Jean Goubault-Larrecq

Believe It Or Not, GOI is a Model of Classical Linear Logic

Research Report LSV-07-03

January 2007

Laboratoire
Spécification
et
Vérification

CENTRE NATIONAL
DE LA RECHERCHE

Ecole Normale Supérieure de Cachan
61, avenue du Président Wilson
94235 Cachan Cedex France

Believe It Or Not, GOI is a Model of Classical Linear Logic

Jean Goubault-Larrecq*
LSV/ CNRS UMR 8643 & INRIA Futurs projet SECSI & ENS Cachan, France
61, avenue du président-Wilson, F-94235 Cachan
`goubault@lsv.ens-cachan.fr`

Abstract

We introduce the Danos-Régnier category $\mathcal{DR}(M)$ of a linear inverse monoid M, a categorical description of geometries of interaction (GOI). The usual setting for GOI is that of a weakly Cantorian linear inverse monoid. It is well-known that GOI is perfectly suited to describe the multiplicative fragment of linear logic, and indeed $\mathcal{DR}(M)$ will be a $*$-autonomous category in this case. It is also well-known that the categorical interpretation of the other linear connectives conflicts with GOI interpretations. We make this precise, and show that $\mathcal{DR}(M)$ has no terminal object, no cartesian product, and no exponential—whatever M is, unless M is trivial. However, a form of coherence completion of $\mathcal{DR}(M)$ à la Hu-Joyal provides a model of full classical linear logic, as soon as M is weakly Cantorian.

1. Introduction

There are by now several families of models for (classical) linear logic. One is the category of *coherence spaces* [16]. Another is given by game models, e.g. [4]. Contrarily to what one might expect, geometry of interaction, in whatever form [13, 14, 15, 3] does not yield models of linear logic. This is surprising, as geometry of interaction was expressly invented to give a solid and original semantics to linear logic. Now by *model* of linear logic we are rather demanding, and mean a *categorical* model. The definition of categorical models of linear logic took some time to emerge, and is certainly posterior to geometry of interaction. We shall consider linear categories [7], LNL categories [6], Lafont and new-Lafont categories [25]. It is remarkable that coherence spaces form a model in all these senses, but other proposals based on games or geometry of interaction do not. The point is subtle: e.g., Baillot *et al.* [4] show that AJM games are a model of MELL proof nets (i.e., without the additives) without box erasure steps, and argue that modeling the latter "seems desperate with games".

In a sense, there are categorical models of a domain-theoretic style, but none coming from the interaction world. This paper bridges the gap. Our main contribution is a categorical model of full classical linear logic, including multiplicative, exponential and additive connectives, based on ideas from geometry of interaction—specifically from Danos and Régnier [10, 9]—and also using the notion of *coherence completion* [20]. So we import from both interaction and domain theory. The thread that unites the two will be *coherence*, which plays a fundamental role in both.

Outline. The material in this paper will import many notions from category theory and linear logic. Most of them are explained in the full version [17], to which we shall also refer the reader for proofs. However we feel that at least some intuition about the roots of this work should be brought forward, and this is the topic of Section 2. We introduce the new concept of a *linear inverse semigroup* M in Section 3, and show in Section 4 how any such M gives rise to a category $\mathcal{DR}(M)$, which we call the *Danos-Régnier category* of M. We shall also see that, provided M is *weakly Cantorian*, $\mathcal{DR}(M)$ is $*$-autonomous, i.e., a model of the multiplicative fragment MLL of linear logic. The purpose of Section 5 is to compare this construction to the $\mathcal{G}$ construction of Abramsky and coauthors, the most prominent categorical interpretation of geometry of interaction. On our way to get a categorical model of the whole of linear logic, we shall then trip on a serious difficulty: we shall show in Section 6 and Section 7 that there is no way to interpret *any* form of additive or exponential connective in $\mathcal{DR}(M)$, whatever M. I.e., changing the languages of paths won't help. Nonetheless, we show in Section 8 that a slight modification of Hu and Joyal's *coherence completion* [20] builds a Lafont category out of any $*$-autonomous category, i.e., a model of full classical linear logic out of any model of just MLL... and this is exactly what $\mathcal{DR}(M)$ provides.

Related Work. We shall heavily discuss related work throughout the paper, notably the construction of compact-

---

*Partially supported by ACI NIM "GeoCal" (geom. of computation).

closed categories from traced monoidal categories [1, 21] in Section 5, and coherence completions [20] in Section 8. The idea of considering inverse monoids is credited to Yves Legrandgérard by Danos and Régnier [10]. As far as the impossibility results mentioned in Sections 6 and 7 are concerned, it was well-known before that trying to add specific new equations between geometry of interaction tokens, aimed at enforcing some categorical identities, resulted in inconsistencies. Our impossibility results are much stronger: we show that *no* change in the underlying inverse monoid M can result in the creation of *any* missing categorical feature (additive, exponential).

Acknowledgments. Heartfelt thanks to P.-A. Melliès, V. Danos, Ph. Scott, and the GeoCal group for advice and support.

2. Motivation.

I came to study inverse monoids following Danos *et al.* [9], where weights from the so-called dynamic algebra arise from an inverse monoid with some added structure (the *bar*, which captures the reduction process). However, my actual initial goal was to try and understand how one may describe Böhm-like trees of lambda-terms up to β- or $\beta\eta$-equivalence, not as trees, but as collections of paths through these trees. (A goal I have not reached yet.)

Let us see what this means on (infinite) trees. By tree we mean some form of infinite first order term: each node t is labeled by some function symbol f of some arity $n \in \mathbb{N}$, and has n successors $t_1, \ldots, t_n$; we then agree to write t as $f(t_1, \ldots, t_n)$. We call Σ the given signature, i.e., the set of all function symbols, together with their respective arities. We write $f/n \in \Sigma$ to state that f is in Σ, with arity n. With each such f/n in Σ, we associate n distinct letters f_1, $\ldots$, f_n. (We need to adjust this when $n = 0$, in all rigor.) This yields the path alphabet $|\mathcal{A}| = \bigcup_{f/n \in \Sigma} \{f_1, \ldots, f_n\}$. Its elements are the path letters. Now call *path* any finite sequence of path letters. Traveling down a tree along any route from the root yields a path in the obvious way. For example, the tree $f(g(t_1, t_2), t_3)$ has (at least) the paths ϵ (the empty path), f_1, $f_1 g_1$, $f_1 g_2$, f_2.

Going from a tree to its set of paths is easy. Recovering a tree from a given set of paths is harder. Foremost, not every set of paths arises as a set of paths of some tree. For example, $\{f_1, g_1\}$ cannot arise as a set of paths of any tree.

The key point to enable this reconstruction process is *coherence*. This was invented under a different name by Harrison and Havel [18]. There is an equivalence relation $\equiv$ on the path alphabet, defined as $f_i \equiv g_j$ if and only if $f = g$. Now define the relation $\stackrel{\frown}{\smile}$ on paths by $w \stackrel{\frown}{\smile} w'$ if and only if, for any strict common prefix w_0 of w and w', writing w as $w_0 a w_1$ and w' as $w_0 a' w'_1$ with $a, a' \in |\mathcal{A}|$, then $a \equiv a'$. The $\stackrel{\frown}{\smile}$ relation is reflexive and symetric, though in general not transitive. When $w \stackrel{\frown}{\smile} w'$, we say that w and w' are *coherent*, and a *clique* is any set of pairwise coherent paths. It is clear that any set of paths of a given tree is a clique. The point is that any clique is a set of paths of some tree. So we *can* describe sets of paths of trees without mentioning the trees themselves, provided we rely on coherence.

A space $X = (|X|, \stackrel{\frown}{\smile})$ where $\stackrel{\frown}{\smile}$ is a reflexive and symmetric relation on $|X|$ is a *coherence space* [16]. So there is a coherence space of paths, $(|\mathcal{A}|^*, \stackrel{\frown}{\smile})$. Coherence spaces are the basis of an elegant semantics of the lambda-calculus, and in fact of the whole of linear logic [16].

Let us refine. Let $\leq$ be the prefix ordering on paths. Then $w \leq w'$ and $w' \stackrel{\frown}{\smile} w''$ implies $w \stackrel{\frown}{\smile} w''$: $(|\mathcal{A}|^*, \leq, \stackrel{\frown}{\smile})$ is a bit more than a coherence space, it is an *event structure*, namely a space $X = (|X|, \leq, \stackrel{\frown}{\smile})$ where $\leq$ is a partial ordering on $|X|$ and $\stackrel{\frown}{\smile}$ is a reflexive and symmetric relation on $|X|$ such that $w \leq w'$ and $w' \stackrel{\frown}{\smile} w''$ implies $w \stackrel{\frown}{\smile} w''$. Event structures are a fundamental model of concurrency [26], where, instead of using $\stackrel{\frown}{\smile}$, a binary irreflexive and symmetric relation $\#$ called *conflict* is used, such that $w \leq w'$ and $w \# w''$ implies $w' \# w''$. (This is equivalent, taking coherence $\stackrel{\frown}{\smile}$ as negation of conflict $\#$.) Then the set of paths in a tree is a *down-closed* clique, and conversely any down-closed clique gives rise to at least one tree.

In the case of lambda-terms, as opposed to infinite first-order terms, there is an additional difficulty: lambda-terms β-reduce to other lambda-terms, and we would like to be able to define a notion of paths through lambda-terms that is *invariant* under $\beta\eta$-equivalence. The result will be a way to compute sets of paths (in the usual sense) through the Böhm tree of t by just computing sets of paths through t itself—*without* reducing t. This is exactly, in my view, what geometry of interaction is about. Girard's execution formula aims at being such an invariant. Our view is that such an invariant should be a denotational (categorical) model of lambda-calculus, and in fact of linear logic proofs.

3. Linear Inverse Semigroups

Such a calculus of paths for MLL terms is lurking around in [10] and [9], where the notion of a (bar) inverse monoid is crucial. The essential quantity that remains invariant through reduction is the set of all weights of paths through a proof net. Note that this cannot be defined in a modular way: if you know the weights of all paths in (the proof net of) a λ-term M and also that for a λ-term N, you cannot infer the weights of paths through the application MN. The reason is that not all paths can be considered: we must only consider those paths that are *legal* and *straight*. The latter condition, in particular cannot be defined on weights alone; the paths themselves have to be taken into account. Our aim here and in Section 4 is to define a semantics $[\![.]\!]$ of

MLL proof nets (which we do by building a $*$-autonomous category) in terms of weights, eliminating the pollution of paths, which only reflect some form of syntax. The key is to collect, not sets, but least upper bounds of cliques in the inverse monoid of weights.

Recall that an *inverse semigroup* is a triple $(M, \cdot, \_^*)$ where $(M, \cdot)$ is a semigroup (i.e., $\cdot$ is associative) and $\_^*$ is a unary operation that satisfies

$$
\begin{array}{ll}
(u^*)^* = u & (uv)^* = v^* u^* \\
uu^*u = u & uu^*vv^* = vv^*uu^*
\end{array}
$$

where the notation uv abbreviates $u \cdot v$. An *inverse monoid* is an inverse semigroup with a multiplicative unit 1.

A paradigmatic example is the space of all partial injections $\mathsf{PI}(E)$ on a given set E. A *partial injection* on E is any binary relation u on E such that every $x \in E$ is mapped to at most one $y \in E$ such that $(x, y) \in u$, and every $y \in E$ is mapped to at most one $x \in E$ such that $(x, y) \in u$. Alternatively, this is a bijection between two subsets of E, called the *domain* $\{x | \exists y \cdot (x, y) \in u\}$ and the *codomain* $\{y | \exists x \cdot (x, y) \in u\}$ of u. $\mathsf{PI}(E)$ is an inverse monoid with 1 the identity on E, composition as multiplication, and star defined as inversion: $u^* = \{(y, x) | (x, y) \in u\}$.

Following [9], write $\langle u \rangle = uu^*$. In $\mathsf{PI}(E)$, this is the identity between the codomain of u and itself, which we shall identify with the codomain of u. Similarly, we think of $\langle u^* \rangle$ as the domain of u, even when M is not of the form $\mathsf{PI}(E)$. Space does not permit us to include most of the proofs; all can be found in the full report [17]. However, we shall often given intuitions of proofs on inverse semigroups in terms of $\mathsf{PI}(E)$. This is all the more justified as, by the Preston-Wagner Theorem, every inverse semigroup M embeds into some inverse monoid of the form $\mathsf{PI}(E)$.

Recall that an *idempotent* in M is any element such that $uu = u$. In any inverse semigroup, the idempotents are exactly the terms of the form $\langle u \rangle$, and every idempotent u satisfies $u = \langle u \rangle = u^* = \langle u^* \rangle$. The defining equation $uu^*vv^* = vv^*uu^*$, equivalently $\langle u \rangle \langle v \rangle = \langle v \rangle \langle u \rangle$, states that idempotents commute.

The *natural ordering* $\leq$ on M may be defined in a variety of ways. The idea is that it should correspond to inclusion between graphs of relations in the case of the inverse monoid $\mathsf{PI}(E)$. Equivalent ways are to define $u \leq v$ iff $vu^* = uu^*$, or $uv^* = uu^*$, or $\langle u \rangle v = u$, or $v \langle u^* \rangle = u$, or $u^*v = u^*u$, or $v^*u = u^*u$. Then $\leq$ is a partial ordering, and multiplication and inverse are monotonic.

This is well-known, see [27, 24]. The main import of this Section is that every inverse semigroup also has a coherence relation that makes it an event structure. The intuition is that, if $u, v \in \mathsf{PI}(E)$ and there is an element x which is mapped by u and v to different elements, either forward (for some $y \neq y'$, $(x, y) \in u$ and $(x, y') \in v$) or backward (for some $y \neq y'$, $(y, x) \in u$ and $(y', x) \in v$), then u and v should be in conflict, in notation $u \# v$. Recall that $\overset{\frown}{\smile}$ is the negation of conflict. This can be defined algebraically:

Definition 3.1 (Coherence) *Let M be an inverse semigroup. The relations $\overset{\frown}{\smile}_0$, $\overset{\frown}{\smile}_1$ and $\overset{\frown}{\smile}$ on M are defined by:*

$$
u \overset{\frown}{\smile}_0 v \textit{ iff } u \langle v^* \rangle = v \langle u^* \rangle \qquad u \overset{\frown}{\smile}_1 v \textit{ iff } \langle v \rangle u = \langle u \rangle v
$$
$$
u \overset{\frown}{\smile} v \textit{ iff } u \overset{\frown}{\smile}_0 v \text{ and } u \overset{\frown}{\smile}_1 v
$$

We can show that $u \overset{\frown}{\smile} v$ iff $u^* \overset{\frown}{\smile} v^*$, and more importantly:

Lemma 3.2 *Let M be an inverse semigroup. Then $(M, \leq, \overset{\frown}{\smile})$ is an event structure: if $u \leq v$ and $v \overset{\frown}{\smile} w$, then $u \overset{\frown}{\smile} w$.*

Proof. We show that, more precisely: if $u \leq v$ and $v \overset{\frown}{\smile}_0 w$, then $u \overset{\frown}{\smile}_0 w$. By passing to inverses, we will deduce that $u \leq v$ and $v \overset{\frown}{\smile}_1 w$ imply $u \overset{\frown}{\smile}_1 w$, whence the claim. So assume $u \leq v$ and $v \overset{\frown}{\smile}_0 w$. Since $u \leq v$, $v \langle u^* \rangle = u$ and $v^*u = \langle u^* \rangle$, so: (a) $\langle v^* \rangle \langle u^* \rangle = v^*u = \langle u^* \rangle$. Since $v \overset{\frown}{\smile}_0 w$, $v \langle w^* \rangle = w \langle v^* \rangle$. Since $u \leq v$, $v \langle u^* \rangle = u$, so $u \langle w^* \rangle = v \langle u^* \rangle \langle w^* \rangle = v \langle w^* \rangle \langle u^* \rangle = w \langle v^* \rangle \langle u^* \rangle = w \langle u^* \rangle$ by (a), whence $u \overset{\frown}{\smile}_0 w$. ☐

It follows that any two elements that have an upper bound in M are coherent; also, that $u \leq v$ implies $u \overset{\frown}{\smile} v$. This is as in any other event structure. Additionally, multiplication preserves coherence: $u_0 \overset{\frown}{\smile} v_0$ and $u_1 \overset{\frown}{\smile} v_1$ imply $u_0u_1 \overset{\frown}{\smile} v_0v_1$. As can be expected from the intuitive description of $\overset{\frown}{\smile}$, if $u \overset{\frown}{\smile} v$ in M, then u and v have an greatest lower bound $u \wedge v$, and $u \wedge v = u \langle v^* \rangle = v \langle u^* \rangle = \langle v \rangle u = \langle u \rangle v$.

Definition 3.3 *An inverse semigroup M is* linear *if and only if: (1) every clique $(u_i)_{i \in I}$ has a least upper bound $\sum_{i \in I} u_i$, and (2) multiplication distributes over least upper bounds of cliques, i.e., for every clique $(u_i)_{i \in I}$, for every element v, $\left(\sum_{i \in I} u_i\right) v = \sum_{i \in I} u_i v$.*

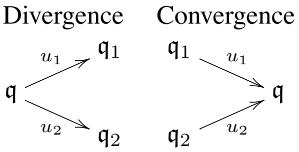
Observe that $\mathsf{PI}(E)$ is always a linear inverse semigroup. The $\sum$ notation for least upper bounds of cliques is justified by the distributivity property (2). One may show that (2) implies that $u \overset{\frown}{\smile} v$ if and only if u and v have a common upper bound; in other words, $\overset{\frown}{\smile}$ coincides with the standard coherence relation associated to the natural ordering $\leq$. Distributivity is equivalent to $v \left(\sum_{i \in I} u_i\right) = \sum_{i \in I} vu_i$ (take inverses, observing that $\sum$ and inverse commute). The empty clique has a least upper bound, which we write 0 (the empty relation in $\mathsf{PI}(E)$), and distributivity implies that $0.v = v.0 = 0$. Moreover, the set of all idempotents is a clique, and its least upper bound 1 is a unit. So any linear inverse semigroup is an inverse monoid.

The construction of the Preston-Wagner Theorem establishes that any inverse semigroup M actually embeds into some linear inverse monoid: $\mathsf{PI}(M)$ itself. The embedding i_M maps $u \in M$ to the partial injection $\{(v, uv) | v \in M, v = \langle u^* \rangle v\}$. This preserves products, inverses, unit (if any), and preserves and reflects order. Coherence is also preserved: indeed coherence is defined by equations, which are preserved by the embedding.

Various linear inverse semigroups are at the heart of several works in geometry of interaction. Danos and Régnier use $\mathsf{PI}(\mathbb{N})$ at the end of [10] as an example. Girard [15] uses sets of *rudimentary clauses*, up to deletion of subsumed clauses and tautologies (see [17, Section 2.4.1] for details). Rudimentary clauses are pairs of first-order terms $s \leftarrow t$ with the same free variables. Multiplying two such clauses $s \leftarrow t$ and $s' \leftarrow t'$ yields their resolvent $s\sigma \leftarrow t'\sigma$, where σ is the mgu of t and s' if it exists, or the empty set otherwise. Inversion is given by $(s \leftarrow t)^* = (t \leftarrow s)$.

We end this section by noticing that linear inverse monoids afford us a nice graphical notation for elements, which we call *automata*. These are oriented graphs with an initial state $\mathfrak{q}_I$ and a final state $\mathfrak{q}_F$, where each state $\mathfrak{q}^A$ is labeled with an idempotent A, and each transition $\mathfrak{q}^A \xrightarrow{u} \mathfrak{q}'^B$ satisfies $\langle u^* \rangle \leq A$ and $\langle u \rangle \leq B$. (We sometimes drop the superscript, and in fact also the state name, replacing the latter by symbols such as $\bullet$ or $\circ$.) The path $\mathfrak{q}_0 \xrightarrow{u_1} \mathfrak{q}_1 \xrightarrow{u_2} \dots \mathfrak{q}_{n-1} \xrightarrow{u_n} \mathfrak{q}_n$ denotes the product $u_n \dots u_2 u_1$. (We reverse products, as in [10].) We then read the automaton $\mathcal{A}$ as the sup of all paths from $\mathfrak{q}_I$ to $\mathfrak{q}_F$.

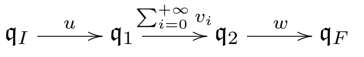
For this to make sense, the paths should form a clique. It is enough to require that $u_1 u_2^* = 0$ for any *divergence* (*forward determinacy*), and $u_1^* u_2 = 0$ for any *convergence* (*backward determinacy*). Such *bideterminacy conditions* should be expected [11, 2]. In $\mathsf{PI}(E)$, note that forward determinacy means that no element of E can be both in the domain of u_1 and in that of u_2. It is helpful to think of elements of E as tokens n that wait at some state $\mathfrak{q}^A$, and can travel along the transition $\mathfrak{q}^A \xrightarrow{u} \mathfrak{q}'^B$ if n is in the domain of u, arriving at state $\mathfrak{q}'^B$ with the new value $u(n)$. Forward determinacy means that tokens can only travel along one path at most. Tokens may also travel backwards (an important feature of the geometry of interaction), and backward determinacy imposes determinacy on backwards paths, too.

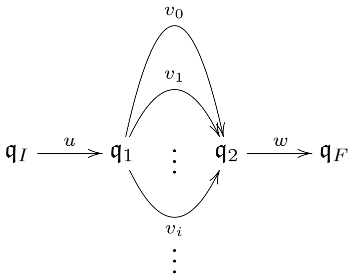


Product is concatenation, and inversion _* is given by exchanging initial and final states and replacing each transition $\mathfrak{q}^A \xrightarrow{u} \mathfrak{q}'^B$ by $\mathfrak{q}'^B \xrightarrow{u^*} \mathfrak{q}^A$.

The determinacy conditions are sufficient (but not necessary) for automata to make sense: if $u_1 u_2^* = u_1^* u_2 = 0$, then $u_1 \mathrel{\substack{\frown \\ \smile}} u_2$. In particular, we allow for non-straight paths. E.g., the path $\mathfrak{q}_0^A \xrightarrow{u} q_1^B \xrightarrow{v} \mathfrak{q}_2^C \xrightarrow{v^*} q_1^B \xrightarrow{w} \mathfrak{q}_4^D$ denoting $w \langle v^* \rangle u$ is not straight: we go from $\mathfrak{q}_1^B$ to $\mathfrak{q}_2^C$ and back through the same edge. If this path exists at all, then the path $\mathfrak{q}_0^A \xrightarrow{u} q_1^B \xrightarrow{w} \mathfrak{q}_4^D$ denoting wu is here, too. These contribute $wu + w \langle v^* \rangle u$ to the value of the whose automaton (assuming here that $\mathfrak{q}_0^A$ is initial and $\mathfrak{q}_4^D$ final). But $\langle v^* \rangle \leq 1$, so $w \langle v^* \rangle u \leq wu$. Since $+$ is least upper bound, $wu + w \langle v^* \rangle u = wu$: we don't have to forbid non-straight paths as in [10, 9]. Keep them, and their value will just not count. Similarly, illegal paths, i.e., those of value 0, do not count, since 0 is the least element of M.

Distributivity (2) allows us to graft entire automata in place of single transitions and preserve the reading of the automaton. For example, the topmost automaton on the right reads as $w \left(\sum_{i=0}^{+\infty} v_i \right) u$. This is the same reading as the bottom one, namely, $\sum_{i=0}^{+\infty} w v_i u$.



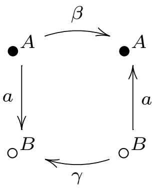


4. The Danos-Régnier Category of a Linear Inverse Monoid

The most standard construction of a category from an inverse monoid M is the *inductive groupoid* $\mathcal{IG}(M)$. Its objects are the idempotents of M, and its morphisms $A \xrightarrow{u} B$ are the elements $u \in M$ such that $\langle u^* \rangle = A$ and $\langle u \rangle = B$. There is a rich theory of inductive groupoids, which we will however not delve into. See for example [28]; be aware that the direction of arrows we adopt is the converse of what Steinberg uses.

We shall however be more interested in the following novel construction: The *Danos-Régnier category* $\mathcal{DR}(M)$ of M has all idempotents A of M as objects; its morphisms from A to B are all triples $(\beta, a, \gamma) \in M^3$ such that:

1. $aA = Ba = a$, $\beta A = A\beta = \beta$, $\gamma B = B\gamma = \gamma$;
2. $\beta^* = \beta$, $\gamma^* = \gamma$;
3. $a\beta = 0$, $\gamma a = 0$;



(1)

We represent such morphisms as automata of the form shown in Diagram (1), with four distinguished states (2 $\bullet$s, 2 $\circ$s). To guide intuition, imagine that β, a, and γ are partial injections on some set, i.e., $M = \mathsf{PI}(E)$. Then A, B can be thought as sets, and condition 1 states that the domain of a is contained in A, its codomain is contained in B, the domain and the codomain of β are contained in A, and the domain and codomain of γ are contained in B. Condition 2 is a symmetry condition on the horizontal arrows. Condition 3 is a forward determinacy condition on the upper left state and a backward determinacy condition on the lower right state. Remember that we want our model to represent sets of paths through linear λ-terms (up to normalisation). Now a λ-term t of type B with a free variable x of type A contains paths from A to B (a), but also from B to B (β) and from A to A (γ).

The *identity morphism* id_A at object A is drawn in Diagram (2). In the sequel, we shall just not draw those arrows labeled 0.

$$\begin{tikzcd}
\bullet^{A} \arrow[r, bend left, "0"] \arrow[d, "A"'] & \bullet^{A} \\
\circ^{A} & \circ^{A} \arrow[l, bend left, "0"] \arrow[u, "A"']
\end{tikzcd} \qquad (2)$$

Composition is obtained by vertical pasting. Formally, the composition of

$$\begin{tikzcd}
\bullet^{B} \arrow[r, bend left, "\beta'"] \arrow[d, "a'"'] & \bullet^{B} \\
\circ^{C} & \circ^{C} \arrow[l, bend left, "\gamma'"] \arrow[u, "a'^*"']
\end{tikzcd}$$

with

$$\begin{tikzcd}
\bullet^{A} \arrow[r, bend left, "\beta"] \arrow[d, "a"'] & \bullet^{A} \\
\circ^{B} & \circ^{B} \arrow[l, bend left, "\gamma"] \arrow[u, "a^*"']
\end{tikzcd}$$

is:

$$\begin{tikzcd}
\bullet^{A} \arrow[r, bend left, "\beta"] \arrow[d, "a"'] & \bullet^{A} \\
B \arrow[r, bend left, "\beta'"] \arrow[d, "a'"'] & B \arrow[u, "a^*"'] \arrow[l, bend left, "\gamma"] \\
\circ^{C} & \circ^{C} \arrow[l, bend left, "\gamma'"] \arrow[u, "a'^*"']
\end{tikzcd} \qquad (3)$$

where the middle Bs no longer label any of the four distinguished states. To ease reading, think of it as a condensed representation of four automata: top left to bottom left, top left to top right, bottom right to bottom left, and bottom right to top right.

This is well-defined: Condition 3 ensures that the middle B states have only forward and backward deterministic transitions. However, explicit formulae are horrible. E.g., the top left to bottom left automaton denotes $\sum_{n \in \mathbb{N}} a'(\gamma\beta')^n a$ ("go down a, loop as many times as you wish through the $\gamma\beta'$ loop, then go down a'). Note that composition is trivially associative, if we look at (3). Using explicit sums would make it a mess. However, sums such as $\sum_{n \in \mathbb{N}} a'(\gamma\beta')^n a$ are interesting: they are the incarnation of Girard's execution formula [13, 10] in our framework.

$\mathcal{DR}(M)$ is a nice category in some respects. E.g., $\mathcal{DR}(M)$ has an epi-mono factorization system (see right), all epis and all monos are split, and every morphism that is both an epi and a mono is iso.

Epis

$$\begin{tikzcd}
\bullet^{A} \arrow[r, bend left, "\beta"] \arrow[d, "a"'] & \bullet^{A} \\
\circ^{B} & \circ^{B} \arrow[u, "a^*"']
\end{tikzcd}$$

$(\langle a \rangle = B)$

Monos

$$\begin{tikzcd}
\bullet^{A} \arrow[d, "a"'] & \bullet^{A} \\
\circ^{B} & \circ^{B} \arrow[l, bend left, "\gamma"] \arrow[u, "a^*"']
\end{tikzcd}$$

$(\langle a^* \rangle = A)$

Moreover, the isos in $\mathcal{DR}(M)$ are exactly the morphisms of the form

$$\begin{tikzcd}
\bullet^{A} \arrow[d, "a"'] & \bullet^{A} \\
\circ^{B} & \circ^{B} \arrow[u, "a^*"']
\end{tikzcd}$$

with $\langle a^* \rangle = A$ and $\langle a \rangle = B$, meaning that the groupoid of $\mathcal{DR}(M)$ is exactly the inductive groupoid $\mathcal{IG}(M)$ [17, Section 5.1.3]: these morphisms are indeed just the morphisms $A \xrightarrow{u} B$ of $\mathcal{IG}(M)$, drawn twice and vertically. There is also a *dualizing functor* $\_^{\perp} : \mathcal{DR}(M) \to \mathcal{DR}(M)^{op}$, defined by $A^{\perp} = A$, and, on morphisms, by rotating them 180 degrees:

$$\left(\begin{tikzcd}
\bullet^{A} \arrow[r, bend left, "\beta"] \arrow[d, "a"'] & \bullet^{A} \\
\circ^{B} & \circ^{B} \arrow[l, bend left, "\gamma"] \arrow[u, "a^*"']
\end{tikzcd}\right)^{\perp} = \begin{tikzcd}
\bullet^{B} \arrow[r, bend left, "\gamma"] \arrow[d, "a^*"'] & \bullet^{B} \\
\circ^{A} & \circ^{A} \arrow[l, bend left, "\beta"] \arrow[u, "a"']
\end{tikzcd}$$

To get a model of MLL, we define:

Definition 4.1 *A linear inverse monoid M is* weakly Cantorian *if and only if it contains two elements p and q such that $p^*q = 0$, $\langle p^* \rangle = \langle q^* \rangle = 1$.*

One usually assumes other elements, e.g., a monoid morphism ! such that $!0 = 0$, and other constants to model the exponential connectives of linear logic. As we said in the introduction, we won't get a categorical model of linear logic this way. However, the situation with multiplicatives is fine.

In $\mathsf{PI}(\mathbb{N})$, think of p as $\{(n, 2n) | n \in \mathbb{N}\}$, and q as $\{(n, 2n+1) | n \in \mathbb{N}\}$. In the rudimentary clause setting, think of p as the clause $X \leftarrow \mathsf{p}(X)$ and q as $X \leftarrow \mathsf{q}(X)$, where p and q are two distinct function symbols.

The weakly Cantorian structure allows us to define a *tensor product* $A_1 \otimes A_2$ of objects A_1, A_2 as $pA_1p^* + qA_2q^*$. (In $\mathsf{PI}(\mathbb{N})$, reading each idempotent as a set, $A_1 \otimes A_2 = \{2n | n \in A_1\} \cup \{2n+1 | n \in A_2\}$ is the disjoint sum of A_1 and A_2.) On morphisms, given

$$f_1 = \begin{tikzcd}
\bullet^{A_1} \arrow[r, bend left, "\beta_1"] \arrow[d, "a_1"'] & \bullet^{A_1} \\
\circ^{B_1} & \circ^{B_1} \arrow[l, bend left, "\gamma_1"] \arrow[u, "a_1^*"']
\end{tikzcd}$$

and

$$f_2 = \begin{tikzcd}
\bullet^{A_2} \arrow[r, bend left, "\beta_2"] \arrow[d, "a_2"'] & \bullet^{A_2} \\
\circ^{B_2} & \circ^{B_2} \arrow[l, bend left, "\gamma_2"] \arrow[u, "a_2^*"']
\end{tikzcd}$$

, similarly, their tensor product is $f_1 \otimes f_2$ is

$$\begin{tikzcd}
\bullet^{A_1 \otimes A_2} \arrow[r, bend left, "p\beta_1p^*+q\beta_2q^*"] \arrow[d, "pa_1p^*+qa_2q^*"'] & \bullet^{A_1 \otimes A_2} \\
\circ^{B_1 \otimes B_2} & \circ^{B_1 \otimes B_2} \arrow[l, bend left, "p\gamma_1p^*+q\gamma_2q^*"] \arrow[u, "pa_1^*p^*+qa_2^*q^*"']
\end{tikzcd} .$$

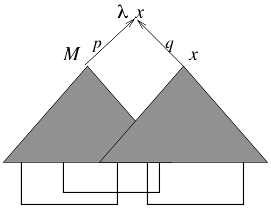
The tensor unit I is just 0. These turn $\mathcal{DR}(M)$ into a symmetric monoidal category [17, Section 5.2.1].

Categorical models of (classical) MLL are $*$-autonomous categories, which one may describe as being equipped with a *linear application* morphism $\mathsf{app}_{A,B} : (A \multimap B) \otimes A \rightarrow B$ (the counit of the adjunction), and a *linear abstraction* operator $\lambda^{C}_{A,B}$ such that $\lambda^{C}_{A,B}(f) : C \rightarrow (A \multimap B)$ for each $f : C \otimes A \rightarrow B$, satisfying:

- β-equivalence: $\mathsf{app}_{A,B} \circ (\lambda^{C}_{A,B}(f) \otimes g) = f \circ (\mathrm{id}_C \otimes g) : C \otimes D \rightarrow B$ for every $f : C \otimes A \rightarrow B$ and $g : D \rightarrow A$;
- η-equivalence: $\lambda^{A \multimap B}_{A,B}(\mathsf{app}_{A,B}) = \mathrm{id}_{A \multimap B}$;
- substitution: $\lambda^{C}_{A,B}(f) \circ g = \lambda^{D}_{A,B}(f \circ (g \otimes \mathrm{id}_A))$ for every $f : C \otimes A \rightarrow B$ and $g : D \rightarrow C$.

While this axiomatization is non-standard, it has the merit of displaying the underlying linear λ-calculus at work, in a style resembling categorical combinators [8]. These are given for $\mathcal{DR}(M)$ in Figure 1.

Let us try to give some intuition. Recall that the idea behind $\mathcal{DR}(M)$ is to describe, as morphisms, the set of paths in linear λ-terms. Represent a linear λ-term in normal form as a portion of the infinite binary tree, with axiom links between leaves. A λ-abstraction $\lambda x \cdot M$ is then represented as on the right, where the left son is the root to the body M of the λ-abstraction, and the right son points to the unique occurrence of x in the unique head application $xN_1 \dots N_k$ in M. (For now, imagine the right triangle consists just of one link connecting x to its use in the left triangle.) The paths from the root of $\lambda x \cdot M$ are as follows. First, go down left (p^*, or rather Bp^*), then enter M (the inner square in the definition of the λ-abstraction). We may then: either exit M at the root of M, and go up right (p, more precisely pB); or exit M through the variable x; this means selecting x from the bunch of variables free in M (the curved q^* starting from $C \otimes A$), then going up left to the root of $\lambda x \cdot M$ (qA); or exit M through some other variable y; this means selecting the set of those free variables of M that are not x (the Cp^* transition). We can similarly explore the other paths in $\lambda x \cdot M$, and thus justify the definition of λ-abstraction given above.



With these constructions, $\mathcal{DR}(M)$ is symmetrical monoidal closed, i.e., a model of intuitionistic MLL. Let $\perp$ be the 0 object, and define intuitionistic negation $\sim A$ as $A \multimap \perp$. It is easy to see that $\sim A$ is isomorphic to $A^{\perp} = A$. The morphism $\mathcal{C}_A =$

$$\begin{tikzcd}
\bullet^{\sim\sim A} \arrow[d, "Aq^{*2}"'] & \bullet^{\sim\sim A} \\
\circ^{A} & \circ^{A} \arrow[u, "q^2A"']
\end{tikzcd}$$

is inverse to $\lambda^{A}_{\sim A,\perp}(\mathsf{app}_{A,\perp} \circ c_{A,\sim A})$, where $c_{A_1,A_2} = qA_1p^* + pA_2q^* : A_1 \otimes A_2 \rightarrow A_2 \otimes A_1$ is the commutativity natural transformation. $\mathcal{C}_A$ is a morphism from $\sim\sim A$ to A, and acts as a linear form of Felleisen *et al.*'s control operator $\mathcal{C}$ [12].

It is easy to see that $A \multimap B$ is isomorphic to $A^{\perp} \otimes B$, and that these constructs turn $\mathcal{DR}(M)$ into a *compact-closed* category. Recall that $*$*-autonomous* categories are symmetric monoidal closed categories with a dualizing object $\perp$, i.e., one such that $\lambda^{A}_{\sim A,\perp}(\mathsf{app}_{A,\perp} \circ c_{A,\sim A})$ is iso; such categories are models of classical MLL [5]. Compact-closed categories [22] are $*$-autonomous categories such that there is a natural iso between $A \otimes B$ and $A \parr B$, where $A \parr B = \sim (\sim A \otimes \sim B)$. Then [17, Theorem 5.2.7]:

Theorem 4.2 *Let M be a weakly Cantorian linear inverse monoid. $\mathcal{DR}(M)$ is a compact-closed category.*

5. Retracing Some Paths in $\mathcal{DR}(M)$

Every compact-closed category has a canonical *trace* [21]. The prototypical example of a compact-closed category is the category whose objects are $\mathbb{R}^n$, $n \in \mathbb{N}$, and whose morphisms are linear maps, i.e., morphisms from $\mathbb{R}^m$ to $\mathbb{R}^n$ are $n \times m$ matrices. The notion of trace in a category then generalizes the usual notion of trace in linear algebra. One may compute the canonical trace of the compact-closed category $\mathcal{DR}(M)$ [17, Proposition 5.2.8]:

Proposition 5.1 *The canonical trace on the compact-closed category $\mathcal{DR}(M)$, $Tr^{X}_{A,B}$*

$$Tr^{X}_{A,B}\left(\begin{tikzcd}
\bullet^{A\otimes X} \arrow[r, "\beta"] \arrow[d, "a"'] & A\otimes X_{\bullet} \\
\circ^{B\otimes X} & B\otimes X_{\circ} \arrow[u, "a^*"'] \arrow[l, "\gamma"]
\end{tikzcd}\right)$$

is:

$$\begin{tikzcd}
\bullet^{A} \arrow[dr, "pA"'] & & & A_{\bullet} \\
& A\otimes X \arrow[r, "\beta"] \arrow[d, "a"'] & A\otimes X \arrow[ur, "Ap^*"'] \arrow[d, bend left=60, "qXq^*"] & \\
& B\otimes X \arrow[u, bend left=60, "qXq^*"] \arrow[dl, "Bp^*"'] & B\otimes X \arrow[u, "a^*"'] \arrow[l, "\gamma"] & \\
\circ^{B} & & & B_{\circ} \arrow[ul, "pB"']
\end{tikzcd}$$

Now consider the subcategory $\mathcal{PIG}(M)$ of $\mathcal{DR}(M)$ whose morphisms are of the form

$$\begin{tikzcd}
\bullet^{A} \arrow[d, "a"'] & \bullet^{A} \\
\circ^{B} & \circ^{B} \arrow[u, "a^*"']
\end{tikzcd}$$

. Equivalently, the morphisms are $A \xrightarrow{u} B$ with $\langle a^* \rangle \leq A$ and $\langle a \rangle \leq B$. (Note the difference with $\mathcal{IG}(M)$, where we require $\langle a^* \rangle = A$, $\langle a \rangle = B$.) The trace operator on $\mathcal{DR}(M)$ then induces one on $\mathcal{PIG}(M)$ (but not on $\mathcal{IG}(M)$, as trace does not preserve isos), by: $Tr^{X}_{A,B}(\ A \otimes X \xrightarrow{a} B \otimes X\ ) =$

$$A \multimap B = qAq^* + pBp^*$$

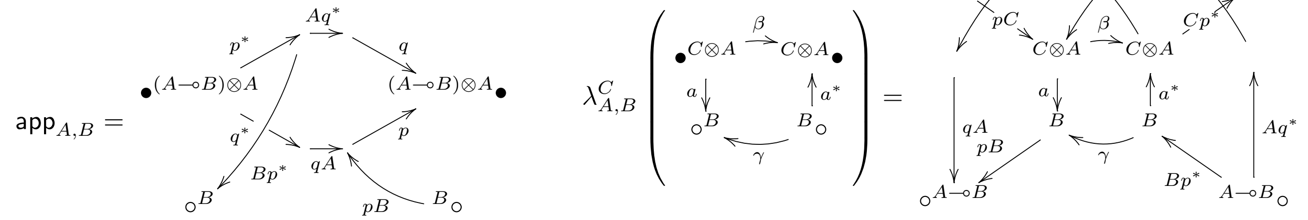
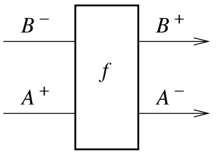


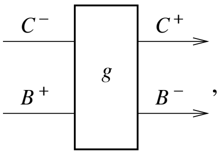
Figure 1. Linear implication, application, abstraction

$$\begin{tikzcd} A \arrow[r, "pA"] & A\otimes X \arrow[r, "a"] & B\otimes X \arrow[r, "Bp^*"] \arrow[l, bend right, "qXq^*"'] & B \end{tikzcd}$$. This exhibits the familiar feedback loop typical of several trace operators.

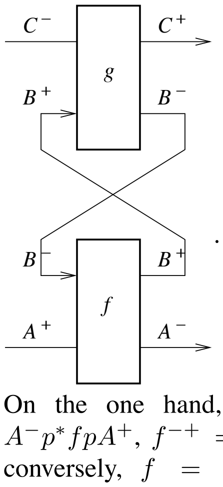
It is then interesting to compare $\mathcal{DR}(M)$ to the construction of a compact-closed category $\mathcal{G}(\mathcal{C})$ from any traced symmetrical monoidal category $\mathcal{C}$ [1, 21]. The objects of $\mathcal{G}(\mathcal{C})$ are pairs (A^+, A^-) of objects of $\mathcal{C}$. A morphism $f : (A^+, A^-) \to (B^+, B^-)$ in $\mathcal{G}(\mathcal{C})$ is a morphism $f : A^+ \otimes B^- \to A^- \otimes B^+$ in $\mathcal{C}$. The identity on (A^+, A^-) is the commutativity c_{A^+,A^-}. Composition is given by *symmetric feedback*. Given $f : (A^+, A^-) \to (B^+, B^-)$ and $g : (B^+, B^-) \to (C^+, C^-)$ in $\mathcal{G}(\mathcal{C})$, i.e., $f : A^+ \otimes B^- \to A^- \otimes B^+$ and $g : B^+ \otimes C^- \to B^- \otimes C^+$ in $\mathcal{C}$, the composition $g \circ f$ in $\mathcal{G}(\mathcal{C})$ is the trace $Tr^{B^-\otimes B^+}_{A^+\otimes C^-, A^-\otimes C^+}(\cong \circ (f \otimes g) \circ \cong)$, where $\cong$ denotes obvious isos built from associativity and commutativity, in both places above. There is an elegant box notation from Kelly and Laplaza [22] that makes this more readable. Eventually, further notational conventions [1] allow one to define the composition of



and



, as:



.

Expanding the definitions for $\mathcal{G}(\mathcal{PIG}(M))$ yields the following. Morphisms $f : (A^+, A^-) \to (B^+, B^-)$ in $\mathcal{G}(\mathcal{PIG}(M))$, i.e., morphisms $A^+ \otimes B^- \xrightarrow{f} A^- \otimes B^+$ in $\mathcal{PIG}(M)$, can be described equivalently by giving four morphisms $A^+ \xrightarrow{f^{++}} B^+$, $A^+ \xrightarrow{f^{+-}} A^-$, $B^- \xrightarrow{f^{-+}} B^+$, and $B^- \xrightarrow{f^{--}} A^-$ in $\mathcal{PIG}(M)$. On the one hand, $f^{++} = B^+q^*fpA^+$, $f^{+-} = A^-p^*fpA^+$, $f^{-+} = B^+q^*fqB^-$, $f^{--} = A^-p^*fqB^-$; conversely, $f = qB^+f^{++}A^+p^* + pA^-f^{+-}A^+p^* + qB^+f^{-+}B^-q^* + pA^-f^{--}B^-q^*$. The two constructions, from f to the four-tuple f^{++}, f^{+-}, f^{-+}, f^{--} and back, are inverse of each other. These can be conveniently written

$$\begin{tikzcd} \bullet A^+ \arrow[r, bend left, "f^{+-}"] \arrow[d, "f^{++}"'] & \bullet A^- \\ \circ B^+ & \circ B^- \arrow[l, bend left, "f^{-+}"] \arrow[u, "f^{--}"'] \end{tikzcd}$$

by imitation with $\mathcal{DR}(M)$. The only condition on such diagrams is Condition 1: $f^{++}A^+ = B^+f^{++} = f^{++}$, $f^{+-}A^+ = A^-f^{+-} = f^{+-}$, $f^{--}B^- = A^-f^{--} = f^{--}$, $f^{-+}B^- = B^+f^{-+} = f^{-+}$. Conditions 2 and 3 are dropped. Then identities, composition, tensor product are defined exactly in the same way in $\mathcal{DR}(M)$ and in $\mathcal{G}(\mathcal{PIG}(M))$. In other words, $\mathcal{DR}(M)$ is essentially a symmetric form of the $\mathcal{G}$ construction over $\mathcal{PIG}(M)$.

We can then define weak GOI situations [19, 3] on $\mathcal{PIG}(M)$, and the construction of a weak linear category from it, i.e., of a categorical model for linear combinatory algebra, carries over to $\mathcal{DR}(M)$. We only need to make sure M comes with a linear inverse semigroup endomorphism $! : M \to M$, and elements $\mathfrak{d}$, $\mathfrak{e}$, $\underline{\mathfrak{d}}$ verifying certain equations [17, Section 6.3]. A typical example is when $M = \mathsf{PI}(\mathbb{N})$, $\langle \_, \_ \rangle$ is any injection from $\mathbb{N}^2$ to $\mathbb{N}$, $!f\langle k, n\rangle = \langle k, f(n)\rangle$, $\mathfrak{d}\langle k_1, \langle k_2, n\rangle\rangle = \langle\langle k_1, k_2\rangle, n\rangle$, $\mathfrak{e}(n) = \langle 1, n\rangle$, and $\underline{\mathfrak{d}} = rp^* + sq^*$, where $r\langle k, n\rangle = \langle 2k, n\rangle$, $s\langle k, n\rangle = \langle 2k + 1, n\rangle$. We shall not pursue this, since our goal here is to find linear, not weak linear categories.

6. $\mathcal{DR}(M)$ Contains No Additive

Surprisingly, there is no way to have $\mathcal{DR}(M)$ contain *any* additive connective, in a very strong sense, as we now show. One might have hoped that enriching M with new constants g, d as in [23] for example, or as in [15] (where M is a linear inverse monoid of rudimentary clauses) would provide a solution. And indeed it does, provided we are ready to forego some natural proof conversion rules. If we are not, there is no way. First, we cannot interpret any of the additive units $\top$ (which would be a terminal object) and **0** (an initial object):

Proposition 6.1 *The following statements are equivalent: (1) $\mathcal{DR}(M)$ has a terminal object; (2) 0 is terminal in $\mathcal{DR}(M)$; (3) $\mathcal{DR}(M)$ has an initial object; (4) 0 is initial in $\mathcal{DR}(M)$; (5) $M = \{0\}$.*

Proof. (1) and (3), (2) and (4) are equivalent through duality $\_^{\perp}$. (1) $\Rightarrow$ (2): Let $\top$ be a terminal object in $\mathcal{DR}(M)$ (i.e., for every object A, there is a unique morphism from A to $\top$). So there is a unique morphism from 0 to $\top$. Here are two,

$$\begin{tikzcd} \bullet^0 & \bullet^0 \\ \circ^\top & \circ^\top \end{tikzcd}$$

and

$$\begin{tikzcd} \bullet^0 & \bullet^0 \\ \circ^\top & \circ^\top \arrow[l, bend left, "\top"] \end{tikzcd}$$

. By uniqueness, $\top = 0$. (2) $\Rightarrow$ (5): Let A be any idempotent of M, i.e., an object of $\mathcal{DR}(M)$. There is a unique morphism from A to 0. Here are two:

$$\begin{tikzcd} \bullet^A \arrow[r, bend left, "A"] & \bullet^A \\ \circ^0 & \circ^0 \end{tikzcd}$$

and

$$\begin{tikzcd} \bullet^A & \bullet^A \\ \circ^0 & \circ^0 \end{tikzcd}$$

. So $A = 0$. For each $u \in M$, take $A = \langle u \rangle$, then $u = \langle u \rangle u = 0 \cdot u = 0$. ☐

Additive units are usually not considered in most models, including game models, of linear logic. However, there is no additive conjunction $\&$ (product $\times$) or disjunction $\oplus$ (coproduct $+$) either!

Proposition 6.2 *Let A and B be any two objects of $\mathcal{DR}(M)$. The following conditions are equivalent: (1) $A \times B$ exists; (2) $A + B$ exists; (3) $M = \{0\}$.*

Proof. The proof proceeds along similar lines as Proposition 6.1, only with added subtleties. See Appendix A, or [17, Proposition 5.1.16] for details. ☐

This is pretty definitive: if M is non-trivial, there is no product, and no coproduct in $\mathcal{DR}(M)$, whatever the constructs (g, d, etc.) we may invent in M.

7. $\mathcal{DR}(M)$ Has No Exponential

Not having additives in $\mathcal{DR}(M)$ is no great loss. To interpret the λ-calculus, we only need to interpret MELL, the multiplicative-exponential fragment of linear logic.

The key to our next impossibility result is the notion of (co)commutative comonoid in a symmetric monoidal category $\mathcal{C}$. The central role of such objects is made explicit in Melliès [25]. A *comonoid* in $\mathcal{C}$ is any triple (A, d_A, e_A) where A is an object, $d_A : A \to A \otimes A$ (*comultiplication*) and $e_A : A \to I$ (*counit*) are morphisms in $\mathcal{C}$ satisfying: (coassociativity) $\alpha_{A,A,A} \circ (d_A \otimes \mathrm{id}_A) \circ d_A = (\mathrm{id}_A \otimes d_A) \circ d_A : A \to A \otimes (A \otimes A)$, where $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is associativity; (left counit) $(e_A \otimes \mathrm{id}_A) \circ d_A : A \to I \otimes A$ is the obvious iso; and (right counit) $(\mathrm{id}_A \otimes e_A) \circ d_A : A \to A \otimes I$ is the obvious iso again. It is *cocommutative* iff $c_{A,A} \circ d_A = d_A : A \to A \otimes A$, where $c_{A,B} : A \otimes B \to B \otimes A$ is commutativity. Note that (cocommutative) comonoids in $\mathcal{S}et^{op}$, the opposite category of $\mathcal{S}et$, are exactly the (commutative) monoids.

It is well-known that there is a category $co\mathcal{M}on(\mathcal{C})$ of cocommutative comonoids, whose morphisms $f : (A, d_A, e_A) \to (B, d_B, e_B)$ are morphisms $f : A \to B$ in $\mathcal{C}$ that preserve comultiplication $d_.$ and counits $e_.$. Moreover, $co\mathcal{M}on(\mathcal{C})$ always has all finite products [25].

A particularly nice notion of model of (classical) linear logic developed by Melliès [25], that of *new-Lafont category*, is defined as a ($*$-)autonomous category $\mathcal{C}$, with a full sub-monoidal category $\mathcal{M}$ of $co\mathcal{M}on(\mathcal{C})$, such that the obvious forgetful functor $U : \mathcal{M} \to \mathcal{C}$ has a right adjoint $F : \mathcal{C} \to \mathcal{M}$. We now show that $\mathcal{DR}(M)$ is never a new-Lafont category, unless M is trivial. To this end, we characterize comonoids in $\mathcal{DR}(M)$. Say that two idempotents A_p and A_q of M form a *partition* of A if and only if $A_p + A_q = A$ and $A_p A_q = 0$.

Theorem 7.1 *Let M be weakly Cantorian. Let (A, d_A, e_A) be a triple verifying the left and right counit laws (e.g., a comonoid of $\mathcal{DR}(M)$). Then there is a partition A_p, A_q of A such that:*

$$d_A = \begin{tikzcd} \bullet^A \arrow[d, dashed, "pA_p + qA_q"] & {}^A\bullet \\ \circ^{A\otimes A} & {}^{A\otimes A}\circ \arrow[u, dashed, "A_p p^* + A_q q^*"] \arrow[l, bend left, "q\beta_0 p^* + p\beta_0 q^*"] \end{tikzcd} \qquad e_A = \begin{tikzcd} \bullet^A \arrow[r, bend left, "\beta_0+\beta_0^*"] & {}^A\bullet \\ \circ^0 & {}^0\circ \end{tikzcd}$$

where β_0 is an iso between A_p and A_q, i.e., $\langle {\beta_0}^* \rangle = A_p$ and $\langle \beta_0 \rangle = A_q$.

Conversely, if d_A and e_A are defined as above, then (A, d_A, e_A) is a comonoid in $\mathcal{DR}(M)$.

Proof. This is [17, Theorem 6.2.5]. The proof is complex, but not particularly deep. In a sense, it is a souped-up version of the proofs of Propositions 6.1 and 6.2: see Appendix A. ☐

Curiously, note that coassociativity is for free once (A, d_A, e_A) obeys the left and the right counit laws. Cocommutativity is an entirely different matter:

Theorem 7.2 *Let M be weakly Cantorian. The only cocommutative comonoid in $\mathcal{DR}(M)$ is $(I, \ell_I^{-1}, \mathrm{id}_I)$, or explicitly $(0, d_0, e_0)$, where $d_0 : 0 \to 0 \otimes 0$ and $e_0 : 0 \to 0$ are the all zero morphisms.*

Proof. Let (A, d_A, e_A) be some cocommutative comonoid in $\mathcal{DR}(M)$, written as in Theorem 7.1. $c_{A,A} \circ d_A$ equals

$$\begin{tikzcd}
\bullet^{A} \arrow[d, dashed, "pA_p + qA_q"'] & & A_{\bullet} & & & & \\
A \otimes A \arrow[d, dashed, "pAq^* + qAp^*"'] & & A \otimes A \arrow[ll, bend left, "q\beta_0 p^* + p\beta_0 q^*"'] \arrow[u, dashed, "A_p p^* + A_q q^*"'] & = & \bullet^{A} \arrow[d, "pA_q + qA_p"'] & & A_{\bullet} \\
{}_{\circ}A \otimes A & & A \otimes A_{\circ} \arrow[u, dashed, "pAq^* + qAp^*"'] & & {}_{\circ}A \otimes A & & A \otimes A_{\circ} \arrow[ll, bend left, "p\beta_0 q^* + q\beta_0 p^*"] \arrow[u, "pA_q + qA_p"']
\end{tikzcd}$$

While the bottom arrow always coincides with that of d_A, the vertical arrows only coincide provided that $pA_q + qA_p = pA_p + qA_q$. Multiply by A_p on the right: since $A_q A_p = 0$, $qA_p = pA_p$. Multiply by A_q on the right: $pA_q = qA_q$. So $qA = q(A_p + A_q) = qA_p + qA_q = pA_p + pA_q = p(A_p + A_q) = pA$. Multiply by p^* on the left: $p^* qA = 0$, while $p^* pA = A$, so $A = 0$. $\square$

Corollary 7.3 ($\mathcal{DR}(M)$ Is Not New-Lafont) *$\mathcal{DR}(M)$ is (the $\mathcal{C}$ component of) a new-Lafont category if and only if M is the trivial semigroup $\{0\}$.*

Proof. Assume $\mathcal{DR}(M)$ is new-Lafont. Since $U \dashv F$, for any object A of $\mathcal{DR}(M)$, there is a bijection between the set of morphisms from $(0, d_0, e_0)$ to $F(A)$ in $\mathcal{M}$ and the set of morphisms from $U(0, d_0, e_0) = 0$ to A in $\mathcal{DR}(M)$. But the first set only contains one morphism, since the only morphism in $\mathcal{M}$ is $0 : (0, d_0, e_0) \to (0, d_0, e_0)$, by Theorem 7.2. So there is exactly one morphism from 0 to A in $\mathcal{DR}(M)$. As in Proposition 6.1, this implies $A = 0$ for every idempotent A, so M is trivial. $\square$

We won't recall the definitions of linear category [7] or that of an LNL category [6]. The deep connections between these and new-Lafont categories [25] then allow us to conclude that $\mathcal{DR}(M)$ is a linear category, resp. an LNL category, if and only if $M = \{0\}$ [17, Theorem 6.2.15, Theorem 6.2.16]. Again, this is definitive: if M is non-trivial, then $\mathcal{DR}(M)$ *cannot* be a categorical model of linear logic. This includes any attempt to invent boxes, dereliction, weakening and promotion constants in M. In particular, there is no way to turn the constructions of e.g. [10, 9, 23, 15] into models of linear logic.

8. Coherence Completions

However, we can build a category *on top* of $\mathcal{DR}(M)$, so that the existing multiplicative structure $(\otimes, I)$ is preserved, while adding all exponentials and additives.

Following Melliès [25, Definition 7], call a (classical) *Lafont category* any ($*$-)autonomous category with finite products where for each object A there is a free cocommutative comonoid $!A$. Lafont categories are probably the strongest categorical notion of a categorical model of linear logic: any Lafont category is new-Lafont, linear, and LNL in particular. The category $\mathcal{C}oh$ of coherence spaces has coherence spaces as objects (Section 2), and *linear maps* $f : X \to Y$ as morphisms. Assuming $X = (|X|, \frown_X)$ and $Y = (|Y|, \frown_Y)$, a linear map f is a binary relation between the *webs* $|X|$ and $|Y|$, such that whenever $(x, y), (x', y') \in f$ and $x \frown_X x'$ then $[y \frown_Y y'$, and $y = y'$ implies $x = x']$. (Brackets added for precision.) It is well-known that $\mathcal{C}oh$ is Lafont (this implies that we are taking the multiclique interpretation of $!A$, not the clique interpretation used e.g. in [16]).

The main construction we use now is the *coherence completion* $\mathcal{COH}(\mathcal{C})$ of a $*$-autonomous category $\mathcal{C}$, due to Hu and Joyal [20]. Interestingly, this is the second place in this work where coherence plays a crucial role, after the definition of linear inverse semigroups. While the original notion of coherence completion only preserves existing exponentials, we show that a simple modification of the construction *creates* them. To obtain a comonad $(!, \epsilon, \delta)$ on $\mathcal{COH}(\mathcal{C})$ giving meaning to the exponential connectives, Hu and Joyal assume a comonad $(!_{\mathcal{C}}, \epsilon_{\mathcal{C}}, \delta_{\mathcal{C}})$ so that for each object A of $\mathcal{C}$, $!_{\mathcal{C}}A$ is a cocommutative comonoid and $!_{\mathcal{C}}(A \times B) = !_{\mathcal{C}}A \otimes !_{\mathcal{C}}B$. (Hu and Joyal assume finite products in $\mathcal{C}$ at this point.) If $\mathcal{C} = \mathcal{DR}(M)$, we will have none of that... in a very strong sense, as we have seen. Instead, take $!_{\mathcal{C}}$ to be the identity comonad.

To repair a slight ambiguity in Hu and Joyal's original construction, we consider a subcategory $\mathcal{CCOH}$ of $\mathcal{COH}$, the full subcategory of so-called *concrete* coherence spaces, where a coherence space $X = (|X|, \frown_X)$ is concrete if and only if $|X|$ is a set of ground first-order terms built on top of some fixed signature $\Sigma_{\mathcal{C}oh}$. We won't expand on this, since all constructions in $\mathcal{CCOH}$ work exactly as in $\mathcal{COH}$, since the actual technical details are unimportant (see [17, Section 6.4.1], where it is shown that $\mathcal{CCOH}$ is a Lafont category), and since the only purpose of this is to obtain a total ordering on $|X|$ for any concrete cohence space, inherited from a fixed total ordering $\preceq$ on all first-order terms. This is needed to define ! formally below.

Now, for any category $\mathcal{C}$, the *concrete coherence completion* $\mathcal{CCOH}(\mathcal{C})$ has as objects all pairs $(X, (A_i)_{i \in |X|})$ where the *base* X is a concrete coherence space $X = (|X|, \frown_X)$, and the *fiber* $(A_i)_{i \in |X|}$ is a family of objects of $\mathcal{C}$, indexed, by the web $|X|$; and as morphisms from $(X, (A_i)_{i \in |X|})$ to $(Y, (B_j)_{j \in |Y|})$ all pairs $(f, (a_{ij})_{(i,j) \in f})$ where f is a linear map from X to Y, and a_{ij} is a morphism from A_i to B_j in $\mathcal{C}$. Coherence and linearity on the base are crucial for composition to make sense [20]. Then $\mathcal{CCOH}(\mathcal{C})$ is ($*$-)autonomous as soon as $\mathcal{C}$ is. Also, $\mathcal{CCOH}(\mathcal{C})$ has all finite products and coproducts, whatever $\mathcal{C}$ is. For example, the binary product ("with") $(X, (A_i)_{i \in |X|}) \times (Y, (B_j)_{j \in |Y|})$ is $(X \times Y, (A_i)_{\mathbf{i}_1(i) \in |X \times Y|} \cup (B_j)_{\mathbf{i}_2(j) \in |X \times Y|})$, where $|X \times Y|$ is the disjoint sum of $|X|$ and $|Y|$, and $\mathbf{i}_1$, $\mathbf{i}_2$ are the canonical injections. The first projection, from $(X, (A_i)_{i \in |X|}) \times$

$(Y, (B_j)_{j \in |Y|})$ to $(X, (A_i)_{i \in |X|})$, is $(\pi_1, (\mathrm{id}_{A_i})_{(\mathbf{i}_1(i), i) \in \pi_1})$, where π_1 denotes the first projection operator in $\mathcal{CCOH}$. Note that morphisms in the fibers are just identities id_{A_i} — this is why we don't need any structure from $\mathcal{C}$ for products to exist in $\mathcal{CCOH}(\mathcal{C})$.

The ! comonad is slightly more complex (Appendix C). Since in $\mathcal{CCOH}$ all webs $|X|$ are totally ordered by $\preceq$, we may represent any multiclique $e = \{\!\{i_1, \ldots, i_k\}\!\}$ (i.e., any finite multiset of pairwise coherent elements; we enclose multiset elements between $\{\!\{$ and $\}\!\}$) as a sorted list $[i_1, \ldots, i_k]$. Assuming $\mathcal{C}$ to be symmetric monoidal, with tensor unit $\mathbb{I}$, we may then define $\bigotimes_{i \in e} A_i$ as $A_{i_1} \otimes (A_{i_2} \otimes \ldots (A_{i_k} \otimes \mathbb{I}) \ldots)$. For any object $(X, (A_i)_{i \in |X|})$ of $\mathcal{CCOH}(\mathcal{C})$, let $!(X, (A_i)_{i \in |X|}) = \left(!X, \left(\bigotimes_{i \in e} A_i\right)_{e \in |!X|}\right)$. For any morphism $(f, (a_{ij})_{(i,j) \in f}) : (X, (A_i)_{i \in |X|}) \rightarrow (Y, (B_j)_{j \in |Y|})$, first recall that in $\mathcal{CCOH}$ (as in $\mathcal{COH}$, with the multiclique ! comonad), $!f$ is the linear map of all pairs of multicliques (e_1, e_2) such that we may write e_1 as $\{\!\{i_1, \ldots, i_k\}\!\}$, $e_2 = \{\!\{j_1, \ldots, j_k\}\!\}$ (up to permutation of elements) with $(i_\ell, j_\ell) \in f$ for every ℓ, $1 \leq \ell \leq k$. Write $\{e_1\}f\{e_2\}$ for $\{\!\{(i_1, j_1), \ldots, (i_k, j_k)\}\!\}$. Then define $!(f, (a_{ij})_{(i,j) \in f}) \ :!(X, (A_i)_{i \in |X|}) \ \rightarrow !(Y, (B_j)_{j \in |Y|})$ as $\left(!f, \left(\bigotimes_{(i,j) \in \{e_1\}f\{e_2\}} a_{ij}\right)_{(e_1, e_2) \in !f}\right)$. Note that this again assumes no extra structure from $\mathcal{C}$, contrarily to [20]. Calculation then shows [17, Theorem 6.4.21]:

Theorem 8.1 *Let $\mathcal{C}$ be any ($*$-)autonomous category. Then $\mathcal{CCOH}(\mathcal{C})$ is a (classical) Lafont category, hence a (classical) new-Lafont category.*

Corollary 8.2 *Let M be any weakly Cantorian linear inverse semigroup. Then $\mathcal{CCOH}(\mathcal{DR}(M))$ is a classical Lafont category, hence also a classical new-Lafont category.*

In other words, $\mathcal{CCOH}(\mathcal{DR}(M))$ is a categorical model of full classical linear logic, in the strongest known sense. Note how coherence completion $\mathcal{CCOH}(\_)$ and the $\mathcal{DR}$ construction complement each other nicely: $\mathcal{CCOH}(\_)$ requires a $*$-autonomous category, and this is exactly what the $\mathcal{DR}(M)$ construction provides, no less, no more.

References

[1] S. Abramsky. Retracing some paths in process algebra. In *CONCUR*, pages 1–17, 1996.

[2] S. Abramsky. A structural approach to reversible computation. *Th. Comp. Sci.*, 347(3):441–464, 2005.

[3] S. Abramsky, E. Haghverdi, and P. Scott. Geometry of interaction and linear combinatory algebra. *Math. Struct. Comp. Sci.*, 12:625–665, 2002.

[4] P. Baillot, V. Danos, T. Ehrhard, and L. Regnier. Believe it or not, AJM's games model is a model of classical linear logic. In G. Winskel, editor, *LICS*, pages 68–75, 1997.

[5] M. Barr. $*$-autonomous categories and linear logic. *Math. Struct. Comp. Sci.*, 1:159–178, 1991.

[6] P. N. Benton. A mixed linear and non-linear logic: Proofs, terms and models (extended abstract). In *CSL*, pages 121–135. Springer-Verlag LNCS 933, 1995.

[7] G. Bierman. What is a categorical model of intuitionistic linear logic? In *TLCA*, pages 73–93. Springer-Verlag LNCS 902, 1995.

[8] P.-L. Curien. *Categorical Combinators, Sequential Algorithms, and Functional Programming*. Birkhäuser, 1993.

[9] V. Danos, M. Pedicini, and L. Régnier. Directed virtual reductions. In *CSL*, pages 76–88. Springer Verlag LNCS 1258, 1997.

[10] V. Danos and L. Régnier. Local and asynchronous beta-reduction. In *LICS*, pages 296–306, 1993.

[11] V. Danos and L. Régnier. Reversible, irreversible and optimal lambda-machines. *Th. Comp. Sci.*, 227(1–2):77–97, 1999.

[12] M. Felleisen, D. P. Friedman, E. Kohlbecker, and B. Duba. A syntactic theory of sequential control. *Th. Comp. Sci.*, 52(3):205–237, 1987.

[13] J.-Y. Girard. Towards a geometry of interaction. In J. W. Gray and A. Scedrov, editors, *Categories in Comp. Science and Logic*, volume 92, pages 69–108. AMS, 1989.

[14] J.-Y. Girard. Geometry of interaction II: Deadlock-free algorithms. In *COLOG'88*, pages 76–93. Springer-Verlag LNCS 417, 1990.

[15] J.-Y. Girard. Geometry of interaction III: Accomodating the additives. In *Advances in Linear Logic*, pages 329–389. Camb. U. Press LMS LNCS 222, 1995.

[16] J.-Y. Girard, Y. Lafont, and P. Taylor. *Proofs and Types*. Camb. U. Press, 1989.

[17] J. Goubault-Larrecq. Musing around the geometry of interaction, and coherence. Draft, `http://www.lsv.ens-cachan.fr/~goubault/isg.pdf`, Jan. 2007.

[18] M. Harrison and I. Havel. Strict deterministic grammars. *J. Comp. Sys. Sciences*, 7:273–277, 1973.

[19] F. Honsell and M. Lenisa. "Wave-style" geometry of interaction models in Rel are graph-like λ-models. In *TYPES*, pages 242–258. Springer-Verlag LNCS 3085, 2004.

[20] H. Hu and A. Joyal. Coherence completions of categories. *Th. Comp. Sci.*, 227(1-2):153–184, 1999.

[21] A. Joyal, R. Street, and D. Verity. Traced monoidal categories. *Math. Proc. Camb. Phil. Soc.*, 119(3):447–468, 1996.

[22] G. M. Kelly and M. L. Laplaza. Coherence for compact closed categories. *J. Pure Applied Alg.*, 19:193–213, 1980.

[23] O. Laurent. A token machine for full geometry of interaction. In *TLCA*, pages 283–297. Springer-Verlag LNCS 2044, 2001.

[24] M. V. Lawson. *Inverse Semigroups: The Theory of Partial Symmetries*. World Scientific, Singapore, 1999.

[25] P.-A. Melliès. Categorical models of linear logic revisited. *Theoretical Computer Science*, July 2002. To appear.

[26] M. Nielsen, G. Plotkin, and G. Winskel. Petri nets, event structures, and domains, part I. *TCS*, 13:85–108, 1981.

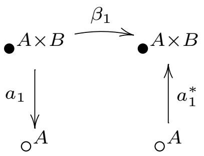
[27] M. Petrich. *Inverse Semigroups*. Pure and Applied Mathematics. John Wiley and sons, New York, 1984.

[28] B. Steinberg. A topological approach to inverse and regular semigroups. *Pacific J. Math.*, 208(2):367–396, 2003.

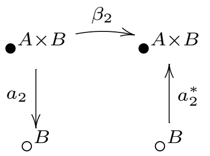
A. Proofs of the Impossibility Theorems

Proposition 6.2 *Let A and B be any two objects of $\mathcal{DR}(M)$. The following conditions are equivalent: (1) $A \times B$ exists; (2) $A + B$ exists; (3) $M = \{0\}$.*

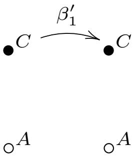
Proof. Write π_1 and π_2 for the two projections from $A \times B$, and $\langle f_1, f_2\rangle : C \to A \times B$ the pairing of $f_1 : C \to A$ and $f_2 : C \to B$. We first show that 1 implies 3. Note that the projections are epi, so we may write $\pi_1 : A\times B \to C$ as



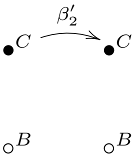
, and $\pi_2 : A\times B \to B$ as



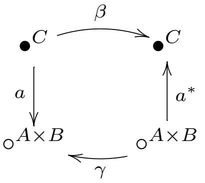
. Consider $f_1 =$



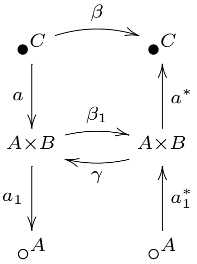
, $f_2 =$



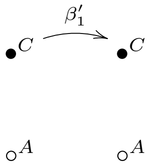
. Let us look for morphisms f such that $\pi_1 \circ f = f_1$ and $\pi_2 \circ f = f_2$. Let f be



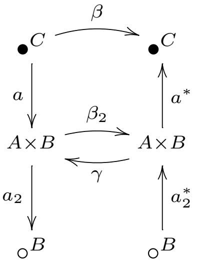
, then we try to satisfy



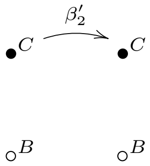
$=$



(4)



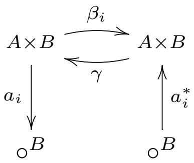
$=$



(5)

Recall that, since we assume $A\times B$ is a product, these equations should have a unique solution in a, β, γ, whatever β_1' and β_2'.

When $\beta_1' = \beta_2' = C$, we may take $\beta = C$, $a = 0$, and we show that even then, the solutions are not unique unless the only idempotent $D \leq A \times B$ such that $D\left\langle a_1{}^*\right\rangle = 0$ and $D\left\langle a_2{}^*\right\rangle = 0$ is 0. Taking $\beta = C$ and $a = 0$ allows us to reduce equations (4) and (5) to finding γ such that:

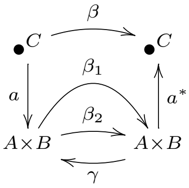


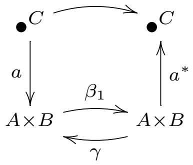
$= 0$

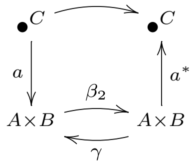
for i equal to 1 and to 2. Taking $\gamma = 0$ is one solution. In general, taking γ as being any idempotent $D \leq A \times B$ such that $D\left\langle a_i{}^*\right\rangle = 0$ $(i = 1, 2)$ gives us a solution. Indeed, recall that $a_i\beta_i = 0$ and $\beta_i^* = \beta_i$ (Conditions 2 and 3), so $\beta_i a_i^* = (a_i\beta_i^*)^* = (a_i\beta_i)^* = 0$; since $D \leq A \times B$, it follows that $\beta_i D a_i^* \leq \beta_i(A\times B)a_i^* = \beta_i a_i^* = 0$; so the left-hand side of the equation above is $\sum_{i\in\mathbb{N}} a_i(D\beta_i)^n D a_i^* = a_i D a_i^* + \sum_{i\in\mathbb{N}} a_i(D\beta_i)^n D\beta_i D a_i^* = a_i D a_i^*$. Now use the fact that $D\left\langle a_i{}^*\right\rangle = 0$: $a_i D a_i^* = a_i D\left\langle a_i{}^*\right\rangle a_i^* = 0$. Since the solutions to (4) and (5) must be unique, it must be the case that the only idempotent $D \leq A \times B$ such that $D\left\langle a_1{}^*\right\rangle = 0$ and $D\left\langle a_2{}^*\right\rangle = 0$ is 0.

It follows that: (a) $\left\langle \beta_1{}^*\right\rangle\left\langle \beta_2{}^*\right\rangle = 0$. Indeed, recall that $\beta_i a_i^* = 0$, so $\left\langle \beta_i{}^*\right\rangle\left\langle a_i{}^*\right\rangle = 0$. Let $D = \left\langle \beta_1{}^*\right\rangle\left\langle \beta_2{}^*\right\rangle$. Then $D(\left\langle a_1{}^*\right\rangle + \left\langle a_2{}^*\right\rangle) = \left\langle \beta_2{}^*\right\rangle\left\langle \beta_1{}^*\right\rangle\left\langle a_1{}^*\right\rangle + \left\langle \beta_1{}^*\right\rangle\left\langle \beta_2{}^*\right\rangle\left\langle a_2{}^*\right\rangle = 0$, whence $D\left\langle a_i{}^*\right\rangle \leq D(\left\langle a_1{}^*\right\rangle + \left\langle a_2{}^*\right\rangle) = 0$ for each i, so $D = 0$ by the above.

We claim this entails: (b) for every self-inverse element β' such that $\langle\beta'\rangle \leq C$, then $\beta' \leq C$. Indeed, fix any two self-inverse elements β_1' and β_2' with $\langle\beta_1'\rangle \leq C$, $\langle\beta_2'\rangle \leq C$, and consider any solution of equations (4) and (5). By (a) $\beta_1\beta_2^* = \beta_1\left\langle \beta_1{}^*\right\rangle\left\langle \beta_2{}^*\right\rangle\beta_2^* = 0$, hence by Condition 2, $\beta_1^*\beta_2 = 0$. So the automaton shown right is bideterministic. This implies that the two elements below are coherent, since they are both less than or equal to the latter in the natural ordering $\leq$.







But these are precisely β_1' and β_2'. We have shown that any two self-inverse elements β_1' and β_2' such that $\langle\beta_1'\rangle \leq C$, $\langle\beta_2'\rangle \leq C$, are coherent. Taking $\beta_2' = C$ itself, and $\beta_1' = \beta'$, we obtain that any self-inverse element β' such that $\langle\beta'\rangle \leq C$ is such that $\beta' \frown C$. By definition of $\frown_0$, $\beta'\left\langle C^*\right\rangle = C\left\langle \beta'^*\right\rangle$, i.e., $\beta' C = C\left\langle \beta'^*\right\rangle$ since C is idempotent. Since $\langle\beta'\rangle \leq C$, $\beta' = \beta'\left\langle \beta'^*\right\rangle = \beta'\langle\beta'\rangle \leq \beta' C = C\left\langle \beta'^*\right\rangle \leq C$.

In turn, by (b) that any self-inverse element u (i.e., with $u = u^*$) is idempotent: for any self-inverse element u, take $\beta' = u$, $C = \langle u \rangle$, so that $u \le \langle u \rangle$ by (b). Since $v \le w$ iff $\langle v \rangle w = v$, we obtain $\langle u \rangle \langle u \rangle = u$, hence u is idempotent.

This allows us to simplify the equations (4) and (5) considerably. Indeed, β_1, β_2, β'_1, β'_2 are idempotents, as well as β and γ. In particular the loops $\gamma\beta_1$ and $\gamma\beta_2$ are just idempotents, so $(\gamma\beta_1)^n, (\gamma\beta_2)^n \le A \times B$ for any $n \in \mathbb{N}$. It follows that no turn through these loops count in the corresponding sums. For example, the equation (4) means in particular that $\sum_{n \in \mathbb{N}} a_1(\gamma\beta_1)^n a = 0$ (left arrow, top left to bottom left). All terms of the sum are less than or equal to the first, $a_1 a$, so $\sum_{n \in \mathbb{N}} a_1(\gamma\beta_1)^n a = 0$ is in fact equivalent to $a_1 a = 0$. The argument is similar for all other equalities described by equations (4) and (5). Therefore, they simplify to: $\beta'_i = \beta + a^*\beta_i a$, $a_i \gamma a_i^* = 0$, and $a_i a = 0$, for $i = 1, 2$.

When $\beta'_1 = \beta'_2 = 0$, any triple $\beta = 0$, $a = 0$, and γ (with γ idempotent) such that $\gamma \le A \times B$ and $\gamma \langle a_1{}^* \rangle = \gamma \langle a_2{}^* \rangle = 0$ is a solution. Since solutions are unique: (c) the only idempotent D with $D \langle a_1{}^* \rangle = D \langle a_2{}^* \rangle = 0$ is 0.

Now fix arbitrary values for β'_1 and β'_2, and take any solution β, a, γ. Let $D = \langle a \rangle \beta_1$, a product of two idempotents, hence an idempotent. We have $D \langle a_1{}^* \rangle = 0$ since $\beta_1 \langle a_1{}^* \rangle = \beta_1 a_1^* a_1 = 0$, by Condition 2. We also have $D \langle a_2{}^* \rangle = 0$, since $D \langle a_2{}^* \rangle = \langle a \rangle \langle a_2{}^* \rangle \beta_1$ (β_1 is idempotent, and idempotents commute) $= aa^* a_2^* a_2 \beta_1 = 0$. Indeed, $a_2 a = 0$ since β, a, γ is a solution. By (c) it follows that $D = 0$, i.e., $\langle a \rangle \beta_1 = 0$. So $a^*\beta_1 a = a^* \langle a \rangle \beta_1 a = 0$. Since β, a, γ is a solution, $\beta'_1 = \beta + a^*\beta_1 a = \beta$. Hence necessarily $\beta = \beta'_1$. In a symmetric way, $\beta = \beta'_2$, so $\beta'_1 = \beta'_2$. Now β'_1 and β'_2 were arbitrary idempotents less than or equal to C. Take $\beta'_1 = 0$, $\beta'_2 = C$, then $C = 0$.

Since C is arbitrary, every idempotent is 0. We have already noticed that this entailed $M = \{0\}$ in Proposition 6.1.

We conclude, since 1 and 2 are equivalent by duality, and 3 clearly implies both. ☐

Theorem 7.1 *Let M be weakly Cantorian. Let (A, d_A, e_A) be a triple verifying the left and right counit laws (e.g., a comonoid of $\mathcal{DR}(M)$). Then there is a partition A_p, A_q of A such that:*

$$d_A = \begin{tikzcd} \bullet^{A} \arrow[d, dashed, "pA_p + qA_q"] & & {}^{A}\bullet \\ \circ^{A\otimes A} & & {A\otimes A}_{\circ} \arrow[u, dashed, "A_p p^* + A_q q^*"] \arrow[ll, bend left, "q\beta_0 p^* +"', "p\beta_0 q^*"] \end{tikzcd} \qquad e_A = \begin{tikzcd} \bullet^{A} \arrow[r, bend left, "\beta_0 + \beta_0^*"] & {}^{A}\bullet \\ \circ^{0} & {}^{0}\circ \end{tikzcd}$$

where β_0 is an iso between A_p and A_q, i.e., $\langle \beta_0{}^* \rangle = A_p$ and $\langle \beta_0 \rangle = A_q$.

Conversely, if d_A and e_A are defined as above, then (A, d_A, e_A) is a comonoid in $\mathcal{DR}(M)$.

Proof. Let (A, d_A, e_A) be a comonoid in $\mathcal{DR}(M)$. Since $\ell_A \circ (e_A \otimes \mathrm{id}_A) \circ d_A = \mathrm{id}_A$ (left counit law), where $\ell_A : I \otimes A \to A$, d_A is mono, so d_A is of the form

$$\begin{tikzcd} \bullet^{A} \arrow[d, "a"'] & & {}^{A}\bullet \\ \circ^{A\otimes A} & & {A\otimes A}_{\circ} \arrow[u, "a^*"'] \arrow[ll, bend left, "\gamma"] \end{tikzcd}$$

with $\langle a^* \rangle = A$. Since e_A is a morphism from A to 0, e_A is of the form

$$\begin{tikzcd} \bullet^{A} \arrow[r, bend left, "\beta"] & {}^{A}\bullet \\ \circ^{0} & {}^{0}\circ \end{tikzcd}.$$

Let us write down the left counit law. Using the fact that

$$\ell_A = \begin{tikzcd} \bullet^{0\otimes A} \arrow[d, "Aq^*"'] & {}^{0\otimes A}\bullet \\ \circ^{A} & {A}_{\circ} \arrow[u, "qA"'] \end{tikzcd}$$

, and simplifying:

$$\begin{tikzcd} \bullet^{A} \arrow[d, "a"'] & {}^{A}\bullet \\ A\otimes A \arrow[r, bend left, "p\beta p^*"] \arrow[d, "Aq^*"'] & A\otimes A \arrow[u, "a^*"'] \arrow[l, bend left, "\gamma"] \\ \circ^{A} & {A}_{\circ} \arrow[u, "qA"'] \end{tikzcd} = \begin{tikzcd} \bullet^{A} \arrow[d, "A"'] & \bullet^{A} \\ \circ^{A} & \circ^{A} \arrow[u, "A"'] \end{tikzcd} \tag{6}$$

Similarly, the right counit law yields

$$\begin{tikzcd} \bullet^{A} \arrow[d, "a"'] & {}^{A}\bullet \\ A\otimes A \arrow[r, bend left, "q\beta q^*"] \arrow[d, "Ap^*"'] & A\otimes A \arrow[u, "a^*"'] \arrow[l, bend left, "\gamma"] \\ \circ^{A} & {A}_{\circ} \arrow[u, "pA"'] \end{tikzcd} = \begin{tikzcd} \bullet^{A} \arrow[d, "A"'] & \bullet^{A} \\ \circ^{A} & \circ^{A} \arrow[u, "A"'] \end{tikzcd} \tag{7}$$

Before we start the formal proof, let us explain how it works, intuitively. To this end, assume M is of the form $\mathsf{PI}(E)$ for some set E. (By the Preston-Wagner Theorem, this would be enough to establish all equations. Unfortunately, the i_M embedding of M into $\mathsf{PI}(M)$ used in the Preston-Wagner Theorem does not preserve 0, which in fact invalidates this approach.) Recall that any element of E is a *token*, and that a token n at B *travels* to $a(n)$ at C along a transition $B \xrightarrow{a} C$ if and only if n is in the domain of a. Otherwise we say that n is *thrown away* by the transition. We explain this along with the formal proof; the explanation will always be in square brackets [...].

Let $A_q = a^* qAq^* a$, $A_p = a^* pAp^* a$. [Look at the top left a transition going downwards in (6). The target $A \otimes A$ is the disjoint sum of pAp^* and qAq^*; A_q is the set of tokens n that travel along a to the right summand qAq^*, A_p is the set of tokens n that travel along a to the left summand pAp^*. So A_p and A_q are disjoint, that is, $A_p A_q = 0$. By (6), every token n at the top left A of the right-hand side of the equation travels to itself at the bottom left A of the right-hand side, so the same happens on the left-hand side of the equation. In particular, no token at the top left A is thrown away by the a transition, so $A = A_p + A_q$.] A_q is idempotent, since $A_q = \langle a^* qA \rangle$; similarly, A_p is idempotent since $A_p = \langle a^* pA \rangle$. Then $A_p A_q = a^* pAp^* aa^* qAq^* a \le$

$a^*pAp^*qAq^*a = 0$ (because $aa^* \leq 1$), so $A_pA_q = 0$. And $A_p + A_q = a^*(pAp^* + qAq^*)a = a^*(A \otimes A)a = a^*a$ (since $(A \otimes A)a = a$) $= \langle a^* \rangle = A$. So A_p, A_q is a partition of A.

Let β_0 be βA_p. We have to show: (a) $a = pA_p + qA_q$, (b) $\beta = \beta_0 + \beta_0^*$, (c) $\langle {\beta_0}^* \rangle = A_p$, (d) $\langle \beta_0 \rangle = A_q$, and (e) $\gamma = q\beta_0^*p^* + p\beta_0q^*$.

[We are starting to show that (a) $a = pA_p + qA_q$. Consider a token n from the top left A. If n is in A_q, it will travel to $a(n)$ in the right summand qAq^* of $A \otimes A$. This is thrown away by the $p\beta p^*$ transition. Since it must eventually travel along some transition to exit as n at the bottom left A—because this is what it does on the right-hand side of the equation, $a(n)$ must travel along the Aq^* transition, and $Aq^*(a(n)) = n$, so $a(n) = q(n)$. Similarly, if n is in A_p, $a(n) = p(n)$, using (7) instead. This describes a as the function mapping every $n \in A_p$ to $p(n)$ and every $n \in A_q$ to $q(n)$, i.e., as $pA_p + qA_q$.]

Consider the path from the top left A to the bottom left A in either side of (6): since they are equal, $\sum_{n\in\mathbb{N}} Aq^*(\gamma p\beta p^*)^n a = A$. Multiply by $A_q = a^*qAq^*a$ on the right, then $\sum_{n\in\mathbb{N}} Aq^*(\gamma p\beta p^*)^n \langle a \rangle qAq^*a = AA_q$. Since $A_p + A_q = A$, $A_q \leq A$, so $AA_q = A_q$. Also, the terms $Aq^*(\gamma p\beta p^*)^n \langle a \rangle qAq^*a$ with $n \geq 1$ are zero, since they are less than or equal to $Aq^*(\gamma p\beta p^*)^n qAq^*a = Aq^*(\gamma p\beta p^*)^{n-1}\gamma p\beta p^*qAq^*a = 0$. So only the term with $n = 0$ remains, and the equation simplifies to $Aq^* \langle a \rangle qAq^*a = A_q$, i.e., $\boxed{\textit{(i)}}$ $Aq^*aA_q = A_q$. The similar path in (7) multiplied by A_p yields $\boxed{\textit{(ii)}}$ $Ap^*aA_p = A_p$. By multiplying *(i)* by a^*q on the left, $a^*qAq^*aA_q = a^*qA_q$, i.e., $\boxed{\textit{(iii)}}$ $A_q = a^*qA_q$. Similarly, $\boxed{\textit{(iv)}}$ $A_p = a^*pA_p$.

Summing *(iii)*, *(iv)*, $A_p + A_q = a^*(pA_p + qA_q)$. Since $A_p + A_q = A$ and $aA = a$, we obtain $a = aA = aa^*(pA_p + qA_q) \leq pA_p + qA_q$.

Conversely, using *(i)* and *(ii)*, $pA_p + qA_q = pAp^*aA_p + qAq^*aA_q \leq pAp^*aA + qAq^*aA$ (since $A_p, A_q \leq A$) $= pAp^*a + qAq^*a$ (since $aA = a$) $= (pAp^* + qAq^*)a = (A \otimes A)a = a$. Together with $a \leq pA_p + qA_q$, we obtain $\boxed{(a)\ a = pA_p + qA_q}$.

[If $n \in A_p$ travels from the top left A of (6), it must go through the a transition to $a(n)$ in the left summand pAp^* of $A \otimes A$. Since $a = pA_p + qA_q$, $a(n) = p(n)$. Now $a(n) = p(n)$ cannot travel down along the Aq^* transition, so it must go through the $p\beta p^*$ transition to $p(\beta(n))$. Then $p(\beta(n))$ cannot travel up along the a^* transition, otherwise n would have traveled to $a^*(p(\beta(n)))$ from the top left A to the top right A on the right-hand side of (6), too. The domain of a^* is $pA_pp^* + qA_qq^* = A_p \otimes A_q$; since $p(\beta(n))$ is not in this domain, $\beta(n)$ is not in A_p, therefore $\beta(n)$ is in A_q. Since n is an arbitrary element of A_p, β maps A_p to A_q. Moreover, since every token at the top left A eventually reaches the bottom left A, no $n \in A_p$ is thrown away by β. Recall that $\beta_0 = \beta A_p$, the restriction of β to A_p. We have just shown that β_0 was total, i.e., the domain of β_0 is A_p. This is (c). Similar reasoning on (7) shows that the restriction of β to A_q is total, too. Since $\beta^* = \beta$, β is an involution, so the restriction of β to A_q is necessarily β_0^*. The equations (b) and (d) follow readily.]

Look again at the path from the top left A to the bottom left A in either side of (6): $A = \sum_{n\in\mathbb{N}} Aq^*(\gamma p\beta p^*)^n a$. By (a), $A = \sum_{n\in\mathbb{N}} Aq^*(\gamma p\beta p^*)^n (pA_p + qA_q) = A_q + \sum_{n\geq 1} Aq^*(\gamma p\beta p^*)^{n-1}\gamma p\beta A_p$. This time, multiply by A_p on the right. Since $A_qA_p = 0$ and $AA_p = A_p$, and since $\beta_0 = \beta A_p$, $\boxed{\textit{(v)}}$ $A_p = \sum_{n\geq 1} Aq^*(\gamma p\beta p^*)^{n-1}\gamma p\beta_0$. Multiplying by $\langle {\beta_0}^* \rangle$ on the right, $A_p \langle {\beta_0}^* \rangle = \sum_{n\geq 1} Aq^*(\gamma p\beta p^*)^{n-1}\gamma p\beta_0 \langle {\beta_0}^* \rangle = \sum_{n\geq 1} Aq^*(\gamma p\beta p^*)^{n-1}\gamma p\beta_0$ (as $u \langle u^* \rangle = u$ for every u) $= A_p$. We have just shown $A_p \langle {\beta_0}^* \rangle = A_p$, so $A_p^* \langle {\beta_0}^* \rangle = A_p^*A_p$, since $A_p^*A_p = A_p$. Recall that $u \leq v$ iff $u^*v = u^*u$. So $A_p \leq \langle {\beta_0}^* \rangle$. On the other hand, $\langle {\beta_0}^* \rangle = A_p\beta^*\beta A_p \leq A_p$ since $\beta^*\beta \leq 1$. So $\boxed{(c)\ \langle {\beta_0}^* \rangle = A_p}$.

Similarly, using (7) and multiplying by A_q, we get $\boxed{\textit{(vi)}}$ $A_q = \sum_{n\geq 1} Ap^*(\gamma q\beta q^*)^{n-1}\gamma q\beta A_q$.

Let us now look at the path from the top left A to the top right A in either side of (6): $\sum_{n\in\mathbb{N}} a^*(p\beta p^*\gamma)^n p\beta p^*a = 0$. By (a) and simplifying, $\boxed{\textit{(vii)}}$ $\sum_{n\in\mathbb{N}} A_pp^*(p\beta p^*\gamma)^n p\beta A_p = 0$. So the term with $n = 0$ is zero, too, namely, $A_p\beta A_p = 0$. Since $\beta = A\beta$ and $A = A_p + A_q$, $\beta_0 = A\beta A_p = (A_p + A_q)\beta A_p = A_q\beta A_p$. Similarly, using (7), $A_q\beta A_q = 0$, so $\beta_0 = A_q\beta A_p = A_q\beta A_p + A_q\beta A_q = A_q\beta(A_p + A_q) = A_q\beta$. Taking converses, and since $\beta^* = \beta$, $\boxed{\textit{(viii)}}$ $\beta_0^* = \beta A_q$.

Let $u_1 = \beta_0$, $u_2 = \beta_0^*$. Using the definition of β_0 for u_1, and property *(viii)* for u_2, we obtain $u_1u_2^* = \beta A_pA_q\beta^* = 0$ (forward determinacy) because $A_pA_q = 0$. Also, $u_1^*u_2 = u_2u_1^* = \beta A_qA_p\beta^* = 0$ (backward determinacy). We have seen in the main text that the bideterminacy condition $u_1u_2^* = u_2u_1^* = 0$ implied $u_1 \mathrel{\frown\mkern-13mu\smile} u_2$, that is, $\beta_0 \mathrel{\frown\mkern-13mu\smile} \beta_0^*$. So it makes sense to consider $\beta_0 + \beta_0^*$. Since $\beta_0 = \beta A_p$ by definition and $\beta_0^* = \beta A_q$ by *(viii)*, $\beta_0 + \beta_0^* = \beta(A_p + A_q) = \beta A = \beta$, whence $\boxed{(b)}$.

By *(vi)* and *(viii)*, $A_q = \sum_{n\geq 1} Ap^*(\gamma q\beta q^*)^{n-1}\gamma q\beta_0^*$. Multiplying by $\langle \beta_0 \rangle$ on the right, $A_q \langle \beta_0 \rangle = \sum_{n\geq 1} Ap^* (\gamma q\beta q^*)^{n-1}\gamma q\beta_0^* \langle \beta_0 \rangle = \sum_{n\geq 1} Ap^*(\gamma q\beta q^*)^{n-1}\gamma q\beta_0^* = A_q$, so $A_q \leq \langle \beta_0 \rangle$. Since $\langle \beta_0 \rangle = A_q\beta^*\beta A_q$ by *(viii)*, $\langle \beta_0 \rangle \leq A_q$, so $\boxed{(d)\ \langle \beta_0 \rangle = A_q}$.

[Consider again an arbitrary token $n \in A_p$ traveling from the top left A of (6). It travels down along a to $A \otimes A$ as $p(n)$, then rightwards along $p\beta p^*$ to $p(\beta_0(n))$. Since the range of β_0 is A_q and n is arbitrary in A_p, $p(\beta_0(n))$ is arbitrary in pA_qp^*. Since every such n eventually exits at the bottom left A, $p(\beta_0(n))$ cannot be thrown away by γ, so the domain of γ contains pA_qp^*. Similarly, using (7), the domain of γ also contains qA_pq^*. No element in the domain of γ can be in pA_pp^* or in qA_qq^*, otherwise these elements would also be in the domain of a^*, and would travel up at this point. So the domain of γ is exactly $pA_qp^*+qA_pq^* = A_q \otimes A_p$. Since $\gamma^* = \gamma$, $pA_qp^*+qA_pq^*$ is also the range of γ.]

[As we have seen above, $p(\beta_0(n))$ (at the rightmost $A \otimes A$ of the left-hand side) cannot travel up along a^*, so it must travel leftwards along γ to $\gamma(p(\beta_0(n)))$. If $\gamma(p(\beta_0(n)))$ was in pA_qp^*, i.e., if it was of the form $p(m)$ with $m \in A_q$, then it would travel again rightwards along $p\beta p^*$, to $p(\beta_0^*(m)) \in pA_pp^*$, then upwards along a^* to $\beta_0^*(m)$, which is impossible. So $\gamma(p(\beta_0(n)))$ is in qA_pq^*, i.e., it is of the form $q(m)$ with $m \in A_p$, and exits as m at the bottom left A. But it can only exit as n, so $m = n$, and therefore $\gamma(p(\beta_0(n))) = q(n)$. Since β_0 is total from A_p to A_q, $p(\beta_0(n))$ is arbitrary in pA_qp^*, therefore γ maps every $m \in pA_qp^*$ to $q(\beta_0^*(p^*(m)))$. Since $\gamma = \gamma^*$, γ also maps every $m \in qA_pq^*$ to $p(\beta_0(q(m)))$. In short, $\gamma = q\beta_0^*p^* + p\beta_0q$, i.e., (e) holds.]

By Condition 3 of the definition of morphisms in $\mathcal{DR}(M)$, $\gamma a = 0$, so by (a) $\gamma pA_p + \gamma qA_q = 0$, whence $\gamma pA_p = 0$ and $\gamma qA_q = 0$. Since $A = A_p + A_q$, we get $A \otimes A = pAp^* + qAq^* = pA_pp^* + pA_qp^* + qA_pq^* + qA_qq^*$. Since $\gamma = \gamma(A \otimes A)$, $\boxed{\textit{(ix)}}$ $\gamma = \gamma(pA_qp^* + qA_pq^*)$. Because the idempotent $pA_qp^* + qA_pq^*$ is less than 1, $\boxed{\textit{(x)}}$ $\langle\gamma^*\rangle \leq pA_qp^* + qA_pq^*$.

Note that, since the codomain $\langle{\beta_0}^*\rangle$ of β_0^* is A_p by (c), $\langle\gamma p\beta_0^*\rangle = \gamma p \langle{\beta_0}^*\rangle p^*\gamma^* = \gamma pA_pp^*\gamma^* = \gamma(pA_qp^* + qA_pq^*)pA_pp^*\gamma^* = 0$, since $A_qA_p = 0$; so $\gamma p\beta_0^* = 0$. In particular, $\gamma p\beta p^* = \gamma p(\beta_0 + \beta_0^*)p^* = \gamma p\beta_0p^*$, using (b). The domain of $\gamma p\beta p^*$ is then $p\beta_0^*p^*\gamma^*\gamma p\beta_0p^* \leq p\beta_0^*p^*(pA_qp^* + qA_pq^*)p\beta_0p^*$ (by *(x)*) $= p\beta_0^*A_q\beta_0p^* = p\beta_0^*\beta_0p^*$ (by (d)) $= p\langle{\beta_0}^*\rangle p^*$; by (c), it follows that $\boxed{\textit{(xi)}}$ $\langle(\gamma p\beta p^*)^*\rangle \leq pA_pp^*$. Similarly, $\gamma q\beta_0 = 0$, $\gamma q\beta q^* = \gamma q\beta_0^*q^*$, so $\boxed{\textit{(xii)}}$ $\langle(\gamma q\beta q^*)^*\rangle \leq qA_qq^*$.

Compute $(\gamma p\beta p^*)\gamma p$. First, $\langle((\gamma p\beta p^*)\gamma p)^*\rangle = p^*\gamma^*\langle(\gamma p\beta p^*)^*\rangle\gamma p \leq p^*\gamma^*pA_pp^*\gamma p$ (by *(xi)*), and $\gamma^*pA_p = \gamma pA_p$ since γ is self-inverse. But $\gamma pA_p = 0$, as we have noticed above (or as can be deduced from *(ix)*), so $\langle((\gamma p\beta p^*)\gamma p)^*\rangle = 0$. So $(\gamma p\beta p^*)\gamma p = 0$. It follows that *(v)* $A_p = \sum_{n\geq 1} Aq^*(\gamma p\beta p^*)^{n-1}\gamma p\beta_0$ simplifies to $\boxed{\textit{(v')}}$ $A_p = Aq^*\gamma p\beta_0$, since all summands vanish except for $n = 1$. Similarly, using *(xii)* we obtain $(\gamma q\beta q^*)\gamma q = 0$, so *(vi)* $A_q = \sum_{n\geq 1} Ap^*(\gamma q\beta q^*)^{n-1}\gamma q\beta A_q$ simplifies to $A_q = Ap^*\gamma q\beta A_q = Ap^*\gamma q\beta_0^*$ (by *(viii)*); that is, $\boxed{\textit{(vi')}}$ $A_q = Ap^*\gamma q\beta_0^*$.

Multiply *(v')* by $\beta_0^*p^*$ on the right and qA_p on the left. Using (d), $qA_p\beta_0^*p^* = qA_pq^*\gamma p\langle\beta_0\rangle p^* = qA_pq^*\gamma pA_qp^*$. By (c), $A_p\beta_0^* = \langle{\beta_0}^*\rangle\beta_0^* = \beta_0^*$, so $\boxed{\textit{(v'')}}$ $q\beta_0^*p^* = qA_pq^*\gamma pA_qp^*$. By taking inverses and remembering that $\gamma^* = \gamma$, $\boxed{\textit{(vi'')}}$ $p\beta_0q^* = pA_qp^*\gamma qA_pq^*$.

Look at the $n = 1$ summand in *(vii)*: $A_p\beta p^*\gamma p\beta A_p = 0$. Since $\beta_0 = \beta A_p$, and $A_p\beta = A_p\beta^* = (\beta A_p)^* = \beta_0^*$, we obtain $\beta_0^*p^*\gamma p\beta_0 = 0$. Multiplying by $p\beta_0$ on the left, by $\beta_0^*p^*$ on the right, and using (d), $\boxed{\textit{(v''')}}$ $pA_qp^*\gamma pA_qp^* = 0$. Similarly, looking at the $n = 1$ summand in the path from the top left A to the top right A in (7), we get $\boxed{\textit{(vi''')}}$ $qA_pq^*\gamma qA_pq^* = 0$.

By *(ix)* and $\gamma^* = \gamma$, $\gamma = (pA_qp^* + qA_pq^*)\gamma$, so using *(ix)* again, $\gamma = (pA_qp^* + qA_pq^*)\gamma(pA_qp^* + qA_pq^*) = pA_qp^*\gamma pA_qp^* + pA_qp^*\gamma qA_pq^* + qA_pq^*\gamma pA_qp^* + qA_pq^*\gamma qA_pq^* = 0 + p\beta_0q^* + q\beta_0^*p^* + 0$ by *(v''')*, *(vi'')*, *(v'')*, *(vi''')*. Therefore $\boxed{(e)}$ obtains.

Conversely, assume that A_p, A_q form a partitition of A, and that (a)–(e) hold. The left counit law (6) means:

$$\begin{tikzcd}
\bullet A \arrow[d, "pA_p+qA_q"'] & & A\bullet & & \bullet A \arrow[dd, "A"'] & \bullet A \\
A\otimes A \arrow[rr, bend left, "p(\beta_0+\beta_0^*)p^*"] \arrow[d, "Aq^*"'] & & A\otimes A \arrow[ll, bend left, "q\beta_0^*p^*+p\beta_0q^*"] \arrow[u, "A_pp^*+A_qq^*"'] & = & & \\
\circ A & & A\circ \arrow[u, "qA"'] & & \circ A & \circ A \arrow[uu, "A"']
\end{tikzcd}$$

which is straightforward from the conditions. (Note that the A down arrow is obtained from the left-hand side as one A_q going straight down from the top left A to the bottom left A, plus one A_p obtained by going once through the loop; no contribution arises from looping twice or more.)

The verification of the right counit laws proceeds by similar means. It remains to establish coassociativity.

$$\begin{tikzcd}
 & \bullet A \arrow[d, "pA_p+qA_q"'] & & A\bullet \\
(\mathrm{id}_A \otimes d_A) \circ d_A = & A\otimes A \arrow[d, "pAp^*+qpA_pq^*+q^2A_qq^*"'] & & A\otimes A \arrow[u, "A_pp^*+A_qq^*"'] \arrow[ll, bend left, "q\beta_0^*p^*+p\beta_0q^*"] \\
 & \circ A\otimes(A\otimes A) & & A\otimes(A\otimes A)\circ \arrow[u, "pAp^*+qA_pp^*q^*+qA_qq^{*2}"'] \arrow[ll, bend left, "q^2\beta_0^*p^*q^*+qp\beta_0q^{*2}"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \bullet A \arrow[d, "pA_p+q^2A_q"'] & & A\bullet \\
= & \circ A\otimes(A\otimes A) & & A\otimes(A\otimes A)\circ \arrow[u, "A_pp^*+A_qq^{*2}"'] \arrow[ll, bend left, "{p\beta_0p^*q^*+qp\beta_0^*p^* +q^2\beta_0^*p^*q^*+qp\beta_0q^{*2}}"]
\end{tikzcd}$$

where we have used $A_pA_q = 0$, $\beta_0A_q = 0$, $\beta_0A_p = \beta_0$, $A_p\beta_0 = 0$, $A_q\beta_0 = \beta_0$ several times to simplify the sums on arrows; while $\alpha_{A,A,A} \circ (d_A \otimes \mathrm{id}_A) \circ d_A$ equals:

$$\begin{tikzcd}
\bullet A \arrow[d, "pA_p+qA_q"'] & & A\bullet \\
A\otimes A \arrow[d, "p^2A_pp^*+pqA_qp^*+qAq^*"'] & & A\otimes A \arrow[u, "A_pp^*+A_qq^*"'] \arrow[ll, bend left, "q\beta_0^*p^*+p\beta_0q^*"] \\
(A\otimes A)\otimes A \arrow[d, "pAp^{*2}+qpAq^*p^*+q^2Aq^*"'] & & (A\otimes A)\otimes A \arrow[u, "pA_pp^{*2}+pA_qq^*p^*+qAq^*"'] \arrow[ll, bend left, "pq\beta_0^*p^{*2}+p^2\beta_0q^*p^*"] \\
\circ A\otimes(A\otimes A) & & A\otimes(A\otimes A)\circ \arrow[u, "p^2Ap^*+pqAp^*q^*+qAq^{*2}"']
\end{tikzcd}$$

which simplifies to the same value, using the same equations. □

B. Formal Definition of Concrete Coherence Spaces

Let $\Sigma_{\mathcal{C}oh}$ be the signature $\{\mathbf{1}/0, \langle\_,\_\rangle/2, \mathbf{i}_1/1, \mathbf{i}_2/1, \mathbf{nil}/0, ::/2\}$. Write $\langle s, t\rangle$ for $\langle\_,\_\rangle$ applied to s and t, $s::t$ for $::$ applied to s and t, and $[s_1, s_2, \ldots, s_n]$ for $s_1::(s_2::\ldots(s_n::\mathbf{nil})\ldots)$. Fix, once and for all, a total ordering $\preceq$ on ground terms built on $\Sigma_{\mathcal{C}oh}$.

A *concrete coherence space* $X = (|X|, \coh_X)$ is any coherence space whose web $|X|$ is a tree language over $\Sigma_{\mathcal{C}oh}$.

Let $\mathcal{CCOH}$ be the full subcategory of $\mathcal{COH}$ consisting of concrete coherence spaces.

The category $\mathcal{CCOH}$ inherits all structure from $\mathcal{COH}$. Recall that its morphisms from $X = (|X|, \coh_X)$ to $Y = (|Y|, \coh_Y)$ are linear maps. Here, these are finite binary relations $f \subseteq |X| \times |Y|$ such that whenever $(x, y) \in f$, $(x', y') \in f$, and $x \coh_X x'$, then $y \coh_Y y'$ and if moreover $y = y'$ then $x = x'$.

- $\mathcal{CCOH}$ is symmetric monoidal. The tensor unit is $\mathbb{I} = (\{\mathbf{1}\}, \coh_{\mathbb{I}})$ where $\coh_{\mathbb{I}}$ relates $\mathbf{1}$ to itself. Tensor product of $X = (|X|, \coh_X)$ and $Y = (|Y|, \coh_Y)$ is $X \otimes Y = (|X \otimes Y|, \coh_{X\otimes Y})$, where $|X \otimes Y| = |X| \times |Y| = \{\langle i, j\rangle | i \in |X|, j \in |Y|\}$, and coherence on $X \otimes Y$ is given by $\langle i, j\rangle \coh_{X\otimes Y} \langle i', j'\rangle$ iff $i \coh_X i'$ and $j \coh_Y j'$.

The associativity $\alpha_{X,Y,Z}$ is $\{(\langle\langle i, j\rangle, k\rangle, \langle i, \langle j, k\rangle\rangle) | i \in |X|, j \in |Y|, k \in |Z|\} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, the commutativity $c_{X,Y}$ is $\{(\langle i, j\rangle, \langle j, i\rangle | i \in |X|, j \in |Y|\} : X \otimes Y \to Y \otimes X$, the left neutral is $\ell_X = \{(\langle \mathbf{1}, i\rangle, i) | i \in |X|\} : \mathbb{I} \otimes X \to X$, and the right neutral is $r_X = \{(\langle i, \mathbf{1}\rangle, i) | i \in |X|\} : X \otimes \mathbb{I} \to X$.

- $\mathcal{CCOH}$ is autonomous. The linear function space $X \multimap Y$ of $X = ([m], \coh_X)$ and $Y = ([n], \coh_Y)$ is given by $|X \multimap Y| = \{\langle i, j\rangle | i \in |X|, j \in |Y|\}$, and coherence on $X \multimap Y$ is given by $\langle i, j\rangle \coh_{X\multimap Y} \langle i', j'\rangle$ if and only if, when $i \coh_X i'$ then $j \coh_Y j'$ and if moreover $j = j'$ then $i = i'$.

- $\mathcal{CCOH}$ is $*$-autonomous. The dual, a.k.a. the linear negation of $X = (|X|, \coh_X)$ is $X^\perp = (|X|, \incoh_X)$, where $i \incoh_X i'$ if and only if, when $i \coh_X i'$ then $i = i'$; equivalently, if $i \not\coh_X i'$ or $i = i'$.

The natural transformation $\mathcal{C}_X :\sim\,\sim X \to X$ (linear control operator), where $\sim X = X \multimap \perp \cong X^\perp$ is the linear trace $\{(\langle\langle i, \mathbf{1}\rangle, \mathbf{1}\rangle, i) | i \in |X|\}$.

- $\mathcal{CCOH}$ has finite products and coproducts. The terminal object $\top$ is $(\emptyset, \coh_\top)$ where $\coh_\top$ is the empty relation. This is also the initial object $\mathbb{0}$.

The binary product $X \times Y$ of $X = (|X|, \coh_X)$ and $Y = (|Y|, \coh_Y)$ is defined as $(|X \times Y|, \coh_{X\times Y})$, where $|X \times Y| = \{\mathbf{i}_1(i) | i \in |X|\} \cup \{\mathbf{i}_2(j) | j \in |Y|\}$, and coherence is defined by: $\mathbf{i}_1(i) \coh_{X\times Y} \mathbf{i}_1(i')$ if and only if $i \coh_X i'$, $\mathbf{i}_2(j) \coh_{X\times Y} \mathbf{i}_2(j')$ if and only if $j \coh_Y j'$, and $\mathbf{i}_1(i) \coh_{X\times Y} \mathbf{i}_2(j)$ for every i, j. (In other words, $|X \times Y|$ is the disjoint sum of $|X|$ and $|Y|$; coherence is inherited from X and Y, and each element of $|X|$ is coherent with any element of $|Y|$.) The first projection is $\pi_1 : X \times Y \to X = \{(\mathbf{i}_1(i), i) | i \in |X|\}$, the second projection is $\pi_2 : X \times Y \to Y = \{(\mathbf{i}_2(j), j) | j \in |Y|\}$, and pairing of $f : Z \to X$ and $g : Z \to Y$ is $\langle f, g\rangle : Z \to X \times Y = \{(k, \mathbf{i}_1(i)) | (k, i) \in f\} \cup \{(k, \mathbf{i}_2(j)) | (k, j) \in g\}$.

The binary coproduct $X + Y$ is given by the same web, $|X + Y| = \{\mathbf{i}_1(i) | i \in |X|\} \cup \{\mathbf{i}_2(j) | j \in |Y|\}$, this time with coherence defined by $\mathbf{i}_1(i) \coh_{X+Y} \mathbf{i}_1(i')$ if and only if $i \coh_X i'$, $\mathbf{i}_2(j) \coh_{X+Y} \mathbf{i}_2(j')$ if and only if $j \coh_Y j'$, and $\mathbf{i}_1(i) \coh_{X+Y} \mathbf{i}_2(j)$ for no i, j. (In other words, this time no element of $|X|$ is coherent with any element of $|Y|$.) The first injection is $\mathbb{i}_1 : X \to X + Y = \{(i, \mathbf{i}_1(i)) | i \in |X|\}$, the second injection is $\mathbb{i}_2 : Y \to X + Y = \{(j, \mathbf{i}_2(j)) | j \in |Y|\}$, and the case analysis construct $[f, g] : X + Y \to Z$ (where $f : X \to Z$, $g : Y \to Z$) is $\{(\mathbf{i}_1(i), k) | (i, k) \in f\} \cup \{(\mathbf{i}_2(j), k) | (j, k) \in g\}$.

- The ! functor. There are two choices here. The only one that makes the concrete coherence completion construction work is to take $|!X|$ to be the set of all multicliques of X, where a *multiclique* is a finite multiset of pairwise coherent elements. In concrete coherence spaces, multicliques can be encoded as lists of pairwise coherent elements. To ensure uniqueness of representation, let these lists be sorted.

Let therefore a *concrete multiclique* of $X = (|X|, \coh_X)$ be any sorted list $[i_1, i_2, \ldots, i_k]$, where by sorted we mean $i_1 \preceq i_2 \preceq \ldots \preceq i_k$, and $i_1, i_2, \ldots, i_k$ form a clique in $|X|$. We shall abuse notation: if e is a concrete multiclique $[i_1, i_2, \ldots, i_k]$, we shall understand e ambiguously as the multiset $\{\!|i_1, i_2, \ldots, i_k|\!\}$; we retrieve the concrete multiclique from the multiset by sorting. In particular, multiset union makes sense on concrete multicliques.

The functor ! of $\mathcal{CCOH}$ maps $X = (|X|, \coh_X)$ to $!X = (|!X|, \coh_{!X})$, where $|!X|$ is the set of concrete multicliques of X, and $e \coh_{!X} e'$ if and only if $e \uplus e'$ is again a (concrete) multiclique, i.e., if and only if every element of e is coherent with any element of e'. Given any morphism $f : X \to Y$, where $X = (|X|, \coh_X)$ and $Y = (|Y|, \coh_Y)$, and two multicliques $e = [i_1, i_2, \ldots, i_k]$ in X and $e' = [j_1, j_2, \ldots, j_k]$ in Y, of the same *length* k, we call an *f-matching* of e with e' any permutation π of $\{1, 2, \ldots, k\}$ such that $(i_\ell, j_{\pi(\ell)}) \in f$ for every ℓ, $1 \leq \ell \leq k$. Then $!f : !X \to !Y$ is defined by

$$\begin{aligned} !f \quad = \quad & \{(e, e') \in |!X| \times |!Y| \\ & \quad \text{there is an } f\text{-matching of } e \text{ and } e'\} \end{aligned} \qquad (8)$$

Given two concrete multicliques $e = [i_1, i_2, \ldots, i_k]$ in X and $e' = [j_1, j_2, \ldots, j_k]$ in Y, an f-matching π need not be unique. For example, if $i_1 = j_1$, $i_2 = j_2$, and $(i_1, j_1) \in f$, then $\{1 \mapsto 1, 2 \mapsto 2\}$ and $\{1 \mapsto 2, 2 \mapsto 1\}$ are two f-matchings. However, the multiset $\{e\}f\{e'\}$ defined as

$\{\!\!\{(i_\ell, j_{\pi(\ell)}) | 1 \leq \ell \leq k\}\!\!\}$ is independent of π, and only depends on f, e and e'. This is by linearity of f.

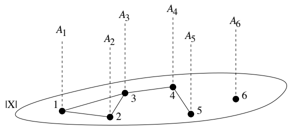
- The $(!, \delta, \epsilon)$ comonad. Let $\delta_X :!X \to !!X$ be the linear trace $\{(e, \{\!\!\{e_1, \ldots, e_n\}\!\!\}) | e \in |!X|, e = e_1 \uplus \ldots \uplus e_n\}$, and $\epsilon_X :!X \to X$ be $\{(\{\!\!\{x\}\!\!\}, x) | x \in |X|\}$.
- The cocommutative comonoid $(!X, \underline{d}_X, \underline{e}_X)$. Let $\underline{d}_X :!X \to !X \otimes !X$ be the linear trace $\{(e_1 \uplus e_2, \langle e_1, e_2 \rangle) \mid e_1, e_2 \in |!X|, e_1 \coh_{!X} e_2\}$, and let $\underline{e}_X :!X \to \mathbb{I}$ be the linear trace $\{(\{\!\!\{\}\!\!\}, \mathbf{1})\}$.

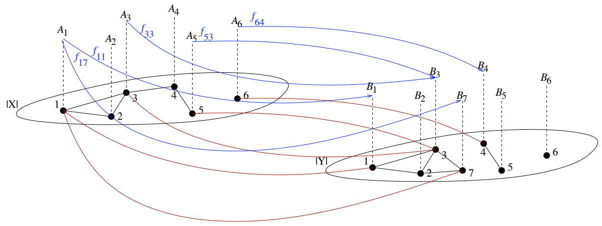
Note that $(!X, \underline{d}_X, \underline{e}_X)$ is the free cocommutative comonoid over X, as shown by Jan van de Wiele [25]. That is, the functor U mapping each cocommutative comonoid (X, d, e) to X in $\mathcal{CCOH}$ has a right adjoint. The non-trivial part of the proof is the construction of the unit of the adjunction: for any cocommutative comonoid (X, d_X, e_X) in $\mathcal{CCOH}$, η_X is the set of all pairs $(a, \{\!\!\{a_1, \ldots, a_n\}\!\!\}) \in |X| \times |!X|$, for every $n \in \mathbb{N}$ such that $(a, \langle a_1, \langle a_2, \ldots, \langle a_n, \mathbf{1} \rangle \ldots \rangle \rangle) \in d_X^n$, where $d_X^n : X \to X \otimes (X \otimes \ldots \otimes (X \otimes \mathbb{I}) \ldots)$ is defined by: $d_X^0 = e_X$, $d_X^{n+1} = (\mathrm{id}_X \otimes d_X^n) \circ d_X$.

C. Formal Definition of Concrete Coherence Completions

For any category $\mathcal{C}$, the *concrete coherence completion* $\mathcal{CCOH}(\mathcal{C})$ has as objects all pairs $(X, (A_i)_{i \in |X|})$ where X is a concrete coherence space $X = (|X|, \coh_X)$; and as morphisms from $(X, (A_i)_{i \in |X|})$ to $(Y, (B_j)_{j \in |Y|})$ all pairs $(f, (a_{ij})_{(i,j) \in f})$ where f is a linear trace from X to Y, and a_{ij} is a morphism from A_i to B_j in $\mathcal{C}$.

In pictures, an object $(X, (A_i)_{i \in |X|})$ is a trivial fibration over the web $|X|$ of the coherence space X: (I would really have liked to include these pictures in the main text—I had to take decisions.) Morphisms are then given as linear traces between the webs forming the base of objects (in brown) plus corresponding maps in $\mathcal{C}$, one atop each link drawn between bases (in blue):





The tricky part of this is to actually realize that this is a category, an observation due to Hu and Joyal [20]. The identity on $(X, (A_i)_{i \in |X|})$ is $(\mathrm{id}_X, (\mathrm{id}_{A_i})_{(i,i) \in \mathrm{id}_X})$. The composition of $(f, (a_{ij})_{(i,j) \in f}) : (X, (A_i)_{i \in |X|}) \to (Y, (B_j)_{j \in |Y|})$ with $(g, (a_{jk})_{(j,k) \in g}) : (Y, (B_j)_{j \in |Y|}) \to (Z, (C_k)_{k \in |Z|})$ is $(g \circ f, (c_{ik})_{(i,k) \in g \circ f})$, where $c_{ik} = b_{jk} \circ a_{ij}$ with j the unique index such that $(i, j) \in f$ and $(j, k) \in g$—it is unique because f and g are linear, and this is the crucial point.

For any (symmetric) monoidal category $\mathcal{C}$, $\mathcal{CCOH}(\mathcal{C})$ is (symmetric) monoidal, with tensor unit $(\mathbb{I}, (I))$ (where (I) denotes the family of just one object, I) and tensor product $(X, (A_i)_{i \in |X|}) \otimes (Y, (B_j)_{j \in |Y|})$ defined as $(X \otimes Y, (A_i \otimes B_j)_{(i,j) \in |X \otimes Y|}$.

Associativity, commutativity, neutrals are given by:

$$\begin{array}{rcl}
\multicolumn{3}{l}{\alpha_{(X,(A_i)_{i \in |X|}),(Y,(B_j)_{j \in |Y|}),(Z,(C_k)_{k \in |Z|})}} \\
 & = & (\alpha_{X,Y,Z}, (\alpha_{A_i,B_j,C_k})_{(\langle\langle i,j \rangle,k\rangle,\langle i,\langle j,k\rangle\rangle) \in \alpha_{X,Y,Z}}) \\
\ell_{(X,(A_i)_{i \in |X|})} & = & (\ell_X, (\ell_{A_i})_{(\langle \mathbf{1},i \rangle,i) \in \ell_X}) \\
r_{(X,(A_i)_{i \in |X|})} & = & (r_X, (r_{A_i})_{(\langle i,\mathbf{1} \rangle,i) \in r_X}) \\
\multicolumn{3}{l}{c_{(X,(A_i)_{i \in |X|}),(Y,(B_j)_{j \in |Y|})}} \\
 & = & (c_{X,Y}, (c_{A_i,B_j})_{(\langle i,j \rangle,\langle j,i \rangle) \in c_{X,Y}})
\end{array}$$

where α, ℓ, r, c are the respective associativity, left neutral, right neutral and commutativity in the corresponding underlying categories $\mathcal{C}$ and $\mathcal{CCOH}$.

If additionally $\mathcal{C}$ is autonomous, then so is $\mathcal{CCOH}(\mathcal{C})$, see Figure 2. If additionally $\mathcal{C}$ is $*$-autonomous with dualizing object $\perp$, then so is $\mathcal{CCOH}(\mathcal{C})$, and (recall that $\perp = \mathbb{I} = (\{\mathbf{1}\}, \coh_{\mathbb{I}})$ in $\mathcal{CCOH}$):

$$\begin{array}{rcl}
\perp & = & (\perp, (\perp)) \\
\mathcal{C}_{(X,(A_i)_{i \in |X|})} & = & (\mathcal{C}_X, (\mathcal{C}_{A_i})_{(\langle\langle i,\mathbf{1} \rangle,\mathbf{1}\rangle,i) \in \mathcal{C}_X})
\end{array}$$

By the way, even when $\mathcal{C}$ is compact-closed, as $\mathcal{DR}(M)$ is when M is weakly Cantorian, $\mathcal{CCOH}(\mathcal{C})$ is not in general compact-closed, because $X \parr Y$ is in general different from $X \otimes Y$ in $\mathcal{CCOH}$.

For any category $\mathcal{C}$, $\mathcal{CCOH}(\mathcal{C})$ has all finite products and coproducts. Finite products are defined in Figure 3, finite coproducts in Figure 4. In the definition of terminal and initial object, $()$ is the $\emptyset$-indexed family of objects. We use the same notation, e.g., π_1, for first projection in $\mathcal{CCOH}$ and for first projection in $\mathcal{CCOH}(\mathcal{C})$. The latter is defined (see above) as $(\pi_1, (\mathrm{id}_{A_i})_{(\mathbf{i}_1(i),i) \in \pi_1})$, where the first component is the former first projection. We hope this won't cause confusion.

Finally, when $\mathcal{C}$ is any symmetric monoidal category, we may define exponentials in $\mathcal{CCOH}(\mathcal{C})$ as follows. This is where we need to be able to totally order the elements of webs, by $\preceq$. For any finite family $(A_i)_{i \in |X|}$ of objects of $\mathcal{C}$, and any concrete multiclique $e = [i_1, i_2, \ldots, i_n]$ of X, where $i_1 \preceq i_2 \preceq \ldots \preceq i_n$, define $\bigotimes_{i \in e} A_i$ as the object $A_{i_1} \otimes (A_{i_2} \otimes \ldots (A_{i_k} \otimes \mathbb{I}) \ldots)$. The key observation we make here, compared to [20] is that we need no additional structure from $\mathcal{C}$ for this to work.

$$\begin{aligned}
(X,(A_i)_{i\in|X|}) \multimap (Y,(B_j)_{j\in|Y|}) &= (X \multimap Y, (A_i \multimap B_j)_{\langle i,j\rangle\in|X\multimap Y|}) \\
\mathsf{app}_{(X,(A_i)_{i\in|X|}),(Y,(B_j)_{j\in|Y|})} &= (\mathsf{app}_{X,Y}, (\mathsf{app}_{A_i,B_j})_{(\langle\langle i,j\rangle,i\rangle,j)} \in \mathsf{app}_{X,Y}) \\
\lambda^{(X,(A_i)_{i\in|X|})}_{(Y,(B_j)_{j\in|Y|}),(Z,(C_k)_{k\in|Z|})}(f,(a_{\langle i,j\rangle\ k})_{(\langle i,j\rangle,k)\in f}) &= (\lambda^X_{Y,Z}(f), (\lambda^{A_i}_{B_j,C_k}(a_{\langle i,j\rangle\ k}))_{(i,\langle j,k\rangle)\in\lambda^X_{Y,Z}(f)})
\end{aligned}$$

Figure 2. Autonomous structure on $\mathcal{CCOH}(\mathcal{C})$

$$\begin{aligned}
\mathsf{T} &= (\top, ()) \qquad \text{(terminal object; remember, } \top = (\emptyset, \smile_{\top})\text{)} \\
(X,(A_i)_{i\in|X|}) \times (Y,(B_j)_{j\in|Y|}) &= (X \times Y, (A_i)_{\mathbf{i}_1(i)\in|X\times Y|} \cup (B_j)_{\mathbf{i}_2(j)\in|X\times Y|}) \\
\pi_1 &: (X,(A_i)_{i\in|X|}) \times (Y,(B_j)_{j\in|Y|}) \to (X,(A_i)_{i\in|X|}) \\
&= (\pi_1, (\mathrm{id}_{A_i})_{(\mathbf{i}_1(i),i)\in\pi_1}) \\
\pi_2 &: (X,(A_i)_{i\in|X|}) \times (Y,(B_j)_{j\in|Y|}) \to (Y,(B_j)_{j\in|Y|}) \\
&= (\pi_2, (\mathrm{id}_{B_j})_{(\mathbf{i}_2(j),j)\in\pi_2}) \\
\langle (f,(a_{ki})_{(k,i)\in f}), (g,(b_{kj})_{(k,j)\in g}) \rangle &= \left(\langle f,g\rangle, \left(\text{case } \ell \text{ of} \left\{ \begin{array}{l} \iota_1 i \mapsto a_{ki} \\ \iota_2 j \mapsto b_{kj} \end{array} \right\} \right)_{(k,\ell)\in\langle f,g\rangle} \right) \\
\text{for every} \quad & (f,(a_{ki})_{(k,i)\in f}) : (Z,(C_k)_{k\in|Z|}) \to (X,(A_i)_{i\in|X|}), \\
& (g,(b_{kj})_{(k,j)\in g}) : (Z,(C_k)_{k\in|Z|}) \to (Y,(B_j)_{j\in|Y|})
\end{aligned}$$

Figure 3. Products in $\mathcal{CCOH}(\mathcal{C})$

$$\begin{aligned}
0 &= (\mathbb{0}, ()) \qquad \text{(initial object; remember, } \mathbb{0} = (\emptyset, \smile_{\mathbb{0}})\text{)} \\
(X,(A_i)_{i\in|X|}) + (Y,(B_j)_{j\in|Y|}) &= (X + Y, (A_i)_{\mathbf{i}_1(i)\in|X+Y|} \cup (B_j)_{\mathbf{i}_2(j)\in|X+Y|}) \\
\mathring{\mathbb{i}}_1 &: (X,(A_i)_{i\in|X|}) \to (X,(A_i)_{i\in|X|}) \times (Y,(B_j)_{j\in|Y|}) \\
&= (\mathring{\mathbb{i}}_1, (\mathrm{id}_{A_i})_{(i,\mathbf{i}_1(i))\in\mathring{\mathbb{i}}_1}) \\
\mathring{\mathbb{i}}_2 &: (Y,(B_j)_{j\in|Y|}) \to (X,(A_i)_{i\in|X|}) \times (Y,(B_j)_{j\in|Y|}) \\
&= (\mathring{\mathbb{i}}_2, (\mathrm{id}_{B_j})_{(j,\mathbf{i}_2(j))\in\mathring{\mathbb{i}}_2}) \\
[(f,(a_{ki})_{(k,i)\in f}), (g,(b_{kj})_{(k,j)\in g})] &= ([f,g], (a_{ik})_{(\iota_1 i,k)\in[f,g]} \cup (b_{jk})_{(\iota_2 j,k)\in[f,g]}) \\
\text{for every} \quad & (f,(a_{ik})_{(i,k)\in f}) : (X,(A_i)_{i\in|X|}) \to (Z,(C_k)_{k\in|Z|}), \\
& (g,(b_{jk})_{(j,k)\in g}) : (Y,(B_j)_{j\in|Y|}) \to (Z,(C_k)_{k\in|Z|})
\end{aligned}$$

Figure 4. Coproducts in $\mathcal{CCOH}(\mathcal{C})$

For any object $(X, (A_i)_{i\in|X|})$ of $\mathcal{CCOH}(\mathcal{C})$, let

$$!(X, (A_i)_{i\in|X|}) \quad = \quad \left(!X, \left(\bigotimes_{i\in e} A_i \right)_{e\in|!X|} \right)$$

For any morphism $(f, (a_{ij})_{(i,j)\in f}) : (X, (A_i)_{i\in|X|}) \to (Y, (B_j)_{j\in|Y|})$, let

$$!(f, (a_{ij})_{(i,j)\in f}) :!(X, (A_i)_{i\in|X|}) \to !(Y, (B_j)_{j\in|Y|})$$
$$= \left(!f, \left(\bigotimes_{(i,j)\in\{e_1\}f\{e_2\}} a_{ij} \right)_{(e_1,e_2)\in !f} \right)$$

where the notation $\bigotimes_{(i,j)\in\{e_1\}f\{e_2\}} a_{ij}$ is an abuse of language for $\bigotimes_{1\leq\ell\leq n} a_{i_\ell j_{\pi(\ell)}}$, where π is any f-matching of $e_1 = [i_1, i_2, \ldots, i_k]$ and $e_2 = [j_1, j_2, \ldots, j_k]$.

Then ! is an endofunctor of $\mathcal{CCOH}(\mathcal{C})$, which can be turned into a comonad by defining comultiplication $\delta_{(X,(A_i)_{i\in|X|})} :!(X, (A_i)_{i\in|X|}) \to !!(X, (A_i)_{i\in|X|})$ as:

$$\left(\delta_X, (\cong_{e_1,\ldots,e_n})_{e\in|!X|, e=e_1\uplus\ldots\uplus e_n} \right)$$

where $\cong_{e_1,\ldots,e_n}$ denotes the obvious natural iso from $\bigotimes_{i_1\in e_1} A_{i_1} \otimes \ldots \otimes \bigotimes_{i_n\in e_n} A_{i_n} \otimes \mathbb{I}$ to $\bigotimes_{i\in e_1\uplus\ldots\uplus e_n} A_i$, defined from associativity, commutativity and the neutrals of the tensor product $\otimes$; and counit $\epsilon_{(X,(A_i)_{i\in|X|})} : !(X, (A_i)_{i\in|X|}) \to (X, (A_i)_{i\in|X|})$ as:

$$\left(\epsilon_X, (r_{A_i})_{i\in|X|} \right)$$

where $r_{A_i} : A_i \otimes I \to A_i$ is right neutral in $\mathcal{C}$.

The cocommutative comonoid structure on $!(X, (A_i)_{i\in|X|})$ is given as follows. First, $\underline{d}_{(X,(A_i)_{i\in|X|})} : !(X, (A_i)_{i\in|X|}) \to !(X, (A_i)_{i\in|X|}) \otimes !(X, (A_i)_{i\in|X|})$ is:

$$\left(\underline{d}_X, (\cong_{e_1,e_2})_{e_1,e_2\in|!X|, e_1\frown_{!X} e_2} \right)$$

where $\cong_{e_1,e_2}$ denotes the obvious natural iso from $\bigotimes_{i\in e_1\uplus e_2} A_i$ to $\bigotimes_{i\in e_1} A_i \otimes \bigotimes_{i\in e_2} A_i$, defined from associativity, commutativity and the neutrals of the tensor product $\otimes$. Then, $\underline{e}_{(X,(A_i)_{i\in|X|})} :!(X, (A_i)_{i\in|X|}) \to (\mathbb{I}, (I))$ is:

$$(\underline{e}_X, (\mathrm{id}_I))$$

where (id_I) is the one-element family only containing $\mathrm{id}_I : \bigotimes_{i\in\{\!\{\}\!\}} A_i \to I$.

To show that this defines a Lafont category, we only need to show that ! is the free cocommutative comonoid comonad on $\mathcal{CCOH}(\mathcal{C})$. The counit of the adjunction is ϵ, and the unit η is defined from the corresponding unit η in $\mathcal{CCOH}$ (see end of Appendix B).

The forgetful functor $U : co\mathcal{M}on(\mathcal{CCOH}(\mathcal{C})) \to \mathcal{CCOH}(\mathcal{C})$ maps each cocommutative comonoid $((X, (A_i)_{i\in|X|}), \widetilde{d}, \widetilde{e})$ to $(X, (A_i)_{i\in|X|})$. Its right adjoint is $F : \mathcal{CCOH}(\mathcal{C}) \to co\mathcal{M}on(\mathcal{CCOH}(\mathcal{C}))$, which maps each object $(X, (A_i)_{i\in|X|})$ to $(!(X, (A_i)_{i\in|X|}), \underline{d}_{(X,(A_i)_{i\in|X|})}, \underline{e}_{(X,(A_i)_{i\in|X|})})$, and each morphism $(f, (a_{ij})_{(i,j)\in f}) : (X, (A_i)_{i\in|X|}) \to (Y, (B_j)_{j\in|Y|})$ to $!(f, (a_{ij})_{(i,j)\in f})$. The counit of the adjunction $U \dashv F$ is $\epsilon_{(X,(A_i)_{i\in|X|})} : !(X, (A_i)_{i\in|X|}) \to (X, (A_i)_{i\in|X|})$. The unit is the most challenging construct. For each object $((X, (A_i)_{i\in|X|}), \widetilde{d}, \widetilde{e})$ of $co\mathcal{M}on(\mathcal{CCOH}(\mathcal{C}))$, let $\eta_{((X,(A_i)_{i\in|X|}),\widetilde{d},\widetilde{e})} : ((X, (A_i)_{i\in|X|}), \widetilde{d}, \widetilde{e}) \to (!(X, (A_i)_{i\in|X|}), \underline{d}_{(X,(A_i)_{i\in|X|})}, \underline{e}_{(X,(A_i)_{i\in|X|})})$ be the following morphism in $\mathcal{CCOH}(\mathcal{C})$ from $(X, (A_i)_{i\in|X|})$ to $!(X, (A_i)_{i\in|X|})$:

$$\left(\eta_X, (\cong \circ a^n_{i,i_1,\ldots,i_n})_{(i,\{\!\{i_1,\ldots,i_n\}\!\})\in|X|\times|!X|, (i,\langle i_1,\langle i_2,\ldots,\langle i_n,\mathbf{1}\rangle\ldots\rangle\rangle)\in d^n_X} \right)$$

where:

- $\widetilde{d} = (d_X, (a_{ijk} : A_i \to A_j \otimes A_k)_{(i,\langle j,k\rangle)\in d_X})$;
- $\widetilde{e} = (e_X, (b_i : A_i \to I)_{(i,\mathbf{1})\in e_X})$;
- $a^k_{i,i_{n-k+1},\ldots,i_n} : A_i \to A_{i_{n-k+1}} \otimes (A_{i_{n-k+2}} \otimes \ldots (A_{i_n} \otimes I) \ldots)$ is defined whenever $0 \leq k \leq n$ and $(i, \langle i_{n-k+1}, \ldots, \langle .i_n, \mathbf{1}\rangle \ldots\rangle) \in d^k_X$, by: $a^0_i = b_i$, and $a^{k+1}_{i,i_{n-k},\ldots,i_n} = (\mathrm{id}_{A_{i_{n-k}}} \otimes a^k_{j,i_{n-k+1},\ldots,i_n}) \circ a_{ii_{n-k}j}$ for some j such that $(i, \langle i_{n-k}, j\rangle) \in d_X$ and $(j, \langle i_{n-k+1}, \ldots, \langle .i_n, \mathbf{1}\rangle \ldots\rangle) \in d^k_X$.
- $\cong$ is the obvious natural iso from $A_{i_1} \otimes (A_{i_2} \otimes \ldots (A_{i_n} \otimes I) \ldots)$ to $\bigotimes_{j\in\{\!\{i_1,i_2,\ldots,i_n\}\!\}} A_j$ defined from associativity, commutativity and the neutrals of the tensor product $\otimes$.