Extending the Rackoff technique to affine nets

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Abstract

We study the possibility of extending the Rackoff technique to Affine nets, which are Petri nets extended with affine functions. The Rackoff technique has been used for establishing Expspace upper bounds for the coverability and boundedness problems for Petri nets. We show that this technique can be extended to strongly increasing Affine nets, obtaining better upper bounds compared to known results. The possible copies between places of a strongly increasing Affine net make this extension non-trivial. One cannot expect similar results for the entire class of Affine nets since coverability is Ackermann-hard and boundedness is undecidable. Moreover, it can be proved that model checking a logic expressing generalized coverability properties is undecidable for strongly increasing Affine nets, while it is known to be Expspace-complete for Petri nets.

1 Introduction

Context Petri nets are infinite state models and have been used for modelling and verifying properties of concurrent systems. Various extensions of Petri nets that increase the power of transitions have been studied, for example Reset/Transfer (Petri) nets [8], Self-Modifying nets [21] and Petri nets with inhibitory arcs [17]. In [9], Well Structured nets are defined as another extension where transitions can be any non-decreasing function. The same paper also defines Affine Well Structured nets (shortly: Affine nets) that can be seen as the affine restriction of Well Structured nets, or as a restriction of the Self-Modifying nets of [21] to matrices with only non-negative integers.

While reachability is decidable for Petri Nets [11, 15, 13], it is undecidable for extensions with at least two extended transitions like Double/Reset/Transfer/Zero-test arcs [8]. However, it remains decidable for Petri Nets with one such extended arc [2] or even with hierarchical zero-tests [17]. The framework of Well-Structured Transition Systems [10] provides the decidability of coverability, termination and boundedness for Petri nets and some of its monotonic extensions [8, 9]. However, boundedness is undecidable for Reset nets (and hence for Affine nets) [8]. Complexity results on Petri nets extensions are scarce, two notable results being that coverability is Ackermann-complete for Reset nets [20, 19] (while reachability and boundedness are undecidable) and boundedness is Expspace-complete for a subclass of (strongly increasing) Affine nets [7].

Other extensions of Petri nets increase the state space of the transition system. These are for example branching Vector Addition Systems [6], ν-Petri nets [18], or Data nets [12]

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(equivalent to Timed Petri nets [3]). As ν-Petri nets and Data nets subsume Reset nets, boundedness and reachability are undecidable, and coverability is Ackermann-hard. On the other hand, while reachability is an (important) open problem for branching Vector Addition Systems, coverability and boundedness are known to be ALTExpspace-complete by a proof that uses the Rackoff technique [6].

Finally, we note that some recent papers [5, 1] have extended the Rackoff technique to show EXPSPACE upper bounds for the model-checking of some logics (that generalizes the notion of coverability and boundedness) for Petri nets.

**Our contribution** The goal is to exhibit a class of extensions of Petri nets for which the Rackoff technique can be extended in order to give an Expspace upper bound for coverability and boundedness. We do not look at extensions that change state spaces, as the complexity of coverability and boundedness for those is known to be either ALTExpspace-complete (branching Vector Addition Systems) or Ackermann-hard (ν-Petri nets, Data nets). Moreover, as this technique relies heavily on the monotonicity of Petri Nets, it is natural to consider only monotonic extensions. The largest classes of such extensions are Affine nets and Well Structured nets, that both include most of the usually studied Petri net extensions. As we know that coverability and boundedness are respectively Ackermann-hard and undecidable for Reset nets, we must forbid resets in order for our generalization to work. This is done by disallowing any 0 in the diagonal of the matrices associated with the functions of Affine nets, yielding again the class of strongly increasing Affine nets, as defined in [9], that are equivalent to the Post-Self-Modifying nets (PSM) defined by Valk in [21]. This class is interesting because it strictly subsumes Petri nets. For example, PSM can recognize the language of palindromes, which Petri nets can not. More generally, all recursively enumerable languages are recognized by PSM [21], while boundedness (and other properties) is still decidable [21].

While the complexity of the reachability problem for Petri nets is unknown, the complexity of coverability and boundedness has been shown to be EXPSPACE-complete (lower bound of SPACE(O(2^n√n)) by Lipton [14] and SPACE(O(2^{c n log n})) upper bound by Rackoff [16], where n is the size of the net). In [7], the boundedness problem is shown to be in SPACE(O(2^{cn log n})) for Post-Self-Modifying nets: the proof associates a standard Petri net that weakly simulates the original Post-Self-Modifying net and then applies the Rackoff theorem [16] as a black box (EXpspace upper bound for coverability could also be shown by the same construction).

We give two results: (1) We extend the Rackoff technique to work directly on strongly increasing Affine nets, improving the upper bounds for coverability and boundedness (from SPACE(O(2^{cn log n})) to SPACE(O(2^{cn log n}))). (2) We state the limit of strongly increasing Affine nets by proving that model checking a fragment of CTL (which can express generalizations of boundedness and various other problems) is undecidable for strongly increasing Affine nets, while it is EXPSPACE-complete for Petri nets [1].

Following are the three main difficulties in extending the Rackoff technique to strongly increasing Affine nets.

1. Showing upper bounds for the lengths of sequences certifying coverability or unboundedness is not enough — short sequences can give rise to large numbers.

2. We can no longer rely on ignoring places that go above some value. The effect of a transition on a place will depend on the exact value at other places.

3. The effect of firing a sequence of transitions can not be determined by the Parikh image of the sequence.

To overcome the first difficulty, we define transition systems that abstract the real ones, where markings from short sequences of transitions will have either small numbers or ω. This also overcomes the second difficulty, since ignored places will have the value ω in the abstract transition systems. If such a place affects another place, the affected place will also get the value
ω. To overcome the third difficulty, we classify the set of places according to the way they affect each other. Places that have high values and interfere among themselves will always be unbounded so that they can be “ignored” (implemented by introducing another abstraction), leaving behind places that are amenable to analysis by Petri net techniques. Since this technique depends on the observation that places interfering with one another are unbounded, it can not be used for problems that require more precise answers than unboundedness. Model checking the fragment of CTL mentioned above does require such precise answers and turns out to be undecidable for strongly increasing Affine nets.

The following table summarises the complexity of various problems on Petri nets and strongly increasing Affine nets, with the contributions of this paper in bold. Abbreviations used in the table: SIAN for Strongly increasing Affine nets, MC(eiPrCTL) for model checking eventually increasing Presburger CTL, \( \text{Sp}(2^{\sqrt{n}} : 2^{cn \log n}) \) for \( \text{SPACE}(2^{\sqrt{n}}) \) lower bound and \( \text{SPACE}(2^{cn \log n}) \) upper bound.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Petri nets</th>
<th>SIAN</th>
<th>Affine nets</th>
</tr>
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<tbody>
<tr>
<td>Coverability</td>
<td>( \text{Sp}(2^{\sqrt{n}} : 2^{cn \log n}) ) [14, 16]</td>
<td>( \text{Sp}(2^{\sqrt{n}} : 2^{cn \log n}) )</td>
<td>Ackermann-hard [20]</td>
</tr>
<tr>
<td>Boundedness</td>
<td>( \text{Sp}(2^{\sqrt{n}} : 2^{cn \log n}) ) [14, 16]</td>
<td>( \text{Sp}(2^{\sqrt{n}} : 2^{cn \log n}) )</td>
<td>Undecidable [8]</td>
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<tr>
<td>MC(eiPrCTL)</td>
<td>EXPSPACE-complete [1]</td>
<td><strong>Undecidable</strong></td>
<td>Undecidable [8]</td>
</tr>
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</table>

### 2 Preliminaries

Let \( \mathbb{Z} \) be the set of integers, \( \mathbb{N} \) be the set of non-negative integers and \( \mathbb{N}^+ \) be the set of positive integers. For any set \( P \), \( \text{card}(P) \) is the cardinality of \( P \).

A transition system \( S = (S, \rightarrow) \) is a set \( S \) endowed with a transition relation “\( \rightarrow \)”, i.e., with a binary relation on the set \( S \). We write \( s \overset{t}{\rightarrow} t \) to mean that there exist \( r \in \mathbb{N}^+ \) and a sequence of states \( s_0 = s, s_1, \ldots, s_r = t \) such that \( s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_r \). We write \( s \overset{*}{\rightarrow} t \) to mean that \( s \overset{t}{\rightarrow} \) or \( s = t \). A state \( t \in S \) of a transition system \( S = (S, \rightarrow) \) is reachable from a state \( s \) if \( s \overset{+\cdot \cdot \cdot}{\rightarrow} t \). The reachability set of \( S \) from the state \( s_0 \) is denoted by \( \text{RS}(S, s_0) \) and is defined to be the set of states reachable from \( s_0 \).

Let \( P \) be a finite non-empty set of places with \( \text{card}(P) = m \in \mathbb{N}^+ \) and let \( \langle p_1, \ldots, p_m \rangle \) be an arbitrary but fixed order on the set of places. A function \( M : P \rightarrow \mathbb{N} \) is called a marking. We denote by \( \mathbf{0} \) the marking such that \( \mathbf{0}(p) = 0 \) for all \( p \in P \). Given a subset \( Q \subseteq P \) and markings \( M_1, M_2 \), we write \( M_1 = Q M_2 \) (resp. \( M_1 \geq Q M_2 \)) if \( M_1(p) = M_2(p) \) (resp. \( M_1(p) \geq M_2(p) \)) for all \( p \in Q \). We write \( M_1 > M_2 \) if \( M_1 \geq M_2 \) and \( M_1 \neq M_2 \). We denote by \( \text{id} \) the identity matrix, whose dimension will be clear from context. We denote by \( A_1 \geq A_2 \), where \( A_1, A_2 \in \mathbb{N}^{m \times m} \), the condition that \( A_1(p_1, p_2) \geq A_2(p_1, p_2) \) for all \( p_1, p_2 \in P \). We consider (positive) affine functions from \( \mathbb{N}^m \) into \( \mathbb{N}^m \) defined by \( f(M) = AM + B \), where \( A \) is a (positive) matrix in \( \mathbb{N}^{m \times m} \) and \( B \) is a vector in \( \mathbb{Z}^m \). It can be verified that for every affine function \( f(M) = AM + B \) with an upward closed domain (i.e., \( dom f \subseteq \mathbb{N}^P \) such that \( M_1 \in dom f \) and \( M_1 \leq M_2 \) imply \( M_2 \in dom f \)), there exists a finite set of vectors \( \{ C_1, \ldots, C_k \} \subseteq \mathbb{Z}^m \) such that \( dom f = \cup_{1 \leq i \leq k} \{ M \in \mathbb{N}^m \mid AM + B \geq 0 \} \). With \( A \in \mathbb{N}^{m \times m} \) and \( B, C \in \mathbb{Z}^m \), we denote by \( f \triangleq (A, B, C) \) the affine function such that \( f(M) = AM + B \) and \( dom f = \{ M \in \mathbb{N}^m \mid AM + B \geq 0 \} \) and \( M + C \geq 0 \). In terms of Petri nets, the vector \( C \) restricts the markings to which the transition can be applied. For example, if the transition should not subtract anything from a place \( p \) but should only be applicable to markings \( M \) with \( M(p) \geq 1 \), we can set \( A(p, p) = 1, B(p) = 0 \) and \( C(p) = -1 \). In the following, we just write \( (A, B, C) \) if the name \( f \) is not important.

**Definition 1.** An Affine net \( \mathcal{N} \) (of dimension \( m \)) is a tuple \( \mathcal{N} = (m, F) \) where \( m \in \mathbb{N}^+ \) and \( F \) is a finite set of affine functions with upward closed domains in \( \mathbb{N}^m \).
The application of the transition function \( f \) to \( M_1 \) resulting in \( M_2 \) is denoted by \( M_1 \xrightarrow{f} M_2 \). The associated Affine transition system \( S_N = (S, f^\rightarrow) \) is naturally defined by \( S = \mathbb{N}^P \) and \( M_1 \xrightarrow{f} M_2 \) iff \( M_1 \in \text{dom} f \) and \( f(M_1) = M_2 \). If there is a sequence \( \sigma = f_1 f_2 \cdots f_r \) of transition functions such that \( M \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_r} M_r \), we denote it by \( M \xrightarrow{\sigma} M_r \). The markings \( M, M_1, \ldots, M_r \) are called intermediate markings arising while firing \( \sigma \) from \( M \). We say a sequence \( \sigma \) of transition functions is enabled at a marking \( M \) if \( M \xrightarrow{\sigma} M' \) for some marking \( M' \). We denote the length of \( \sigma \) by \( |\sigma| \). We denote the set of transition functions of \( N \) occurring in \( \sigma \) by \( \text{alph}(\sigma) \). A sequence \( \sigma' \) is called a sub-sequence of \( \sigma \) if \( \sigma' \) can be obtained from \( \sigma \) by removing some transition functions.

**Definition 2.** An affine function \((A, B, C)\) is strongly increasing \([9, \text{Section 2.2}]\) if \( A \geq \text{Id} \). An Affine net \( N = (m, F) \) is strongly increasing if each of its functions is strongly increasing.

Note that if \( M_1 \xrightarrow{f} M_2 \) and \( M_1' > M_1 \), then the fact that \( f \) is strongly increasing implies that \( M_1' \xrightarrow{f} M_2' \) for some \( M_2' > M_2 \) and for every \( p \in P \), \( M_1'(p) > M_1(p) \) implies \( M_2'(p) > M_2(p) \).

**Definition 3.** Given an Affine net \( N \) with an initial marking \( M_{\text{init}} \) and a target marking \( M_{\text{cov}} \), the coverability problem is to determine if there exists a marking \( M \in RS(S_N, M_{\text{init}}) \) such that \( M \geq M_{\text{cov}} \). The boundedness problem is to determine if there exists a number \( B \in \mathbb{N} \) such that for all markings \( M \in RS(S_N, M_{\text{init}}) \), \( M(p) \leq B \) for all \( p \in P \).

For an Affine net \( N \), \( R_N \) will denote the maximum absolute value of any entry in \( A, B \) or \( C \) for any transition function \((A, B, C)\) of \( N \). When \( N \) is clear from context, we skip the subscript \( N \) and write \( R \). We also write function instead of transition function when it is clear from context that it is a transition function in an Affine net. The size of \( N \) with initial marking \( M_{\text{init}} \) is defined to be \((\text{card}(F))(m^2 + m)\log R + m\log\|M_{\text{init}}\|_\infty\) where \( \|M_{\text{init}}\|_\infty \) is the maximum entry in \( M_{\text{init}} \). If \( A = \text{Id} \) for each function \((A, B, C)\) of \( N \), then \( N \) is a Petri net.

In Affine nets, markings cannot decrease too much.

**Proposition 4.** If \( M_1 \xrightarrow{\sigma} M_2 \), then \( M_2(p) \geq M_1(p) - R|\sigma| \) for all \( p \in P \).

**Proof.** By induction on \( |\sigma| \). For the base case \( |\sigma| = 1 \), let \( \sigma = f \) and \( f = (A, B, C) \). We have \( M_2 = f(M_1) = AM_1 + B \) and the result follows since \( A \geq \text{Id} \). For the induction step, let \( \sigma = f \sigma' \) and \( M_1 \xrightarrow{f=(A,B,C)} M_1' \xrightarrow{\sigma'} M_2 \). Since \( A \geq \text{Id} \), \( M_1'(p) \geq M_1(p) - R \) for all \( p \in P \). By induction hypothesis, \( M_2(p) \geq M_1'(p) - R|\sigma'| \). Combining the two inequalities, we get \( M_2(p) \geq M_1(p) - R|\sigma| \). \( \square \)

### 3 Value Abstracted Semantics

In Affine nets, a short sequence of functions can generate markings with large values. Beyond some value, it is not necessary to store the exact values of a marking to decide coverability and boundedness. Let \( \mathbb{N}_\omega = \mathbb{N} \cup \{\omega\} \) and \( \mathbb{Z}_\omega = \mathbb{Z} \cup \{\omega\} \) where addition, multiplication and order are as usual with the extra definition of \( \omega \times 0 = 0 \times \omega = 0 \). To avoid using excessive memory space to store large values of markings, we introduce extended markings \( W : P \rightarrow \mathbb{N}_\omega \). The domains of transitions functions are extended to include extended markings: for a function \( f = (A, B, C) \) and an extended marking \( W \), we have \( W \in \text{dom} f \) iff \( W + C \geq 0 \) and \( AW + B \geq 0 \). The result \( W' \) of applying \( f \) to \( W \in \text{dom} f \) is given by \( W' = AW + B \), denoted by \( W \xrightarrow{f} W' \).

For an extended marking \( W : P \rightarrow \mathbb{N}_\omega \), let \( \omega(W) = \{p \in P \mid W(p) = \omega\} \) and \( \omega(W) = P \setminus \omega(W) \). For a function \( t : \{0, \ldots, m\} \rightarrow \mathbb{N} \) (which will be used to denote thresholds) and...
an extended marking $W$, we define $[W]_{t \to \omega}$ and $[W]_{\omega \to t}$ by:

$$
(W(p) \quad \text{if } W(p) < t(\text{card}(\omega(W))),
\omega \quad \text{otherwise}.
$$

$$
(W(p) \quad \text{if } W(p) \in \mathbb{N},
t(\text{card}(\omega(W)) + 1) \quad \text{otherwise}.
$$

The threshold function $t$ gives the threshold beyond which numbers can be abstracted. In the extended marking $[W]_{t \to \omega}$, values beyond the threshold given by $t$ are abstracted by $\omega$. In the marking $[W]_{\omega \to t}$, abstraction is reversed by replacing $\omega$ with the corresponding threshold value.

**Definition 5.** Let $t : \{0, \ldots, m\} \to \mathbb{N}$ be a threshold function and $\mathcal{N}$ be a strongly increasing Affine net. The associated $t$-transition system $S_{\mathcal{N}, t} = (S_1, \xrightarrow{\mathcal{F}}_1)$ is defined by $S_1 = \mathbb{N}_0^P$ and $W_1 \xrightarrow{(A,B,C)}_t W_2$ iff $W_1 \geq C$ and $W_2 = [(A_1W_1 + B)]_{t \to \omega} \in \mathbb{N}_0^P$. We write $W_0 \xrightarrow{\sigma}_t W_r$ if $\sigma = f_1 \cdots f_r$ and $W_1 \xrightarrow{f_1}_t W_i$ for each $i$ between $1$ and $r$. The extended markings $W_0, \ldots, W_r$ are called intermediate extended markings in the run $W_0 \xrightarrow{\sigma}_t W_r$.

Note that for any $W_1 \xrightarrow{\sigma'}_t W_2$, $\omega(W_2) \geq \omega(W_1)$. In the $t$-transition system, a place having the value $\omega$ will retain it after the application of any function. The following propositions establish some relationships between $t$-transition systems and natural transition systems.

**Proposition 6.** Let $W_1 \xrightarrow{\sigma}_t W_2$, $\text{card}(\omega(W_2)) = \text{card}(\omega(W_1)) < m$ and $t(\text{card}(\omega(W_1)) + 1) \geq R|\sigma| + x$ for some $x \in \mathbb{N}$. Then $[W_1]_{\omega \to t} \xrightarrow{\sigma} M_2$ such that $M_2 = \frac{\omega}{\omega(W_2)} W_2$ and $M_2(p) \geq x$ for all $p \in \omega(W_2)$.

**Proof.** By induction on $|\sigma|$. The base case where $|\sigma| = 0$ follows from the definition of $[W_1]_{\omega \to t}$ since $M_2 = [W_1]_{\omega \to t}$ and $W_2 = W_1$.

For the induction step, let $\sigma = f \sigma'$, $f = (A, B, C)$ and $W_1 \xrightarrow{f}_t W_3 \xrightarrow{\sigma'}_t W_2$. Since $\omega(W_3) = \omega(W_1)$, we have $A(p, p') = 0$ for all $p \in \omega(W_1)$ and $p' \in \omega(W_1)$. Therefore, $[W_1]_{\omega \to t} \xrightarrow{f} M_3$ with $M_3(p) = \sum_{p' \in \omega(W_1)} A(p, p') W_1(p') + B(p) = W_3(p)$ for all $p \in \omega(W_1)$ and $M_3(p') \geq R|\sigma| + x$ for all $p' \in \omega(W_1)$. Let $t'$ be the same as $t$ except that $t'(\text{card}(\omega(W_1)) + 1) = t(\text{card}(\omega(W_1)) + 1) - R$. Now we have $M_3 \geq [W_3]_{\omega \to t'}$, $M_3 = \frac{\omega(W_3)}{\omega(W_2)} W_3$ and $W_3 \xrightarrow{\sigma'}_t W_2$. Since $t'(\text{card}(\omega(W_2)) + 1) \geq R|\sigma'| + x$, we have by induction hypothesis that $[W_3]_{\omega \to t'} \xrightarrow{\sigma'}_t M_2$ with $M_2 = \frac{\omega(W_2)}{\omega(W_2)} W_2$ and $M_2(p) \geq x$ for all $p \in \omega(W_2)$. We have $A'(p, p') = 0$ for all $p \in \omega(W_3)$, $p' \in \omega(W_3)$ and $f' = (A', B', C')$ occurring in $\sigma'$. Hence, $[W_1]_{\omega \to t} \xrightarrow{f} M_3 \xrightarrow{\sigma'}_t M_2$ with $M_2 \geq M_2$ and $M_2 = \frac{\omega(W_2)}{\omega(W_2)} M_2$. This completes the induction step and hence the proof.

A routine induction on $|\sigma|$ allows to prove the following:

**Proposition 7.** If $M_1 \xrightarrow{\sigma} M_2$, then $M_1 \xrightarrow{\sigma}_t W_2 \geq M_2$ for any $M_1' \geq M_1$.

## 4 Coverability

In this section, we give a $\text{SPACE}(\mathcal{O}(2^{n \cdot \log n}))$ upper bound for the coverability problem in strongly increasing Affine nets, for some constant $c_2$. Let $R' = \max\{\{M_{\text{cov}}(p) \mid p \in P\} \cup \{R\}\}$, where $M_{\text{cov}}$ is the marking to be covered. In the rest of this section, we fix a strongly increasing Affine net $\mathcal{N} = (m, F)$ with an initial marking $M_{\text{init}}$ and the marking to be covered $M_{\text{cov}}$. The set of places is $P = \{p_1, \ldots, p_m\}$. 


We briefly recall the Rackoff technique for the Expspace upper bound for the coverability problem in Petri nets. The idea is to define a function $\ell : \mathbb{N} \to \mathbb{N}$ and prove that for a Petri net with $m$ places, coverable markings can be covered with sequences of transitions of length at most $\ell(m)$. This is done by induction on the number of places. In a Petri net with $i + 1$ places, suppose $M_{\text{init}} \xrightarrow{\sigma} M' \geq M_{\text{cov}}$ and $M$ is the first intermediate marking where one of the values is more than $R(\ell(i) + R') - 1$ (this is the intuition behind the definition of the threshold function $t_1$ below). If there is no such marking, all intermediate markings have small values and it is easy to bound the length of $\sigma$ by $(R(\ell(i) + R') + 1)^{i+1}$. Otherwise, let $\sigma = \sigma_1 \sigma_2$ such that $M_{\text{init}} \xrightarrow{\sigma_1} M \xrightarrow{\sigma_2} M' \geq M_{\text{cov}}$. The length of $\sigma_1$ is bounded by $(R(\ell(i) + R') + 1)^{i+1}$.

Temporarily forgetting the existence of place $p$ (where $M(p) \geq R\ell(i) + R'$), we conclude by induction hypothesis that starting from $M$, $M_{\text{cov}}$ can be covered (in all places except $p$) with a sequence $\sigma_2'$ of length at most $\ell(i)$. Since $M(p) \geq R\ell(i) + R'$ and $\sigma_2'$ reduces the value in $p$ by at most $R\ell(i)$, $\sigma_2'$ in fact covers all places, including $p$. Hence, $\sigma_1 \sigma_2'$ covers $M_{\text{cov}}$ from $M_{\text{init}}$ and its length is at most $(R\ell(i) + R')^{i+1} + \ell(i) + 1$. This is the intuition behind the definition of the length function $\ell_1$ below. The counterpart of “temporarily forgetting $p$” is assigning it the value $\omega$.

**Definition 8.** The functions $\ell_1, t_1 : \mathbb{N} \to \mathbb{N}$ are as follows.

$$t_1(0) = 0 \quad \ell_1(0) = 0$$

$$t_1(i + 1) = R\ell_1(i) + R' \quad \ell_1(i + 1) = (t_1(i + 1))^{i+1} + \ell_1(i) + 1$$

**Definition 9.** A covering sequence enabled at $M$ is a sequence $\sigma$ of functions such that $M \xrightarrow{\sigma} M'$ and $M' \geq M_{\text{cov}}$. A $t_1$-covering sequence enabled at $W$ is a sequence $\sigma$ of functions such that $W \xrightarrow{\sigma} W'$ and $W' \geq M_{\text{cov}}$.

The following lemma shows that even after abstracting values that are above the ones given by the threshold function $t_1$, there is still enough information to check coverability.

**Lemma 10.** If a $t_1$-covering sequence $\sigma$ is enabled at $W$, then $M_{\text{cov}}$ is coverable from $[W]_{\omega \rightarrow t_1}$.

**Proof.** By induction on $\text{card}(\omega(W))$. For the base case $\text{card}(\omega(W)) = 0$, we have $[W]_{\omega \rightarrow t_1}(p) = R' \geq M_{\text{cov}}(p)$ for all $p \in P$. So $[W]_{\omega \rightarrow t_1} \geq M_{\text{cov}}$ and we are done.

For the induction step, let $\text{card}(\omega(W)) = i + 1$ and $W \rightarrow_{t_1} W' \geq M_{\text{cov}}$. First consider the case where $\text{card}(\omega(W')) = i + 1$. For any intermediate extended marking $W_1$ in the run $W \rightarrow_{t_1} W'$, $W_1(p) < t_1(i + 1)$ for all $p \in \omega(W)$. Remove all functions between any two identical intermediate extended markings in the run $W \rightarrow_{t_1} W'$ to obtain the sequence $\sigma'$. We have $|\sigma'| \leq (t_1(i + 1))^{i+1}$ and $W \rightarrow_{t_1} W'$. Since $[W]_{\omega \rightarrow t_1}(p) \geq R\ell_1(i + 1) + R' \geq R|\sigma'| + R'$ for all $p \in \omega(W)$, we infer from Prop. 6 that $[W]_{\omega \rightarrow t_1} \rightarrow_{t_1} M'$ where $M' = \omega(W')$. We are done since $W' \geq M_{\text{cov}}$ and $R' \geq M_{\text{cov}}(p)$ for all $p \in P$.

Next consider the case of the induction step where $\text{card}(\omega(W')) < i + 1$. Let $W_1$ be the last intermediate extended marking such that $\text{card}(\omega(W_1)) = i + 1$. Let $W \xrightarrow{\sigma_1} W_1 \xrightarrow{\sigma_1} W_2 \xrightarrow{\sigma_1} W'$. For any intermediate extended marking $W_3$ in the run $W \xrightarrow{\sigma_1} W_1, W_3(p) < t_1(i + 1)$ for all $p \in \omega(W)$. Remove all functions between any two identical intermediate extended markings in the run $W \xrightarrow{\sigma_1} W_1$ to obtain the sequence $\sigma'_1$. We have $|\sigma'_1| \leq (t_1(i + 1))^{i+1}$ and $W \xrightarrow{\sigma'_1} W_1$. Since $[W]_{\omega \rightarrow t_1}(p) = R\ell_1(i + 1) + R' \geq R|\sigma'_1| + R + R\ell_1(i) + R'$ for all $p \in \omega(W)$, we infer from Prop. 6 that $[W]_{\omega \rightarrow t_1} \rightarrow_{t_1} M_1$ where $M_1 = \omega(W_1)$ and $M_1(p) \geq R + R\ell_1(i) + R'$ for all $p \in \omega(W_1)$. Since $W_1 \xrightarrow{f=(A,B,C)} W_2$, we have $M_1 \rightarrow_{f} M_2, W_2(p) = \sum_{p' \in \omega(W_1)} A(p, p')W_1(p') + B(p) = \sum_{p' \in \omega(W_2)} A(p, p')M_1(p') + B(p) = M_2(p)$ for all $p \in \omega(W_2)$ and $M_2(p) \geq R\ell_1(i) + R'$ for all $p \in \omega(W_2)$. Hence, $M_2 \geq [W_2]_{\omega \rightarrow t_1}$. By induction hypothesis, there is a covering sequence $\sigma'_2$ enabled at $[W_2]_{\omega \rightarrow t_1}$. Therefore, $\sigma'_1 f \sigma'_2$ is a covering sequence enabled at $[W]_{\omega \rightarrow t_1}$. This completes the induction step and hence the proof. \[\square\]
Lemma 11. If there is a covering sequence $\sigma$ enabled at $M_{init}$, there is a $t_1$-covering sequence $\sigma'$ enabled at $M_{init}$ such that $|\sigma'| \leq \ell_1(m)$ (recall that $m = \text{card}(P)$).

Proof. Let $M_{init} \xrightarrow{\sigma} M \geq M_{cov}$. We infer from Prop. 7 that $M_{init} \xrightarrow{\sigma \cdot t_i} W \geq M$. Let $\sigma'$ be the sequence obtained from $\sigma$ by removing all functions between any two identical intermediate extended markings in the run $M_{init} \xrightarrow{\sigma \cdot t_i} W$. We have $M_{init} \xrightarrow{\sigma' \cdot t_i} W \geq M_{cov}$, so $\sigma'$ is a $t_1$-covering sequence enabled at $M_{init}$. It remains to prove that $|\sigma'| \leq \ell_1(m)$.

For every $i$ between $\text{card}(\omega(W))$ and $m$, let $W_i$ be the first extended marking in the run $M_{init} \xrightarrow{\sigma' \cdot t_i} W$ such that $\text{card}(\omega(W_i)) = i$ ( $W_m = M_{init}$) and let $\sigma'_j$ be the suffix of $\sigma'$ that appears after $W_i$ ( $\sigma'_m = \sigma'$). We prove by induction $i$ that $|\sigma'_i| \leq \ell_1(i)$. For the base case $i = \text{card}(\omega(W))$, every intermediate extended marking $W'$ in the run $W_i \xrightarrow{\sigma' \cdot t_i} W$ satisfies $W'(p) < t_1(i)$ for all $p \in \omega(W)$. Hence, $|\sigma'_i| \leq (t_1(i))^i \leq \ell_1(i)$.

For the induction step, let $\sigma_{i+1}$ be the prefix of $\sigma'_{i+1}$ that appears between $W_{i+1}$ and $W_i$. Every intermediate extended marking $W'$ except $W_i$ in the run $W_{i+1} \xrightarrow{\sigma_{i+1} \cdot t_i} W_i$ satisfies $W'(p) < t_1(i+1)$ for all $p \in \omega(W_{i+1})$. Hence, $|\sigma_{i+1}| \leq (t_1(i+1))^{i+1} + 1$. By induction hypothesis, $|\sigma'_i| \leq \ell_1(i)$. Hence, $|\sigma'_{i+1}| = |\sigma_{i+1}| + |\sigma'_i| \leq (t_1(i+1))^{i+1} + 1 + \ell_1(i) = \ell_1(i+1)$. This completes the induction step and hence the proof.

Lemma 12. For all $i \in \mathbb{N}$, $\ell_1(i) \leq (6RR')^{(i+1)!}$.

Proof. By induction on $i$. For the base case $i = 0$, we have $\ell_1(0) = 0 \leq (6RR')^1$.

The following inequalities prove the induction step.

$$\ell_1(i+1) = (R\ell_1(i) + R')^{i+1} + \ell_1(i) + 1$$

[Definition 8]

$$\leq (2RR'\max(\ell_1(i), 1))^{i+1} + \ell_1(i) + 1$$

$$\leq 3(2RR' \times \max(\ell_1(i), 1))^{i+1}$$

$$\leq (6RR')^{i+1}(\max(\ell_1(i), 1))^{i+1}$$

[Induction hypothesis]

$$\leq (6RR')^{(i+1)!}(6RR')^{(i+1)!}$$

$$= (6RR')^{(i+2)!}$$

Theorem 13. For some constant $c_1$, the coverability problem for strongly increasing Affine nets is in $\text{NSPACE}(O(2^{c_1m\log m}(\log R + \log \|M_{init}\|_\infty)))$.

Proof. The hardness follows from EXPSPACE-hardness of the coverability problem for Petri nets [14].

For the upper bound, it follows from Lemma 10 and Lemma 11 that a covering sequence is enabled at $M_{init}$ iff there is a $t_1$-covering sequence enabled at $M_{init}$ of length at most $\ell_1(m)$. The existence of such a sequence can be verified by a non-deterministic Turing machine that maintains one intermediate extended marking and a counter that counts up to a maximum of $\ell_1(m)$. Any entry in the intermediate extended marking to be maintained is either less than $t_1(m)$ or equal to $\omega$. From the bounds given by Lemma 12, we infer that the memory space needed by the above non-deterministic algorithm is $O(m \log \|M_{init}\|_\infty + (m+1)!(\log R + \log R'))$. This can be simplified to $O(2^{c_1m\log m}(\log R + \log \|M_{init}\|_\infty))$. 

Taking $n = (\text{card}(F))(m^2 + m) \log R + m \log \|M_{init}\|_\infty) + m \log \|M_{cov}\|_\infty$ as the size of the input to the coverability problem, we can infer from the above theorem an upper bound of $\text{SPACE}(O(2^{2n\log n}))$. 


5 Boundedness

In this section, we give a \(\text{SPACE}(O(2^{c_4 n \log n}))\) upper bound for the boundedness problem in strongly increasing Affine nets, for some constant \(c_4\). In the rest of this section, we fix a strongly increasing Affine net \(N = (m, F)\) with an initial marking \(M_{\text{init}}\). The set of places is \(P = \{p_1, \ldots, p_m\}\).

**Definition 14.** A self-covering pair enabled at \(M\) is a pair \((\sigma_1, \sigma_2)\) of sequences of functions such that \(M \xrightarrow{\sigma_1} M_1 \xrightarrow{\sigma_2} M_2 \) and \(M_2 > M_1\).

Since all transition functions are strongly increasing and their domains are upward closed, we can infer from the above definition that \(M_i \xrightarrow{\sigma_2} M_{i+1}\) and \(M_{i+1} > M_i\) for all \(i \in \mathbb{N}^+\). Hence, if a self covering pair is enabled at \(M_{\text{init}}\), then \(N\) is unbounded. Conversely, if \(N\) is unbounded, infinitely many distinct markings can be reached from \(M_{\text{init}}\). Since there are only finitely many transition functions, König’s lemma implies that there is an infinite sequence of functions enabled at \(M_{\text{init}}\) such that all intermediate markings are distinct. We infer from Dickson’s lemma that there are two markings \(M_1, M_2\) along this sequence such that \(M_2 > M_1\). Let \(M_{\text{init}} \xrightarrow{\sigma_1} M_1 \xrightarrow{\sigma_2} M_2\). By Definition 14, \((\sigma_1, \sigma_2)\) is a self-covering pair enabled at \(M_{\text{init}}\).

The Rackoff technique again defines a length function \(\ell' : \mathbb{N} \to \mathbb{N}\) and shows that if a Petri net with \(m\) places is unbounded, there is a self-covering pair of total length at most \(\ell'(m)\). As an example, let us consider giving an upper bound for \(\ell'(2)\). Consider the sequence of markings shown below, produced by a self-covering pair. Let \(\sigma_2\) be the portion occurring after the first marking where \(p_1\) has the value 100. Since \(p_5, p_6\) have low values throughout, we would like to abstract the remaining places and reduce the length of \(\sigma_2\) to get an upper bound on \(\ell'(2)\). We denote \(\{p_5, p_6\}\) by \(P_{\geq 2}^\sigma\). In the block of intermediate markings shown above enclosed in [ ], the first and last markings are identical when projected to \(p_5, p_6\). Since this block does not change \(p_5, p_6\), we can remove this block, provided that after removal, the abstracted places \(p_1, p_2, p_3, p_4\) will still have values at least 100, 150, 100, 200 respectively. To decide whether this is the case, the effect of the block on \(p_1, p_2, p_3, p_4\) is calculated in a Petri net by simply summing up the effect of each transition in the block. In a strongly increasing Affine net, this is however not possible since the effect of the block depends not only on the transitions in it, but also on the values in the marking at the beginning of the block. In addition, affine functions can copy the value of one place to another one.

If some transition copies the value of some place among \(p_1, p_2, p_3, p_4\) into \(p_5\) or \(p_6\), a large value will result in \(p_5\) or \(p_6\), so that they too can be abstracted, letting us use induction hypothesis to deal with the remaining fewer number of non-abstracted places. To deal with the other case, we assume that no transition in \(\sigma_2\) does this kind of copying (\(\sigma_2\) isolates \(\{p_5, p_6\}\) from \(\{p_1, p_2, p_3, p_4\}\)). Next, suppose a function \(f = (A, B, C)\) occurs inside the block, where \(A\) and \(B\) are as shown in Fig. 1 in the next page. Let the rows and columns of \(A\) correspond to \(p_1, p_2, \ldots, p_6\) in that order. The function \(f\) doubles the value in \(p_2\) and copies the value of \(p_3\) to \(p_1\), but isolates \(\{p_3, p_4\}\) from \(\{p_1, p_2, p_3, p_4\}\). In the following, we will say \(f\) “crosses” \(P_{\geq 2}^\sigma = \{p_1, p_2\}\) and isolates \(P_{\sim 2}^\sigma = \{p_3, p_4\}\) from \(P \setminus P_{\geq 2}^\sigma = \{p_1, p_2, p_3, p_4\}\). Since \(p_1, p_2\) had large values to begin with, they will have even larger values after crossed by \(f\). We will see that this will result in crossed places becoming unbounded, and so we can forget the exact
effect of a block on such places, and just remember that they are crossed by some function. The forgetting part is done in Definition 16 by simplifying the matrix \( A \), and the remembering part is done by setting the corresponding entry in \( B \) to 1. To decide whether a block can be removed or not, it remains to compute the effect of \( P_{<\omega}^2 \) on \( P_{is}^2 \). This can be achieved since we know the exact values in \( p_5, p_6 \). This is formalised in the proof of Lemma 20.

Definition 15. Let \( f = (A, B, C) \) be a function.

- \( f \) multiplies a place \( p \) if \( A(p, p) \geq 2 \).
- \( f \) copies \( p' \) to \( p \) if \( A(p, p') \geq 1 \) for any two places \( p \neq p' \).
- \( f \) isolates \( Q_1 \) from \( Q_2 \), where \( Q_1, Q_2 \subseteq P \), if \( A(p, p') = 0 \) for all \( p \in Q_1 \) and \( p' \in Q_2 \setminus \{p\} \).

Although the sets \( P_{<\omega}^2, P_{<\omega}^2 \) and \( P_{<\omega}^2 \) are determined by \( \sigma_2 \), we avoid heavy notation in the following definition and instead use an arbitrary partition of \( P \) into \( P_{<\omega}, P_{<\omega} \) and \( P_{is} \).

Definition 16. Let \( \rho = (P_{<\omega}, P_{<\omega}, P_{is}) \) be a triple such that the sets \( P_{<\omega}, P_{<\omega} \) and \( P_{is} \) partition the set of places \( P \). To each function \( f = (A, B, C) \), we associate another function \( f[\rho] = (A[\rho], B[\rho], C) \), where \( A[\rho] \) and \( B[\rho] \) are as follows.

\[
A[\rho](p, p') = \begin{cases} 
B(p) & \text{if } p \notin P_{<\omega} \\
1 & \text{if } f \text{ multiplies } p \in P_{<\omega} \\
1 & \text{if } f \text{ copies } p \text{ to } p \in P_{<\omega} \setminus \{p\} \\
0 & \text{otherwise}
\end{cases}
\]

\[
B[\rho](p) = \begin{cases} 
0 & \text{if } p \notin P_{<\omega} \\
-1 & \text{if } f \text{ multiplies } p \in P_{<\omega} \\
-1 & \text{if } f \text{ copies } p \text{ to } p \in P_{<\omega} \setminus \{p\} \\
0 & \text{otherwise}
\end{cases}
\]

Associated with the triple \( \rho \) is the \( \rho \)-transition system \( S_{\rho} = (\rho, \rightarrow) \), defined by \( S_{\rho} = \mathbb{N}_{\omega}^f \) and \( W_1 \rightarrow_{(A, B, C)} W_2 \) iff \( W_1 + C \geq 0 \) and \( W_2 = A[\rho]W_1 + B[\rho] \in \mathbb{N}_{\omega}^f \). We write \( W_0 \rightarrow_{\rho} W_r \) if \( \sigma = f_1 \cdots f_r \) and \( W_i -1 \rightarrow_{\rho} W_i \) for each \( i \) between 1 and \( r \). We write \( \sigma[\rho] \) to denote the sequence of functions obtained from \( \sigma \) by replacing each function \( f \) of \( \sigma \) by replacing each function \( \rho \).

For any extended marking \( W \), if \( W(p) \in \mathbb{N} \) for all \( p \in P \) (i.e., if \( W \) is a marking), then applying any function to \( W \) in the \( \rho \)-transition system will result in another marking (new \( \omega \) values are not introduced). In the example given before Definition 15, partition the set of places into \( P_{<\omega} = \{p_1, p_2\} \), \( P_{is} = \{p_3, p_4\} \) and \( P_{<\omega} = \{p_5, p_6\} \). The corresponding matrices \( A[\rho] \) and \( B[\rho] \) defining \( f[\rho] \) are as shown below. We obtain \( A[\rho] \) from \( A \) by replacing the first

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example_transition_functions.png}
\caption{Examples of transition functions}
\end{figure}

\[
A = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
\end{bmatrix}
\quad \quad
B = \begin{bmatrix}
0 \\
-1 \\
2 \\
0 \\
0 \\
-1 \\
\end{bmatrix}
\quad \quad
A[\rho] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
\end{bmatrix}
\quad \quad
B[\rho] = \begin{bmatrix}
1 \\
1 \\
-1 \\
2 \\
0 \\
-1 \\
\end{bmatrix}
\]
be unbounded and this technique can not be applied for problems that need more precise answers. For example, eventually increasing existential Presburger CTL (ePrECTL\(_<\)\(U\)), introduced in \cite{1}, can express the presence of sequences along which the value of one place grows unboundedly while another place remains bounded. Model checking ePrECTL\(_<\)\(U\) is shown to be EXPSPACE-complete \cite{1} for Petri nets by extending the Rackoff technique. We show in Section 6 that model checking ePrECTL\(_>\)\(U\) is undecidable for strongly increasing Affine nets.

**Definition 17.** Let \(\rho = (P_{<\omega}, P_\times, P_{is})\) be a triple such that the sets \(P_{<\omega}\), \(P_\times\) and \(P_{is}\) partition the set of places \(P\). A sequence of functions \(\sigma\) is a \(\rho\)-pumping sequence enabled at a marking \(M_0\) if

1. \(M_0 \xrightarrow{\sigma} \rho M_1\) and \(M_1 > M_0\),
2. for all places \(p \in P_\times\), some function \((A, B, C)\) in \(\sigma\) has \(B[p](p) = 1\),
3. each function in \(\sigma\) isolates \(P_{is} \cup P_{<\omega}\) from \(P_\times \cup P_{is}\) and
4. no function in \(\sigma\) multiplies \(p\) for any place \(p \in P_{is}\).

Next we develop formalisms needed to show that if there are \(\rho\)-pumping sequences, there are short ones. Suppose \(W_1 \xrightarrow{\sigma} t W_2\) and \(\bar{\omega}(W_1) = \bar{\omega}(W_2)\). Then each function in \(\sigma\) isolates \(\bar{\omega}(W_1)\) from \(\omega(W_1)\) (otherwise, we could not have \(\bar{\omega}(W_1) = \bar{\omega}(W_2)\)). The following proposition establishes a relationship between \(t\)-transition systems and \(\rho\)-transition systems.

**Proposition 18.** Let \(t\) be a threshold function, \(W_1 \xrightarrow{\sigma} t W_2\) and \(\bar{\omega}(W_1) = \bar{\omega}(W_2)\). Let \(\rho = (\bar{\omega}(W_1), P_\times, P_{is})\). If \([W_1]\omega \rightarrow t \xrightarrow{\sigma} M_2\), then \(M(p) < t(\text{card}(\bar{\omega}(W_1)))\) for every intermediate marking \(M\) arising while firing \(\sigma[p]\) from \([W_1]\omega \rightarrow t\) and every \(p \in \omega(W_1)\).

**Proof.** We claim that for any prefix \(\sigma_1\) of \(\sigma\), \(W_1 \xrightarrow{\sigma_1} t W\) implies \([W_1]\omega \rightarrow t \xrightarrow{\sigma_1} \rho M\) where \(M = \omega(W)\). Assume for contradiction’s sake that the claim is false and let \(\sigma_1\) be the shortest prefix of \(\sigma\) demonstrating this falsity. \(\sigma_1\) can not be the empty sequence — if it was, then \(W = W_1\) and \(M = [W_1]\omega \rightarrow\), contradicting the assumption that \(\sigma_1\) demonstrates the falsity of our claim. Let \(M'\) occur immediately before \(M\) in the run \([W_1]\omega \rightarrow t \xrightarrow{\sigma_1} \rho M\) and \(W'\) occur immediately before \(W\) in the run \(W_1 \xrightarrow{\sigma_1} t W\). Since \(\sigma_1\) is the shortest prefix of \(\sigma\) violating our claim, we have \(M' = \omega(W')\). Let \(f = (A, B, C)\) be the last function of \(\sigma_1\). Since \(f\) isolates \(\omega(W')\) from \(\omega(W)\), we have \(M(p) = \sum_{p' \in \omega(W')} A(p, p') M'(p') + B(p)\) for all \(p \in \omega(W)\). Since \(M' = \omega(W')\), \(\sum_{p' \in \omega(W')} A(p, p') M'(p') + B(p) = \sum_{p' \in \omega(W')} A(p, p') W'(p') + B(p) = W(p)\) for all \(p \in \omega(W')\). Hence, \(M = \omega(W)\), contradicting the assumption that \(\sigma_1\) violates our claim. Hence the claim is true and the result follows.

The following definition of loops will be used only in the proof of Lemma 20.

**Definition 19.** Suppose \(W_1\) is an extended marking such that \(\bar{\omega}(W_1) = P_{<\omega}\) and \(\sigma\) is a sequence such that all functions in \(\sigma[p]\) isolate \(P_{<\omega}\) from \(P \setminus P_{<\omega}\). Suppose \(\sigma\) can be decomposed as \(\sigma = \pi_1 \pi_2\) and \(W_1 \xrightarrow{\pi_1 \rho} L \xrightarrow{\pi_2 \rho} W_2\). The pair \((\pi, L)\) is a \(P_{<\omega}\)-loop if all extended markings (except the last one) arising while firing \(\pi[p]\) from \(L\) are distinct from one another.

**Lemma 20.** There exists a constant \(d\) such that for any strongly increasing Affine net \(\mathcal{N}\) and for every \(\rho\)-pumping sequence \(\sigma\) enabled at some marking \(M_0\), there exists a \(\rho\)-pumping sequence \(\sigma'\) enabled at \(M_0\) such that \(|\sigma'| \leq (2eR)^{dn^3}\), where:

- \(\rho = (P_{<\omega}, P_\times, P_{is})\) is a triple such that \(P_{<\omega}\), \(P_\times\) and \(P_{is}\) partition the set of places \(P\),
- \(e = 1 + \max\{M(p) \mid p \in P_{<\omega}\}\) and \(M\) is an intermediate marking occurring while firing \(\sigma\) from \(M_0\)
• $M'_0$ is any marking such that $M'_0 = P_{<\omega} M_0$ and $M'_0(p) \geq R |\sigma'|$ for all $p \in P_{is} \cup P_x$.

Proof. Let $M_0 \xrightarrow{\sigma}_{\rho} M_k$. From Definition 17, all functions in $\sigma$ isolate $P_{<\omega}$ from $P_x \cup P_{is}$. Hence we can remove all the $P_{<\omega}$-loops from $\sigma$ to get $\pi'$, $M'_0 \xrightarrow{\pi'}_{\rho} M'_k = P_{<\omega} M_k$. However, $\pi'$ is not necessarily a $\rho$-pumping sequence. We extend the technique used for Petri nets [16, 4] that use the existence of small solutions to linear Diophantine equations to show that it is enough to retain a small number of loops to get a shorter $\rho$-pumping sequence. Some intermediate steps of the proof deal with sub-sequences of $\sigma$ that may not be enabled at $M'_0$. They will however be enabled at the extended marking $W_0$ where $W_0 = P_{<\omega} M_0$ and $W_0(p) = \omega$ for all $p \in P_x \cup P_{is}$. The proof is organised into the following steps.

Step 1: We first associate a vector with each sub-sequence of $\sigma$ to measure the effect of the sub-sequence on $P_x \cup P_{is}$.

Step 2: Next we remove some $P_{<\omega}$-loops from $\sigma$ to obtain $\sigma''$ such that for every intermediate extended marking $W$ arising while firing $\sigma[\rho]$ from $W_0$, $W$ also arises while firing $\sigma''[\rho]$ from $W_0$.

Step 3: The sequence $\sigma''$ obtained above is not a $\rho$-pumping sequence. With the help of the vectors defined in step 1, we formulate a set of linear Diophantine equations that encode the fact that the effects of $\sigma''$ and the $P_{<\omega}$-loops that were removed combine to give the effect of a $\rho$-pumping sequence.

Step 4: Then we use the result about existence of small solutions to linear Diophantine equations to construct a sequence $\sigma'$ that meets the length constraint of the lemma.

Step 5: Finally, we prove that $\sigma'$ is a $\rho$-pumping sequence enabled at $M'_0$.

Now we explain each of the above steps in detail.

Step 1: Suppose $\pi \in \text{alph}(\sigma)^*$ is a sequence consisting of functions occurring in $\sigma$. Suppose $W_1$ is an extended marking such that $W_1(p) \in \mathbb{N}$ for all $p \in P_{<\omega}$ and $W_1(p') = \omega$ for all $p' \in P_x \cup P_{is}$. Also suppose that $W_1 \xrightarrow{\pi}_{\rho} W_r$. We want to measure the effect of $\pi$ on places in $P_x \cup P_{is}$ when we replace $\omega$ by large enough values in $W_1$. We define a vector $\Delta[\pi, W_1]$ of integers for this measurement. For a place $p \in P_x$, all that a function ($A[\rho], B[\rho], C$) can do to $p$ is add 0 or 1 (this is due to the way $A[\rho]$ and $B[\rho]$ are defined in Definition 16). So we take $\Delta[\pi, W_1](p)$ to be the sum of all $B[\rho](p)$ for all functions ($A, B, C$) occurring in $\pi$. For a place $p \in P_{is}$, we have to take into account the “interference” from other places. Since from Definition 17, each function in $\pi$ isolates $P_{is}$ from $P_x \cup P_{is}$, the only places that can interfere with $p$ are those in $P_{<\omega}$. Let $\pi = f_1 f_2 \cdots f_r$ and $W_1 \xrightarrow{f_1}_{\rho} W_2 \xrightarrow{f_2}_{\rho} \cdots \xrightarrow{f_{r-1}}_{\rho} W_r$. For each $i$ between 1 and $r - 1$, let $f_i = (A_i, B_i, C_i)$. Following is the formal definition of $\Delta[\pi, W_1]$:

$$\Delta[\pi, W_1](p) = \sum_{i=1}^{r-1} \left( \sum_{p' \in P_{<\omega}} A_i(p, p') W_i(p') + B_i(p) \right)$$

for all $p \in P_{is}$

$$\Delta[\pi, W_1](p) = \sum_{i=1}^{r-1} B_i[\rho](p)$$

for all $p \in P_x$

Since all functions in $\pi$ isolate $P_{<\omega}$ from $P_x \cup P_{is}$, we infer that $\omega(W_r) = \omega(W_1) = P_{<\omega}$. It is routine to infer the following two facts from the definition of $\Delta[\pi, W_1]$.

• If $M_1$ is any marking such that $M_1 = P_{<\omega} W_1$ and $M_1 \xrightarrow{\pi}_{\rho} M_2$, then $M_2(p) - M_1(p) = \Delta[\pi, W_1](p)$ for all $p \in P_x \cup P_{is}$. 

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• Suppose $\pi = \pi_1\pi_2\pi_3$, $W_1 \xrightarrow{\pi_1} \rho W' \xrightarrow{\pi_2} \rho W' \xrightarrow{\pi_3} \rho W_r$ and $(\pi_2, W')$ is a $P_{<\omega}$-loop. Then $\Delta[\pi, W_1] = \Delta[\pi_1\pi_3, W_1] + \Delta[\pi_2, W']$.

The above two facts are sufficient to extend the technique used in [16] to $\rho$-transition systems. This technique was developed for Petri nets, where the effect of a sequence of functions can be determined from its Parikh image. This is not true in general for strongly increasing Affine nets, but the core idea can be lifted to $\rho$-transition systems.

**Step 2:** Now we remove some $P_{<\omega}$-loops from $\sigma$ to obtain $\sigma''$. To obtain some bounds in the next step, we first make the following observations on $P_{<\omega}$-loops. Let $card(P_{<\omega}) = m_1$. Recall that $W_0$ is the extended marking such that $W_0 = P_{<\omega} M_0$ and $W_0(\rho) = \omega$ for all $p \in P_x \cup P_a$. Suppose $(\pi, L)$ is a $P_{<\omega}$-loop arising while firing $\sigma[\rho]$ from $W_0$. There can be at most $e^{m_1}$ function occurrences in $\pi$, so $-e^{m_1}R \leq \Delta[\pi, L](p) \leq e^{m_1}R + m_1 e^{m_1}Re$ for any $p \in P_x \cup P_a$. Let $\Delta$ be the matrix whose set of columns is equal to $\{\Delta[\pi, L] \mid (\pi, L) \text{ is a } P_{<\omega}\text{-loop occurring while firing }\sigma[\rho]\text{ from }W_0\}$. There are at most $(m_1 + 2 )e^{m_1+1}R + 1)^m$ columns in $\Delta$. We use $\delta$ to denote the columns of $\Delta$.

Now we remove $P_{<\omega}$-loops from $\sigma$ according to the following steps. Let $x_0 = 0$ be the zero vector whose dimension is equal to the number of columns in $\Delta$. Begin the following steps with $i = 0$ and $\sigma_i = \sigma$.

a. Think of the first $(e^{m_1} + 1)^2$ functions of $\sigma_i$ as $e^{m_1} + 1$ blocks of length $e^{m_1} + 1$ each.

b. There is at least one block in which all extended markings also occur in some other block.

c. Let $(\pi, L)$ be a $P_{<\omega}$-loop occurring in the above block while firing $\sigma_i$ from $W_0$.

d. Let $\sigma_{i+1}$ be the sequence obtained from $\sigma_i$ by removing $\pi$.

e. Let $x_{i+1}$ be the vector obtained from $x_i$ by incrementing $x_i(\Delta[\pi, L])$ by 1.

f. Increment $i$ by 1.

g. If the length of the remaining sequence is more than or equal to $(e^{m_1} + 1)^2$, go back to step a. Otherwise, stop.

Let $n$ be the value of $i$ when the above process stops. Let $\sigma'' = \sigma_n$ and $x = x_n$. We remove a sequence $\tau$ starting at an extended marking $L$ only if all the intermediate extended markings occurring while firing $\tau[\rho]$ from $L$ occur at least once more in the remaining sequence. Hence, for every extended marking $W$ arising while while firing $\sigma[\rho]$ from $W_0$, $W$ also arises while firing $\sigma''[\rho]$ from $W_0$. We have $|\sigma''| \leq (e^{m_1} + 1)^2$. For each column $\delta$ of $\Delta$, $x(\delta)$ contains the number of occurrences of $P_{<\omega}$-loops $(\pi, L)$ removed from $\sigma$ such that $\Delta[\pi, L] = \delta$.

**Step 3:** From Definition 17, there are two reasons for $\sigma''$ not being a $\rho$-pumping sequence. The first is that none of the functions $(A, B, C)$ occurring in $\sigma''$ may have $B[\rho](p) = 1$ for some $p \in P_x$. The second reason is that if $M'_0 \xrightarrow{\sigma''} \rho M'_k$, we may not have $M'_k > M'_0$. Recall that $M_0 \xrightarrow{\rho} M_k$. The following definition of a vector $b$ whose dimension is equal to $card(P_x \cup P_{is})$ states the above two requirements:

$$b(p) = \begin{cases} 1 & p \in P_x, \\ 1 & p \in P_{is}, M_k(p) > M_0(p) \\ 0 & p \in P_{is}, M_k(p) = M_0(p) \end{cases}$$

Recall that for each column $\delta$ of $\Delta$, $x(\delta)$ contains the number of occurrences of $P_{<\omega}$-loops $(\pi, L)$ removed from $\sigma$ such that $\Delta[\pi, L] = \delta$ and that $\sigma''$ is the sequence remaining after all
removals. Hence, $\Delta[\sigma, W_0] = \Delta x + \Delta[\sigma'', W_0]$. Since $\sigma$ is a $\rho$-pumping sequence, we have
\begin{align*}
\Delta[\sigma, W_0] &\ge b \\
\Rightarrow \Delta x + \Delta[\sigma'', W_0] &\ge b \\
\Rightarrow \Delta x &\ge b - \Delta[\sigma'', W_0].
\end{align*}

(1)

**Step 4:** We use the following result about the existence of small integral solutions to linear equations [4], which has been used by Rackoff to give EXPSPACE upper bound for the boundedness problems in Petri nets [16, Lemma 4.4].

Let $d_1, d_2 \in \mathbb{N}^+$, let $B$ be a $d_1 \times d_2$ integer matrix and let $b$ be an integer vector of dimension $d_1$. Let $d \ge d_2$ be an upper bound on the absolute value of the integers in $B$ and $b$. Suppose there is a vector $x \in \mathbb{N}^{d_2}$ such that $Bx \ge b$. Then for some constant $c$ independent of $d, d_1, d_2$, there exists a vector $y \in \mathbb{N}^{d_2}$ such that $By \ge b$ and $y(i) \le d^{cd_1}$ for all $i$ between 1 and $d_2$.

We apply the above result to (1). Each entry of $\Delta[\sigma'', M_0]$ is of absolute value at most $(e^{m_1+1}R + m_1 e^{m_1+1}R + 1)^m$ columns in $\Delta$, with the absolute value of each entry at most $e^{m_1} + 1$ and the sum of all entries in $y$ is at most $(2eR)^{dm^3}$ for some constant $d'$ (this expression is obtained from simplifying $(m_1 + 2) e^{m_1+1} R + 1)^m (2Re(m_1 + 1)e^{m_1+1} + 1)^2$ for some constant $d''$).

For each column $\delta$ of $\Delta$, let $(\pi_\delta, L_\delta)$ be a $P_{\prec \omega}$-loop occurring while firing $\sigma[\rho]$ from $W_0$ such that $\Delta[\pi_\delta, L_\delta] = \delta$. Recall from step 2 that there is some intermediate extended marking $W_4$ occurring while firing $\sigma''$ from $W_0$ such that $W_4 = L_\delta$. Let $i_\delta$ be the position in $\sigma''$ where $M_0$ occurs. Let $\sigma'$ be the sequence obtained from $\sigma''$ by inserting $\gamma(\delta)$ copies of $\pi_\delta$ into $\sigma''$ at the column $i_\delta$ for each column $\delta$ of $\Delta$. Since we insert at most $(2eR)^{dm^3}$ loops, each of length at most $e^{m_1} + \sigma'' i_\delta = (2eR)^{dm^3}(e^{m_1} + 1)^2$. Choose the constant $d$ such that $(2eR)^{dm^3} e^{m_1} + (e^{m_1} + 1)^2 \le (2eR)^{dm^3}$ Now we have $|\sigma'| \le (2eR)^{dm^3}$.

Step 5: Now we prove that $\sigma'$ is a $\rho$-pumping sequence enabled at $M_0'$. Recall that $M_0 \xrightarrow{\sigma'} P_{\prec \omega} M_k$ and that $\sigma'$ is obtained from $\sigma$ by removing or adding extra copies of some $P_{\prec \omega}$-loops. Since all functions in $\sigma$ isolate $P_{\prec \omega}$ from $P_\prec \cup P_{\prec \omega}$, $M_0' = P_{\prec \omega} M_0$ and $M_0(p) \ge R|\sigma'|$ for all $p \in P_\prec \cup P_{\prec \omega}$ we infer that $\sigma'$ is enabled at $M_0'$. Let $M_0' \xrightarrow{\sigma'} M_k'$. Since $\sigma'$ is obtained from $\sigma$ by removing $P_{\prec \omega}$-loops and all functions in $\sigma$ isolate $P_{\prec \omega}$ from $P_\prec \cup P_{\prec \omega}$, we have $M_k' = P_{\prec \omega} M_k$. In addition, $\sigma'$ consists of $\sigma''$ and $\gamma(\delta)$ copies of $\pi_\delta$ for each column $\delta$ of $\Delta$. Hence, $\Delta[\sigma', W_0] = \Delta y + \Delta[\sigma'', W_0]$. Since $\Delta y \ge b - \Delta[\sigma'', W_0]$ (recall that $y$ is a small solution that replaces $x$ in (1)), we conclude that $\Delta[\sigma', W_0] \ge b$.

Now we show that $\sigma'$ satisfies each condition of Definition 17. For the first condition, first recall that $M_k > M_0, M_0' = P_{\prec \omega} M_0$ and $M_k = P_{\prec \omega} M_k$. Now $M_k(p) > M_0(p)$ implies $M_k'(p) > M_0'(p)$ for all $p \in P_{\prec \omega}$. For any $p \in P_{\prec \omega}$, $M_k(p) > M_0(p)$ implies $b(p) = 1$ and hence $M_k'(p) - M_0'(p) = \Delta[\sigma', W_0](p) \ge b(p) = 1$. Finally, for any $p \in P_\prec$, $b(p) = 1$ and hence $M_k'(p) - M_0'(p) = \Delta[\sigma', W_0](p) \ge b(p) = 1$. Hence, $M_k' > M_0'$.

For the second condition of Definition 17, we have $b(p) = 1$ for all $p \in P_\prec$ and hence $\Delta[\sigma', W_0](p) \ge 1$. Hence, there is at least one function $(\tilde{A}, \tilde{B}', C')$ in $\sigma'$ with $B'[\rho](p') = 1$.

The last two conditions of Definition 17 are met by $\sigma'$ since it is a sub-sequence of $\sigma$. Therefore, $\sigma'$ satisfies all conditions of Definition 17. Hence, $\sigma'$ is a $\rho$-pumping sequence enabled at $M_0'$ and this concludes the proof.

The following lemma establishes what value is "large enough" at the initial marking to ensure that crossed places are unbounded. The bound given in this lemma could probably be further optimized using a slightly more intricate proof, but the current bound is good enough for our upper bound.
Lemma 21. Let \( \rho = (P_{<\omega}, P, P_{is}) \) be a triple such that the sets \( P_{<\omega}, P \) and \( P_{is} \) partition the set of places \( P \). Suppose \( \sigma \) is a \( p \)-pumping sequence enabled at \( M_0 \). If \( M_0(p) \geq 3R|\sigma| + R + 1 \) for all \( p \in P \cup P_{is} \), then \( M_0 \overset{\rho}{\rightarrow} M_1 \) and \( M_1 > M_0 \).

Proof. Let \( M_0 \overset{\rho}{\rightarrow} M_2 \). First we claim that for every prefix \( \sigma_1 \) of \( \sigma \), \( M_0 \overset{\sigma_1}{\rightarrow} M_1 \) and \( M_1 = P_{<\omega} M_1 \). Suppose for contradiction’s sake that this claim is false and let \( \sigma_1 = \sigma' f \). Since \( \sigma_1 \) is the shortest prefix of \( \sigma \) violating our claim, we have \( M_0 \overset{\sigma_1}{\rightarrow} M_1 \) and \( M_1 = P_{<\omega} M_1 \). By Prop. 4, \( M_1(p) \geq R \) for all \( p \in P \cup P_{is} \). Since \( f = (A, B, C) \) isolates \( P_{<\omega} \) from \( P \cup P_{is} \), we have \( M_1 = P_{<\omega} M_1 \). By Prop. 4, we have \( M(p) \geq M_0(p) - R|\sigma_1| \) and \( M(p') \geq M_0(p') - R|\sigma_1| \). Hence, either \( M'(p) \geq 2M_0(p) - 2R|\sigma_1| - R \geq M_0(p) + R|\sigma| + 1 \) (in case \( A \) multiplies \( p \)) or \( M'(p) \geq M_0(p) - R|\sigma_1| + M_0(p') - R|\sigma_1| - R \geq M_0(p) + R|\sigma| + 1 \) (in case \( A \) transfers \( p' \) to \( p \)). Again by Prop. 4, \( M_2(p) \geq M'(p) - R|\sigma| \geq M_0(p) + 1 \).

Definition 22. Let \( c = 2d \). The functions \( \ell_2, t_2 : \mathbb{N} \rightarrow \mathbb{N} \) are as follows:

\[
\begin{align*}
t_2(0) &= 0 & \ell_2(0) &= (2R)^{cm^3} \\
t_2(i + 1) &= 4R\ell_2(i) + R + 1 & \ell_2(i + 1) &= (2t_2(i + 1)R)^{cm^3}
\end{align*}
\]

Due to the selection of the constant \( c \) above, we have \((2xR)^{cm^3} \geq x^i + (2Rx)^{dm^3}\) for all \( x \in \mathbb{N} \) and all \( i \in \{0, \ldots, m\} \).

Definition 23. A \( t_2 \)-pumping pair enabled at \( W \) is a pair \( (\sigma_1, \sigma_2) \) of sequence of functions satisfying the following conditions.

1. \( W_1 \overset{\sigma_1}{\rightarrow} t_2 W_1 \overset{\sigma_2}{\rightarrow} t_2 W_2 \).
2. \( \omega(W_2) = \omega(W_1) \) and
3. for some partition of \( \omega(W_1) \) into \( P \) and \( P_{is} \), \( \sigma_2 \) is a \( \rho \)-pumping sequence enabled at \( [W_1]_{\omega \rightarrow t_2} \), where \( \rho = (\omega(W_1), P, P_{is}) \).

The following two lemmas prove that unboundedness in strongly increasing Affine nets is equivalent to the presence of a short \( t_2 \)-pumping pair enabled at \( M_{init} \). The ideas of these two lemmas are similar to those of Lemma 10 and Lemma 11 respectively. Lemma 20 is used in Lemma 25.

Lemma 24. If \( (\sigma_1, \sigma_2) \) is a \( t_2 \)-pumping pair enabled at \( W \), there is a self-covering sequence \( (\sigma_1', \sigma_2') \) enabled at \( [W]_{\omega \rightarrow t_2} \).

Proof. Let \( W_1 \overset{\sigma_1}{\rightarrow} t_2 W_1 \overset{\sigma_2}{\rightarrow} t_2 W_2 \). The proof is by induction on \( \text{card}(\omega(W)) \).

For the base case \( \text{card}(\omega(W)) = 0 \), we have \( \omega(W) = P \), so \( W_1 = W \) and \( [W]_{\omega \rightarrow t_2} = [W_1]_{\omega \rightarrow t_2} \). By Definition 23, \( \sigma_2 \) is a \( (P, P_{is}, \emptyset) \)-abstract pumping sequence that is enabled at \( [W_1]_{\omega \rightarrow t_2} \). By setting \( e = 1 \) in Lemma 20, we conclude that there is a \( (P, P_{is}, \emptyset) \)-abstract pumping sequence \( \sigma_2' \) enabled at \( [W_1]_{\omega \rightarrow t_2} \) such that \( |\sigma_2'| \leq \ell_2(0) \). From Definition 17,
$|W_1|_{\omega \rightarrow t_2} \sigma_{i\rho} M'$ such that $M' > |W_1|_{\omega \rightarrow t_2}$. From Definition 22, $|W_1|_{\omega \rightarrow t_2}(p) \geq 4R|\sigma'_2| + R + 1$ for all $p \in P$ and hence from Lemma 21, $|W_1|_{\omega \rightarrow t_2} \sigma_{i\rho}$ $M'$ such that $M > |W_1|_{\omega \rightarrow t_2}$. Therefore, $(\epsilon, \sigma'_2)$ is a self-covering pair enabled at $|W_1|_{\omega \rightarrow t_2}$.

For the induction step, suppose $\text{card}(\omega(W)) = i + 1$. First consider the case where $\text{card}(\omega(W_1)) = \text{card}(\omega(W_2)) = i + 1$. By Definition 23, $\sigma_2$ is a $(P_x, P_{M_1}, \omega(W_1))$-abstract pumping sequence enabled at $|W_1|_{\omega \rightarrow t_2}$. By Prop. 18, any intermediate marking $M'$ in the run $|W_1|_{\omega \rightarrow t_2} \sigma_{i\rho}$ $M_2$ will satisfy $M'(p) < t_2(i + 1)$ for all $p \in \omega(W_1)$. Hence by Lemma 20, there is a $(P_x, P_{M_1}, \omega(W_1))$-abstract pumping sequence $\sigma'_2$ enabled at $|W_1|_{\omega \rightarrow t_2}$ such that $|\sigma'_2| \leq (2t_2(i + 1) + 1)R^{dm_3}$. Let $\sigma'_1$ be the sequence obtained from $\sigma_1$ by removing all functions between any two identical intermediate extended markings occurring in the run $W \sigma_{i\rho} W_1$. We have $|\sigma'_1| \leq (t_2(i + 1))^{i+1}$ and by Definition 22, $|\sigma_1'| + |\sigma'_2| \leq \ell_2(i + 1)$.

Since $t_2(i + 2) = 4Rt_2(i + 1) + R + 1$, Prop. 6 implies that $|W_1|_{\omega \rightarrow t_2} \sigma_{i\rho}$ $M_1' \sigma_{i\rho}$ $M_2'$ such that $M_1'(p) \geq 3Rt_2(i + 1) + R + 1$ for all $p \in \omega(W_1)$ and $M_1' = \omega(W_1)$. Now by Lemma 21, $M_2' > M_1'$. Therefore, $(\sigma_1', \sigma_2')$ is a self-covering pair enabled at $|W_1|_{\omega \rightarrow t_2}$.

Now consider the case of the induction step where $\text{card}(\omega(W)) = i + 1$ and $\text{card}(\omega(W_1)) < i + 1$. Let $W_3$ be the first intermediate extended marking in the run $W \sigma_{i\rho} W_1$ such that $\text{card}(\omega(W_3)) < i + 1$. Let $W \sigma_{i\rho} W_3' \triangleleft W_3 \sigma_{i\rho} W_1$ and $\sigma_3$ be the sequence obtained from $\sigma_3$ by removing all functions between any two identical intermediate extended markings in the run $W \sigma_{i\rho} W_3'$. We have $|\sigma'_3| \leq (t_2(i + 1))^{i+1}$, $W \sigma_{i\rho} W_3'$, $\omega(W_3') = \omega(W)$ and if $i + 1 < m$, $t_2(i + 2) \geq R|\sigma'_3| + R + t_2(i + 1)$ by Definition 22. By Prop. 6, $|W_1|_{\omega \rightarrow t_2} \sigma_{i\rho} M_3'such that $M_3' = \omega(W_3')$ and $M_3'(p) \geq t_2(i + 1) + R$ for all $p \in \omega(W_3')$.

Since $W_3' \sigma_{i\rho} W_3$, $A(p, p') = 0$ for all $p \in \omega(W_3)$ and $p' \in \omega(W_3)$. Hence, $M_3' \sigma_{i\rho} M_3$ such that $M_3(p) = \sum_{p' \in \omega(W_3)} A(p, p') M_3'(p') + B(p) = W_3(p)$ for all $p \in \omega(W_3)$. Since $([A|B])_{t_2} = W_3$, we have $\sum_{p' \in P} A(p, p') W_3'(p) + B(p) \geq t_2(i + 1)$ for all $p \in \omega(W_3)$. Since $M_3'(p) \geq t_2(i + 1) + R$ for all $p \in \omega(W_3)$, $M_3(p) = \sum_{p' \in P} A(p, p') M_3'(p') + B(p) \geq t_2(i + 1)$. Therefore, $M_3 \geq W_3_{\omega \rightarrow t_2}$. Since $(\sigma_4, \sigma_2)$ is an abstract $t_2$-pumping pair enabled at $W_3$, by induction hypothesis there is a self-covering pair $(\sigma_4', \sigma_2')$ enabled at $[W_3|_{\omega \rightarrow t_2}$ and hence at $M_3$. Therefore, $(\sigma_3', \sigma_4', \sigma_2')$ is a self-covering pair enabled at $|W_1|_{\omega \rightarrow t_2}$.

**Lemma 25.** If a self covering pair is enabled at $M_{init}$, there is a $t_2$-pumping pair $(\sigma_1', \sigma_2')$ enabled at $M_{init}$ such that $|\sigma_1'| + |\sigma_2'| \leq \ell_2(m)$.

**Proof.** Let $\sigma = \sigma_1\sigma_2$ be a self-covering pair enabled at $M_{init}$. Suppose $M_{init} \sigma_{i\rho} M_1 \sigma_{i\rho} M_2$ and $M_2 > M_1$ By Prop. 6, $M_{init} \sigma_{i\rho} W_1 \sigma_{i\rho} W_2$ for some extended markings $W_1, W_2$. For each $i \in \mathbb{N}^+$, let $W_i \sigma_{i\rho} W_{i+1}$ and $M_i \sigma_{i\rho} M_{i+1}$. Since $M_2 > M_1$, $M_{i+1} > M_i$ and $W_{i+1} \geq W_i$ for all $i \in \mathbb{N}^+$. Let $j \geq 2$ be the first number such that $\omega(W_j) = \omega(W_{j-1})$. Now we have $M_{init} \sigma_{i\rho} W_{j-1} \sigma_{i\rho} W_{j-1} \sigma_{i\rho} W_j$, $W_{j-1} \geq M_{j-1}$ and $\omega(W_{j-1}) = \omega(W_j)$. Each function in $\sigma_2$ isolates $\omega(W_{j-1})$ from $\omega(W_{j-1})$ (otherwise, we could not have $\omega(W_{j-1}) = \omega(W_j)$). Let $P_x \subseteq \omega(W_{j-1})$ be the set of places $p$ in $\omega(W_{j-1})$ such that some function in $\sigma_2$ either multiplies $p$ or transfers $p'$ to $p$ for some $p' \in \omega(W_{j-1}) \setminus \{p\}$. Let $P_{is} = \omega(W_{j-1}) \setminus P_x$. Let $p = (\omega(W_{j-1}), P_x, P_{is})$. Now $\sigma_2$ is a $p$-pumping sequence enabled at $M_{j-1}$. For any intermediate marking $M$ arising while firing $\sigma_2$ from $M_{j-1}$, $M(p) < t_2(\text{card}(\omega(W_{j-1})))$ for all $p \in \omega(W_{j-1})$. Hence, by Lemma 20, there is a $p$-pumping sequence $\sigma'_2$ enabled at $[W_{j-1}, \omega \rightarrow t_2$ with $|\sigma'_2| \leq (2\text{max}(t_2(\text{card}(\omega(W_{j-1}))), 1))_{R^{dm_3}}$, since $[W_{j-1}, \omega \rightarrow t_2 \omega(W_{j-1}) = W_{j-1}$ and $[W_{j-1}, \omega \rightarrow t_2 \omega(W_{j-1}) = W_{j-1}$. If any two intermediate extended markings in the run $M_{init} \sigma_{i\rho} W_{j-1}$ are identical, remove all the functions between them. Let $\sigma'_1$ be the sequence left after all such removals. Now, we have $M_{init} \sigma_{i\rho} W_{j-1}$. 


(σ′₁, σ′₂) is the required abstract ℓ₂-pumping pair enabled at M_{init}.

\[\Box\]

**Lemma 26.** Let k = 8c. Then \(\ell_2(i) \leq (2R)^{k+1}m^{3(i+1)}\) for all \(i \in \mathbb{N}\).

**Proof.** By induction on \(i\). For the base case \(i = 0\), the result is obvious since by Definition 22, \(\ell_2(0) = (2R)^{cm^3}\).

Induction step:

\[
\ell_2(i + 1) = (2\ell_2(i + 1)R)^{cm^3} \\
\leq (2 \cdot 6R\ell_2(i) \cdot R)^{cm^3} \\
= (12R^2)^{cm^3} (\ell_2(i))^{cm^3} \\
\leq (2R)^{3cm^3} (\ell_2(i))^{cm^3} \\
\leq (2R)^{3cm^3} ((2R)^{k+1}m^{3(i+1)})^{cm^3} \\
= (2R)^{3cm^3} (2R)^{ck+1}m^{3(i+2)} \\
\leq (2R)^{6ck+1}m^{3(i+2)} \\
= (2R)^{k+2}m^{3(i+2)}
\]

\[\Box\]

**Theorem 27.** For some constant \(c_3\), the Boundedness problem for strongly increasing Affine nets is in \(\text{NSPACE}(O(\omega^m \log m (\log R + \log \|M_{init}\|_{\infty}) ))\).

**Proof.** \(\text{EXPSPACE}\)-hardness follows from \(\text{EXPSPACE}\)-hardness for Petri nets [14].

For the upper bound, we have from Lemma 24 and Lemma 25 that an Affine net is unbounded iff a \(\ell_2\)-pumping pair \((σ_1, σ_2)\) is enabled at the initial marking \(M_{init}\) such that \(|σ'_1| + |σ'_2| \leq \ell_2(m)\). The following non-deterministic Turing machine can guess and verify the existence of such a sequence.

1. Initialize a counter \(ctr\) to 0 and extended marking \(W\) to \(M_{init}\).

2. Non-deterministically choose some function \(f = (A, B, C)\) and apply to \(W\). If \(W \notin \text{dom} f\), reject and stop. Otherwise, update \(W\) to \([AW + B]_{t_2 \rightarrow \omega}\) and increment \(ctr\).

3. If \(ctr\) has value at most \(\ell_2(m)\), non-deterministically go to step 2 or step 4. Otherwise, reject and stop.

4. The sequence of functions guessed till now is \(σ_1\) of the \(t_2\)-pumping pair \((σ_1, σ_2)\). Store two markings \(M_1, M_2\) initialised to \([W]_{\omega \rightarrow t_2}\). Guess some partition \(P_x, P_is\) of \(\omega(W)\).

5. Non-deterministically choose some function \(f = (A, B, C)\) that isolates \(\overline{ω(W)} \cup P_is\) from \(P_x \cup P_is\) and does not multiply any place in \(P_is\). If \(M_x \notin \text{dom} f[ρ]\), reject and stop. Otherwise, update \(M_2\) to \(A[ρ]M_2 + B[ρ]\) and increment \(ctr\). If \(M_2(p) \geq t_2(\text{card}(\overline{ω(W)}))\) for any \(p \in \overline{ω(W)}\), reject and stop. Otherwise, continue to the next step.

6. If \(ctr\) has value at most \(\ell_2(m)\), non-deterministically go to step 5 or step 7. Otherwise, reject and stop.

7. If \(M_2 > M_1\) and \(M_2(p) > M_1(p)\) for all \(p \in P_x\), accept and stop. Otherwise, reject and stop.

If a \(t_2\)-pumping pair \((σ_1, σ_2)\) is enabled at the initial marking \(M_{init}\) such that \(|σ'_1| + |σ'_2| \leq \ell_2(m)\), then the run of the above algorithm that guesses \(σ_1\) in steps 1–4 and guesses \(σ_2\) in
steps 4–7 will accept. If there is no \( t_2 \)-pumping pair \((\sigma_1, \sigma_2)\) enabled at the initial marking \( M_{\text{init}} \) such that \( |\sigma'_1| + |\sigma'_2| \leq t_2(m) \), then none of the runs of the above algorithm accept.

The space required to store \( ctr \) is at most \( \log(t_2(m)) \). The space required to store \( W \) and \( M_1 \) is at most \( m \log(t_2(m)) \). The space required to store \( M_2 \) is at most \( m \log((mRt_2(m) + R)t_2(m)) \). Using the upper bound given by Lemma 26, we conclude that the memory space required by the above algorithm is \( O(m \log(\|M_{\text{init}}\|_\infty) + k^{m+1}m^{3m+4} \log R) \). This can be simplified to \( O(2^{2m} \log m (\log R + \log\|M_{\text{init}}\|_\infty)) \).

With the size of the input \( n = (\text{card}(F))(m^2 + m) \log R + m \log\|M_{\text{init}}\|_\infty) \), we can infer from the above theorem an upper bound of \( \text{SPACE}(O(2^{2m} \log m)) \) for the boundedness problem.

6 Model Checking

In this section, we show that model checking a logic, which is \( \text{EXPSPACE} \)-complete for Petri nets [1], is undecidable for strongly increasing Affine nets.

Following is the syntax of existential Presburger CTL (PrECTL\( \geq (U) \)), introduced in [1].

\[
\phi ::= \top | \bot | \phi \lor \phi | \phi \land \phi | \mathbf{E}(\phi U \psi) | \mu(p) \geq c,
\]

where \( p \in P, c \in \mathbb{N} \) and \( \psi \) is a quantifier-free Presburger formula with \( m \) variables from the following syntax:

\[
\psi ::= \top | \neg \psi | \psi \lor \psi | \alpha, \alpha ::= \tau \geq \tau | \tau \equiv q \tau, \tau ::= 0 | 1 | x | \tau + \tau,
\]

where \( x \) is a variable and \( q \geq 2 \). Given a formula \( \psi \) with variables \( x_1, \ldots, x_k \) and a valuation \( v : \{x_1, \ldots, x_k\} \to \mathbb{Z} \), we write \( \psi(v) \) for the closed formula with \( v(x_j) \) substituted for \( x_j \) for each \( j \) between 1 and \( k \). Given a closed Presburger formula \( \psi \), we write \( PA \models \psi \) if the formula is valid. Given two markings \( M_1 \) and \( M_2 \), \( M_1 - M_2 : P \to \mathbb{Z} \) is the function such that \( (M_1 - M_2)(p) = M_1(p) - M_2(p) \) for all \( p \in P \). In [1], the semantics of PrECTL\( \geq (U) \) is defined on coverability graphs of Petri nets, where the goal is to use PrECTL\( \geq (U) \) to express interesting properties. The atomic formula \( \mu(p) \geq c \) could have \( c = \omega \) as defined in [1]. Here, our goal is to prove undecidability. We restrict \( c \) to be in \( \mathbb{N} \) in the atomic formula \( \mu(p) \geq c \) and define the semantics directly on the transition system. A marking \( M \) of an Affine net \( \mathcal{N} \) satisfies a formula \( \phi \), written as \( M \models \phi \), in the following inductive cases:

- \( M \models \top \) always,
- \( M \models \bot \) never,
- \( M \models \phi_1 \lor \phi_2 \) if \( M \models \phi_1 \) or \( M \models \phi_2 \),
- \( M \models \phi_1 \land \phi_2 \) if \( M \models \phi_1 \) and \( M \models \phi_2 \),
- \( M \models \mathbf{E}(\phi U \psi) \) if there is a sequence of functions \( \sigma \) such that \( M \xrightarrow{\sigma} M', M' \models \phi', PA \models \psi(M' - M) \) and for every intermediate marking \( M'' \neq M' \) arising while firing \( \sigma \) from \( M, M'' \models \phi \),
- \( M \models \mu(p) \geq c \) if \( M(p) \geq c \).

We can restrict ourselves to finite tree shaped models for PrECTL\( \geq (U) \) formulas. Such a model \( \mathcal{T} \) has a root \( M \) and a number of leaves \( M_1, \ldots, M_n \), each satisfying some coverability constraint sub-formula \( \gamma \) of the form

\[
\gamma ::= \top | \bot | \gamma \land \gamma | \gamma \lor \gamma | \mu(p) \geq c,
\]

where \( p \in P \) and \( c \in \mathbb{N} \). This model is increasing if \( M_i \geq M \) for all \( i \) between 1 and \( n \). A formula \( \phi \) of PrECTL\( \geq (U) \) is increasing if all its tree-shaped models are increasing. An eventually increasing formula is a formula of the form \( \mathbf{E}(\top U \top \phi) \) for some increasing formula \( \phi \). The set of eventually increasing formulas is denoted by eiPrECTL\( \geq (U) \). It is shown in [1] that model checking eiPrECTL\( \geq (U) \) on coverability graphs of Petri nets is in \( \text{EXPSPACE} \).
Theorem 28. Model checking $\text{eiPrECTL} \geq (U)$ formulas on strongly increasing Affine nets is undecidable.

Proof. We reduce the halting problem in 2-counter deterministic Minsky machines to an instance of model checking $\text{eiPrECTL} \geq (U)$ formula on strongly increasing Affine nets.

Let $\mathcal{M}$ be a 2-counter deterministic Minsky machine with set of states $Q$. Let $q_{\text{init}}$ be the initial state of $\mathcal{M}$. We assume without loss of generality that if $\mathcal{M}$ halts, it halts in the state $q_{\text{halt}}$. We also assume that no transition of $\mathcal{M}$ is enabled in the state $q_{\text{halt}}$. Let $\mathcal{N}$ be a strongly increasing Affine net with the set of places $\{p_q \mid q \in Q\} \cup \{p_1, p_2, p'_1, p'_2\}$. For convenience, we call the places corresponding to $q_{\text{init}}$ and $q_{\text{halt}}$ as $p_{\text{init}}$ and $p_{\text{halt}}$ respectively. The places $p_1, p_2$ are used to simulate the counters $\text{ctr}_1, \text{ctr}_2$ of $\mathcal{M}$. The places $p'_1, p'_2$ are used for zero testing.

For every transition “$q: \text{ctr}_i \leftarrow \text{ctr}_i + 1$; goto $q'$” of $\mathcal{M}$, $\mathcal{N}$ has the function $f = (A, B, C)$ where $A = Id$, $C = 0$ and

$$B(p) = \begin{cases} -1 & \text{if } p = p_q, \\ +1 & \text{if } p = p'_q, \\ +1 & \text{if } p = p_i, \\ 0 & \text{otherwise.} \end{cases}$$

For any transition of the Minsky machine that increments one of the two counters, $\mathcal{N}$ can apply the corresponding function given above. For any function of the above form applied by $\mathcal{N}$, the Minsky machine can execute the corresponding incrementing transition.

For every transition “$q: \text{ctr}_i \leftarrow 0$ then goto $q'$ else $\text{ctr}_i \leftarrow \text{ctr}_i - 1$; goto $q''$” of $\mathcal{M}$, $\mathcal{N}$ has two functions $f_0 = (A_0, B_0, C_0)$ and $f_{>0} = (A_{>0}, B_{>0}, C_{>0})$. Define $A_{>0} = Id$, $C_{>0} = 0$ and

$$B_{>0}(p) = \begin{cases} -1 & \text{if } p = p_q, \\ +1 & \text{if } p = p'_q, \\ -1 & \text{if } p = p_i, \\ 0 & \text{otherwise.} \end{cases}$$

Define $A_0(p'_i, p_i) = 1$ and $A_0(p, p') = Id(p, p')$ for all $(p, p') \neq (p'_i, p_i)$. Define $C_0 = 0$ and

$$B_0(p) = \begin{cases} -1 & \text{if } p = p_q, \\ +1 & \text{if } p = p'_q, \\ 0 & \text{otherwise.} \end{cases}$$

For any “else” branch of a zero-testing transition executed by $\mathcal{M}$, $\mathcal{N}$ can apply the corresponding function $f_{>0}$. For any “zero” branch of a zero-testing transition executed by $\mathcal{M}$, $\mathcal{N}$ can apply the corresponding function $f_0$. For any function of the form $f_{>0}$ applied by $\mathcal{N}$, $\mathcal{M}$ can execute the corresponding “else” branch of a zero-testing transition. For a function of the form $f_0$ applied by $\mathcal{N}$, $\mathcal{M}$ can execute the “zero” branch of a zero-testing transition, provided the corresponding counter has the value 0. The function $f_0$ also adds the value in the place $p_i$ to $p'_i$ so that if this function was applied erroneously when $\text{ctr}_i$ had some value greater than 0, it is recorded in $p'_i$ to be tested later with a $\text{eiPrECTL} \geq (U)$ formula.

Finally, $\mathcal{N}$ has a function $f_h = (A_h, B_h, C_h)$ with $B_h = 0$, $C_h(p_{\text{halt}}) = -1$, $C_h(p) = 0$ for all $p \in P \setminus \{p_{\text{halt}}\}$, $A_h(p'_1, p'_1) = A_h(p'_2, p'_2) = A_h(p_{\text{halt}}, p_{\text{halt}}) = 2$ and $A_h(p, p') = Id(p, p')$ for all $(p, p') \notin \{(p'_1, p'_1), (p'_2, p'_2), (p_{\text{halt}}, p_{\text{halt}})\}$. An execution of $f_h$ will result in the values of $p'_1$, $p'_2$ and $p_{\text{halt}}$ being multiplied by 2. The initial marking $M_{\text{init}}$ is defined as $M_{\text{init}}(p_{\text{init}}) = 1$ and $M_{\text{init}}(p) = 0$ for all $p \in P \setminus \{p_{\text{halt}}\}$.

Let $x_h$ be a variable corresponding to $p_{\text{halt}}$ and $x_p$ be a variable corresponding to $p \in P \setminus \{p_{\text{halt}}\}$. Let $\psi_{\text{faithful}}$ be the quantifier-free Presburger formula $\psi_{\text{faithful}} = x_h \geq 2 \land \ldots$
$\bigwedge_{p \in P \setminus \{x_h\}} (x_p \geq 0 \land \neg(x_p \geq 1))$. If $M_1, M_2$ are markings, then $PA \models \psi_{\text{faithful}}(M_2 - M_1)$ iff $M_2(p_{\text{halt}}) \geq M_1(p_{\text{halt}}) + 2$ and $M_2(p) = M_1(p)$ for all $p \in P \setminus \{p_{\text{halt}}\}$. Note that $E(TU\psi_{\text{faithful}} \top)$ is an increasing formula and hence $\phi_M = E(TU\psi_{\text{faithful}} \top)$ is an eventually increasing formula.

Now we will show that $M_{\text{init}} \models \phi_M$ iff $M$ halts. Suppose $M$ halts. Then there is a run of $N$ that faithfully simulates the halting run of $M$ to reach a marking $M_h$ such that $M_h(p_{\text{halt}}) = 1$ and $M_h(p'_1) = M_h(p'_2) = 0$. Let $M_h \xrightarrow{f_h} M_2$. It is easy to verify that $PA \models \psi_{\text{faithful}}(M_2 - M_h)$ so $M_h = E(TU\psi_{\text{faithful}} \top)$ and hence $M_{\text{init}} \models \phi_M$.

Now suppose $M_{\text{init}} \models \phi_M$. There is a marking $M_h$ reachable from $M_{\text{init}}$ such that $M_h \models E(TU\psi_{\text{faithful}} \top)$. Since no transitions of $M$ are enabled at $q_{\text{halt}}$, the only function enabled at $M_h$ is $f_h$, which must be fired to ensure that $M_h \models E(TU\psi_{\text{faithful}} \top)$. Hence, $M_h(p_{\text{halt}}) \geq 1$ and $M_h(p'_1) = M_h(p'_2) = 0$. Since $M_h$ is reachable from $M_{\text{init}}$, there must be a halting run of $M$, since that is the only way to reach any marking $M$ with $M(p_{\text{halt}}) \geq 1$ and $M(p'_1) = M(p'_2) = 0$.

## 7 Conclusions and Perspectives

We proved that coverability and boundedness are in $\text{SPACE}(O(2^{cn \log n}))$ for strongly increasing Affine nets. The main difficulty in adapting Rackoff technique is that one cannot simply ignore places that have large enough values, as transitions may copy values from one place to another. From this result, we may immediately deduce the same result for the termination problem as one can add a new place $p_{\text{time}}$ which is incremented by every transition. Then, the system terminates iff it is bounded. A natural question is to identify the properties that could be proved (with Rackoff techniques) to be $\text{EXPSPACE}$-complete for strongly increasing Affine nets. At least two (recent) classes of properties are candidates: the generalized unboundedness properties of Demri [5] and the CTL fragment of Blockelet and Schmitz [1]. As this last logic is proved undecidable for strongly increasing Affine nets, a natural restriction of this logic would be defined. We conjecture that replacing the predicates on the effect of a path by predicates on the Parikh image of the path would put the model checking problem for the logic in $\text{EXPSPACE}$.

As we have limited our study to Affine nets, another question would be to consider not only affine functions, but to find classes of recursive functions (that still forbids resets) for which the Rackoff techniques can still be applied. It is likely that the proof of coverability could be adapted by altering Prop. 6 to take into account the “maximum reduction” that functions can perform. For example, if we allow functions to halve the value in a place, it would suffice to say that the initial value is $2E(i)$ times higher (of course, this would change the final upper bound obtained). However, the proof of boundedness relies heavily on the fact that a place is either fully copied, or not at all, so how to generalize it is unclear.

Coverability and boundedness for Petri nets with an unique Reset/Transfer/Zero-test extended arc have been recently proved to be decidable [2]. For the Zero-test case, the complexity of coverability is at least as hard as reachability for Petri Nets, so there is not much hope of applying this technique. We conjecture that it could be applied to the one Reset or Transfer case, even if it would yield an upper bound greater than $\text{EXPSPACE}$.

## References


