.*

*Proof.* It is in fact profitable to solve a more general problem. Fix $X$, a ludic transition system $\sigma = (\sigma_\ell)_{\ell \in L}$ on $X$, and $(V, E)$ an automaton, $E \subseteq V \times L \times V$. (We do not assume $V$ or $E$ to be finite. However, we do assume that given $q \in V$ and $\ell \in L$, there are only finitely many internal states $q'$ such that $(q, \ell, q') \in E$.) Write $q, q'$ the elements of $V$ (internal states), and $q \xrightarrow{\ell} q'$ whenever $(q, \ell, q') \in E$. We define a general class of functional equations, of which (4) is a special case.

Let $\mathsf{R}$ be a set of so-called *reward parameters*, $\mathsf{G}$ a set of so-called *discount parameters*.

Define the grammars of *functional $V$-expressions* $f \in \mathcal{F}(\mathsf{R}, \mathsf{G}, \mathsf{X})$ on $\mathsf{R}$, $\mathsf{G}$, and the set of unknowns $\mathsf{X}$, and of *functional $E$-expressions* $f^{\rightarrow} \in \mathcal{F}^{\rightarrow}(\mathsf{R}, \mathsf{G}, \mathsf{X})$ on $\mathsf{R}$, $\mathsf{G}$, and the set of unknowns $\mathsf{X}$, by:

| | | |
|---|---|---|
| $f ::=$ | $\mathsf{r}$ | reward ($\mathsf{r} \in \mathsf{R}$) |
| | $\mid v$ | variable ($v \in \mathsf{X}$) |
| | $\mid \max(f, f)$ | maximum |
| | $\mid \min(f, f)$ | minimum |
| | $\mid f + f$ | sum |
| | $\mid \mathsf{g} \times f$ | product by a constant ($\mathsf{g} \in \mathsf{G}$) |
| | $\mid \Box_A f^{\rightarrow}$ | lub on outgoing transitions ($A \in \mathbb{P}(L)$) |
| | $\mid \Diamond_A f^{\rightarrow}$ | glb on outgoing transitions ($A \in \mathbb{P}_{\text{fin}}(L)$) |
| $f^{\rightarrow} ::=$ | $\mathsf{r}$ | reward ($\mathsf{r} \in \mathsf{R}$) |
| | $\mid \max(f^{\rightarrow}, f^{\rightarrow})$ | maximum |
| | $\mid \min(f^{\rightarrow}, f^{\rightarrow})$ | minimum |
| | $\mid f^{\rightarrow} + f^{\rightarrow}$ | sum |
| | $\mid \mathsf{g} \times f^{\rightarrow}$ | product by a constant ($\mathsf{g} \in \mathsf{G}$) |
| | $\mid \int f$ | average on next state |

A *context* $\mathcal{C}$ is the following data: functions $\mathcal{C}_q$ ($q \in V$) and $\mathcal{C}_{q \xrightarrow{\ell} q'}$ ($(q, \ell, q') \in E$) from $\mathsf{R}$ to the space of all continuous function from $X$ to $\mathbb{R}$; functions, again written $\mathcal{C}_q$ ($q \in V$) and $\mathcal{C}_{q \xrightarrow{\ell} q'}$ ($(q, \ell, q') \in E$), from $\mathsf{G}$ to $\mathbb{R}^+$.

An *environment* $\rho$ is a family of functions $\rho_q$, $q \in V$, mapping each variable $v \in \mathsf{X}$ to a continuous function from $X$ to $\mathbb{R}$.

The semantics of functional expressions is given by two families of continuous functions $\mathcal{C}\,[\![f]\!]_q\,\rho$ from $X$ to $\mathbb{R}$ ($q \in V$, $f \in \mathcal{F}(\mathsf{R}, \mathsf{G}, \mathsf{X})$) and $\mathcal{C}\,[\![f^{\rightarrow}]\!]_{q \xrightarrow{\ell} q'}\,\rho$ from $X$ to $\mathbb{R}$ ($(q, \ell, q') \in E$, $f^{\rightarrow} \in \mathcal{F}^{\rightarrow}(\mathsf{R}, \mathsf{G}, \mathsf{X})$), defined as follows. The most important clauses are:

$$\mathcal{C}\left[\!\left[\Box_A f^{\rightarrow}\right]\!\right]_q \rho(x) = \sup_{\ell\in A, q'\in V/q\xrightarrow{\ell}q'} \mathcal{C}\left[\!\left[f^{\rightarrow}\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x)$$

$$\mathcal{C}\left[\!\left[\Diamond_A f^{\rightarrow}\right]\!\right]_q \rho(x) = \min_{\ell\in A, q'\in V/q\xrightarrow{\ell}q'} \mathcal{C}\left[\!\left[f^{\rightarrow}\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x)$$

$$\mathcal{C}\left[\!\left[\int f\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x) = \oint_{y\in X} \mathcal{C}\left[\!\left[f\right]\!\right]_{q'} \rho(y) d\sigma_\ell(x)$$

where, in the first and second cases, we shall assume by convention that the lub and the min of a family of reals indexed by the empty set (when there is no $\ell \in A$ and no $q' \in V$ such that $q\xrightarrow{\ell}q'$) is 0. The other cases are trivial:

$$\mathcal{C}\left[\!\left[\mathsf{r}\right]\!\right]_q \rho(x) = \mathcal{C}_q(\mathsf{r})(x)$$

$$\mathcal{C}\left[\!\left[v\right]\!\right]_q \rho(x) = \rho_q(v)$$

$$\mathcal{C}\left[\!\left[\max(f_1, f_2)\right]\!\right]_q \rho(x) = \max(\mathcal{C}\left[\!\left[f_1\right]\!\right]_q \rho(x), \mathcal{C}\left[\!\left[f_2\right]\!\right]_q \rho(x))$$

$$\mathcal{C}\left[\!\left[\min(f_1, f_2)\right]\!\right]_q \rho(x) = \min(\mathcal{C}\left[\!\left[f_1\right]\!\right]_q \rho(x), \mathcal{C}\left[\!\left[f_2\right]\!\right]_q \rho(x))$$

$$\mathcal{C}\left[\!\left[f_1 + f_2\right]\!\right]_q \rho(x) = \mathcal{C}\left[\!\left[f_1\right]\!\right]_q \rho(x) + \mathcal{C}\left[\!\left[f_2\right]\!\right]_q \rho(x)$$

$$\mathcal{C}\left[\!\left[\mathsf{g} \times f\right]\!\right]_q \rho(x) = \mathcal{C}_q(\mathsf{g}).\mathcal{C}\left[\!\left[f\right]\!\right]_q \rho(x)$$

and also:

$$\mathcal{C}\left[\!\left[\mathsf{r}\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x) = \mathcal{C}_{q\xrightarrow{\ell}q'}(\mathsf{r})(x)$$

$$\mathcal{C}\left[\!\left[\max(f_1^{\rightarrow}, f_2^{\rightarrow})\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x) = \max(\mathcal{C}\left[\!\left[f_1^{\rightarrow}\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x), \mathcal{C}\left[\!\left[f_2^{\rightarrow}\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x))$$

$$\mathcal{C}\left[\!\left[\min(f_1^{\rightarrow}, f_2^{\rightarrow})\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x) = \min(\mathcal{C}\left[\!\left[f_1^{\rightarrow}\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x), \mathcal{C}\left[\!\left[f_2^{\rightarrow}\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x))$$

$$\mathcal{C}\left[\!\left[f_1^{\rightarrow} + f_2^{\rightarrow}\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x) = \mathcal{C}\left[\!\left[f_1^{\rightarrow}\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x) + \mathcal{C}\left[\!\left[f_2^{\rightarrow}\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x)$$

$$\mathcal{C}\left[\!\left[\mathsf{g} \times f^{\rightarrow}\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x) = \mathcal{C}_{q\xrightarrow{\ell}q'}(\mathsf{g}).\mathcal{C}\left[\!\left[f^{\rightarrow}\right]\!\right]_{q\xrightarrow{\ell}q'} \rho(x)$$

A *functional equation* is an expression of the form $v \approx f$, where $v \in \mathsf{X}$ and $f$ is a functional $V$-expression. A *functional system* $\Sigma$ is a finite family of functional equations $v_i \approx f_i$, $1 \leq i \leq n$, where the $v_i$s are pairwise distinct.

A *solution* of a functional system $\Sigma = \{\mathsf{v}_i \approx \mathsf{f}_i, 1 \leq i \leq n\}$ in a context $\mathcal{C}$ is an environment $\rho$ such that $\rho_q(\mathsf{v}_i) = \mathcal{C}\left[\!\left[f_i\right]\!\right]_q \rho$ for every $q \in V$.

We need to put some restrictions on the objects above so that the notion of solution makes sense. A context $\mathcal{C}$ is *bounded* iff there are two constants $a, b \in \mathbb{R}$ such that $a \leq \mathcal{C}_q(\mathsf{r})(x), \mathcal{C}_{q\xrightarrow{\ell}q'}(\mathsf{r})(x), \mathcal{C}_q(\mathsf{g}), \mathcal{C}_{q\xrightarrow{\ell}q'}(\mathsf{g}) \leq q$ for every vertex $q \in V$, every transition $(q, \ell, q') \in E$, every parameter $\mathsf{r} \in \mathsf{R}$, $\mathsf{g} \in \mathsf{G}$, and for every $x \in X$. It is *bounded positive* iff one may choose $a \in \mathbb{R}^+$.

Similarly, say that an environment $\rho$ is *bounded* iff there are two constants $a, b \in \mathbb{R}$ such that $a \leq \rho_q(v)(x) \leq b$ for every $q \in V$, $v \in \mathsf{X}$, $x \in X$. It is *bounded positive* iff one may choose $a \in \mathbb{R}^+$.

Claim L. *Let $\mathcal{C}$ and $\rho$ be bounded. Assume that $\sigma$ is standard, or that $\mathcal{C}$ and $\rho$ are bounded positive. Then for every vertex $q \in V$, every transition $q\xrightarrow{\ell}q'$ of $(V, E)$,* *for every functional expression $f$, resp. $f^{\rightarrow}$, $\mathcal{C}\,[\![f]\!]_q\,\rho$ and $\mathcal{C}\,[\![f^{\rightarrow}]\!]_{q\xrightarrow{\ell}q'}\,\rho$ are well defined, continuous from $X$ to $\mathbb{R}$, and bounded in the sense that there are two constants $a, b \in \mathbb{R}$ such that $a \leq \mathcal{C}\,[\![f]\!]_q\,\rho(x), \mathcal{C}\,[\![f^{\rightarrow}]\!]_{q\xrightarrow{\ell}q'}\,\rho(x) \leq b$ for every $q \in V$, $(q, \ell, q') \in E$, $x \in X$. Furthermore, if $\mathcal{C}$ and $\rho$ are bounded positive, then $a \in \mathbb{R}^+$.*

*Proof.* By structural induction on $f$, resp. $f^{\rightarrow}$. In case $f = \Box_A f_1$, note that the least upper bound of an arbitrary family of continuous functions from $X$ to $[a, b]$ (with the Scott topology, remember) exists, and is again bounded and continuous. Let indeed $f = \sup_{i\in I} f_i$. If $I = \emptyset$, this is trivial. Otherwise, $f$ is bounded by $[a, b]$, and as far as continuity is concerned, $f^{-1}(t, +\infty) = \{x \in X |\sup_{i\in I} f_i(x) > t\} = \{x \in X|\exists i \in I \cdot f_i(x) > t\} = \bigcup_{i\in I} f_i^{-1}(t, +\infty)$ is open.

In case $f = \Diamond_A f_1$, recall that $A$ is finite. Given $q$, there are only finitely many transitions $(q, \ell, q')$ such that $\ell \in A$. This is why we assumed that given $q \in V$ and $\ell \in L$, there are only finitely many internal states $q'$ such that $(q, \ell, q') \in E$. Next, the min of a finite family of continuous functions from $X$ to $[a, b]$ is bounded and continuous. Let $f(x) = \min_{i\in I} f_i(x)$. If $I = \emptyset$, this is clear. Otherwise, $f$ is bounded by $a$ and $b$, and continuous since $f^{-1}(t, +\infty) = \bigcap_{i\in I} f_i^{-1}(t, +\infty)$ is a finite intersection of opens.

In case $f^{\rightarrow} = \int f$, note that $\mathcal{C}\,[\![f]\!]_{q\xrightarrow{\ell}q'}\,\rho$ is defined, continuous and bounded (say in $[a, b]$, where $a$ and $b$ are independent of $q$) by induction hypothesis, so its Choquet integral along $\sigma_\ell(x)$ exists. So the function $\mathcal{C}\,[\![\int f]\!]_{q\xrightarrow{\ell}q'}\,\rho$ is well defined.

If $\mathcal{C}$ and $\rho$ are bounded positive, then $\mathcal{C}\,[\![f]\!]_{q'}\,\rho$ takes its values in $\mathbb{R}^+$, so $\mathcal{C}\,[\![\int f]\!]_{q\xrightarrow{\ell}q'}\,\rho$ is bounded by $0$ and $b$. Since the weak topology is also the coarsest one that makes continuous all functionals mapping $\nu \in Y$ to the integral of $f$ along $\nu$, for all $f : X \to \mathbb{R}^+$ bounded continuous (note that $f$ takes its values in $\mathbb{R}^+$, not $\mathbb{R}$!), $\mathcal{C}\,[\![\int f]\!]_{q\xrightarrow{\ell}q'}\,\rho$ is also continuous.

Otherwise, by assumption $\sigma$ is standard. Assuming wlog. that $a \leq 0$ and $b \geq 0$, the function $\mathcal{C}\,[\![\int f]\!]_{q\xrightarrow{\ell}q'}\,\rho$ takes its values in $[a, b]$, since $\sigma_\ell(x)(X) \leq 1$ for every $\ell \in L$ and every $x \in X$. Let $F$ be the set of all deadlock states. By assumption, $F$ is open. On the other hand, $F$ is the complement of $\sigma_\ell^{-1}[X > 1/2]$, and is therefore closed. So $F$, and its complement $U$, are clopen. Then:

$$\mathcal{C}\left[\!\!\left[\int f\right]\!\!\right]_{q\xrightarrow{\ell}q'}\rho(x) = \oint_{y\in Y}\left(\mathcal{C}\,[\![f]\!]_{q'}\,\rho(y) - a\right) d\sigma_\ell(x) + a \quad \text{if } x \in U$$

$$\mathcal{C}\left[\!\!\left[\int f\right]\!\!\right]_{q\xrightarrow{\ell}q'}\rho(x) = 0 \quad \text{if } x \in F$$

and we notice that the function $\mathcal{C}\,[\![f]\!]_{q'}\,\rho - a$ takes its values in $\mathbb{R}^+$. So, as above, the function mapping $\nu \in \mathbf{J}_{\leq 1\ wk}(X)$ to the Choquet integral of $\mathcal{C}\,[\![f]\!]_{q'}\,\rho(y) - a$ along $\nu$ is continuous. Since $\sigma_\ell$ is continuous from $X$ to $\mathbf{J}_{\leq 1\ wk}(X)$, the function $g$ mapping $x$ to:

$$\oint_{y\in Y}\left(\mathcal{C}\,[\![f]\!]_{q'}\,\rho(y) - a\right) d\sigma_\ell(x) + a$$

is continuous. For every $t \in \mathbb{R}$, let's compute the inverse image of the open $]t, +\infty[$ by $\mathcal{C} \left[\!\left[ \int f \right]\!\right]_q \rho$. If $t \geq 0$, this is $\{x \in X | g(x) > t \text{ et } x \in U\} = g^{-1}]t, +\infty[ \cap U$, which is open. Otherwise, this is $F \cup (g^{-1}]t, +\infty[ \cap U))$, which is also open since $F$ is open.
The other cases are clear. □

As an example, Equation (4) expresses that $(V_q)_{q \in V}$ is a solution of the functional system $\Sigma$ which contains only the following equation:

$$v \approx \Box_L \left( \mathsf{r} + \mathsf{g} \times \int v \right) \tag{16}$$

in the context $\mathcal{C}$ described by:

$$\mathcal{C}_{q \xrightarrow{\ell} q'}(\mathsf{r})(x) = r_{q \xrightarrow{\ell} q'}(x)$$
$$\mathcal{C}_{q \xrightarrow{\ell} q'}(\mathsf{g}) = \gamma_{q \xrightarrow{\ell} q'}$$

Next, we remark that the equation (16), is *guarded*, i.e., that every variable ($v$, here) only occurs under the scope of a □ operator.

Say that $q \in V$ is at *finite depth* iff there is a constant $k \in \mathbb{N}$ such that, for every path $q = q_0 \xrightarrow{\ell_1} q_1 \xrightarrow{\ell_2} \ldots \xrightarrow{\ell_i} q_i$, we have $i \leq k$. The least such constant $k$ is the *depth* of $q$ in $(V, E)$. Say that $(V, E)$ itself *has finite depth* iff every $q \in V$ is at finite depth, and the depths of all $q$s is bounded from above by a constant $k \in \mathbb{N}$. The *depth* of $(V, E)$ is the smallest such $k$.

The finite horizon case is then a consequence of the following claim.

Claim M. *Let $\mathcal{C}$ be bounded. Assume that $\sigma$ is standard, or that $\mathcal{C}$ is bounded positive. If $(V, E)$ has finite depth, then every guarded functional system $\Sigma = \{v_i \approx f_i | 1 \leq i \leq n\}$ has a bounded solution $\rho$, which is bounded positive if $\mathcal{C}$ is bounded positive. If the only variables occurring in the $f_i$s are among $v_1, \ldots, v_n$, then this solution is unique in the sense that the functions $\rho_q(v_i)$, $q \in V$, $1 \leq i \leq n$, are unique.*
This is in particular the case when $V$ and the automaton has no loop.
*Proof.* Necessarily, for every $i$, $\rho_q(v_i) = \mathcal{C} [\![f_i]\!]_q \rho$. We build $\rho_q(v)$ for each $v \in \mathsf{X}$ by induction on the depth of $q$ in $(V, E)$. I.e., we may assume (this is the induction hypothesis) that $\rho_{q'}(v)$ is already defined (in a unique way if the only variables occurring in the $f_i$s are among $v_1, \ldots, v_n$) for every $q'$ at strictly smaller depths. Since $f_i$ is guarded, this implies that $\mathcal{C} [\![f_i]\!]_q \rho$ is itself well-defined (and unique if the only variables occurring in the $f_i$s are among $v_1, \ldots, v_n$), by an easy structural induction proof on $f_i$. This defines (in a unique way again if the only variables occurring in the $f_i$s are among $v_1, \ldots, v_n$) the values $\rho_q(v_i)$, $1 \leq i \leq n$. For every variable $v$ outside $v_1, \ldots, v_n$, define $\rho_q(v)$ arbitrarily, say $\rho_q(v)(x) = 0$.
An easy induction on the depth of $q$ shows that if $\mathcal{C}$ is bounded by $[a, b]$, where wlog. $a \leq 0, b \geq 0$, then $\rho_q(v)$ is bounded by an interval that is independent of the function expressions involved and of the depth of $q$, and only depends on $q$ itself. Moreover, if $\mathcal{C}$ is bounded positive, we may choose $a = 0$, therefore $\rho_q(v)$ takes only non-negative values. □

In the discounted case, define the *expansion factor* $Exp_\gamma^{v_0}(f)$ of the function expression $f$ in the variable $v_0$ by:

$$Exp_\gamma^{v_0}(\mathsf{r}) = 0 \quad Exp_\gamma^{v_0}(v) = \begin{cases} 1 \text{ if } v = v_0 \\ 0 \text{ otherwise} \end{cases}$$
$$Exp_\gamma^{v_0}(f_1 + f_2) = Exp_\gamma^{v_0}(f_1) + Exp_\gamma^{v_0}(f_2) \quad Exp_\gamma^{v_0}(\max(f_1, f_2)) = \max(Exp_\gamma^{v_0}(f_1), Exp_\gamma^{v_0}(f_2))$$
$$Exp_\gamma^{v_0}(\mathsf{g} \times f) = \gamma.Exp_\gamma^{v_0}(f) \quad Exp_\gamma^{v_0}(\min(f_1, f_2)) = \max(Exp_\gamma^{v_0}(f_1), Exp_\gamma^{v_0}(f_2))$$
$$Exp_\gamma^{v_0}(\Box_A f^{\rightarrow}) = Exp_\gamma^{v_0}(f^{\rightarrow}) \quad Exp_\gamma^{v_0}\left(\int f\right) = Exp_\gamma^{v_0}(f)$$
$$Exp_\gamma^{v_0}(\Diamond_A f^{\rightarrow}) = Exp_\gamma^{v_0}(f^{\rightarrow})$$

Claim N. *Let $\mathcal{C}$ be bounded, $\rho$ and $\rho'$ be two bounded environments, and assume $\sigma$ is standard, or $\mathcal{C}$, $\rho$, $\rho'$ are bounded positive. Let $\gamma \in \mathbb{R}^+$ be a constant such that $\mathcal{C}_q(\mathsf{g}) \leq \gamma$ for every $\mathsf{g} \in \mathsf{G}$ and $q \in V$. Assume that $\rho_q(v) = \rho'_q(v)$ for every variable $v \neq v_0$. Then for every functional expression $f$, for every $q \in V$, every $(q, \ell, q') \in E$, for every $x \in X$:*

$$\mathcal{C}\,[\![f]\!]_q\,\rho(x) - \mathcal{C}\,[\![f]\!]_q\,\rho'(x) \leq Exp_\gamma^{v_0}(f).\max(\rho_q(v_0)(x) - \rho'_q(v_0)(x), 0)$$
$$\mathcal{C}\,[\![f^{\rightarrow}]\!]_{q\xrightarrow{\ell}q'}\,\rho(x) - \mathcal{C}\,[\![f^{\rightarrow}]\!]_{q\xrightarrow{\ell}q'}\,\rho'(x) \leq Exp_\gamma^{v_0}(f^{\rightarrow}).\max(\rho_{q\xrightarrow{\ell}q'}(v_0)(x) - \rho'_{q\xrightarrow{\ell}q'}(v_0)(x), 0)$$

*where $d_{\mathbb{R}}(x, y) = \max(x - y, 0)$.*
In other words, the evaluation function of $f$ is $Exp_\gamma^{v_0}(f)$-Lipschitz in $\rho(v_0)(x)$, where a function $g : \mathbb{R} \to \mathbb{R}$ is $A$-Lipschitz iff $d_{\mathbb{R}}(g(x), g(y)) \leq A d_{\mathbb{R}}(x, y)$ for every $x, y \in \mathbb{R}$.
*Proof.* By structural induction on $f$, $f^{\rightarrow}$, we show that:

$$\mathcal{C}\,[\![f]\!]_q\,\rho(x) \leq \mathcal{C}\,[\![f]\!]_q\,\rho'(x) + Exp_\gamma^{v_0}(f).\max(\rho_q(v_0)(x) - \rho'_q(v_0)(x), 0)$$
$$\mathcal{C}\,[\![f^{\rightarrow}]\!]_{q\xrightarrow{\ell}q'}\,\rho(x) \leq \mathcal{C}\,[\![f^{\rightarrow}]\!]_{q\xrightarrow{\ell}q'}\,\rho'(x) + Exp_\gamma^{v_0}(f^{\rightarrow}).\max(\rho_{q\xrightarrow{\ell}q'}(v_0)(x) - \rho'_{q\xrightarrow{\ell}q'}(v_0)(x), 0)$$

This is clear when $f$, resp. $f^{\rightarrow}$, is a reward parameter $\mathsf{r}$, a variable $v$ (equal to $v_0$ or not). In the other cases, let $a_q = \max(\rho_q(v_0)(x) - \rho'_q(v_0)(x), 0)$, and $a_{q\xrightarrow{\ell}q'} = \max(\rho_{q\xrightarrow{\ell}q'}(v_0)(x) - \rho'_{q\xrightarrow{\ell}q'}(v_0)(x), 0)$, to lighten up notation somewhat.
If $f$ is of the form $\max(f_1, f_2)$, this is a consequence of: $(*)$ $\max(a + b, c + d) \leq \max(a, c) + \max(b, d)$. (Because $a + b$ and $c + d$ are both less than or equal to the right-hand side.) Indeed,

$$\begin{aligned}
\mathcal{C}\,[\![\max(f_1, f_2)]\!]_q\,\rho(x) &= \max\left(\mathcal{C}\,[\![f_1]\!]_q\,\rho(x) + \mathcal{C}\,[\![f_2]\!]_q\,\rho(x)\right) \\
&\leq \max\left(\mathcal{C}\,[\![f_1]\!]_q\,\rho'(x) + Exp_\gamma^{v_0}(f_1).a_q, \mathcal{C}\,[\![f_2]\!]_q\,\rho'(x) + Exp_\gamma^{v_0}(f_2).a_q\right) \\
&\qquad \text{by induction hypothesis} \\
&\leq \max\left(\mathcal{C}\,[\![f_1]\!]_q\,\rho'(x), \mathcal{C}\,[\![f_2]\!]_q\,\rho'(x)\right) + \max(Exp_\gamma^{v_0}(f_1).a_q, Exp_\gamma^{v_0}(f_2).a_q) \\
&\qquad \text{par } (*) \\
&= \mathcal{C}\,[\![\max(f_1, f_2)]\!]_q\,\rho'(x) + Exp_\gamma^{v_0}(\max(f_1, f_2)).a_q
\end{aligned}$$

since $a_q \geq 0$, and by definition.
If $f$ is of the form $\min(f_1, f_2)$, proceed similarly, using the inequality: $\min(a + b, c+d) \leq \min(a, c) + \max(b, d)$. The latter is proved by noting that if $b \geq d$, then $\min(a+b, c+d) \leq \min(a+b, c+b) = \min(a, c) + b = \min(a, c) + \max(b, d)$, and similarly if $b \leq d$.
If $f$ is of the form $f_1 + f_2$, we use the equality $(a+b)+(c+d) = (a+c)+(b+d)$.
If $f$ is of the form g $\times f_1$:

$$\begin{aligned}
\mathcal{C}\llbracket \mathrm{g} \times f_1 \rrbracket_q \rho(x) &= \mathcal{C}_q(\mathrm{g}).\mathcal{C}\llbracket f_1 \rrbracket_q \rho(x) \\
&\leq \mathcal{C}_q(\mathrm{g}).(\mathcal{C}\llbracket f_1 \rrbracket_q \rho'(x) + Exp_\gamma^{v_0}(f_1).a_q) \quad \text{since } \mathcal{C}_q(\mathrm{g}) \geq 0 \\
&= \mathcal{C}_q(\mathrm{g}).\mathcal{C}\llbracket f_1 \rrbracket_q \rho'(x) + \mathcal{C}_q(\mathrm{g}).Exp_\gamma^{v_0}(f_1).a_q \\
&\leq \mathcal{C}_q(\mathrm{g}).\mathcal{C}\llbracket f_1 \rrbracket_q \rho'(x) + \gamma.Exp_\gamma^{v_0}(f_1).a_q \\
&\qquad \text{by assumption, and since } Exp_\gamma^{v_0}(f_1) \geq 0, a_q \geq 0 \\
&= \mathcal{C}\llbracket \mathrm{g} \times f_1 \rrbracket_q \rho'(x) + Exp_\gamma^{v_0}(\mathrm{g} \times f_1).a_q
\end{aligned}$$

When $f = \Box_A f^{\rightarrow}$,

$$\begin{aligned}
\mathcal{C}\llbracket \Box_A f^{\rightarrow} \rrbracket_q \rho(x) &= \sup_{\ell \in A, q' \in V / q \xrightarrow{\ell} q'} \mathcal{C}\llbracket f^{\rightarrow} \rrbracket_{q \xrightarrow{\ell} q'} \rho(x) \\
&\leq \sup_{\ell \in A, q' \in V / q \xrightarrow{\ell} q'} \left[\mathcal{C}\llbracket f^{\rightarrow} \rrbracket_{q \xrightarrow{\ell} q'} \rho'(x) + Exp_\gamma^{v_0}(f^{\rightarrow}).a_q\right] \\
&\leq \sup_{\ell \in A, q' \in V / q \xrightarrow{\ell} q'} \mathcal{C}\llbracket f^{\rightarrow} \rrbracket_{q \xrightarrow{\ell} q'} \rho'(x) + Exp_\gamma^{v_0}(f^{\rightarrow}).a_q \\
&= \mathcal{C}\llbracket \Box_A f^{\rightarrow} \rrbracket_q \rho'(x) + Exp_\gamma^{v_0}(\Box_A f^{\rightarrow}).a_q
\end{aligned}$$

We conclude similarly when $f = \Diamond_A f^{\rightarrow}$. Finally, let us deal with the case $\int f$:

$$\begin{aligned}
\mathcal{C}\left\llbracket \int f \right\rrbracket_{q \xrightarrow{\ell} q'} \rho(x) &= \oint_{y \in X} \mathcal{C}\llbracket f \rrbracket_{q'} \rho(y) d\sigma_\ell(x) \\
&\leq \oint_{y \in X} \left[\mathcal{C}\llbracket f \rrbracket_{q'} \rho'(y) + Exp_\gamma^{v_0}(f).a_q\right] d\sigma_\ell(x) \\
&\leq \oint_{y \in X} \mathcal{C}\llbracket f \rrbracket_{q'} \rho'(y) d\sigma_\ell(x) + Exp_\gamma^{v_0}(f).a_q \\
&\qquad \text{since Choquet integration is additive on comonotonic functions,} \\
&\qquad \text{any constant function is comonotonic with any function, and } \sigma_\ell(x)(X) \leq 1 \\
&= \mathcal{C}\left\llbracket \int f \right\rrbracket_{q \xrightarrow{\ell} q'} \rho'(x) + Exp_\gamma^{v_0}\left(\int f\right).a_q
\end{aligned}$$

□

<u>Claim O.</u> *Let $\mathcal{C}$ be bounded, and assume $\sigma$ standard or $\mathcal{C}$ bounded positive. Let $\Sigma = \{v_i \approx f_i | 1 \leq i \leq n\}$ a functional system where the only variables that occur are $v_1, \ldots, v_n$, and assume that there are three constants $a \leq 0$, $b \geq 0$, and $\gamma \geq 0$ such that for every vertex $q \in V$ and every transition $(q, \ell, q') \in E$:*

– *for every* $\mathsf{r} \in \mathsf{R}$, $\mathcal{C}_q(\mathsf{r})(x) \in [a, b]$ *and* $\mathcal{C}_{q \xrightarrow{\ell} q'}(\mathsf{r})(x) \in [a, b]$*;*
– *for every* $i$, $1 \leq i \leq n$, $\sum_{j=1}^{n} Exp_\gamma^{v_j}(f_i) < 1$.

*Then* $\Sigma$ *has a unique solution* $\rho$*, which is bounded positive as soon as* $\mathcal{C}$ *is bounded positive.*

*Proof.* Let $\mathcal{C}^\downarrow$ be the context mapping every reward parameter $\mathsf{r}$ to $a$, at every internal state $q$ and every transition $q \xrightarrow{\ell} q'$. Similarly, let $\mathcal{C}^\uparrow$ be the context mapping every reward parameter $\mathsf{r}$ to $b$. Clearly, $\mathcal{C}^\downarrow \llbracket f \rrbracket_q \rho(x) \leq \mathcal{C} \llbracket f \rrbracket_q \rho(x) \leq \mathcal{C}^\downarrow \llbracket f \rrbracket_q \rho(x)$ et $\mathcal{C}^\downarrow \llbracket f \rrbracket_{q \xrightarrow{\ell} q'} \rho(x) \leq \mathcal{C} \llbracket f \rrbracket_{q \xrightarrow{\ell} q'} \rho(x) \leq \mathcal{C}^\downarrow \llbracket f \rrbracket_{q \xrightarrow{\ell} q'} \rho(x)$.

Let $c = \max_{1 \leq i \leq n} \sum_{j=1}^{n} Exp_\gamma^{v_j}(f_i) < 1$.

We first show: $(*)$ there is an environment $\rho$ such that $\mathcal{C}^\downarrow \llbracket f_i \rrbracket_q \rho(x) \geq \rho_q(v_i)(x)$ for every $q \in V$, every $x \in X$, and every $i$, $1 \leq i \leq n$. If $\mathcal{C}$ is bounded positive, this is obvious: let $\rho_q(v_i)(x) = 0$. Otherwise, let $\rho[t]$ be the environment defined by $\rho[t]_q(v)(x) = t$, for every $t \in \mathbb{R}$, and let $g_q(t) = \min_{1 \leq i \leq n} \mathcal{C}^\downarrow \llbracket f_i \rrbracket_q \rho[t](x)$, for some arbitrary $x \in X$. Note indeed that the latter quantity is independent of $x$. Now use Claim N with $v_0$ being $v_1$, then $v_2$, then $\ldots$, then $v_n$; we obtain $g_q(0) - g_q(t) \leq Exp_\gamma^{v_1}(f_i). \max(\rho[0]_q(v_1)(x) - \rho[t]_q(v_1)(x), 0) + \ldots + Exp_\gamma^{v_n}(f_i). \max(\rho[0]_q(v_n)(x) - \rho[t]_q(v_n)(x), 0)$ (for every $i$, $1 \leq i \leq n$) $\leq c. \max(-t, 0)$. If $t < 0$, therefore, $g_q(t) \geq g_q(0) + ct$. Note that $g_q(0)$ is bounded from below, independently of $q$, by the non-positive real $\min_{1 \leq i \leq n} lo(f_i)$, where $lo(f)$ is defined by:

$$
\begin{aligned}
lo(\mathsf{r}) &= a \\
lo(v) &= 0 \\
lo(\max(f_1, f_2)) &= \max(lo(f_1), lo(f_2)) \\
lo(\min(f_1, f_2)) &= \min(lo(f_1), lo(f_2)) \\
lo(f_1 + f_2) &= lo(f_1) + lo(f_2) \\
lo(\mathsf{g} \times f) &= lo(f) \\
lo(\Box_A f^\rightarrow) &= lo(f^\rightarrow) \\
lo(\Diamond_A f^\rightarrow) &= lo(f^\rightarrow) \\
lo\left(\int f\right) &= lo(f)
\end{aligned}
$$

So $g_q(t) \geq t$ as soon as $t \leq \min_{1 \leq i \leq n} lo(f_i)/(1-c)$. Fix such a negative real $t$. Then $\rho = \rho[t]$ satisfies $(*)$.

Let now $\rho_0$ be an arbitrary environment $\rho$ satisfying $(*)$. By induction on $k \in \mathbb{N}$, define an environment $\rho_k$ by $\rho_{(k+1)q}(v_i) = \mathcal{C} \llbracket f_i \rrbracket_q \rho_k$ for every $i$, $1 \leq i \leq n$, and every $q \in V$. By $(*)$, $\rho_{1q}(v_i)(x) = \mathcal{C} \llbracket f_i \rrbracket_q \rho_0(x) \geq \mathcal{C}^\downarrow \llbracket f_i \rrbracket_q \rho_0(x) \geq \rho_{0q}(v_i)(x)$ for every $i$, $1 \leq i \leq n$, and every $x \in X$. It is easy to see that $\mathcal{C} \llbracket f \rrbracket_q \rho(x)$ is a monotonic function of $\rho(v_i)(x)$, $1 \leq i \leq n$. So, by induction on $k \in \mathbb{N}$, $\rho_{(k+1)q}(v_i) \geq \rho_{kq}(v_i)$ for every $k$, $q$, $i$. It follows that the sequence $(\rho_k)_{k \in \mathbb{N}}$ is increasing.

Then, $\rho_k$ is bounded from above, at each $q \in V$ and each $x \in X$. Indeed, for every $k \geq 1$:

$$\begin{aligned}
\rho_{(k+1)q}(v_i)(x) - \rho_{kq}(v_i)(x) &= \mathcal{C}\,[\![f_i]\!]_q\,\rho_k(x) - \mathcal{C}\,[\![f_i]\!]_q\,\rho_{k-1}(x) \\
&\leq Exp_\gamma^{v_1}(f_i).\max(\rho_{kq}(v_1)(x) - \rho_{(k-1)q}(v_1)(x), 0) + \\
&\quad \ldots + Exp_\gamma^{v_n}(f_i).\max(\rho_{kq}(v_n)(x) - \rho_{(k-1)q}(v_n)(x), 0) \\
&\qquad \text{by Claim N} \\
&\leq c.\max_{1\leq j\leq n}(\rho_{kq}(v_j)(x) - \rho_{(k-1)q}(v_j)(x))
\end{aligned}$$

So by induction on $k \in \mathbb{N}$, we infer:

$$\rho_{(k+1)q}(v_i)(x) - \rho_{kq}(v_i)(x) \leq c^k.\max_{1\leq j\leq n}(\rho_{1q}(v_j)(x) - \rho_{0q}(v_j)(x))$$

Summing up, and observing that $1 + c + c^2 + \ldots + c^{k-1} \leq 1/(1-c)$, we obtain:

$$\rho_{kq}(v_i)(x) \leq \rho_{0q}(v_i)(x) + \frac{\max_{1\leq j\leq n}(\rho_{1q}(v_j)(x) - \rho_{0q}(v_j)(x))}{1-c}$$

So $\rho_q(v_i)(x) = \sup_{k\in\mathbb{N}} \rho_{kq}(v_i)(x)$ is well defined. An easy structural induction on $f$, $f^{\rightarrow}$ shows that:

$$\begin{aligned}
\mathcal{C}\,[\![f]\!]_q\,\rho(x) &= \sup_{k\in\mathbb{N}} \mathcal{C}\,[\![f]\!]_q\,\rho_k(x) \\
\mathcal{C}\,[\![f^{\rightarrow}]\!]_{q\xrightarrow{\ell}q'}\,\rho(x) &= \sup_{k\in\mathbb{N}} \mathcal{C}\,[\![f^{\rightarrow}]\!]_{q\xrightarrow{\ell}q'}\,\rho_k(x)
\end{aligned}$$

Indeed, all operators $\max$, $\min$, $+$, $\sup_{\ell\in A,(q,\ell,q')\in E}$, $\min_{\ell\in A,(q,\ell,q')\in E}$ are Scott-continuous, and the Choquet integral is also Scott-continuous in the function to integrate. Therefore:

$$\begin{aligned}
\rho_q(v_i)(x) &= \sup_{k\geq 1} \rho_{kq}(v_i)(x) \\
&= \sup_{k\in\mathbb{N}} \mathcal{C}\,[\![f_i]\!]_q\,\rho_k(x) = \mathcal{C}\,[\![f_i]\!]_q\,\rho(x)
\end{aligned}$$

So $\rho$ is a solution of $\Sigma$.
Finally, let us show that this solution is unique. If $\rho$ and $\rho'$ are two solutions, then, still using Claim N:

$$\begin{aligned}
\rho_q(v_i)(x) - \rho'_q(v_i)(x) &= \mathcal{C}\,[\![f_i]\!]_q\,\rho(x) - \mathcal{C}\,[\![f_i]\!]_q\,\rho'(x) \\
&\leq c.\max_{1\leq j\leq n}\max(\rho_q(v_j)(x) - \rho'_q(v_j)(x), 0)
\end{aligned}$$

for every $i$, $1 \leq i \leq n$. Since $c < 1$, it follows that $\max_{1\leq i\leq n}(\rho_q(v_i)(x) - \rho'_q(v_i)(x)) \leq 0$. By symmetry, $\max_{1\leq i\leq n}(\rho_q(v_i)(x) - \rho'_q(v_i)(x)) = 0$, so $\rho = \rho'$. □

The second part of the Theorem then follows from Claim N. □