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On aperiodic and star-free formal power series in partially commuting variables*

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Abstract. Formal power series over non-commuting variables have been investigated as representations of the behavior of automata with multiplicities. Here we introduce and investigate the concepts of aperiodic and of star-free formal power series over semirings and partially commuting variables. We prove that if the semiring K is idempotent and commutative, or if K is idempotent and the variables are non-commuting, then the product of any two aperiodic series is again aperiodic. We also show that if K is idempotent and the matrix monoids over K have a Burnside property (satisfied, e.g. by the tropical semiring), then the aperiodic and the star-free series coincide. This generalizes a classical result of Schützenberger (1961) for aperiodic regular languages and subsumes a result of Guaiana, Restivo and Salemi (1992) on aperiodic trace languages.

1 Introduction

In the theory of automata, Kleene's fundamental theorem on the coincidence of regular and rational languages in free monoids has been extended in many ways. Schützenberger [24] investigated formal power series over arbitrary semirings (e.g., like the natural numbers) and with non-commuting variables and showed that the recognizable formal power series, which represent precisely the behavior of automata with multiplicities (cf. Eilenberg [9]), coincide with the rational series. This was the starting point for a large amount of work on formal power series, cf. [23, 15, 1, 14] for surveys. Special cases of automata with multiplicities are networks with capacities (weights) and have been also investigated in operations research for algebraic optimization problems, cf. [29] and in the 'max-plus-community' [10].

Schützenberger [25] also showed that in free monoids the aperiodic regular languages coincide with the star-free languages. Such languages are important for

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and arise from counter-free automata, and have been also investigated intensively due to characterizations using first order logic (McNaughton and Papert [20]) or temporal logic (Kamp [13]).

An algebraic characterization of the sub-algebra of series over commutative fields generated by letters and geometric series was announced by Reutenauer [22] as an analogue of Schützenberger's theorem. However, in the case of the boolean semiring, his class of series restricts to a proper subclass of the aperiodic languages (dot-depth 3/2).

It is the aim of this paper to introduce and investigate the concepts of aperiodicity and star-freeness for formal power series over arbitrary semirings. In fact, we will allow the variables to be partially commutative. A recognizable series is called aperiodic if, when iterating any complex task in a representing automaton with multiplicities, there is a fixed bound of iterations after which the computed value (weight) remains stable. This is an assumption often made in optimization problems, cf. [29]. A series is star-free, if it can be constructed from finitely many polynomials using the operations sum, product and complement, with the latter being applied only to characteristic series. This generalizes the concepts of aperiodic and of star-free languages, respectively.

Before stating our results, let us recall the notion of partially commuting variables. A trace alphabet (Σ, I) consists of a finite alphabet Σ and an irreflexive symmetric relation I indicating when two elements a, b of Σ commute, e.g. can occur independently of each other in a given concurrent system. A trace monoid $\mathbb{M}$ is therefore defined as the quotient of the free monoid Σ^* modulo the congruence generated by the relations $ab \sim ba$ if $a \; I \; b$. These monoids were introduced by Mazurkiewicz [18, 19] as an important mathematical model for the behavior of concurrent systems, see also [3, 5, 4] for their well-developed theory.

Now let K be an arbitrary semiring, and let $K\langle\langle\mathbb{M}\rangle\rangle$ be the collection of all formal power series $S = \sum_{m\in\mathbb{M}}(S, m) \cdot m$. These can also be viewed as series with entries (S, m) from K in which certain of the variables (= elements of Σ) are allowed to commute, as indicated by the relation I.

Whereas formal power series over non-commuting variables represent the behavior of sequential systems with weights, series over partially commuting variables can be viewed as the behavior of concurrent systems with multiplicities ('weights' for the actions). For an investigation of recognizable and rational formal power series over trace monoids, we refer the reader to [7].

Let us now give a summary of our main results. They all require that the semiring K of coefficients be idempotent. In applications, this is satisfied when 'addition' means the operation of taking minimum or maximum. We first show that the product of any two aperiodic series with non-commuting variables is again aperiodic. This means that the sequentialization of two such aperiodic systems stays aperiodic. We then show that this remains true even with partially commuting variables, but under the additional assumption that K be commutative. With an example, we show that the idempotence of K is necessary.

For our further results we need that the matrix monoids $K^{n\times n}$ of $(n \times n)$-matrices over K have a Burnside property: each finitely generated torsion submonoid of $K^{n\times n}$ be finite. We note that since Burnside's question in 1902, this property has been deeply investigated in group and semigroup theory, cf. the surveys of Simon [27] and Pin [21] for its relevance to automata theory. By a deep result of Simon [26], the important tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +)$ (and several others) satisfies this property. We can then show that any series over K and partially commuting variables is aperiodic if and only if it is star-free. If here $K = \mathbb{B}$, the Boolean semiring, we obtain as a consequence that in trace monoids the aperiodic languages are precisely the star-free ones, a result of Guaiana, Restivo and Salemi [12] which in turn contains Schützenberger's classical result for aperiodic languages of words.

Note that this provides a syntactic construction of how to obtain the behavior of a given aperiodic concurrent system by combining singleton automata using the operations parallel sum, sequentialization, and complementation.

A preliminary version of this work appeared in the extended abstract [8].

2 Preliminaries

Here we recall the necessary notation and background for formal power series and for trace theory. For more details, we refer the reader to [23, 1, 3, 5].

Let M be any monoid and $(K, +, \cdot, 0, 1)$ any semiring, i.e., $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, multiplication distributes over addition, and $0 \cdot x = x \cdot 0 = 0$ for each $x \in K$. If multiplication is commutative, we say that K is *commutative*. If the addition is idempotent, then the semiring is called *idempotent*. For instance, the Boolean semiring $\mathbb{B} = (\{0, 1\}, +, \cdot, 0, 1)$ is both commutative and idempotent. The semiring of natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$ is commutative but not idempotent. The semiring $(\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{1\})$ of languages over some alphabet Σ is idempotent but not commutative. The semiring $\mathbb{N}^{n\times n}$ of $(n \times n)$-matrices is neither commutative nor idempotent. Other semirings useful in computer science (and also in optimization problems of operations research [29]) are the min-plus (or max-plus or min-max) semirings over the integers or the reals. For instance the min-plus semiring over the reals $(\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$ is both commutative and idempotent.

Also, the semiring K is locally finite if any finitely generated subsemiring of K is finite. For instance, if both sum and product are commutative and idempotent then it is easy to see that the semiring is locally finite. This is in particular the case for the min-max semiring $(\mathbb{R} \cup \{-\infty, +\infty\}, \min, \max, +\infty, -\infty)$ or for semirings which are boolean algebras like $(\mathcal{P}(\Sigma^*), \cup, \cap, \emptyset, \{1\})$.

A formal power series is a mapping

$$\begin{aligned} S : M &\longrightarrow K \\ m &\longmapsto (S, m) \end{aligned}$$

It is usually denoted as a formal sum $S = \sum_{m\in M}(S, m).m$. The set $\mathrm{Im}(S) = \{(S, m) \mid m \in M\}$ is called the *image* of S. The set $\mathrm{Supp}(S) = \{m \in M \mid (S, m) \neq 0\}$ is called the *support* of S. If $\mathrm{Supp}(S)$ is finite, then S is called a *polynomial*. The collection of all formal power series is denoted by $K\langle\langle M\rangle\rangle$, and its subset of all polynomials by $K\langle M\rangle$. We consider elements of K also as polynomials in the natural way, having a non-zero entry only at $1 \in M$. If $L \subseteq M$, we define the *characteristic series* 1_L of L by letting $1_L(m) = 1$ if $m \in L$ and $1_L(m) = 0$ otherwise.

Let $n \geq 1$ and $[n] = \{1, \ldots, n\}$. We let $K^{n\times n}$ be the monoid of all $(n \times n)$-matrices over K (with matrix multiplication as usual). A series $S \in K\langle\langle M\rangle\rangle$ is called *recognizable*, if there exists an integer $n \geq 1$, a monoid morphism $\mu : M \to K^{n\times n}$ and vectors $\lambda \in K^{1\times n}, \gamma \in K^{n\times 1}$ such that

$$(S, m) = \lambda \cdot \mu(m) \cdot \gamma = \sum_{i,j\in[n]} \lambda_i \mu(m)_{ij} \gamma_j$$

for each $m \in M$. In this case, the triple (λ, μ, γ) is called a *representation* of dimension n of the series S, and we often shortly write $S = (\lambda, \mu, \gamma)$ to denote this. We let $K^{\mathrm{rec}}\langle\langle M\rangle\rangle$ denote the set of all recognizable formal power series.

With componentwise addition, $K\langle\langle M\rangle\rangle$ becomes a commutative monoid: the sum of two series $S, S' \in K\langle\langle M\rangle\rangle$ is defined for $m \in M$ by

$$(S + S', m) = (S, m) + (S', m).$$

The *Hadamard product* $S \odot S'$ of two series $S, S' \in K\langle\langle M\rangle\rangle$ is defined by componentwise multiplication, that is, for $m \in M$ let

$$(S \odot S', m) = (S, m) \cdot (S', m).$$

With sum and Hadamard product, $K\langle\langle M\rangle\rangle$ is a semiring. In order to define the *(Cauchy) product* of two series, we assume that M is a finitely generated monoid which carries a length function, that is a morphism ℓ from M to $(\mathbb{N}, +)$ such that $\ell(m) = 0$ iff $m = 1$. This ensures in particular that each element $m \in M$ has only finitely many factorizations $m = m_1 \cdot m_2$. Then, the *(Cauchy) product* of two series S, S' in $K\langle\langle M\rangle\rangle$ is the series defined for $m \in M$ by

$$(S \cdot S', m) = \sum_{m=m_1\cdot m_2} (S, m_1) \cdot (S', m_2).$$

Without any assumptions on M, a product $S_1 \cdot S_2$ of two series $S_1, S_2 \in K\langle\langle M\rangle\rangle$ can also be naturally defined as above if S_1, S_2 have finite image and K is idempotent (using the convention that an infinite sum of a constant value $k \in K$ equals k). Note that if $K = \mathbb{B}$, the Boolean semiring, the mapping $L \mapsto 1_L$ constitutes a bijection between $\mathcal{P}(M)$ and $\mathbb{B}\langle\langle M\rangle\rangle$ with inverse $S \mapsto \mathrm{supp}(S)$. Under this bijection, the operations union, intersection and product for languages correspond to sum, Hadamard product and Cauchy product for series. With sum and Cauchy product, $K\langle\langle M\rangle\rangle$ is again a semiring.

Next we recall basic notions from trace theory. A pair (Σ, I) is called a *trace alphabet*, if Σ is a finite set and I is an irreflexive symmetric binary *independence* relation on Σ. Let $\sim$ denote the smallest congruence on Σ^* containing $\{(ab, ba) \mid a\ I\ b\}$. The quotient monoid $\mathbb{M} = \mathbb{M}(\Sigma, I) := \Sigma^*/\sim$ is called the *trace* *monoid* (or free partially commutative monoid) over (Σ, I) and its elements are called traces. Note that a trace monoid is finitely generated and carries a length function.

If $w \in \Sigma^*$, we let $[w]$ denote the equivalence class of w in $\mathbb{M}$. Also, let $\alpha(w)$ be the set of all letters of Σ occurring in w, called the *alphabet* of w. Since equivalent words have the same alphabet, we may put $\alpha([w]) = \alpha(w)$.

We say that two subsets A, B of Σ are independent and we write $A\ I\ B$ if $A \times B \subseteq I$. We also say that two words w, w' of Σ^* are independent, denoted by $w\ I\ w'$, if $\alpha(w)\ I\ \alpha(w')$. Similarly, we define $[w]\ I\ [w']$, $w\ I\ A$, etc.

The following generalized Levi's factorization is a very useful and classical result in trace theory.

Lemma 2.1 ([2, 3]). *Let $u, v, w_1, \ldots, w_n \in \mathbb{M}$. Then, $uv = w_1 \cdots w_n$ if and only if there are $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{M}$ such that $u = u_1 \cdots u_n$, $v = v_1 \cdots v_n$, $w_i = u_i v_i$ for all $1 \leq i \leq n$ and $v_i\ I\ u_j$ for all $1 \leq i < j \leq n$. Moreover, the traces $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{M}$ with this property are unique.*

A monoid N is said to be *aperiodic* if there exists some integer $m \geq 0$ such that $x^m = x^{m+1}$ for all $x \in N$. As is easy to see, this condition is equivalent to demanding both that any group contained in N as a subsemigroup is trivial and that the sets $\{x^m \mid m \geq 0\}$ are finite for all $x \in N$. The *index* of an aperiodic monoid N is the smallest integer $m \geq 0$ such that $x^m = x^{m+1}$ for all $x \in N$.

A language $L \subseteq M$ is *aperiodic* if there exists a morphism $\varphi : M \to N$ into some finite aperiodic monoid such that $L = \varphi^{-1}(\varphi(L))$. When this holds, we say that the morphism φ recognizes L. We denote by $\mathrm{AP}(M)$ the family of aperiodic languages of M.

Note that by definition, any aperiodic language is recognizable.The collection $\mathrm{SF}(M)$ of all *star-free* languages in M is defined as the smallest set of languages of M containing all finite languages and which is closed under the operations union, product and complement.

A fundamental theorem by Schützenberger [25] states that in free monoids aperiodic languages coincide with star-free languages. It was extended by Guaiana, Restivo and Salemi [12] to trace monoids. It is the aim of this paper to generalize this theorem to formal power series both for free monoids and for trace monoids.

3 Aperiodic series

In this section we introduce *aperiodic* and *weakly aperiodic* series and we study their closure properties.

Definition 3.1. *A recognizable series $S \in K\langle\langle M\rangle\rangle$ is* aperiodic *if there exists a representation $S = (\lambda, \mu, \gamma)$ with $\mu(M)$ aperiodic, i.e. there is some integer $m \geq 0$ such that $\mu(u^m) = \mu(u^{m+1})$ for all $u \in M$. In this case, we say that the morphism μ is aperiodic. The collection of aperiodic series in $K\langle\langle M\rangle\rangle$ is denoted by $K^{\mathrm{ap}}\langle\langle M\rangle\rangle$.*

Also we say that a recognizable series is *weakly aperiodic* if there exists some integer $m \geq 0$ such that $(S, uv^m w) = (S, uv^{m+1} w)$ for all $u, v, w \in M$. Clearly, all aperiodic series are also weakly aperiodic. We will see in the next section that the converse is true when the semiring K is locally finite or a field.

It is well-known [23] that if a language $L \subseteq M$ is recognizable then so is its characteristic series $1_L \in K\langle\langle M \rangle\rangle$. The converse does not hold for arbitrary semirings but it holds for a wide class of semirings such as positive semirings ($\mathbb{B}$,$\mathbb{N}$,...) or locally finite semirings (see next section). We show now that assuming recognizability, the equivalence is true for aperiodic languages and aperiodic characteristic series.

Proposition 3.2. *Let K be an arbitrary semiring, let M be an arbitrary monoid and let $L \subseteq M$ be recognizable. Then L is aperiodic iff its characteristic series $1_L \in K\langle\langle M \rangle\rangle$ is aperiodic.*

Proof. First, let N be a finite aperiodic monoid and let $\varphi : M \to N$ be a morphism recognizing L, i.e. $L = \varphi^{-1}(\varphi(L))$. Let $n = |N|$. We identify N with $[n] = \{1, \ldots, n\}$, 1 being indeed the neutral element of N. We define $\mu : M \to K^{n \times n}$, $\lambda \in K^{1 \times n}$ and $\gamma \in K^{n \times 1}$ by

$$\mu(u)_{i,j} = \begin{cases} 1 & \text{if } j = i \cdot \varphi(u) \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \gamma_j = \begin{cases} 1 & \text{if } j \in \varphi(L) \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that μ is a morphism and that $1_L = (\lambda, \mu, \gamma)$. Indeed,

$$\begin{aligned}(\mu(u) \cdot \mu(v))_{i,j} &= \sum_k \mu(u)_{i,k} \mu(v)_{k,j} = \mu(u)_{i,i\cdot\varphi(u)} \mu(v)_{i\cdot\varphi(u),j} = \mu(v)_{i\cdot\varphi(u),j} \\ &= \begin{cases} 1 & \text{if } j = i \cdot \varphi(u) \cdot \varphi(v) \\ 0 & \text{otherwise} \end{cases} = \mu(uv)_{i,j}\end{aligned}$$

and

$$\begin{aligned}\lambda\mu(w)\gamma &= \sum_{i,j} \lambda_i \mu(w)_{i,j} \gamma_j = \sum_j \mu(w)_{1,j} \gamma_j = \mu(w)_{1,1\cdot\varphi(w)} \gamma_{1\cdot\varphi(w)} = \gamma_{1\cdot\varphi(w)} \\ &= \begin{cases} 1 & \text{if } 1 \cdot \varphi(w) \in \varphi(L) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } w \in \varphi^{-1}\varphi(L) \\ 0 & \text{otherwise} \end{cases} = 1_L(w)\end{aligned}$$

Finally, it remains to show that the morphism μ is aperiodic. Since N is aperiodic, there exists some integer $m \geq 0$ such that $x^m = x^{m+1}$ for all $x \in N$. Then,

$$\mu(u^{m+1})_{i,j} = \begin{cases} 1 & \text{if } j = i \cdot \varphi(u)^{m+1} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } j = i \cdot \varphi(u)^m \\ 0 & \text{otherwise} \end{cases} = \mu(u^m)_{i,j}$$

Conversely, assume that 1_L is aperiodic, then it is also weakly aperiodic which implies the existence of an integer $m \geq 0$ such that $uv^m w \in L$ iff $uv^{m+1}w \in L$ for all $u, v, w \in M$. Therefore, the syntactic monoid of L is aperiodic. Since we assumed L recognizable, the syntactic monoid of L is also finite and we obtain that L is aperiodic. □

From the above result, we deduce that a semiring for which L is recognizable whenever 1_L is recognizable satisfies also that L is aperiodic whenever 1_L is aperiodic. We will see in the next section (Corollary 4.6) that this last statement is true for even more general semirings.

Proposition 3.3. *Let K be an arbitrary semiring and let M be an arbitrary monoid. Aperiodic series in $K\langle\langle M\rangle\rangle$ are closed under sum and external product. If K is commutative, aperiodic series are also closed under Hadamard product.*

Proof. Let $S = (\lambda, \mu, \gamma)$ be an aperiodic series and let $b \in K$. Then $b \cdot S = (b\lambda, \mu, \gamma)$ is also aperiodic.

For $i = 1, 2$, let $S_i = (\lambda^i, \mu^i, \gamma^i)$ be an aperiodic series of dimension n_i. Then $S_1 + S_2 = (\lambda, \mu, \gamma)$ with $\lambda = (\lambda^1, \lambda^2)$, $\gamma = \begin{pmatrix} \gamma^1 \\ \gamma^2 \end{pmatrix}$, and $\mu = \begin{pmatrix} \mu^1 & 0 \\ 0 & \mu^2 \end{pmatrix}$. Now, since μ is a block matrix, it is immediate to check that if μ^1 and μ^2 are aperiodic with indexes m_1 and m_2 then μ is also aperiodic with index $m = \max(m_1, m_2)$.

Now assume K is commutative and S_1, S_2 are aperiodic series as above. Let $Q = [n_1] \times [n_2]$ and define $S = (\lambda, \mu, \gamma)$ with $\lambda, \gamma \in K^Q$ and $\mu : M \to K^{Q \times Q}$ by $\lambda(i_1, i_2) = \lambda^1(i_1) \cdot \lambda^2(i_2)$, $\gamma(j_1, j_2) = \gamma^1(j_1) \cdot \gamma^2(j_2)$, $\mu(w)_{(i_1,i_2),(j_1,j_2)} = \mu^1(w)_{i_1,j_1} \cdot \mu^2(w)_{i_2,j_2}$, so $\mu(w)$ is the Kronecker product of the matrices $\mu^1(w)$ and $\mu^2(w)$. Since K is commutative, μ is a morphism and $S = S_1 \odot S_2$, [23, Thm. II.4.4]. Now let μ^1, μ^2 have indices m_1 and m_2, and put $m = \max\{m_1, m_2\}$. For each $w \in M$ we have $\mu^i(w^m) = \mu^i(w^{m+1})$ for $i = 1, 2$, so $\mu(w^m) = \mu(w^{m+1})$ by definition of μ. □

Recall that polynomials are the series with finite supports and they correspond to finite languages when the semiring is $\mathbb{B}$. In arbitrary monoids, finite languages are not always recognizable. But finite languages are aperiodic in particular when the monoid is finitely generated and carries a length function, which is the case for trace monoids. We show that polynomials are aperiodic series whenever finite languages are aperiodic.

Corollary 3.4. *Let K be an arbitrary semiring and let M be a monoid such that singletons are aperiodic languages. Then, all polynomials in $K\langle\langle M\rangle\rangle$ are aperiodic.*

Proof. Let $S \in K\langle\langle M\rangle\rangle$ be a polynomial and let $L = \{w \in M \mid (S, w) \neq 0\}$ be its finite support. We may write $S = \sum_{w \in L} (S, w) \cdot 1_{\{w\}}$. We can conclude immediately using Propositions 3.2 and 3.3. □

Our aim is now to show that aperiodic series over free monoids are closed under Cauchy product when the semiring K is idempotent. For this, we will use special representations whose existence are proved in the next two lemmas.

When (λ, μ, γ) is a representation of dimension n, we denote by $I = \{i \in [n] \mid \lambda_i \neq 0\}$ the set of so-called initial states and by $F = \{j \in [n] \mid \gamma_j \neq 0\}$ the set of so-called final states.

We say that M is a monoid *without proper subgroups*, if the only group occurring as submonoid of M is the trivial group; equivalently, $x \cdot y = 1$ implies $x = y = 1$ for any $x, y \in M$.

Lemma 3.5. *Let K be an arbitrary semiring, M be an arbitrary monoid without proper subgroups and $S \in K\langle\langle M\rangle\rangle$ be a recognizable series. Then there exists a representation $S = (\lambda, \mu, \gamma)$ of dimension n such that for all $i, j \in [n]$,*

- *j is final iff $\gamma_j = 1$, and*
- *for all $w \in M \setminus \{1\}$, $\mu(w)_{ij} = 0$ whenever i is final.*

Moreover, we may also require that μ is aperiodic if S is an aperiodic series and that there is exactly one final state.

Proof. Let (λ, μ, γ) be a representation of S of dimension n. We define a representation $(\lambda', \mu', \gamma')$ of dimension $n+1$ by $\lambda'_i = \lambda_i$ if $1 \leq i \leq n$, $\lambda'_{n+1} = \sum_{1 \leq j \leq n} \lambda_j \gamma_j$, $\gamma' = (0, \ldots, 0, 1)^t$ and for $w \in M \setminus \{1\}$,

$$\mu'(w)_{ij} = \begin{cases} \mu(w)_{ij} & \text{if } 1 \leq i, j \leq n \\ \sum_{1 \leq q \leq n} \mu(w)_{iq}\gamma_q & \text{if } 1 \leq i \leq n, j = n+1 \\ 0 & \text{otherwise (i.e. if } i = n+1). \end{cases}$$

We first show that μ' is indeed a morphism so that $(\lambda', \mu', \gamma')$ is a well-defined representation. Let $u, v \in M \setminus \{1\}$ and $i, j \in [n+1]$. We have

$$\begin{aligned} (\mu'(u) \cdot \mu'(v))_{ij} &= \sum_{1 \leq k \leq n+1} \mu'(u)_{ik}\mu'(v)_{kj} = \sum_{1 \leq k \leq n} \mu'(u)_{ik}\mu'(v)_{kj} \\ &= \begin{cases} \sum_{1 \leq k \leq n} \mu(u)_{ik}\mu(v)_{kj} & \text{if } 1 \leq i, j \leq n \\ \sum_{1 \leq k, q \leq n} \mu(u)_{ik}\mu(v)_{kq}\gamma_q & \text{if } 1 \leq i \leq n, j = n+1 \\ 0 & \text{otherwise (i.e. if } i = n+1). \end{cases} \\ &= \mu'(uv)_{ij}, \end{aligned}$$

where the last equality uses for the case $i = j = n + 1$ that $uv \neq 1$, which is implied by the hypothesis on M. Clearly, the representation $(\lambda', \mu', \gamma')$ fulfills the requirements of the lemma and we have for all $w \in M \setminus \{1\}$,

$$\lambda'\mu'(w)\gamma' = \sum_{1 \leq i \leq n+1} \lambda'_i \mu'(w)_{i,n+1} = \sum_{1 \leq i,j \leq n} \lambda_i \mu(w)_{ij}\gamma_j = (S, w)$$

and also

$$(S, 1) = \sum_{1 \leq j \leq n} \lambda_j \gamma_j = \lambda'_{n+1} = \lambda'\mu'(1)\gamma'$$

which proves that $S = (\lambda', \mu', \gamma')$.

Finally, if S is aperiodic then we may assume that there exists $m \geq 0$ such that $\mu(w^m) = \mu(w^{m+1})$ for all $w \in M$. From the definition of μ' above, it is clear that $\mu'(w^m) = \mu'(w^{m+1})$ for all $w \in M$ showing that μ' is also aperiodic. ☐

Using a similar proof, we also obtain

Lemma 3.6. *Let K be an arbitrary semiring, M be an arbitrary monoid without proper subgroups and $S \in K\langle\langle M\rangle\rangle$ be a recognizable series. Then there exists a representation $S = (\lambda, \mu, \gamma)$ of dimension n such that for all $i, j \in [n]$,*

- *i is initial iff $\lambda_i = 1$, and*
- *for all $w \in M \setminus \{1\}$, $\mu(w)_{ij} = 0$ whenever j is initial.*

Moreover, we may also require that μ is aperiodic if S is an aperiodic series and that there is exactly one initial state.

Note that it is also possible to find a representation which satisfies both requirements of Lemmas 3.5 and 3.6 if the series S is proper, i.e. if $(S, 1) = 0$.

We are now ready to prove that aperiodic series over the free monoid Σ^* are closed under Cauchy product when the semiring K is idempotent.

Theorem 3.7. *Assume that the semiring K is idempotent. Let $S_1, S_2 \in K\langle\langle \Sigma^*\rangle\rangle$ be aperiodic series, then their product $S = S_1 \cdot S_2$ is also aperiodic.*

Proof. Let $(\lambda^1, \mu^1, \gamma^1)$ be an aperiodic representation of S_1 of dimension n_1 which satisfies the requirements of Lemma 3.5 and let $(\lambda^2, \mu^2, \gamma^2)$ be an aperiodic representation of S_2 of dimension n_2 which satisfies the requirements of Lemma 3.6. Let $n = n_1 \cdot n_2$ and identify $[n]$ with $[n_1] \times [n_2]$. We define the morphism $\mu : \Sigma^* \to K^{n \times n}$ by giving its value over letters $a \in \Sigma$:

$$\mu(a)_{(j_1,j_2)(k_1,k_2)} = \delta_{j_2,k_2}\mu^1(a)_{j_1,k_1} + \delta_{j_1,k_1}\mu^2(a)_{j_2,k_2} f(a, j_1)$$

where

$$\delta_{j,k} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad f(v, j) = \begin{cases} 1 & \text{if } v = 1 \text{ or } j \text{ is final} \\ 0 & \text{otherwise} \end{cases}$$

Note that $f(a, j_1)\mu^1(a)_{j_1,k_1} = 0$, hence at most one of the two terms in the definition of $\mu(a)_{(j_1,j_2)(k_1,k_2)}$ is non-zero.

We first prove by induction that for all $w \in \Sigma^*$ we have

$$\mu(w)_{(i_1,i_2)(j_1,j_2)} = \sum_{w=uv} \mu^1(u)_{i_1 j_1}\mu^2(v)_{i_2 j_2} f(v, j_1).$$

The result is clear for $w = 1$. Assume that the formula above holds for some $w \in \Sigma^*$ and let $a \in \Sigma$. Then, we have

$$\begin{aligned}
\mu(wa)_{(i_1,i_2)(k_1,k_2)} &= \sum_{j_1,j_2} \mu(w)_{(i_1,i_2)(j_1,j_2)}\mu(a)_{(j_1,j_2)(k_1,k_2)} \\
&= \sum_{j_1} \mu(w)_{(i_1,i_2)(j_1,k_2)}\mu^1(a)_{j_1k_1} + \sum_{j_2} \mu(w)_{(i_1,i_2)(k_1,j_2)}\mu^2(a)_{j_2k_2} f(a,k_1) \\
&= \sum_{j_1,w=uv} \mu^1(u)_{i_1j_1}\mu^2(v)_{i_2k_2} f(v,j_1)\mu^1(a)_{j_1k_1} \\
&\qquad + \sum_{j_2,w=uv} \mu^1(u)_{i_1k_1}\mu^2(v)_{i_2j_2} f(v,k_1)\mu^2(a)_{j_2k_2} f(a,k_1) \\
&= \sum_{j_1} \mu^1(w)_{i_1j_1}\mu^1(a)_{j_1k_1}\mu^2(1)_{i_2k_2} + \sum_{w=uv} \mu^1(u)_{i_1k_1}\mu^2(va)_{i_2k_2} f(va,k_1) \\
&= \mu^1(wa)_{i_1k_1}\mu^2(1)_{i_2k_2} + \sum_{w=uv} \mu^1(u)_{i_1k_1}\mu^2(va)_{i_2k_2} f(va,k_1) \\
&= \sum_{wa=uv} \mu^1(u)_{i_1k_1}\mu^2(v)_{i_2k_2} f(v,k_1)
\end{aligned}$$

Note that the fourth equality is justified by the fact that $f(v,j_1)\mu^1(a)_{j_1k_1} = 0$ when $v \neq 1$ because $(\lambda^1,\mu^1,\gamma^1)$ satisfies the requirements of Lemma 3.5. Note also that in the computation above, we have used that 0 and 1 commute with everything in the semiring K.

Now, we define $\lambda \in K^{1\times n}$ by $\lambda_{(i_1,i_2)} = \lambda^1_{i_1}\lambda^2_{i_2}$ and $\gamma \in K^{n\times 1}$ by $\gamma_{(j_1,j_2)} = \gamma^1_{j_1}\gamma^2_{j_2}$. We claim that $S = S_1 \cdot S_2 = (\lambda,\mu,\gamma)$. Note that, from the hypotheses on the representations of S_1 and S_2 we know that γ^1 and λ^2 consist of 0 and 1 only, which commute with any value in the semiring K. Also, for all $j_1 \in [n_1]$ and $v \in \Sigma^*$ we have $\gamma^1_{j_1} = f(v,j_1)\gamma^1_{j_1}$. Therefore, we compute for any $w \in \Sigma^*$

$$\begin{aligned}
(S_1S_2,w) &= \sum_{w=uv} (S_1,u)(S_2,v) \\
&= \sum_{w=uv} \left(\sum_{i_1,j_1} \lambda^1_{i_1}\mu^1(u)_{i_1j_1}\gamma^1_{j_1} \cdot \sum_{i_2,j_2} \lambda^2_{i_2}\mu^2(v)_{i_2j_2}\gamma^2_{j_2} \right) \\
&= \sum_{i_1,i_2,j_1,j_2} \lambda^1_{i_1}\lambda^2_{i_2} \left(\sum_{w=uv} \mu^1(u)_{i_1j_1}\mu^2(v)_{i_2j_2} \right) \gamma^1_{j_1}\gamma^2_{j_2} \\
&= \sum_{i_1,i_2,j_1,j_2} \lambda_{(i_1,i_2)}\mu(w)_{(i_1,i_2)(j_1,j_2)}\gamma_{(j_1,j_2)} = \lambda\mu(w)\gamma.
\end{aligned}$$

Finally, it remains to show that the morphism μ is aperiodic. Let $q > 1$ be such that $\mu^i(w^q) = \mu^i(w^{q-1})$ for all $w \in \Sigma^*$ and $i = 1,2$. We claim that for $m > 2q$ we have $\mu(w^m) = \mu(w^{m+1})$ for all $w \in \Sigma^*$.

Fix $(i_1,i_2),(j_1,j_2) \in [n]$ and let $\varphi(u,v) = \mu^1(u)_{i_1j_1}\mu^2(v)_{i_2j_2} f(v,j_1)$ for all $u,v \in \Sigma^*$. If $m > 2q$ and $w^m = uv$ then we have either $u = w^qu'$ or $v = v'w^q$. In the first case, using the fact that $\mu^1(w^q) = \mu^1(w^{q-1}) = \mu^1(w^{q+1})$ we deduce that $\varphi(u,v) = \varphi(w^{q-1}u',v) = \varphi(w^{q+1}u',v)$. We proceed similarly in the second case and we obtain $\{\varphi(u,v) \mid w^m = uv\} = \{\varphi(u,v) \mid w^{m+1} = uv\}$.

From this, using the fact that sum is idempotent, we deduce

$$\mu(w^m)_{(i_1,i_2)(j_1,j_2)} = \sum_{w^m=uv} \varphi(u,v) = \sum_{w^{m+1}=uv} \varphi(u,v) = \mu(w^{m+1})_{(i_1,i_2)(j_1,j_2)}$$

which completes the proof of the theorem. □

The above proof essentially uses a product of two weighted automata with behaviour S_1 resp. S_2 and defines the weights of the transitions of the product automaton suitably. We just note that one could also take a union of the state sets of the two automata for S_1 and S_2, identifying the final state of the first automaton with the initial state of the second automaton, and keeping the transitions (and their weights) of the two automata; all other transitions get weight 0, cf. [9, proof of Proposition VI.7.8]. Then, if the two automata for S_1 and S_2 are 'aperiodic', so is the union automaton. However, the behaviour of this union automaton is also mimicked exactly in the transitions of our product automaton as given above. Moreover, the use of a product automaton will be essential for extending the result to trace monoids. Therefore, the above proof can also be viewed as a preparation for this extension.

The idempotence of sum is needed to obtain this result, even for the free monoid. Indeed, let $K = (\mathbb{N}, +, \cdot)$ and $\Sigma = \{a\}$. The language Σ^* is clearly aperiodic and therefore its characteristic series $S = 1_{\Sigma^*}$ is aperiodic as well (Proposition 3.2). Now, for all $m \geq 0$, $(S \cdot S, a^m) = \sum_{a^m=uv}(S,u)(S,v) = m+1$. Therefore, the series S^2 is not weakly aperiodic, whence is not aperiodic.

We aim now at establishing a similar result for arbitrary trace monoids. Alphabetic representations have been introduced in [7] in order to study the closure of recognizable series over trace monoids under product and star. We will use a simpler form of alphabetic representations which is sufficient to show that aperiodic series are also closed under product.

We say that a representation (λ, μ, γ) of dimension n is *(past-)alphabetic*, if there exists a function $\overleftarrow{\alpha} : [n] \to \mathcal{P}(\Sigma)$ such that for all $u \in \mathbb{M}$, the following two conditions are satisfied:

(1) Whenever $\mu(u)_{ij} \neq 0$, then $\overleftarrow{\alpha}(j) = \overleftarrow{\alpha}(i) \cup \alpha(u)$

(2) whenever $\lambda_i \neq 0$, then $\overleftarrow{\alpha}(i) = \emptyset$.

We call $(\lambda, \mu, \gamma; \overleftarrow{\alpha})$ an alphabetic representation of S.

Proposition 3.8. *Let K be an arbitrary semiring and $S \in K\langle\langle\mathbb{M}\rangle\rangle$ be an aperiodic series over traces. Then there exists an alphabetic representation $S = (\lambda, \mu, \gamma; \overleftarrow{\alpha})$ with μ aperiodic.*

Proof. The construction of the representation follows the same line as in [7, Proposition 6]. Assume that $S = (\lambda', \mu', \gamma')$ with $\mu' : \mathbb{M} \to K^{n' \times n'}$ an aperiodic morphism. Let $n = n' \cdot 2^{|\Sigma|}$. Subsequently, we identify $[n]$ with $[n'] \times \mathcal{P}(\Sigma)$. We define $\mu : \mathbb{M} \to K^{n \times n}$ and $\lambda \in K^{1 \times n}$, $\gamma \in K^{n \times 1}$ by

$$\mu(u)_{(i,X)(j,Y)} = \begin{cases} \mu'(u)_{ij} & \text{if } Y = X \cup \alpha(u) \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{(i,X)} = \begin{cases} \lambda'_i & \text{if } X = \emptyset \\ 0 & \text{otherwise} \end{cases} \qquad \gamma_{(i,X)} = \gamma'_i$$

Also, we put $\overleftarrow{\alpha}(i, X) = X$.

It can be shown as in [7, Proposition 6] that $(\lambda, \mu, \gamma; \overleftarrow{\alpha})$ is an alphabetic representation of S. Here, we only have to show that μ is aperiodic. Indeed, let $m > 0$ be such that $\mu'(u^m) = \mu'(u^{m+1})$ for all $u \in \mathbb{M}$. Note that since we have taken $m > 0$, we have $\alpha(u^{m+1}) = \alpha(u) = \alpha(u^m)$ for all $u \in \mathbb{M}$. It follows directly from the definition of μ that $\mu(u^m) = \mu(u^{m+1})$ for all $u \in \mathbb{M}$. □

We will now prove that aperiodic series over trace monoids are closed under Cauchy product when the semiring K is idempotent and commutative.

Theorem 3.9. *Assume that the semiring K is idempotent and commutative. Let $S_1, S_2 \in K\langle\langle\mathbb{M}\rangle\rangle$ be aperiodic series, then their product $S = S_1 \cdot S_2$ is also aperiodic.*

Proof. Again, we use the construction of [7, Theorem 7]. Let $(\lambda^1, \mu^1, \gamma^1)$ be an aperiodic representation of S_1 of dimension n_1 and let $(\lambda^2, \mu^2, \gamma^2; \overleftarrow{\alpha})$ be an alphabetic representation of S_2 of dimension n_2 with μ_2 aperiodic (Proposition 3.8). Let $n = n_1 \cdot n_2$ and identify $[n]$ with $[n_1] \times [n_2]$. Next, we define $\mu : \mathbb{M} \to K^{n\times n}$ by

$$\mu(w)_{(i_1,i_2)(j_1,j_2)} = \sum_{w=uv} I(u, i_2)\mu^1(u)_{i_1,j_1}\mu^2(v)_{i_2,j_2}$$

where

$$I(u, i) = \begin{cases} 1 & \text{if } u \; I \; \overleftarrow{\alpha}(i) \\ 0 & \text{otherwise.} \end{cases}$$

It was shown in [7, Theorem 7] that μ is a morphism and $S = (\lambda, \mu, \gamma)$ where $\lambda \in K^{1\times n}$ and $\gamma \in K^{n\times 1}$ are defined by $\lambda_{(i_1,i_2)} = \lambda^1_{i_1}\lambda^2_{i_2}$ and $\gamma_{(k_1,k_2)} = \gamma^1_{k_1}\gamma^2_{k_2}$.

We have assumed μ^1 and μ^2 aperiodic so let $q > 1$ be such that $\mu^1(w^q) = \mu^1(w^{q-1})$ and $\mu^2(w^q) = \mu^2(w^{q-1})$ for all $w \in \mathbb{M}$. We will show that μ is aperiodic. Throughout the rest of the proof, we fix $(i_1, i_2), (j_1, j_2) \in [n]$. For $w \in \mathbb{M}$ and $m \geq 0$, we define

$$X_m(w) = \{(u_1, \dots, u_m, v_1, \dots, v_m) \in \mathbb{M}^{2m} \mid \forall i \in [m], w = u_i v_i \text{ and } \forall 1 \leq i < j \leq m, v_i \; I \; u_j\}.$$

Also, for $x = (u_1, \dots, u_m, v_1, \dots, v_m)$, we set $\varphi(x) = I(u, i_2)\mu^1(u)_{i_1,j_1}\mu^2(v)_{i_2,j_2}$ where $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_m$ (recall that $(i_1, i_2), (j_1, j_2)$ have been fixed). Using the generalized Levi's factorization (Lemma 2.1) and the definition of μ we deduce immediately that

$$\mu(w^m)_{(i_1,i_2)(j_1,j_2)} = \sum_{x\in X_m(w)} \varphi(x).$$

Note that, thanks to the unicity of the factorization in Levi's Lemma, we do not use that sum is idempotent for this result.

Claim. If $m > q(|\Sigma|+1)$ then $\varphi(X_m(w)) = \varphi(X_{m+1}(w))$. Actually, we will prove that for all $x \in X_m(w)$, there exist $y \in X_{m-1}(w)$ and $z \in X_{m+1}(w)$ such that $\varphi(x) = \varphi(y) = \varphi(z)$.

Let $x = (u_1, \ldots, u_m, v_1, \ldots, v_m) \in X_m(w)$. For $1 \le i < j \le m$ we have $v_i \; I \; u_j$, which implies that $\alpha(v_i) \subseteq \alpha(w) \setminus \alpha(u_j) \subseteq \alpha(v_j)$. Therefore we have $\alpha(v_1) \subseteq \cdots \subseteq \alpha(v_m)$ and from the hypothesis on m we deduce that there exists $1 \le p \le m - q$ such that $\alpha(v_p) = \cdots = \alpha(v_{p+q})$.

Let $p < k \le p + q$, we have $\alpha(w) = \alpha(u_k) \cup \alpha(v_k)$ and $\alpha(u_k) \; I \; \alpha(v_{k-1}) = \alpha(v_k)$. Hence we obtain $\alpha(u_k) = \alpha(w) \setminus \alpha(v_k)$. Since there is at most one way to split the trace w into two independent traces whose alphabet are fixed, we deduce that $u_{p+1} = \cdots = u_{p+q} = \bar{u}$ and $v_{p+1} = \cdots = v_{p+q} = \bar{v}$.

Let $u' = u_1 \cdots u_p$, $u'' = u_{p+q+1} \cdots u_m$, $v' = v_1 \cdots v_p$ and $v'' = v_{p+q+1} \cdots v_m$. Then we have

$$\begin{aligned}
\mu^1(u) &= \mu^1(u'\bar{u}^q u'') = \mu^1(u'\bar{u}^{q+1}u'') = \mu^1(u'\bar{u}^{q-1}u'') \\
\mu^2(v) &= \mu^2(v'\bar{v}^q v'') = \mu^2(v'\bar{v}^{q+1}v'') = \mu^2(v'\bar{v}^{q-1}v'') \\
\alpha(u) &= \alpha(u'\bar{u}^q u'') = \alpha(u'\bar{u}^{q+1}u'') = \alpha(u'\bar{u}^{q-1}u'')
\end{aligned}$$

The claim follows since this implies that $\varphi(x) = \varphi(y) = \varphi(z)$ for

$$\begin{aligned}
y &= (u_1, \ldots, u_{p+q-1}, u_{p+q+1}, \ldots, u_m, v_1, \ldots, v_{p+q-1}, v_{p+q+1}, \ldots, v_m) \in X_{m-1}(w) \\
z &= (u_1, \ldots, u_{p+q}, u_{p+q}, \ldots, u_m, v_1, \ldots, v_{p+q}, v_{p+q}, \ldots, v_m) \in X_{m+1}(w).
\end{aligned}$$

Using the claim, we deduce immediately that for $m > q(|\Sigma| + 1)$ it holds

$$\mu(w^m)_{(i_1,i_2)(j_1,j_2)} = \sum_{x \in X_m(w)} \varphi(x) = \sum_{x \in X_{m+1}(w)} \varphi(x) = \mu(w^{m+1})_{(i_1,i_2)(j_1,j_2)}.$$

Here we have used that sum is idempotent in K since $|X_m(w)| < |X_{m+1}(w)|$ as soon as $w \neq 1$. □

Commutativity is needed because we are dealing with trace monoids. An example was given in [7, Section 5] showing that recognizable series over trace monoids are not closed under product in general.

As usual, in the special case of the boolean semiring we obtain the result on trace languages as a corollary.

Corollary 3.10 ([12]). *The product of two aperiodic trace languages is again aperiodic.*

Proof. Let $L_1, L_2 \subseteq \mathbb{M}$ be two aperiodic trace languages. By Proposition 3.2 the characteristic series $1_{L_1}, 1_{L_2} \in \mathbb{B}\langle\langle\mathbb{M}\rangle\rangle$ are aperiodic. Then by Theorem 3.9 we deduce that $1_{L_1} \cdot 1_{L_2} = 1_{L_1 \cdot L_2} \in \mathbb{B}\langle\langle\mathbb{M}\rangle\rangle$ is also aperiodic. Since we are in the boolean semiring this implies that $L_1 \cdot L_2$ is recognizable. Applying again Proposition 3.2 we deduce that $L_1 \cdot L_2$ is aperiodic. □

4 Aperiodic series over semirings with Burnside matrix monoids

In this section we will derive a representation of recognizable and of aperiodic formal power series over semirings with local finiteness conditions which will be crucial later on. We will also briefly investigate the possible weakening of aperiodicity mentioned after Definition 3.1.

Recall that a semiring K is locally finite, if each finitely generated subsemiring is finite. A monoid is called locally finite, if each finitely generated submonoid is finite. Clearly, any commutative idempotent monoid is locally finite. Also, a semiring $(K, +, \cdot, 0, 1)$ is locally finite iff both monoids $(K, +, 0)$ and $(K, \cdot, 1)$ are locally finite. Indeed, if X is a finite subset of K then the submonoid Y of $(K, \cdot, 1)$ generated by X is finite and the submonoid Z of $(K, +, 0)$ generated by Y is also finite. Now, it is easy to check that $Z \cdot Z \subseteq Z$ and we deduce that the subsemiring of $(K, +, \cdot, 0, 1)$ generated by X is the finite set Z.

For example, the max-min semiring $\mathcal{R}_{\max,\min} = (\mathbb{R}_+ \cup \{\infty\}, \max, \min, 0, \infty)$ of positive reals, used in operations research for maximum capacity problems of networks, is locally finite. In fact, any distributive lattice $(L, \vee, \wedge, 0, 1)$ with smallest element 0 and largest element 1 is a locally finite semiring.

Note that if K is a locally finite semiring, then the matrix monoids $K^{n\times n}$ are locally finite for all n. Indeed, let $Y \subseteq K^{n\times n}$ be a finite set of matrices and let X be the subsemiring of K generated by the finite set $\{A_{ij} \mid A \in Y, i, j \in [n]\}$. Then the submonoid of $K^{n\times n}$ generated by Y is contained in $X^{n\times n}$ which is a finite submonoid of $K^{n\times n}$.

Conversely, if $K^{2\times 2}$ is a locally finite monoid then K is a locally finite semiring. Indeed, let $X \subseteq K$ be a finite set. By considering the submonoid generated by the matrices

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in X \right\}$$

we obtain that $(K, \cdot, 1)$ is locally finite. Now, by considering the submonoid generated by the matrices

$$\left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in X \right\}$$

we obtain that $(K, +, 0)$ is locally finite. For this, note that

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}.$$

Proposition 4.1. *Let K be a locally finite semiring, M be a finitely generated monoid and $S = (\lambda, \mu, \gamma) \in K^{\mathrm{rec}}\langle\langle M\rangle\rangle$ be a recognizable series. Then $\mu(M)$ and $\mathrm{Im}(S)$ are finite.*

Proof. Let $S = (\lambda, \mu, \gamma)$ be a representation with $\mu : M \to K^{n\times n}$. Let Σ be a finite set of generators of M. Then, $\mu(M)$ is the submonoid of $K^{n\times n}$ generated by the finite set $\mu(\Sigma)$. Since K is locally finite we deduce that $\mu(M)$ is finite. Therefore, $\mathrm{Im}(S) = \{\lambda \cdot A \cdot \gamma \mid A \in \mu(M)\}$ is also finite. □

For applications on aperiodic series, we use weaker assumptions on K. A monoid N is called *torsion* (or *periodic*), if each of its cyclic submonoids $\{x^n \mid n \geq 0\}$ ($x \in N$) is finite. Each locally finite monoid is torsion. The converse, which was posed as a problem for groups by Burnside in 1902 (and answered negatively by Golod in 1964), leads to deep problems in semigroup theory, see the surveys by Simon [27] and Pin [21] for its relevance to automata theory. We will say that a semiring K has *Burnside matrix monoids*, if in the monoids $K^{n\times n}(n \in \mathbb{N})$ each finitely generated torsion submonoid is finite. These semirings will be important to us because of the following easy but crucial observation.

Proposition 4.2. *Let K be a semiring with Burnside matrix monoids, M be a finitely generated monoid and let $S = (\lambda, \mu, \gamma) \in K\langle\langle M\rangle\rangle$ with μ aperiodic. Then $\mu(M)$ and $\mathrm{Im}(S)$ are finite.*

Proof. Since μ is aperiodic and M is finitely generated, $\mu(M)$ is a finitely generated torsion submonoid of $K^{n\times n}$ and hence finite by assumption on K. Then, $\mathrm{Im}(S) = \{\lambda \cdot A \cdot \gamma \mid A \in \mu(M)\}$ is also finite. □

Before continuing, let us point out examples of such semirings. Clearly, if K is locally finite then K has Burnside matrix monoids. But also all of the following important (not locally finite) semirings have been shown to have Burnside matrix monoids:

Theorem 4.3. *Each of the following semirings has Burnside matrix monoids:*

- $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ *and its completion* $\mathcal{N} = (\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ *(Mandel and Simon [16]),*
- $\mathcal{M} = (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ *(Simon [26]),*
- $\mathcal{P} = (\mathbb{N} \cup \{-\infty, \infty\}, \max, +, -\infty, 0)$ *(Mascle [17]),*
- $\mathcal{R}at = (Rat(a^*), \cup, \cdot, \emptyset, a^*)$ *(Mascle [17]).*
- *any commutative semiring or PI-ring (a ring satisfying a polynomial identity), cf. [21] and [28]*

Now let $S \in K\langle\langle M\rangle\rangle$. If $k \in K$, we let $S^{-1}(k) = \{m \in M \mid (S, m) = k\}$. Now we obtain the following finitary description of recognizable respectively aperiodic series:

Corollary 4.4. *Let M be a finitely generated monoid and $S \in K\langle\langle M\rangle\rangle$. Assume that either S is recognizable and K is locally finite or that S is aperiodic and K has Burnside matrix monoids. Then, $S = \sum_{k\in\mathrm{Im}(S)} k \cdot 1_{S^{-1}(k)}$ is a finite linear combination of characteristic series with each $S^{-1}(k)$ recognizable (resp. aperiodic). In particular,* $\mathrm{Supp}(S)$ *is recognizable (resp. aperiodic).*

Proof. By Proposition 4.1 (resp. 4.2), $\mathrm{Im}(S)$ is finite and clearly, we have $S = \sum_{k\in\mathrm{Im}(S)} k \cdot 1_{S^{-1}(k)}$. It remains to show for $k \in K$ that $S^{-1}(k)$ is recognizable (resp. aperiodic). For this, let (λ, μ, γ) be a representation of S, with μ aperiodic in case S is aperiodic. By Proposition 4.1 (resp. 4.2), $\mu(M)$ is a finite monoid. We show that $\mu : M \to \mu(M)$ recognizes $S^{-1}(k)$. Indeed, if $u, v \in M$ with

$\mu(u) = \mu(v)$ and $u \in S^{-1}(k)$, then $(S, v) = \lambda \cdot \mu(v) \cdot \gamma = \lambda \cdot \mu(u) \cdot \gamma = (S, u) = k$. Therefore, $S^{-1}(k)$ is recognizable in M. Now if S is aperiodic, then $\mu(M)$ is an aperiodic monoid, so $S^{-1}(k)$ is aperiodic in M. □

As a consequence of this description of S, we can represent such an S by a classical deterministic complete M-automaton with weights attached only to the final states. That is, there is a representation (λ, μ, γ) of S with a unique initial state (which has weight 1) such that for each $w \in M$ and each i there is a unique j with $\mu(w)_{ij} \neq 0$, and then $\mu(w)_{ij} = 1$. Moreover, if the monoid M, the semiring K and a representation of S are given in an effective way, all the constituents of the above description of S can be effectively computed. Thus we obtain:

Corollary 4.5. *Let M be a computable finitely generated monoid, K a computable semiring and $S, T \in K\langle\langle M\rangle\rangle$. Assume that either K is locally finite and S, T have effectively given representations, or K has Burnside matrix monoids and S, T have effectively given aperiodic representations. Then from this, the following are decidable:*

1. $S = T$
2. $\mathrm{supp}(S) = \emptyset$
3. $\mathrm{supp}(S) = M$.

Proof. Compute the descriptions of S and T given in Corollary 4.4 and compare the arising languages and the coefficients of their characteristic series. □

The following is a further immediate consequence of Corollary 4.4.

Corollary 4.6. *Let M be a finitely generated monoid, K a semiring and $L \subseteq M$.*

(a) If 1_L is recognizable and K is locally finite, then L is recognizable.
(b) If 1_L is aperiodic and K has Burnside matrix monoids, then L is aperiodic.

Now we turn to the Hadamard product. As a further consequence of Corollary 4.4 we note:

Corollary 4.7. *Let M be a finitely generated monoid.*

(a) If K is locally finite, the Hadamard product of any two recognizable series in $K\langle\langle M\rangle\rangle$ is again recognizable.
(b) If K has Burnside matrix monoids, the Hadamard product of any two aperiodic series in $K\langle\langle M\rangle\rangle$ is again aperiodic.

Proof. (a) Let $S, T \in K^{\mathrm{rec}}\langle\langle M\rangle\rangle$. By Corollary 4.4, we have $S = \sum_{i=1}^{m} k_i \cdot 1_{L_i}$ and $T = \sum_{j=1}^{n} k'_j \cdot 1_{L'_j}$ with $k_i, k'_j \in K$ and recognizable languages $L_i, L'_j \subseteq M$. Then each $L_i \cap L'_j$ is recognizable in M, and $1_{L_i} \odot 1_{L'_j} = 1_{L_i \cap L'_j}$, so $S \odot T = \sum_{i,j} k_i k'_j \cdot 1_{L_i \cap L'_j} \in K^{\mathrm{rec}}\langle\langle M\rangle\rangle$.

(b) We argue as above, obtaining aperiodic languages L_i, L'_j, and apply Proposition 3.3. □

Now we will derive another version of Theorems 3.7 and 3.9 which holds also if K is non-commutative. Note that its proof does not require the (relatively complicated) arguments used for Theorems 3.7 and 3.9. Note also that we do not need to assume that the monoid M has a length function because the semiring is idempotent and the series considered have finite images. The (Cauchy) product of two series is well-defined in this case.

Corollary 4.8. *Let M be any finitely generated monoid in which the product of two aperiodic languages is again aperiodic, and let K be an idempotent semiring with Burnside matrix monoids. Let $S_1, S_2 \in K\langle\langle M\rangle\rangle$ be aperiodic. Then $S_1 \cdot S_2$ is aperiodic.*

Proof. By Corollary 4.4, we have $S_1 = \sum_{k_1 \in I_1} k_1 \cdot 1_{S_1^{-1}(k_1)}$ and $S_2 = \sum_{k_2 \in I_2} k_2 \cdot 1_{S_2^{-1}(k_2)}$ with $I_i = \mathrm{Im}(S_i)$ finite $(i = 1, 2)$ and each $S_1^{-1}(k_1), S_2^{-1}(k_2)$ aperiodic in M. For all $a, b \in K$ and $A, B \subseteq M$ we have $(a \cdot 1_A) \cdot (b \cdot 1_B) = ab \cdot (1_A \cdot 1_B)$ and since K is idempotent we also have $1_A \cdot 1_B = 1_{A \cdot B}$. Hence,

$$S_1 \cdot S_2 = \sum_{k_1 \in I_1, k_2 \in I_2} k_1 \cdot k_2 \cdot 1_{S_1^{-1}(k_1)} \cdot 1_{S_2^{-1}(k_2)} = \sum_{k_1 \in I_1, k_2 \in I_2} k_1 \cdot k_2 \cdot 1_{S_1^{-1}(k_1) \cdot S_2^{-1}(k_2)},$$

and each $S_1^{-1}(k_1) \cdot S_2^{-1}(k_2)$ is aperiodic in M. Now by Propositions 3.2 and 3.3, $S_1 \cdot S_2$ is aperiodic. □

With an analogous proof we also obtain:

Corollary 4.9. *Let M be any finitely generated monoid in which the product of two recognizable languages is again recognizable, and let K be a locally finite idempotent semiring. Let $S_1, S_2 \in K\langle\langle M\rangle\rangle$ be recognizable. Then $S_1 \cdot S_2$ is recognizable.*

Next we show for series over locally finite semirings K the equivalence of three notions of aperiodicity.

Corollary 4.10. *Let M be a finitely generated monoid, let K be a locally finite semiring and let $S \in K^{\mathrm{rec}}\langle\langle M\rangle\rangle$. Assume either S is weakly aperiodic or S has a representation (λ, μ, γ) such that $\mu(M)$ is group-free. Then S is aperiodic.*

Proof. First assume that S is weakly aperiodic. By Corollary 4.4, we have $S = \sum_{k \in \mathrm{Im}(S)} k \cdot 1_{S^{-1}(k)}$ with $\mathrm{Im}(S)$ finite and each $S^{-1}(k) \subseteq M$ recognizable. We show that the languages $S^{-1}(k)$ are also aperiodic. Let $m \geq 0$ be such that $(S, uv^m w) = (S, uv^{m+1} w)$ for all $u, v, w \in M$. Then, for all $k \in K$ we have $uv^m w \in S^{-1}(k)$ iff $uv^{m+1} w \in S^{-1}(k)$ and we deduce that the syntactic monoid of $S^{-1}(k)$ is aperiodic. Therefore, the languages $S^{-1}(k)$ are all aperiodic. From Propositions 3.2 and 3.3, we deduce that S is aperiodic.

Secondly, it is well-known that a finite monoid is aperiodic iff it is group-free. The result follows since by Proposition 4.1 the monoid $\mu(M)$ is finite. □

Next we will consider the case where K is a field and $S \in K\langle\langle M\rangle\rangle$ is weakly aperiodic. First we recall some background. Let K be a semiring and M any monoid. Let $S \in K\langle\langle M\rangle\rangle$. For $x \in M$, let $x^{-1}S \in K\langle\langle M\rangle\rangle$ be the series defined by $(x^{-1}S, y) = (S, xy)$ $(y \in M)$. Note that $K\langle\langle M\rangle\rangle$ with addition and multiplication with scalars from K as usual is a K-semimodule (and a vector space if K is a field). A submodule U of $K\langle\langle M\rangle\rangle$ is called *stable* if $x^{-1}S \in U$ for any $S \in U$ and $x \in K$.

For any series $S \in K\langle\langle M\rangle\rangle$ its *Hankel-matrix* $H(S) \in K^{M\times M}$ is defined by $H(S)_{u,v} = (S, uv)$ $(u, v \in M)$. Note that the rows of $H(S)$ are precisely the series $u^{-1}S$ $(u \in M)$. Furthermore, S is weakly aperiodic iff S is recognizable and there is $m \in \mathbb{N}$ such that $v^{-m}(u^{-1}S) = v^{-(m+1)}(u^{-1}S)$ for all $u, v \in M$.

It is well-known that S is recognizable iff S is contained in a finitely generated stable subsemimodule of $K\langle\langle M\rangle\rangle$ iff the rows of $H(S)$ are contained in some finitely generated stable subsemimodule of $K\langle\langle M\rangle\rangle$ [23, Ch.II.3].

Theorem 4.11. *Let K be a field, M any monoid, and $S \in K\langle\langle M\rangle\rangle$ weakly aperiodic. Then S is aperiodic.*

Proof. Let $Z = \langle u^{-1}S : u \in M\rangle$, the subspace of $K\langle\langle M\rangle\rangle$ generated by the rows of $H(S)$. Since S is recognizable, Z has a finite basis $B = \{F_1, \ldots, F_n\}$, say. Now define a mapping $\mu : M \to K^{n\times n}$ such that for each $v \in M$ and $i \in \{1, \ldots, n\}$, we have $v^{-1}F_i = \sum_{j=1}^{n} \mu(v)_{ij} \cdot F_j$ with uniquely determined scalars $\mu(v)_{ij} \in K$. Furthermore, $S = \sum_{i=1}^{n} \lambda_i F_i$ with $\lambda_i \in K$, and let $\gamma_j = F_j(1)$. Put $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)'$. Then μ is a morphism, and (λ, μ, γ) is a representation of S [23, Ch.II.3].

Now let $u_i \in M$ such that $F_i = u_i^{-1}S$ $(i = 1, \ldots, n)$. Since S is weakly aperiodic, as noted before there is $m \in \mathbb{N}$ such that for all $v \in M$ and $i = 1, \ldots, n$ we have $v^{-m}(u_i^{-1}S) = v^{-(m+1)}(u_i^{-1}S)$. Hence

$$\sum_{j=1}^{n} (\mu v^m)_{ij} \cdot F_j = v^{-m}F_i = v^{-(m+1)}F_i = \sum_{j=1}^{n} (\mu v^{m+1})_{ij} \cdot F_j.$$

Since B is a basis, comparison of coefficients implies $\mu(v^m) = \mu(v^{m+1})$. Thus S is aperiodic. □

5 Star-free series

In a monoid M, the collection $\mathrm{SF}(M)$ of star-free languages is defined as the smallest system of languages in M containing all finite languages and being closed under the operations union, complement and product. We define a corresponding notion for formal power series as follows. If K is a semiring and $L \subseteq M$, the *complement* of the characteristic series 1_L is simply the characteristic series $\overline{1_L} := 1_{\overline{L}}$ of the complement $\overline{L}$ of L. Hence complement is only a partially defined operation on $K\langle\langle M\rangle\rangle$, provided K has more than 2 elements.

Definition 5.1. *Let M be a finitely generated monoid and K a semiring. Assume either that M has a length function or that K is idempotent. The collection $K^{\mathrm{sf}}\langle\langle M\rangle\rangle$ of all star-free series in $K\langle\langle M\rangle\rangle$ is the smallest collection of formal power series containing all polynomials and being closed under the operations sum, product and the partial operation complement.*

Note that if K is idempotent, by structural induction all star-free series in $K\langle\langle M\rangle\rangle$ have finite image and the product operation is well-defined on $K^{\mathrm{sf}}\langle\langle M\rangle\rangle$. Note that if S is star-free and $k \in K$, then the series $k \cdot S$ is star-free, since k is a polynomial.

Proposition 5.2. *Let M be a finitely generated monoid and let K be idempotent and $L \subseteq M$ star-free. Then 1_L is a star-free series.*

Proof. By structural induction on L. If L is finite, 1_L is a polynomial. If $L = L_1 \cup L_2$, we have $1_L = 1_{L_1} + 1_{L_2}$ since $(K, +)$ is idempotent. For the same reason, $L = L_1 \cdot L_2$ implies $1_L = 1_{L_1} \cdot 1_{L_2}$. Finally, $1_{\overline{L}} = \overline{1_L}$ by definition. □

Schützenberger [25] showed that in the free monoid Σ^*, aperiodic languages are star-free. This was generalized by Guaiana [11] to arbitrary finitely generated monoids:

Lemma 5.3 ([11, Thm. 5.1.4]). *Let M be any finitely generated monoid. Then each aperiodic language $L \subseteq M$ is star-free.*

Now we can prove the main result of this section.

Theorem 5.4. *Let M be any finitely generated monoid, and let K be an idempotent semiring with Burnside matrix monoids. Then any aperiodic series in $K\langle\langle M\rangle\rangle$ is star-free. Moreover, the following are equivalent:*

(1) $\mathrm{SF}(M) = \mathrm{AP}(M)$
(2) $K^{\mathrm{sf}}\langle\langle M\rangle\rangle = K^{\mathrm{ap}}\langle\langle M\rangle\rangle$.

Proof. Let $S \in K\langle\langle M\rangle\rangle$ be aperiodic. By Corollary 4.4, we have $S = \sum_{k \in \mathrm{Im}(S)} k \cdot 1_{S^{-1}(k)}$ with $\mathrm{Im}(S)$ finite and each $S^{-1}(k)$ aperiodic in M. By Lemma 5.3, $S^{-1}(k)$ is star-free in M. Now apply Proposition 5.2 to obtain that S is star-free.

(1) $\Rightarrow$ (2) : By the above, it remains to show that each star-free series in $K\langle\langle M\rangle\rangle$ is aperiodic. We proceed by induction. Each singleton in M is star-free, hence aperiodic by (1). Using Corollary 3.4 we deduce that polynomials are aperiodic. By Proposition 3.3 and Corollary 4.8, the sum and the product of two aperiodic series are again aperiodic. Now let 1_L be aperiodic. Then L is aperiodic by Corollary 4.6, hence $\overline{L}$ is aperiodic. Then $\overline{1_L} = 1_{\overline{L}}$ is aperiodic by Proposition 3.2.

(2) $\Rightarrow$ (1) : Let $L \subseteq M$ be star-free. By Proposition 5.2, 1_L is star-free and hence aperiodic by (2). By Corollary 4.6, L is aperiodic. The converse is just Lemma 5.3. □

As a consequence of this and Corollary 4.4, we obtain an analogue of Corollary 4.4 for star-free series and the converse of Proposition 5.2.

Corollary 5.5. *Let K be an idempotent semiring with Burnside matrix monoids. Let M be any finitely generated monoid satisfying* $\mathrm{SF}(M) = \mathrm{AP}(M)$.

(a) If $S \in K\langle\langle M\rangle\rangle$ is star-free, then for any $k \in K$ the set $S^{-1}(k)$ is star-free in M. In particular, $\mathrm{Supp}(S)$ *is star-free.*
(b) If $L \subseteq K$, then L is star-free iff 1_L is star-free.

6 Conclusion

As a conclusion, we would like to summarize the main results obtained in this paper. The following theorem derives from Corollary 4.4, Propositions 3.2 and 3.3 and Theorem 5.4. It holds in particular for all trace monoids and for the semirings $\mathcal{M}$, $\mathcal{P}$, $\mathcal{R}at$ (Theorem 4.3) and $\mathcal{R}_{\max,\min}$, or any distributive lattice with 0 and 1.

Theorem 6.1. *Let K be an idempotent semiring with Burnside matrix monoids and M be any finitely generated monoid satisfying* $\mathrm{SF}(M) = \mathrm{AP}(M)$. *For a series $S \in K\langle\langle M\rangle\rangle$, the following are equivalent:*

(1) S is aperiodic,
(2) S is star-free,
(3) S is a finite linear combination of characteristic series of aperiodic languages.

The condition $\mathrm{SF}(M) = \mathrm{AP}(M)$ is satisfied for all trace monoids by Guaiana, Restivo and Salemi [12]. It also holds, e.g., for a class of particular concurrency monoids (intersecting with trace monoids only in the class of free monoids), see [6].

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