Musings Around the Geometry of Interaction, and Coherence

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Abstract

We introduce the Danos-Régnier category $\mathcal{DR}(M)$ of a linear inverse monoid M, as a categorical description of geometries of interaction (GOI) inspired from the weight algebra. The natural setting for GOI is that of a so-called weakly Cantorian linear inverse monoid, in which case $\mathcal{DR}(M)$ is a kind of symmetrized version of the classical Abramsky-Haghverdi-Scott construction of a weak linear category from a GOI situation. It is well-known that GOI is perfectly suited to describe the multiplicative fragment of linear logic, and indeed $\mathcal{DR}(M)$ will be a $*$-autonomous category in this case. It is also well-known that the categorical interpretation of the other linear connectives conflicts with GOI interpretations. We make this precise, and show that $\mathcal{DR}(M)$ has no terminal object, no cartesian product of any two objects, and no exponential—whatever M is, unless M is trivial. However, a form of coherence completion of $\mathcal{DR}(M)$ à la Hu-Joyal (which for additives resembles a layered approach à la Hughes-van Glabbeek), provides a model of full classical linear logic, as soon as M is weakly Cantorian. One finally notes that Girard's notion of *coherence* is pervasive, and instrumental in every aspect of this work.

Key words: Geometry of interaction, linear logic, inverse monoid, $*$-autonomous category, coherence completion.
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1. Introduction

There are by now several families of models for (classical) linear logic. One is the category of *coherence spaces* [17]. Another is given by game models, e.g. [4]. Contrarily to what one might expect, geometry of interaction, in whatever form [14, 15, 16, 19] does not yield models of linear logic. Now by *model* of linear logic we are rather demanding, and mean a denotational, in fact a *categorical* model. The definition of categorical models of linear logic took some time to emerge, and is certainly posterior to geometry of interaction. We shall consider linear categories [7], LNL categories

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[6], Lafont and new-Lafont categories [27]. It is remarkable that coherence spaces form a model in all these senses, but most proposals based on games or geometry of interaction do not. The point is subtle: e.g., Baillot *et al.* [4] show that AJM games are a model of MELL proof nets (i.e., without the additives) without box erasure steps. Some more recent game semantics, such as Melliès' asynchronous games [28], do provide a categorical model of linear logic.

In a sense, there are categorical models of a domain-theoretic style, but only a few coming from the interaction world, and none from the geometry of interaction. This paper bridges the gap. Our main contribution is a categorical model of full classical linear logic, including multiplicative, exponential and additive connectives, based on ideas from geometry of interaction—specifically from Danos and Régnier [11, 10]—and also using the notion of *coherence completion* [21]. So we import from both interaction and domain theory. *Coherence* plays a fundamental role in both.

A word on the organization of this paper. First, we feel that some intuition about the roots of this work should be brought forward, and we devote Section 2 to it. We introduce the concept of a *linear inverse semigroup M* in Section 3, and show in Section 4 how any such M gives rise to a category $\mathcal{DR}(M)$, which we call the *Danos-Régnier category* of M. We shall also see that, provided M is *weakly Cantorian*, $\mathcal{DR}(M)$ is compact-closed. In particular, it is a model of the multiplicative fragment MLL of linear logic. The purpose of Section 5 is to compare this construction to the $\mathcal{G}$ construction of Abramsky *et al.* [1], a.k.a. Joyal *et al.*'s *Int* construction [23], the most prominent categorical interpretation of geometry of interaction. On our way to get a categorical model of the whole of linear logic, we shall then trip on a serious difficulty: we shall show in Section 6 and Section 7 that there is no way to interpret *any* form of additive or exponential connective in $\mathcal{DR}(M)$, whatever M. I.e., changing the languages of paths won't help. Nonetheless, we show in Section 8 that a slight modification of Hu and Joyal's *coherence completion* [21] builds a Lafont category out of any $*$-autonomous category, i.e., a model of full classical linear logic out of any model of just MLL... and this is exactly what $\mathcal{DR}(M)$ provides, no less, no more.

Another word on related work. We shall heavily discuss related work throughout the paper, notably the construction of compact-closed categories from traced monoidal categories [1, 23] in Section 5, and coherence completions [21] in Section 8. The idea of considering inverse monoids is credited to Yves Legrandgérard by Danos and Régnier [11]. As far as the impossibility results mentioned in Sections 6 and 7 are concerned, it is well-known that trying to add specific new equations between geometry of interaction tokens, aimed at enforcing some categorical identities, resulted in inconsistencies. Our impossibility results are much stronger: we show that *no* change in the underlying inverse monoid M can result in the creation of *any* instance of any missing categorical feature (additive, exponential).

2. Motivation

I came to study inverse monoids following Danos *et al.* [10], where weights from the so-called dynamic algebra arise from an inverse monoid with some added structure (the *bar*, which captures the reduction process). However, my actual initial goal was to try and understand how one may describe Böhm-like trees of λ-terms up to β- or

$\beta\eta$-equivalence, not as trees, but as collections of paths through these trees. (A goal I have not reached yet.)

Let us see what this means on trees. By tree we mean some form of infinite first order term: each node t is labeled by a function symbol f of some arity $n \in \mathbb{N}$, and has n successors $t_1, \ldots, t_n$; we then agree to write t as $f(t_1, \ldots, t_n)$. We call Σ the given signature, i.e., the set of all function symbols, together with their respective arities. We write $f/n \in \Sigma$ to state that f is in Σ, with arity n. With each such f/n in Σ, we associate n distinct letters $f_1, \ldots, f_n$. (We need to adjust this when $n = 0$, in all rigor.) This yields the path alphabet $|\mathcal{A}| = \bigcup_{f/n\in\Sigma}\{f_1, \ldots, f_n\}$. Its elements are the path letters, and a *path* is any finite sequence of path letters. Traveling down a tree along any route from the root yields a path in the obvious way. E.g., the tree $f(g(t_1, t_2), t_3)$ has (at least) the paths ϵ (the empty path), f_1, f_1g_1, f_1g_2, f_2.

Going from a tree to its set of paths is easy. Recovering a tree from a given set of paths is harder. First, not every set of paths arises from some tree, e.g., $\{f_1, g_1\}$. The key point to enable this reconstruction process is *coherence*. This was invented under a different name by Harrison and Havel [20]. Define an equivalence relation $\equiv$ on the path alphabet by $f_i \equiv g_j$ iff $f = g$. Now let $\frown$ be the relation on paths such that $w \frown w'$ iff, for any strict common prefix w_0 of w and w', writing w as w_0aw_1 and w' as $w_0a'w'_1$ with $a, a' \in |\mathcal{A}|$, then $a \equiv a'$; $\frown$ is reflexive and symetric, though in general not transitive. When $w \frown w'$, we say that w and w' are *coherent*, and a *clique* is any set of pairwise coherent paths. Clearly, any set of paths of a given tree is a clique. In general, a space $X = (|X|, \frown)$ where $\frown$ is a reflexive and symmetric relation on $|X|$ is a *coherence space* [17]. So there is a coherence space of paths, $(|\mathcal{A}|^*, \frown)$; this was explored by Reddy [31, Section 5.2]. Coherence spaces form the basis of an elegant semantics of the λ-calculus, and in fact of all of linear logic [17].

Let us refine. Let $\leq$ be the prefix ordering on paths. Then $w \leq w'$ and $w' \frown w''$ implies $w \frown w''$: $(|\mathcal{A}|^*, \leq, \frown)$ is a bit more than a coherence space, it is an *event structure*, i.e., a space $X = (|X|, \leq, \frown)$ where $\leq$ is a partial ordering and $\frown$ is a reflexive and symmetric relation on $|X|$ such that $w \leq w'$ and $w' \frown w''$ implies $w \frown w''$. Then the set of paths in a tree is a *down-closed* clique, and conversely any down-closed clique is the set of paths of a unique tree (except that functions f/n may have less than n subtrees).

Event structures are a fundamental model of concurrency [29], where, instead of using $\frown$, a binary irreflexive and symmetric relation # called *conflict* is used, such that $w \leq w'$ and $w\#w''$ implies $w'\#w''$. (We have also ignored the axiom of so-called finite causes here.) This is equivalent: take coherence $\frown$ as negation of conflict #. The relationship between order $\leq$ and coherence $\frown$ is explained, and generalized to so-called bistructures, by Curien *et al.* [9].

In the case of λ-terms, as opposed to infinite first-order terms, there is an extra difficulty in identifying terms with certain cliques of paths: λ-terms reduce to other λ-terms, and we would like to define a notion of paths through λ-terms that is *invariant* under $\beta\eta$-equivalence. The result will be a way to compute paths through the Böhm tree of t by just computing paths through t itself—*without* reducing t. This is exactly what geometry of interaction is about. Girard's execution formula aims at being such an invariant. Our view is that such an invariant should be a denotational (categorical) model of λ-calculus, and in fact of linear logic proofs.

3. Linear Inverse Semigroups

Such a calculus of paths for MLL terms is lurking around in [11, 10], based on the notion of a (bar) inverse monoid. The quantity that remains invariant through reduction is the set of all weights of paths through a proof net. But this cannot be defined in a modular way: if you know the weights of all paths in (the proof net of) a λ-term M and also that for a λ-term N, you cannot infer the weights of paths through MN. The reason is that not all paths can be considered: we must only consider those paths that are *legal* and *straight*. The latter condition in particular cannot be defined on weights alone; the paths themselves have to be taken into account. Our aim here and in Section 4 is to define a semantics of MLL proof nets (which we do by building a $*$-autonomous category) in terms of weights, eliminating the pollution of paths, which only reflect some form of syntax. The key is to collect, not sets, but *least upper bounds* of cliques in the inverse monoid of weights.

Recall that an *inverse semigroup* is a triple $(M, \cdot, \_^*)$ where $(M, \cdot)$ is a semigroup (i.e., $\cdot$ is associative) and $\_^*$ is a unary operation that satisfies: $(u^*)^* = u$, $(uv)^* = v^*u^*$, $uu^*u = u$, and $uu^*vv^* = vv^*uu^*$ for all $u, v \in M$; we abbreviate $u \cdot v$ as uv. An *inverse monoid* also has a unit 1. A typical example is the set $\mathsf{PI}(E)$ of *partial injections* on a set E, i.e., of (graphs of) bijections u between two subsets of E, the *domain* $\{x \mid \exists y \cdot (x, y) \in u\}$ and the *codomain* $\{y \mid \exists x \cdot (x, y) \in u\}$ of u. $\mathsf{PI}(E)$ is an inverse monoid with 1 the identity on E, composition as multiplication, and star as inversion: $u^* = \{(y, x) | (x, y) \in u\}$.

Following [10], write $\langle u \rangle = uu^*$. In $\mathsf{PI}(E)$, this is the identity on the codomain of u, which we identify with the codomain of u. Similarly, we think of $\langle u^* \rangle$ as the domain of u. It always helps to look at the case $M = \mathsf{PI}(E)$. This is all the more justified as, by the Preston-Wagner Theorem, every inverse semigroup M embeds into some inverse monoid of the form $\mathsf{PI}(E)$, namely $\mathsf{PI}(M)$.

An *idempotent* in M is any u such that $uu = u$. In any inverse semigroup, the idempotents are the terms of the form $\langle u \rangle$, and every idempotent u satisfies $u = \langle u \rangle = u^* = \langle u^* \rangle$. The defining equation $uu^*vv^* = vv^*uu^*$, i.e., $\langle u \rangle \langle v \rangle = \langle v \rangle \langle u \rangle$, states that idempotents commute. The *natural ordering* $\leq$ on M corresponds to inclusion between graphs of relations in the case of $\mathsf{PI}(E)$. Equivalent ways are to define $u \leq v$ iff $vu^* = uu^*$, or $uv^* = uu^*$, or $\langle u \rangle v = u$, or $v \langle u^* \rangle = u$, or $u^*v = u^*u$, or $v^*u = u^*u$. Then $\leq$ is a partial ordering, and multiplication and inverse are monotonic. This is well-known, see [30, 26].

The main import of this Section is that every inverse semigroup also has a *coherence* relation. Intuitively, if $u, v \in \mathsf{PI}(E)$ and there is an element x which is mapped by u and v to different elements, either forward (for some $y \neq y'$, $(x, y) \in u$ and $(x, y') \in v$) or backward, then u and v should be in conflict. Recall that $\frown$ is the negation of conflict. Algebraically:

Definition 3.1 (Coherence). Let M be an inverse semigroup. The relations $\frown_0$, $\frown_1$ and $\frown$ on M are defined by: $u \frown_0 v$ iff $u \langle v^* \rangle = v \langle u^* \rangle$; $u \frown_1 v$ iff $\langle v \rangle u = \langle u \rangle v$; and $u \frown v$ iff $u \frown_0 v$ and $u \frown_1 v$.

We can show that $u \frown v$ iff $u^* \frown v^*$, and more importantly:

Lemma 3.2 (Event Structure). *Let M be an inverse semigroup. Then $(M, \leq, \frown)$ is an event structure: if $u \leq v$ and $v \frown w$, then $u \frown w$.*

Proof. We claim that: if $u \leq v$ and $v \frown_0 w$, then $u \frown_0 w$. By applying $\_^*$, we will deduce that $u \leq v$ and $v \frown_1 w$ imply $u \frown_1 w$. So assume $u \leq v$, $v \frown_0 w$. Since $u \leq v$, $v \langle u^* \rangle = u$ and $v^* u = \langle u^* \rangle$, so: (a) $\langle v^* \rangle \langle u^* \rangle = v^* u = \langle u^* \rangle$. Since $v \frown_0 w$, $v \langle w^* \rangle = w \langle v^* \rangle$. Since $u \leq v$, $v \langle u^* \rangle = u$, so $u \langle w^* \rangle = v \langle u^* \rangle \langle w^* \rangle = v \langle w^* \rangle \langle u^* \rangle = w \langle v^* \rangle \langle u^* \rangle = w \langle u^* \rangle$ by (a), whence $u \frown_0 w$. □

In particular, $u \leq v$ implies $u \frown v$, and any two elements that have an upper bound in M are coherent. This is as in all event structures. Additionally, multiplication preserves coherence: $u_0 \frown v_0$ and $u_1 \frown v_1$ imply $u_0 u_1 \frown v_0 v_1$. As can be expected from the intuitive description of $\frown$, if $u \frown v$ in M, then u and v have an greatest lower bound $u \wedge v$, and $u \wedge v = u \langle v^* \rangle = v \langle u^* \rangle = \langle v \rangle u = \langle u \rangle v$.

Definition 3.3 (Linear Inverse Semigroup). An inverse semigroup M is *linear* iff: (1) every clique $(u_i)_{i \in I}$ has a least upper bound $\sum_{i \in I} u_i$, and (2) multiplication distributes over least upper bounds of cliques, i.e., for every clique $(u_i)_{i \in I}$, for every element v, $\left(\sum_{i \in I} u_i\right) v = \sum_{i \in I} u_i v$.

$\mathsf{PI}(E)$ is always a linear inverse semigroup. The $\sum$ notation for least upper bounds of cliques is justified by the distributivity property (2). One may show that (2) implies that $u \frown v$ iff u and v have a common upper bound; in other words, $\frown$ coincides with the standard coherence relation of $\leq$. Distributivity is equivalent to $v \left(\sum_{i \in I} u_i\right) = \sum_{i \in I} v u_i$ (take inverses, observing that $\sum$ and inverse commute). The empty clique has a least upper bound, which we write 0 (the empty relation in $\mathsf{PI}(E)$), and distributivity implies that $0.v = v.0 = 0$. Moreover, the set of all idempotents is a clique, and its least upper bound 1 is a unit. So any linear inverse semigroup is an inverse monoid.

The construction of the Preston-Wagner Theorem establishes that any inverse semigroup M actually embeds into some *linear* inverse monoid: $\mathsf{PI}(M)$ itself. The embedding i_M maps $u \in M$ to the partial injection $\{(v, uv) | v \in M, v = \langle u^* \rangle v\}$. This preserves products, inverses, unit (if any), and preserves and reflects order. Coherence is also preserved: indeed coherence is defined by equations, which are preserved by the embedding. At least two linear inverse semigroups have been used previously in the literature. Danos and Régnier use $\mathsf{PI}(\mathbb{N})$ at the end of [11] as an example. Girard [16] uses sets of *rudimentary clauses*, up to deletion of subsumed clauses and tautologies (see [18, Section 2.4.1] for details). Rudimentary clauses are pairs of first-order terms $s \leftarrow t$ with the same free variables. Multiplying two such clauses $s \leftarrow t$ and $s' \leftarrow t'$ yields their resolvent $s\sigma \leftarrow t'\sigma$, where σ is the mgu of t and s' if it exists, or the empty set otherwise. Inversion is given by $(s \leftarrow t)^* = (t \leftarrow s)$.

We end this section by noting that linear inverse monoids afford us a nice graphical notation for elements, which we call *automata*. These are oriented graphs with an initial state q_I and a final state q_F, where each state q^A is labeled with an idempotent A, and each transition $\mathsf{q}^A \xrightarrow{u} \mathsf{q}'^B$ satisfies $\langle u^* \rangle \leq A$ and $\langle u \rangle \leq B$. (We sometimes drop the superscript, and in fact also the state name, replacing the latter by symbols such as $\bullet$ or $\circ$.) The path $\mathsf{q}_0 \xrightarrow{u_1} \mathsf{q}_1 \xrightarrow{u_2} \ldots \mathsf{q}_{n-1} \xrightarrow{u_n} \mathsf{q}_n$ denotes the product $u_n \ldots u_2 u_1$. (We reverse products, as in [11].) We then read the automaton $\mathcal{A}$ as the sup of all paths from $\mathfrak{q}_I$ to $\mathfrak{q}_F$.

For this to make sense, the paths should form a clique. It is enough to require that $u_1 u_2^* = 0$ for any *divergence* (*forward determinacy*), and $u_1^* u_2 = 0$ for any *convergence* (*backward determinacy*); such *bideterminacy conditions* are to be expected [12, 2].

Divergence

$$\begin{tikzcd} & \mathfrak{q}_1 \\ \mathfrak{q} \arrow[ur, "u_1"] \arrow[dr, "u_2"'] & \\ & \mathfrak{q}_2 \end{tikzcd}$$

Convergence

$$\begin{tikzcd} \mathfrak{q}_1 \arrow[dr, "u_1"] & \\ & \mathfrak{q} \\ \mathfrak{q}_2 \arrow[ur, "u_2"'] & \end{tikzcd}$$

Then product is concatenation, and inversion $\_^*$ exchanges initial and final states and replaces each transition $\mathfrak{q}^A \xrightarrow{u} \mathfrak{q}'^B$ by $\mathfrak{q}'^B \xrightarrow{u^*} \mathfrak{q}^A$. In $\mathsf{PI}(E)$, forward determinacy says that no element of E can be both in the domains of u_1 and of u_2. Think of elements of E as tokens n that wait at some state $\mathfrak{q}^A$, and can travel along the transition $\mathfrak{q}^A \xrightarrow{u} \mathfrak{q}'^B$ if n is in the domain of u, arriving at state $\mathfrak{q}'^B$ with the new value $u(n)$. Forward determinacy means that tokens travel along one path at most. Tokens may also travel backwards, and backward determinacy imposes determinacy on backwards paths, too. Bideterminacy is sufficient for automata to make sense: if $u_1 u_2^* = u_1^* u_2 = 0$, then $u_1 \frown u_2$. But it is *not* necessary; in particular, we allow for *non-straight* paths. E.g., the path $\mathfrak{q}_0^A \xrightarrow{u} \mathfrak{q}_1^B \xrightarrow{v} \mathfrak{q}_2^C \xrightarrow{v^*} \mathfrak{q}_1^B \xrightarrow{w} \mathfrak{q}_4^D$ denoting $w \langle v^* \rangle u$ is not straight: we go from $\mathfrak{q}_1^B$ to $\mathfrak{q}_2^C$ and back through the same edge. If this path exists at all, then $\mathfrak{q}_0^A \xrightarrow{u} \mathfrak{q}_1^B \xrightarrow{w} \mathfrak{q}_4^D$ denoting wu is here, too. These two contribute $wu + w \langle v^* \rangle u$ to the value of the whole automaton (assuming that $\mathfrak{q}_0^A$ is initial and $\mathfrak{q}_4^D$ final). But $\langle v^* \rangle \leq 1$, so $w \langle v^* \rangle u \leq wu$. Since $+$ is least upper bound, $wu + w \langle v^* \rangle u = wu$: we don't have to forbid non-straight paths as in [11, 10]. Keep them: their value will just not count. Similarly, illegal paths, i.e., those of value 0, do not count, since 0 is the least element of M.

Finally, distributivity (2) allows us to graft entire automata in place of single transitions and preserve the reading of the automaton. This will be essential. E.g., $\mathfrak{q}_I \xrightarrow{u} \mathfrak{q}_1 \xrightarrow{\sum_{i=0}^{+\infty} v_i} \mathfrak{q}_2 \xrightarrow{w} \mathfrak{q}_F$ reads as $w \left(\sum_{i=0}^{+\infty} v_i \right) u$. This is the same reading as the automaton shown on the right, i.e., $\sum_{i=0}^{+\infty} w v_i u$. (Remember an automaton reads as the sup of its paths, i.e., of all $w v_i u$, $i \in \mathbb{N}$, here.)

$$\begin{tikzcd} \mathfrak{q}_I \arrow[r, "u"] & \mathfrak{q}_1 \arrow[r, bend left=80, "v_0"] \arrow[r, bend left=50, "v_1"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right=60, "v_i"'] & \mathfrak{q}_2 \arrow[r, "w"] & \mathfrak{q}_F \end{tikzcd}$$

$\vdots$

4. The Danos-Régnier Category of a Linear Inverse Monoid

The most standard construction of a category from an inverse monoid M is the *inductive groupoid* $\mathcal{IG}(M)$. Its objects are the idempotents of M, and its morphisms $A \xrightarrow{u} B$ are the elements $u \in M$ such that $\langle u^* \rangle = A$ and $\langle u \rangle = B$. There is a rich theory of inductive groupoids, see e.g. Steinberg [32]. We shall be more interested in the following novel construction: The *Danos-Régnier category* $\mathcal{DR}(M)$ of M has all idempotents A of M as objects; its morphisms from A to B are all triples $(\beta, a, \gamma) \in M^3$ such that:

a. $aA = Ba = a$, $\beta A = A\beta = \beta$, $\gamma B = B\gamma = \gamma$;

b. $\beta^* = \beta$, $\gamma^* = \gamma$;

c. $a\beta = 0$, $\gamma a = 0$;

$$\begin{tikzcd} \bullet^{A} \arrow[r, bend left, "\beta"] \arrow[d, "a"'] & \bullet^{A} \\ \circ^{B} & \circ^{B} \arrow[l, bend left, "\gamma"] \arrow[u, "a^*"'] \end{tikzcd}$$

We represent such morphisms as automata of the form shown to the right, with four distinguished states (two $\bullet$s, two $\circ$s).

To guide intuition, imagine that $M = \mathsf{PI}(E)$, β, a, γ are partial injections, and A, B are sets. Condition **a** states that the domain of a is contained in A, its codomain is contained in B, the domain and the codomain of β are contained in A, and similarly for γ and B: this is a typing condition. Condition **b** is a symmetry condition. Under Condition **b**, Condition **c** expresses forward determinacy on the upper left state and backward determinacy on the upper right state, and also backward determinacy on the lower left state and forward determinacy on the lower right state.

The *identity morphism* id_A at object A is $\begin{tikzcd} \bullet^{A} \arrow[r, bend left, "0"] \arrow[d, "A"'] & \bullet^{A} \\ \circ^{A} & \circ^{A} \arrow[l, bend left, "0"] \arrow[u, "A"'] \end{tikzcd}$. (In the sequel, we shall not draw those arrows labeled 0.)

Composition is by vertical pasting, a.k.a., stacking. Formally, the composition of $\begin{tikzcd} \bullet^{B} \arrow[r, bend left, "\beta'"] \arrow[d, "a'"'] & \bullet^{B} \\ \circ^{C} & \circ^{C} \arrow[l, bend left, "\gamma'"] \arrow[u, "a'^*"'] \end{tikzcd}$ with $\begin{tikzcd} \bullet^{A} \arrow[r, bend left, "\beta"] \arrow[d, "a"'] & \bullet^{A} \\ \circ^{B} & \circ^{B} \arrow[l, bend left, "\gamma"] \arrow[u, "a^*"'] \end{tikzcd}$ is shown on the right, where the middle Bs no longer label any distinguished states.

$$\begin{tikzcd} \bullet^{A} \arrow[r, bend left, "\beta"] \arrow[d, "a"'] & \bullet^{A} \\ B \arrow[r, bend left, "\beta'"] \arrow[d, "a'"'] & B \arrow[l, bend left, "\gamma"] \arrow[u, "a^*"'] \\ \circ^{C} & \circ^{C} \arrow[l, bend left, "\gamma'"] \arrow[u, "a'^*"'] \end{tikzcd}$$

To ease reading, think of it as a condensed representation of four automata: top left to bottom left, top left to top right, bottom right to bottom left, and bottom right to top right. This is well-defined: Condition **c** ensures that the middle B states have only forward and backward deterministic transitions. A nice feature of the above diagram is that it displays clearly why composition is associative. Explicit formulae are less readable, e.g., the top left to bottom left automaton denotes $\sum_{n\in\mathbb{N}} a'(\gamma\beta')^n a$ ("go down a, loop as many times as you wish through the $\gamma\beta'$ loop, then go down a'). Still, such sums are interesting, as they incarnate Girard's execution formula [14, 11].

There is also a *dualizing functor* $\_^\perp : \mathcal{DR}(M) \to \mathcal{DR}(M)^{op}$, defined by $A^\perp = A$, and, on morphisms, by rotating them 180 degrees:

$$\left(\begin{tikzcd} \bullet^{A} \arrow[r, bend left, "\beta"] \arrow[d, "a"'] & \bullet^{A} \\ \circ^{B} & \circ^{B} \arrow[u, "a^*"'] \arrow[l, bend left, "\gamma"] \end{tikzcd}\right)^{\perp} = \begin{tikzcd} \bullet^{B} \arrow[r, bend left, "\gamma"] \arrow[d, "a^*"'] & \bullet^{B} \\ \circ^{A} & \circ^{A} \arrow[u, "a"'] \arrow[l, bend left, "\beta"] \end{tikzcd}$$

$\mathcal{DR}(M)$ is a nice category in some respects. E.g., $\mathcal{DR}(M)$ has an epi-mono factorization system, all epis and all monos are split, and every morphism that is both epi and mono is iso. Concretely, (β, a, γ) is epi from A to B iff $\langle a \rangle = B$ and $\gamma = 0$, while it is mono iff $\langle a^* \rangle = A$ and $\beta = 0$. (I.e., if e is epi, then $(0, \langle a \rangle, 0) \circ e = \mathrm{id}_B \circ e$, so $\langle a \rangle = B$. By Conditions **a** and **c**, $\gamma = 0$. Conversely, $e = (\beta, a, 0)$ with $\langle a \rangle = B$ is split epi since $e \circ e^\perp = \mathrm{id}_B$.) Moreover, the isos in $\mathcal{DR}(M)$ are exactly the morphisms of the form $\begin{tikzcd} \bullet^{A} \arrow[d, "a"'] & \bullet^{A} \\ \circ^{B} & \circ^{B} \arrow[u, "a^*"'] \end{tikzcd}$ with $\langle a^* \rangle = A$ and $\langle a \rangle = B$, meaning that the groupoid of $\mathcal{DR}(M)$ is exactly the inductive groupoid $\mathcal{IG}(M)$ [18, Section 5.1.3]: these morphisms are indeed just the morphisms $A \xrightarrow{u} B$ of $\mathcal{IG}(M)$, drawn twice and vertically.

To get a model of MLL, we define:

Definition 4.1 (Weakly Cantorian). A linear inverse monoid M is *weakly Cantorian* iff it contains two elements p and q with $p^*q = 0$, $\langle p^* \rangle = \langle q^* \rangle = 1$.

In $\mathsf{PI}(\mathbb{N})$, think of p as $\{(n, 2n) | n \in \mathbb{N}\}$, and q as $\{(n, 2n+1) | n \in \mathbb{N}\}$. In the rudimentary clause setting, think of p as the clause $X \leftarrow \mathsf{p}(X)$ and q as $X \leftarrow \mathsf{q}(X)$, where p and q are two distinct function symbols. Weak Cantorian structures allow us to define a *tensor product* $A_1 \otimes A_2$ of objects A_1, A_2 as $pA_1p^* + qA_2q^*$. (In $\mathsf{PI}(\mathbb{N})$, reading idempotents as sets, $A_1 \otimes A_2 = \{2n | n \in A_1\} \cup \{2n+1 | n \in A_2\}$ is the disjoint sum of A_1 and A_2.) On morphisms, let $f_1 \otimes f_2 = \begin{tikzcd} \bullet^{A_1 \otimes A_2} \arrow[r, bend left, "p\beta_1 p^* + q\beta_2 q^*"] \arrow[d, "pa_1p^* + qa_2q^*"'] & \bullet^{A_1 \otimes A_2} \\ \circ^{B_1 \otimes B_2} & \circ^{B_1 \otimes B_2} \arrow[u, "pa_1^*p^* + qa_2^*q^*"'] \arrow[l, bend left, "p\gamma_1 p^* + q\gamma_2 q^*"] \end{tikzcd}$ whenever

$f_1 = \begin{tikzcd} \bullet^{A_1} \arrow[r, bend left, "\beta_1"] \arrow[d, "a_1"'] & \bullet^{A_1} \\ \circ^{B_1} & \circ^{B_1} \arrow[u, "a_1^*"'] \arrow[l, bend left, "\gamma_1"] \end{tikzcd}$, $f_2 = \begin{tikzcd} \bullet^{A_2} \arrow[r, bend left, "\beta_2"] \arrow[d, "a_2"'] & \bullet^{A_2} \\ \circ^{B_2} & \circ^{B_2} \arrow[u, "a_2^*"'] \arrow[l, bend left, "\gamma_2"] \end{tikzcd}$. The tensor unit I is 0. One checks easily that these make $\mathcal{DR}(M)$ a symmetric monoidal category [18, Section 5.2.1].

Categorical models of (intuitionistic) MLL are *symmetric monoidal closed* categories, i.e., those having a *linear application* morphism $\mathsf{app}_{A,B} : (A \multimap B) \otimes A \to B$ (the counit of the adjunction), and a *linear abstraction* operator $\lambda^C_{A,B}$ such that $\lambda^C_{A,B}(f) : C \to (A \multimap B)$ for each $f : C \otimes A \to B$, satisfying:

- β-equivalence: $\mathsf{app}_{A,B} \circ (\lambda^C_{A,B}(f) \otimes g) = f \circ (\mathrm{id}_C \otimes g) : C \otimes D \to B$ for every $f : C \otimes A \to B$ and $g : D \to A$;

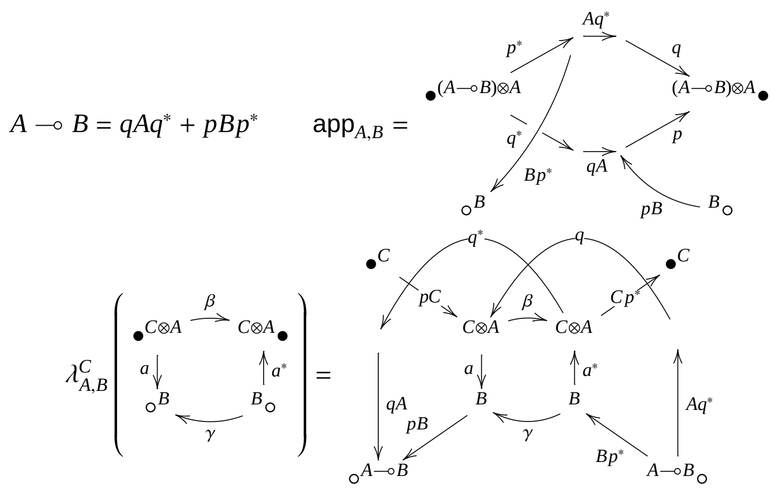
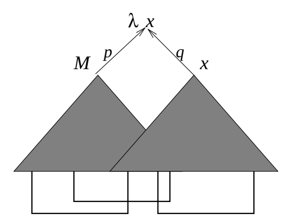


Figure 1: Linear implication, application, abstraction

- η-equivalence: $\lambda^{A\multimap B}_{A,B}(\mathsf{app}_{A,B}) = \mathrm{id}_{A\multimap B}$;
- substitution: $\lambda^{C}_{A,B}(f) \circ g = \lambda^{D}_{A,B}(f \circ (g \otimes \mathrm{id}_A))$ for every $f : C \otimes A \to B$ and $g : D \to C$.

While this axiomatization is non-standard, it has the merit of displaying the underlying linear λ-calculus at work, in a style resembling categorical combinators [8]. These are given for $\mathcal{DR}(M)$ in Figure 1.



Let us give some intuition. Recall that the idea behind $\mathcal{DR}(M)$ is to describe, as morphisms, the set of paths in linear λ-terms. Represent a linear λ-term in normal form as a portion of the infinite binary tree, with axiom links between leaves. A λ-abstraction $\lambda x \cdot M$ is then represented as on the right, where the left son is the root to the body M of the λ-abstraction, and the right son points to the unique occurrence of x in the unique head application $xN_1 \ldots N_k$ in M. (For now, imagine the right triangle consists just of one link connecting x to its use in the left triangle.) The paths from the root of $\lambda x \cdot M$ are as follows. First, go down left (p^*, or rather Bp^*), then enter M (the inner square in the definition of the λ-abstraction). We may then either exit M at the root of M, and go up right (p, more precisely pB); or exit M through the variable x; this means selecting x from the bunch of variables free in M (the curved q^* starting from $C \otimes A$), then going up left to the root of $\lambda x \cdot M$ (qA); or exit M through some other variable y; this means selecting the set of those free variables of M that are not x (the Cp^* transition). We can similarly explore the other paths in $\lambda x \cdot M$, and thus justify the definition of λ-abstraction given above.

With these constructions, one checks that $\mathcal{DR}(M)$ is symmetrical monoidal closed, i.e., a model of intuitionistic MLL. Let $\perp$ be the 0 object, and define intuitionistic negation $\sim A$ as $A \multimap \perp$. It is easy to see that $\sim A$ is isomorphic to $A^{\perp} = A$. The morphism $C_A = \begin{tikzcd} \bullet^{\sim\sim A} \arrow[d, "Aq^{*2}"'] & \bullet^{\sim\sim A} \\ \circ^{A} & \circ^{A} \arrow[u, "q^2A"'] \end{tikzcd}$ is inverse to $\lambda^{A}_{\sim A,\perp}(\mathsf{app}_{A,\perp} \circ c_{A,\sim A})$, where $c_{A_1,A_2} = qA_1p^* + pA_2q^* : A_1 \otimes A_2 \to A_2 \otimes A_1$ is the commutativity natural transformation. C_A is a morphism from $\sim\sim A$ to A, and acts as a linear form of Felleisen *et al.*'s control operator $\mathcal{C}$ [13]. It is easy to see that $A \multimap B$ is isomorphic to $A^{\perp} \otimes B$, and that these constructs turn $\mathcal{DR}(M)$ into a *compact-closed* category. Recall that $*$-*autonomous* categories are symmetric monoidal closed categories with a dualizing object $\perp$, i.e., one such that $\lambda^{A}_{\sim A,\perp}(\mathsf{app}_{A,\perp} \circ c_{A,\sim A})$ is iso; such categories are models of classical MLL [5]. Compact-closed categories [24] are $*$-autonomous categories such that there is a natural iso between $A \otimes B$ and $A \parr B$, where $A \parr B = \sim(\sim A \otimes \sim B)$. Summing up [18, Theorem 5.2.7]:

Theorem 4.2. *Let M be a weakly Cantorian linear inverse monoid. Then $\mathcal{DR}(M)$ is a compact-closed category, i.e., a categorical model of classical MLL.*

5. Retracing Some Paths in $\mathcal{DR}(M)$

Every compact-closed category has a canonical *trace* [23]. The prototypical example of a compact-closed category is the category whose objects are $\mathbb{R}^n$, $n \in \mathbb{N}$, and whose morphisms are linear maps, i.e., morphisms from $\mathbb{R}^m$ to $\mathbb{R}^n$ are $n \times m$ matrices. The notion of trace in a category then generalizes the usual notion of trace in linear algebra. One may compute the canonical trace of the compact-closed category $\mathcal{DR}(M)$ [18, Proposition 5.2.8]:

Proposition 5.1. *The canonical trace on the compact-closed category $\mathcal{DR}(M)$ is given by*

$$Tr^{X}_{A,B}\left(\begin{tikzcd} \bullet^{A\otimes X} \arrow[r, "\beta"] \arrow[d, "a"'] & A\otimes X^{\bullet} \\ \circ_{B\otimes X} & B\otimes X_{\circ} \arrow[l, "\gamma"] \arrow[u, "a^*"'] \end{tikzcd}\right) = \begin{tikzcd} \bullet^{A} \arrow[dr, "pA"'] & & & {}^{A}\bullet \\ & A\otimes X \arrow[r, "\beta"] \arrow[d, "a"'] & A\otimes X \arrow[ur, "Ap^*"'] \arrow[d, bend left=60, "qXq^*"] & \\ & B\otimes X \arrow[dl, "Bp^*"] \arrow[u, bend left=60, "qXq^*"] & B\otimes X \arrow[l, "\gamma"] \arrow[u, "a^*"'] & \\ \circ^{B} & & & {}^{B}\circ \arrow[ul, "pB"] \end{tikzcd}.$$

Consider the subcategory $\mathcal{S}plit(M)$ of $\mathcal{DR}(M)$ whose morphisms are of the form $(0, a, 0)$. In $\mathcal{IG}(M)$-like notation, the morphisms are $A \xrightarrow{u} B$ with $\langle a^* \rangle \leq A$ and $\langle a \rangle \leq B$. (In $\mathcal{IG}(M)$, we would require $\langle a^* \rangle = A$, $\langle a \rangle = B$.) In other words, $\mathcal{S}plit(M)$ is exactly the *Karoubi envelope* of the monoid M, i.e., the category whose objects are idempotents of M, and whose morphisms from A to B are elements a of M such that $BaA = a$.

The trace operator on $\mathcal{DR}(M)$ then induces one on $\mathcal{S}plit(M)$ (not on $\mathcal{IG}(M)$—trace does not preserve isos), by: $Tr^X_{A,B}(A\otimes X \xrightarrow{a} B\otimes X) = $
$$\begin{tikzcd} A \arrow[r, "pA"] & A\otimes X \arrow[r, "a"] & B\otimes X \arrow[l, bend right, "qXq^*"'] \arrow[r, "Bp^*"] & B \end{tikzcd}.$$
This exhibits the familiar feedback loop typical of several trace operators.

The formula for trace in $\mathcal{S}plit(M)$ can also be obtained directly from the standard trace of Haghverdi and Scott [19, Proposition 6], observing that $\mathcal{S}plit(M)$ is a *unique decomposition category* in the sense of Haghverdi. Such categories are symmetric monoidal categories, whose homsets are enriched over Σ-monoids, and having certain quasi-injection and quasi-projection operators. Enrichment means that we may take sums of certain countable families of morphisms between two objects, so that sums distribute over composition on both sides, that any partition $((x_i)_{i\in I_j})_{j\in J}$ of a summable family $(x_i)_{i\in I}$ is summable and $\sum_{j\in J}\sum_{i\in I_j} x_i = \sum_{i\in I} x_i$, and that one-element families are summable, with the obvious sum. The latter two properties are ensured here by the fact that the summable families are the cliques, the former property is due to our requirement of distributivity (Definition 3.3). The quasi-injection $A \to A\otimes B$ is pA, the quasi-injection $B \to A\otimes B$ is qB, and the quasi-projections are their inverses Ap^* and Bq^*.

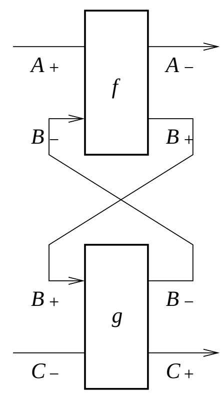
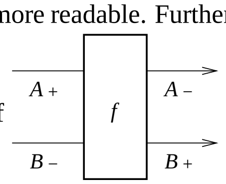
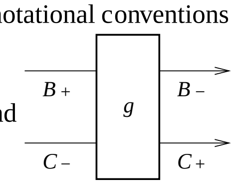


Figure 2: Composition in $\mathcal{G}(\mathcal{C})$.

It is then interesting to compare $\mathcal{DR}(M)$ to the construction of a compact-closed category $\mathcal{G}(\mathcal{C})$ from any traced symmetrical monoidal category $\mathcal{C}$ [1, 23]. While the latter is motivated by geometry of interaction interpretations of multiplicative linear logic, one should however be aware that it is not the gist of Girard's original geometry of interaction. Some aspects of compact-closed categories, such as duality $\_^\perp$ or typing, have no equivalent in Girard's geometry of interaction. Conversely, Hilbert spaces, as initially put forward by Girard, play no role in the $\mathcal{G}$ construction. $\mathcal{DR}(M)$ will appear as a sort of middle ground: although the construction arose from reverse-engineering Danos and Régnier's presentation of the weight algebra [11, 10], we shall see that $\mathcal{DR}(M)$ has much to do with the $\mathcal{G}$ construction.

The objects of $\mathcal{G}(\mathcal{C})$ are pairs (A^+, A^-) of objects of $\mathcal{C}$. A morphism $f : (A^+, A^-) \to (B^+, B^-)$ in $\mathcal{G}(\mathcal{C})$ is a morphism $f : A^+ \otimes B^- \to A^- \otimes B^+$ in $\mathcal{C}$. The identity on (A^+, A^-) is the commutativity c_{A^+,A^-}. Composition is given by *symmetric feedback*. Given $f : (A^+, A^-) \to (B^+, B^-)$ and $g : (B^+, B^-) \to (C^+, C^-)$ in $\mathcal{G}(\mathcal{C})$, i.e., $f : A^+ \otimes B^- \to A^- \otimes B^+$ and $g : B^+ \otimes C^- \to B^- \otimes C^+$ in $\mathcal{C}$, the composition $g \circ f$ in $\mathcal{G}(\mathcal{C})$ is the trace $Tr^{B^-\otimes B^+}_{A^+\otimes C^-, A^-\otimes C^+}(\cong \circ (f \otimes g) \circ \cong)$, where $\cong$ denotes obvious isos built from associativity and commutativity. An elegant box notation due to Kelly and Laplaza [24] makes this more readable. Further notational conventions [1] allow one to define the composition of 

and 

as in Figure 2.

Let us expand the definitions for $\mathcal{G}(\mathcal{S}plit(M))$. Each morphism $f : (A^+, A^-) \to (B^+, B^-)$ in $\mathcal{G}(\mathcal{S}plit(M))$, i.e., each morphism $A^+ \otimes B^- \xrightarrow{f} A^- \otimes B^+$ in $\mathcal{S}plit(M)$ gives rise to four morphisms $A^+ \xrightarrow{f^{++}} B^+$, $A^+ \xrightarrow{f^{+-}} A^-$, $B^- \xrightarrow{f^{-+}} B^+$, and $B^- \xrightarrow{f^{--}} A^-$ in $\mathcal{S}plit(M)$. On the one hand, $f^{++} = B^+ q^* f p A^+$, $f^{+-} = A^- p^* f p A^+$, $f^{-+} = B^+ q^* f q B^-$, $f^{--} = A^- p^* f q B^-$. By imitation with $\mathcal{DR}(M)$, we may organize the latter in a square such as $\begin{tikzcd} \bullet^{A^+} \arrow[r, bend left, "f^{+-}"] \arrow[d, "f^{++}"'] & \bullet^{A^-} \\ \circ^{B^+} & \circ^{B^-} \arrow[u, "f^{--}"'] \arrow[l, bend left, "f^{-+}"] \end{tikzcd}$, obeying just Condition **a**: $f^{++}A^+ = B^+ f^{++} = f^{++}$, $f^{+-}A^+ = A^- f^{+-} = f^{+-}$, $f^{--}B^- = A^- f^{--} = f^{--}$, $f^{-+}B^- = B^+ f^{-+} = f^{-+}$. Conditions **b** and **c** or similar conditions are not required here.

However, one should note that there is in general no way to recover f from the four-tuple f^{++}, f^{+-}, f^{-+}, f^{--}, unless the following Condition **c'** holds: $f^{++} \smile_0 f^{+-}$, $f^{-+} \smile_0 f^{--}$, $f^{+-} \smile_1 f^{--}$, and $f^{-+} \smile_1 f^{++}$. Indeed, then the sum $qB^+ f^{++} A^+ p^* + pA^- f^{+-} A^+ p^* + qB^+ f^{-+} B^- q^* + pA^- f^{--} B^- q^*$ makes sense and equals f. Condition **c'** is clearly entailed by **b** and **c**, and is both necessary to recover f from the above four-tuple, and to make sense of composition as juxtaposition of automata, as in $\mathcal{DR}(M)$.

One may check that identities and tensor product are defined in $\mathcal{DR}(M)$ exactly as in $\mathcal{G}(\mathcal{S}plit(M))$. Composition in $\mathcal{G}(\mathcal{S}plit(M))$ also coincides with the $\mathcal{DR}(M)$ definition by juxtaposition of automata, but only for morphisms satisfying Condition **c'**. In general, composition in $\mathcal{G}(\mathcal{S}plit(M))$ is more complex.

Note finally that the symmetry Condition **b** of $\mathcal{DR}(M)$ is not necessary at all to define composition: we only require Condition **c'**. In particular, four-tuples as above obeying Conditions **a** and **c'** would define a larger compact-closed category than $\mathcal{DR}(M)$, with all operations defined similarly. However, Condition **b** is natural from our intended interpretation of paths in λ-terms, where e.g., for any weight w of a path from the input to the output, w^* is the weight of the converse path, from output to input [11, 10].

We can then define weak GOI situations [19] on $\mathcal{S}plit(M)$, and the construction of a weak linear category from it, i.e., of a categorical model for linear combinatory algebra, carries over to $\mathcal{DR}(M)$. We only need to make sure M comes with a linear inverse semigroup endomorphism $! : M \to M$, and elements $\mathfrak{d}$, $\mathfrak{e}$, $\underline{\mathfrak{d}}$ verifying certain equations [18, Section 6.3]. A typical example is when $M = \mathsf{PI}(\mathbb{N})$, $\langle \_, \_ \rangle$ is any injection from $\mathbb{N}^2$ to $\mathbb{N}$, $!f\langle k, n\rangle = \langle k, f(n)\rangle$, $\mathfrak{d}\langle k_1, \langle k_2, n\rangle\rangle = \langle\langle k_1, k_2\rangle, n\rangle$, $\mathfrak{e}(n) = \langle 1, n\rangle$, and $\underline{\mathfrak{d}} = rp^* + sq^*$, where $r\langle k, n\rangle = \langle 2k, n\rangle$, $s\langle k, n\rangle = \langle 2k+1, n\rangle$. We shall not pursue this, since this is well-known, and our goal here is to find linear, not just weak linear categories.

6. $\mathcal{DR}(M)$ Contains No Additive

Surprisingly, there is no way to have $\mathcal{DR}(M)$ contain *any* additive connective, in a very strong sense, as we now show. One might have hoped that enriching M with new constants g, d as in [25] for example, or as in [16] (where M is a linear inverse monoid of rudimentary clauses) would provide a solution. And indeed it does, provided we are ready to forego some natural proof conversion rules. If we are not, there is no way. First, we cannot interpret any of the additive units $\top$ (which would be a terminal object) and $\mathbf{0}$ (an initial object):

Proposition 6.1. *The following statements are equivalent: (1) $\mathcal{DR}(M)$ has a terminal object; (2) 0 is terminal in $\mathcal{DR}(M)$; (3) $\mathcal{DR}(M)$ has an initial object; (4) 0 is initial in $\mathcal{DR}(M)$; (5) $M = \{0\}$.*

Proof. (1) and (3), (2) and (4) are equivalent through duality $\_^{\perp}$. (1) $\Rightarrow$ (2): Let $\top$ be a terminal object in $\mathcal{DR}(M)$ (i.e., for every object A, there is a unique morphism from A to $\top$). So there is a unique morphism from 0 to $\top$. Here are two,
$$\begin{tikzcd} \bullet^{0} & \bullet^{0} \\ \circ^{\top} & \circ^{\top} \end{tikzcd}$$
and
$$\begin{tikzcd} \bullet^{0} & \bullet^{0} \\ \circ^{\top} & \circ^{\top} \arrow[l, bend left, "\top"] \end{tikzcd}$$
. By uniqueness, $\top = 0$. (2) $\Rightarrow$ (5): Let A be any idempotent of M, i.e., an object of $\mathcal{DR}(M)$. There is a unique morphism from A to 0. Here are two:
$$\begin{tikzcd} \bullet^{A} \arrow[r, bend left, "A"] & \bullet^{A} \\ \circ^{0} & \circ^{0} \end{tikzcd}$$
and
$$\begin{tikzcd} \bullet^{A} & \bullet^{A} \\ \circ^{0} & \circ^{0} \end{tikzcd}$$
. So $A = 0$. For each $u \in M$, take $A = \langle u \rangle$, then $u = \langle u \rangle\, u = 0 \cdot u = 0$. □

Additive units are usually not considered in most models, including game models, of linear logic. However, there is no additive conjunction $\&$ (product $\times$) or disjunction $\oplus$ (coproduct $+$) either:

Proposition 6.2. *Let A and B be any two objects of $\mathcal{DR}(M)$. The following conditions are equivalent: (1) $A \times B$ exists; (2) $A + B$ exists; (3) $M = \{0\}$.*

Proof. Write π_1 and π_2 for the two projections from $A \times B$, and $\langle f_1, f_2 \rangle : C \to A \times B$ the pairing of $f_1 : C \to A$ and $f_2 : C \to B$. We first show that (1) implies (3). Note that the projections are epi, so we may write $\pi_1 : A \times B \to C$ as
$$\begin{tikzcd} \bullet^{A\times B} \arrow[r, bend left, "\beta_1"] \arrow[d, "a_1"'] & \bullet^{A\times B} \\ \circ^{A} & \circ^{A} \arrow[u, "a_1^*"'] \end{tikzcd}$$
, and $\pi_2 : A \times B \to B$ as
$$\begin{tikzcd} \bullet^{A\times B} \arrow[r, bend left, "\beta_2"] \arrow[d, "a_2"'] & \bullet^{A\times B} \\ \circ^{B} & \circ^{B} \arrow[u, "a_2^*"'] \end{tikzcd}$$
. Consider $f_1 =$
$$\begin{tikzcd} \bullet^{C} \arrow[r, bend left, "\beta_1'"] & \bullet^{C} \\ \circ^{A} & \circ^{A} \end{tikzcd}$$
, $f_2 =$
$$\begin{tikzcd} \bullet^{C} \arrow[r, bend left, "\beta_2'"] & \bullet^{C} \\ \circ^{B} & \circ^{B} \end{tikzcd}$$
. Let us

look for morphisms f such that $\pi_1 \circ f = f_1$ and $\pi_2 \circ f = f_2$. Let f be $\begin{tikzcd} \bullet^{C} \arrow[r, bend left, "\beta"] \arrow[d, "a"'] & \bullet^{C} \\ \circ^{A\times B} & \circ^{A\times B} \arrow[l, bend left, "\gamma"] \arrow[u, "a^*"'] \end{tikzcd}$,

then we try to satisfy

$$\begin{tikzcd} \bullet^{C} \arrow[r, bend left, "\beta"] \arrow[d, "a"'] & \bullet^{C} \\ A\times B \arrow[r, bend left, "\beta_1"] \arrow[d, "a_1"'] & A\times B \arrow[l, bend left, "\gamma"] \arrow[u, "a^*"'] \\ \circ^{A} & \circ^{A} \arrow[u, "a_1^*"'] \end{tikzcd} = \begin{tikzcd} \bullet^{C} \arrow[r, bend left, "\beta'_1"] & \bullet^{C} \\ \circ^{A} & \circ^{A} \end{tikzcd} \qquad \begin{tikzcd} \bullet^{C} \arrow[r, bend left, "\beta"] \arrow[d, "a"'] & \bullet^{C} \\ A\times B \arrow[r, bend left, "\beta_2"] \arrow[d, "a_2"'] & A\times B \arrow[l, bend left, "\gamma"] \arrow[u, "a^*"'] \\ \circ^{B} & \circ^{B} \arrow[u, "a_2^*"'] \end{tikzcd} = \begin{tikzcd} \bullet^{C} \arrow[r, bend left, "\beta'_2"] & \bullet^{C} \\ \circ^{B} & \circ^{B} \end{tikzcd} \tag{1}$$

Recall that, since we assume $A \times B$ is a product, these equations should have a unique solution in a, β, γ, whatever β'_1 and β'_2, and also whatever C.

When $\beta'_1 = \beta'_2 = C$, we may take $\beta = C$, $a = 0$, and we show that even then, the solutions are not unique unless the only idempotent $D \leq A \times B$ such that $D\langle a_1{}^*\rangle = 0$ and $D\langle a_2{}^*\rangle = 0$ is 0. Taking $\beta = C$ and $a = 0$ allows us to reduce equations (1) to finding γ such that: $\begin{tikzcd} A\times B \arrow[r, bend left, "\beta_i"] \arrow[d, "a_i"'] & A\times B \arrow[l, bend left, "\gamma"] \\ \circ^{B} & \circ^{B} \arrow[u, "a_i^*"'] \end{tikzcd} = 0$ for i equal to 1 and to 2. Taking $\gamma = 0$ is one solution. In general, taking γ as being any idempotent $D \leq A \times B$ such that $D\langle a_i{}^*\rangle = 0$ $(i = 1, 2)$ gives us a solution. Indeed, recall that $a_i\beta_i = 0$ and $\beta_i^* = \beta_i$ (Conditions **b** and **c**), so $\beta_i a_i^* = (a_i\beta_i^*)^* = (a_i\beta_i)^* = 0$; since $D \leq A \times B$, it follows that $\beta_i D a_i^* \leq \beta_i(A \times B)a_i^* = \beta_i a_i^* = 0$; so the left-hand side of the equation above is $\sum_{i\in\mathbb{N}} a_i(D\beta_i)^n D a_i^* = a_i D a_i^* + \sum_{i\in\mathbb{N}} a_i(D\beta_i)^n D\beta_i D a_i^* = a_i D a_i^*$. Now use the fact that $D\langle a_i{}^*\rangle = 0$: $a_i D a_i^* = a_i D\langle a_i{}^*\rangle a_i^* = 0$. Since the solutions to (1) must be unique, it must be the case that the only idempotent $D \leq A \times B$ such that $D\langle a_1{}^*\rangle = 0$ and $D\langle a_2{}^*\rangle = 0$ is 0.

It follows that: **(a)** $\langle\beta_1{}^*\rangle\langle\beta_2{}^*\rangle = 0$. Indeed, recall that $\beta_i a_i^* = 0$, so $\langle\beta_i{}^*\rangle\langle a_i{}^*\rangle = 0$. Let $D = \langle\beta_1{}^*\rangle\langle\beta_2{}^*\rangle$. Then $D(\langle a_1{}^*\rangle + \langle a_2{}^*\rangle) = \langle\beta_2{}^*\rangle\langle\beta_1{}^*\rangle\langle a_1{}^*\rangle + \langle\beta_1{}^*\rangle\langle\beta_2{}^*\rangle\langle a_2{}^*\rangle = 0$, so $D\langle a_i{}^*\rangle \leq D(\langle a_1{}^*\rangle + \langle a_2{}^*\rangle) = 0$ for each i, so $D = 0$ by the above.

We claim that this entails: **(b)** for every self-inverse element β' (i.e., with $\beta'^* = \beta'$) such that $\langle\beta'\rangle \leq C$, then $\beta' \leq C$. Indeed, fix any two self-inverse elements β'_1 and β'_2 with $\left\langle\beta'_1\right\rangle \leq C$, $\left\langle\beta'_2\right\rangle \leq C$, and consider any solution of equations (1). By **(a)** $\beta_1\beta_2^* = \beta_1\langle\beta_1{}^*\rangle\langle\beta_2{}^*\rangle\beta_2^* = 0$, hence by Condition **b**, $\beta_1^*\beta_2 = 0$. So the automaton shown next is bideterministic. This implies that

$$\begin{tikzcd} \bullet^{C} \arrow[r, bend left, "\beta"] \arrow[d, "a"'] & \bullet^{C} \\ A\times B \arrow[r, bend left=60, "\beta_1"] \arrow[r, bend left=15, "\beta_2"] & A\times B \arrow[l, bend left, "\gamma"] \arrow[u, "a^*"'] \end{tikzcd}$$

the two elements $\begin{tikzcd} \bullet^C \arrow[r, bend left, "\beta"] \arrow[d, "a"'] & \bullet^C \\ A\times B \arrow[r, bend left, "\beta_1"] & A\times B \arrow[u, "a^*"'] \arrow[l, bend left, "\gamma"] \end{tikzcd}$ and $\begin{tikzcd} \bullet^C \arrow[r, bend left, "\beta"] \arrow[d, "a"'] & \bullet^C \\ A\times B \arrow[r, bend left, "\beta_2"] & A\times B \arrow[u, "a^*"'] \arrow[l, bend left, "\gamma"] \end{tikzcd}$ are coherent, since they are both less than or equal to the latter in the natural ordering $\leq$. But these are precisely β'_1 and β'_2. We have shown that any two self-inverse elements β'_1 and β'_2 such that $\left\langle\beta'_1\right\rangle \leq C$, $\left\langle\beta'_2\right\rangle \leq C$, are coherent. Taking $\beta'_2 = C$ itself, and $\beta'_1 = \beta'$, we obtain that any self-inverse element β' such that $\langle\beta'\rangle \leq C$ is such that $\beta' \smile C$. By definition of $\smile_0$, $\beta' \langle C^*\rangle = C \langle\beta'^*\rangle$, i.e., $\beta' C = C \langle\beta'^*\rangle$. Since $\langle\beta'\rangle \leq C$, $\beta' = \beta' \langle\beta'^*\rangle = \beta' \langle\beta'\rangle \leq \beta' C = C \langle\beta'^*\rangle \leq C$.

Next, **(b)** entails that every self-inverse element u is idempotent: for any self-inverse element u, take $\beta' = u$, $C = \langle u\rangle$ (recall that C was arbitrary, too), so that $u \leq \langle u\rangle$ by **(b)**. Since $v \leq w$ iff $\langle v\rangle w = v$, we obtain $\langle u\rangle \langle u\rangle = u$, hence u is idempotent. This allows us to simplify the equations (1) considerably. Indeed, β_1, β_2, β'_1, β'_2, β, γ must now all be idempotent. In particular the loops $\gamma\beta_1$ and $\gamma\beta_2$ are idempotent, so $(\gamma\beta_1)^n, (\gamma\beta_2)^n \leq A \times B$ for any $n \in \mathbb{N}$. No turn through these loops then counts in the corresponding sums. E.g., the left equation of (1) states that $\sum_{n\in\mathbb{N}} a_1(\gamma\beta_1)^n a = 0$ (left arrow, top left to bottom left). All terms of the sum are less than or equal to the first, $a_1 a$, so $\sum_{n\in\mathbb{N}} a_1(\gamma\beta_1)^n a = 0$ iff $a_1 a = 0$. The argument is similar for all other equalities in (1). So they simplify to: $\beta'_i = \beta + a^*\beta_i a$, $a_i\gamma a_i^* = 0$, and $a_i a = 0$, for $i = 1, 2$.

When $\beta'_1 = \beta'_2 = 0$, any triple $\beta = 0$, $a = 0$, and γ (with γ idempotent) such that $\gamma \leq A \times B$ and $\gamma \langle a_1{}^*\rangle = \gamma \langle a_2{}^*\rangle = 0$ is a solution. Since solutions are unique: **(c)** the only idempotent D with $D \langle a_1{}^*\rangle = D \langle a_2{}^*\rangle = 0$ is 0.

Now fix arbitrary values for β'_1 and β'_2, and take any solution β, a, γ. Let $D = \langle a\rangle \beta_1$, a product of two idempotents, hence an idempotent. We have $D \langle a_1{}^*\rangle = 0$ since $\beta_1 \langle a_1{}^*\rangle = \beta_1 a_1^* a_1 = 0$, by Condition **b**. We also have $D \langle a_2{}^*\rangle = 0$, since $D \langle a_2{}^*\rangle = \langle a\rangle \langle a_2{}^*\rangle \beta_1$ (β_1 is idempotent, and idempotents commute) $= aa^* a_2^* a_2 \beta_1 = 0$. Indeed, $a_2 a = 0$ since β, a, γ is a solution. By **(c)** it follows that $D = 0$, i.e., $\langle a\rangle \beta_1 = 0$. So $a^*\beta_1 a = a^* \langle a\rangle \beta_1 a = 0$. Since β, a, γ is a solution, $\beta'_1 = \beta + a^*\beta_1 a = \beta$. Hence necessarily $\beta = \beta'_1$. In a symmetric way, $\beta = \beta'_2$, so $\beta'_1 = \beta'_2$. Now β'_1 and β'_2 were arbitrary idempotents less than or equal to C. Take $\beta'_1 = 0$, $\beta'_2 = C$, then $C = 0$. Since C is arbitrary, every idempotent is 0. We have already noticed that this entailed $M = \{0\}$ in Proposition 6.1. We conclude, since 1 and 2 are equivalent by duality, and 3 clearly implies both. □

This is pretty definitive: if $M \neq \{0\}$, there is no product, and no coproduct in $\mathcal{DR}(M)$ at all, whatever the constructs (g, d, etc.) we may invent in M.

7. $\mathcal{DR}(M)$ Has No Exponential

Additives are not a great loss. To interpret the λ-calculus, we only need to interpret MELL, the multiplicative-exponential fragment of linear logic.

The key to our next impossibility result is the notion of (co)commutative comonoid in a symmetric monoidal category $\mathcal{C}$. The central role of such objects is made explicit in Melliès [27]. A *comonoid* in $\mathcal{C}$ is any triple (A, d_A, e_A) where A is an object, $d_A : A \to A \otimes A$ (*comultiplication*) and $e_A : A \to I$ (*counit*) are morphisms in $\mathcal{C}$ satisfying: (coassociativity) $\alpha_{A,A,A} \circ (d_A \otimes \mathrm{id}_A) \circ d_A = (\mathrm{id}_A \otimes d_A) \circ d_A : A \to A \otimes (A \otimes A)$, where $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is associativity; (left counit) $(e_A \otimes \mathrm{id}_A) \circ d_A : A \to I \otimes A$ is the obvious iso; and (right counit) $(\mathrm{id}_A \otimes e_A) \circ d_A : A \to A \otimes I$ is the obvious iso again. It is *cocommutative* iff $c_{A,A} \circ d_A = d_A : A \to A \otimes A$, where $c_{A,B} : A \otimes B \to B \otimes A$ is commutativity. Note that (cocommutative) comonoids in $\mathcal{S}et^{op}$, the opposite category of $\mathcal{S}et$, are exactly the (commutative) monoids.

It is well-known that there is a category $co\mathcal{M}on(\mathcal{C})$ of cocommutative comonoids, whose morphisms $f : (A, d_A, e_A) \to (B, d_B, e_B)$ are morphisms $f : A \to B$ in $\mathcal{C}$ that preserve comultiplication $d_.$ and counits $e_.$. Moreover, $co\mathcal{M}on(\mathcal{C})$ always has all finite products [27].

A particularly nice notion of model of (classical) linear logic developed by Melliès [27], that of *new-Lafont category*, is defined as a ($*$-)autonomous category $\mathcal{C}$, with a full sub-monoidal category $\mathcal{M}$ of $co\mathcal{M}on(\mathcal{C})$, such that the obvious forgetful functor $U : \mathcal{M} \to \mathcal{C}$ has a right adjoint $F : \mathcal{C} \to \mathcal{M}$. We now show that $\mathcal{DR}(M)$ is never a new-Lafont category, unless M is trivial. To this end, we characterize comonoids in $\mathcal{DR}(M)$. Say that two idempotents A_p and A_q of M form a *partition* of A iff $A_p + A_q = A$ and $A_p A_q = 0$.

Theorem 7.1. *Let M be weakly Cantorian, (A, d_A, e_A) be a triple verifying the left and right counit laws (e.g., a comonoid of $\mathcal{DR}(M)$). Then there is a partition A_p, A_q of A such that*

$$d_A = \begin{tikzcd} \bullet^{A} \arrow[d, "pA_p+qA_q"'] & {}^{A}\bullet \\ \circ^{A\otimes A} & {}^{A\otimes A}\circ \arrow[l, bend left, "q\beta_0 p^*+p\beta_0 q^*"] \arrow[u, "A_pp^*+A_qq^*"'] \end{tikzcd} \quad \text{and } e_A = \begin{tikzcd} \bullet^{A} \arrow[r, bend left, "\beta_0+\beta_0^*"] & {}^{A}\bullet \\ \circ^{0} & {}^{0}\circ \end{tikzcd}$$

where β_0 is an iso between A_p and A_q, i.e., $\langle {\beta_0}^* \rangle = A_p$ and $\langle \beta_0 \rangle = A_q$. Conversely, if d_A and e_A are as above, then (A, d_A, e_A) is a comonoid in $\mathcal{DR}(M)$.

Proof. Let (A, d_A, e_A) be a comonoid in $\mathcal{DR}(M)$. The left counit law is $\ell_A \circ (e_A \otimes \mathrm{id}_A) \circ d_A = \mathrm{id}_A$, where $\ell_A : I \otimes A \to A$ is the left unit of $\otimes$. This entails that d_A is mono, hence of the form

$$\begin{tikzcd} \bullet^{A} \arrow[d, "a"'] & {}^{A}\bullet \\ \circ^{A\otimes A} & {}^{A\otimes A}\circ \arrow[l, bend left, "\gamma"] \arrow[u, "a^*"'] \end{tikzcd}$$

with $\langle a^* \rangle = A$. Since $e_A : A \to I$ and $I = 0$, e_A is of the form

$$\begin{tikzcd} \bullet^{A} \arrow[r, bend left, "\beta"] & {}^{A}\bullet \\ \circ^{0} & {}^{0}\circ \end{tikzcd}$$

. Now

$$\ell_A = \begin{tikzcd} \bullet^{0\otimes A} \arrow[d, "Aq^*"'] & {}^{0\otimes A}\bullet \\ \circ^{A} & {}^{A}\circ \arrow[u, "qA"'] \end{tikzcd}$$

, so the left counit law simplifies to

the equation below left.

$$
\begin{tikzcd}[row sep=small]
\bullet^A \arrow[d, "a"'] & {}^A\bullet \\
A\otimes A \arrow[r, bend left, "p\beta p^*"] \arrow[d, "Aq^*"'] & A\otimes A \arrow[l, bend left, "\gamma"] \arrow[u, "a^*"'] \\
\circ^A & {}^A\circ \arrow[u, "qA"']
\end{tikzcd}
=
\begin{tikzcd}
\bullet^A \arrow[d, "A"'] & \bullet^A \\
\circ^A & \circ^A \arrow[u, "A"']
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small]
\bullet^A \arrow[d, "a"'] & {}^A\bullet \\
A\otimes A \arrow[r, bend left, "q\beta q^*"] \arrow[d, "Ap^*"'] & A\otimes A \arrow[l, bend left, "\gamma"] \arrow[u, "a^*"'] \\
\circ^A & {}^A\circ \arrow[u, "pA"']
\end{tikzcd}
=
\begin{tikzcd}
\bullet^A \arrow[d, "A"'] & \bullet^A \\
\circ^A & \circ^A \arrow[u, "A"']
\end{tikzcd}
\tag{2}
$$

Similarly, the right counit law yields the equation above, right.

Before we start, let us explain how the proof works. To this end, assume M is of the form $\mathsf{PI}(E)$ for some set E. (By the Preston-Wagner Theorem, this would be enough to establish all equations. Unfortunately, the i_M embedding of M into $\mathsf{PI}(M)$ used in the Preston-Wagner Theorem does not preserve 0, which invalidates this approach.) Recall that any element of E is a *token*, and that a token n at B *travels* to $a(n)$ at C along a transition $B \xrightarrow{a} C$ iff n is in the domain of a. Otherwise we say that n is *thrown away* by the transition. We explain this along with the formal proof, using square brackets [...].

Let $A_q = a^*qAq^*a$, $A_p = a^*pAp^*a$. [Look at the top left a transition going downwards in (2, left). The target $A \otimes A$ is the disjoint sum of pAp^* and qAq^*; A_q is the set of tokens n that travel along a to the right summand qAq^*, A_p is the set of tokens n that travel along a to the left summand pAp^*. So A_p and A_q are disjoint, that is, $A_pA_q = 0$. By (2, left), every token n at the top left A of the right-hand side of the equation travels to itself at the bottom left A of the right-hand side, so the same happens on the left-hand side of the equation. In particular, no token at the top left A is thrown away by the a transition, so $A = A_p + A_q$.] A_q is idempotent, since $A_q = \langle a^*qA\rangle$; similarly, A_p is idempotent since $A_p = \langle a^*pA\rangle$. Then $A_pA_q = a^*pAp^*aa^*qAq^*a \le a^*pAp^*qAq^*a = 0$ (because $aa^* \le 1$), so $A_pA_q = 0$. And $A_p + A_q = a^*(pAp^* + qAq^*)a = a^*(A\otimes A)a = a^*a$ (since $(A\otimes A)a = a) = \langle a^*\rangle = A$. So A_p, A_q is a partition of A.

Let β_0 be βA_p. We have to show: **(a)** $a = pA_p + qA_q$, **(b)** $\beta = \beta_0 + \beta_0^*$, **(c)** $\langle {\beta_0}^*\rangle = A_p$, **(d)** $\langle\beta_0\rangle = A_q$, and **(e)** $\gamma = q\beta_0^*p^* + p\beta_0q^*$.

[We are starting to show that **(a)** $a = pA_p + qA_q$. Consider a token n from the top left A. If n is in A_q, it will travel to $a(n)$ in the right summand qAq^* of $A\otimes A$. This is thrown away by the $p\beta p^*$ transition. Since it must eventually travel along some transition to exit as n at the bottom left A—because this is what it does on the right-hand side of the equation, $a(n)$ must travel along the Aq^* transition, and $Aq^*(a(n)) = n$, so $a(n) = q(n)$. Similarly, if n is in A_p, $a(n) = p(n)$, using (2, right) instead. This describes a as the function mapping every $n \in A_p$ to $p(n)$ and every $n \in A_q$ to $q(n)$, i.e., as $pA_p + qA_q$.]

Consider the path from the top left A to the bottom left A on either side of (2, left): since they are equal, $\sum_{n\in\mathbb{N}} Aq^*(\gamma p\beta p^*)^n a = A$. Multiply by $A_q = a^*qAq^*a$ on the right, then $\sum_{n\in\mathbb{N}} Aq^*(\gamma p\beta p^*)^n \langle a\rangle\, qAq^*a = AA_q$. Since $A_p + A_q = A$, $A_q \le A$, so $AA_q = A_q$. Also, the terms $Aq^*(\gamma p\beta p^*)^n \langle a\rangle\, qAq^*a$ with $n \ge 1$ are zero, since they are less than or equal to $Aq^*(\gamma p\beta p^*)^n qAq^*a = Aq^*(\gamma p\beta p^*)^{n-1}\gamma p\beta p^*qAq^*a = 0$. So only the term with $n = 0$ remains, and the equation simplifies to $Aq^*\langle a\rangle\, qAq^*a = A_q$, i.e., *(i)* $Aq^*aA_q = A_q$. The similar path in (2, right) multiplied by A_p yields *(ii)* $Ap^*aA_p = A_p$. By multiplying *(i)* by a^*q on the left, $a^*qAq^*aA_q = a^*qA_q$, i.e., *(iii)* $A_q = a^*qA_q$. Similarly, *(iv)*

$A_p = a^* p A_p$. Summing *(iii)*, *(iv)*, $A_p + A_q = a^*(pA_p + qA_q)$. Since $A_p + A_q = A$ and $aA = a$, we obtain $a = aA = aa^*(pA_p + qA_q) \leq pA_p + qA_q$.

Conversely, using *(i)* and *(ii)*, $pA_p + qA_q = pAp^* aA_p + qAq^* aA_q \leq pAp^* aA + qAq^* aA$ (since $A_p, A_q \leq A$) $= pAp^* a + qAq^* a$ (since $aA = a$) $= (pAp^* + qAq^*)a = (A \otimes A)a = a$. Together with $a \leq pA_p + qA_q$, we obtain **(a)**.

[If $n \in A_p$ travels from the top left A of (2, left), it must go through the a transition to $a(n)$ in the left summand pAp^* of $A \otimes A$. Since $a = pA_p + qA_q$, $a(n) = p(n)$. Now $a(n) = p(n)$ cannot travel down along the Aq^* transition, so it must go through $p\beta p^*$ to $p(\beta(n))$. Then $p(\beta(n))$ cannot travel up along the a^* transition, otherwise n would have traveled to $a^*(p(\beta(n)))$ from the top left A to the top right A on the right-hand side of (2, left), too. The domain of a^* is $pA_p p^* + qA_q q^* = A_p \otimes A_q$; since $p(\beta(n))$ is not in this domain, $\beta(n)$ is not in A_p, therefore $\beta(n)$ is in A_q. Since n is an arbitrary element of A_p, β maps A_p to A_q. Moreover, since every token at the top left A eventually reaches the bottom left A, no $n \in A_p$ is thrown away by β. Recall that $\beta_0 = \beta A_p$, the restriction of β to A_p. We have just shown that β_0 was total, i.e., the domain of β_0 is A_p. This is **(c)**. Similar reasoning on (2, right) shows that the restriction of β to A_q is total, too. Since $\beta^* = \beta$, β is an involution, so the restriction of β to A_q is necessarily β_0^*. The equations **(b)** and **(d)** follow readily.]

Look again at the path from the top left A to the bottom left A on either side of (2, left): $A = \sum_{n \in \mathbb{N}} Aq^*(\gamma p \beta p^*)^n a$. By **(a)**, $A = \sum_{n \in \mathbb{N}} Aq^*(\gamma p \beta p^*)^n (pA_p + qA_q) = A_q + \sum_{n \geq 1} Aq^*(\gamma p \beta p^*)^{n-1} \gamma p \beta A_p$. This time, multiply by A_p on the right. Since $A_q A_p = 0$ and $AA_p = A_p$, and since $\beta_0 = \beta A_p$, *(v)* $A_p = \sum_{n \geq 1} Aq^*(\gamma p \beta p^*)^{n-1} \gamma p \beta_0$. Multiplying by $\langle \beta_0{}^* \rangle$ on the right, $A_p \langle \beta_0{}^* \rangle = \sum_{n \geq 1} Aq^*(\gamma p \beta p^*)^{n-1} \gamma p \beta_0 \langle \beta_0{}^* \rangle = \sum_{n \geq 1} Aq^*(\gamma p \beta p^*)^{n-1} \gamma p \beta_0$ (as $u \langle u^* \rangle = u$ for every u) $= A_p$. We have just shown $A_p \langle \beta_0{}^* \rangle = A_p$, so $A_p^* \langle \beta_0{}^* \rangle = A_p^* A_p$, since $A_p^* A_p = A_p$. Recall that $u \leq v$ iff $u^* v = u^* u$. So $A_p \leq \langle \beta_0{}^* \rangle$. On the other hand, $\langle \beta_0{}^* \rangle = A_p \beta^* \beta A_p \leq A_p$ since $\beta^* \beta \leq 1$. So **(c)** $\langle \beta_0{}^* \rangle = A_p$.

Similarly to *(v)*, using (2, right), and multiplying by A_q, we get *(vi)* $A_q = \sum_{n \geq 1} Ap^* (\gamma q \beta q^*)^{n-1} \gamma q \beta A_q$. Let us now look at the path from the top left A to the top right A on either side of (2, left): $\sum_{n \in \mathbb{N}} a^*(p \beta p^* \gamma)^n p \beta p^* a = 0$. By **(a)** and simplifying, *(vii)* $\sum_{n \in \mathbb{N}} A_p p^*(p \beta p^* \gamma)^n p \beta A_p = 0$. The $n = 0$ term must then cancel, too, so $A_p \beta A_p = 0$. Since $\beta = A\beta$ and $A = A_p + A_q$, $\beta_0 = A\beta A_p = (A_p + A_q)\beta A_p = A_q \beta A_p$. Similarly, using (2, right), $A_q \beta A_q = 0$, so $\beta_0 = A_q \beta A_p = A_q \beta A_p + A_q \beta A_q = A_q \beta (A_p + A_q) = A_q \beta$. Taking converses, and since $\beta^* = \beta$, *(viii)* $\beta_0^* = \beta A_q$.

Let $u_1 = \beta_0$, $u_2 = \beta_0^*$. Using the definition of β_0 for u_1, and property *(viii)* for u_2, we obtain $u_1 u_2^* = \beta A_p A_q \beta^* = 0$ (forward determinacy) since $A_p A_q = 0$. Also, $u_1^* u_2 = u_2 u_1^* = \beta A_q A_p \beta^* = 0$ (backward determinacy). We have seen that the bideterminacy condition $u_1 u_2^* = u_2 u_1^* = 0$ implied $u_1 \smile u_2$, that is, $\beta_0 \smile \beta_0^*$. So it makes sense to consider $\beta_0 + \beta_0^*$. Since $\beta_0 = \beta A_p$ by definition and $\beta_0^* = \beta A_q$ by *(viii)*, $\beta_0 + \beta_0^* = \beta(A_p + A_q) = \beta A = \beta$, whence **(b)**.

By *(vi)* and *(viii)*, $A_q = \sum_{n \geq 1} Ap^*(\gamma q \beta q^*)^{n-1} \gamma q \beta_0^*$. Multiplying by $\langle \beta_0 \rangle$ on the right, $A_q \langle \beta_0 \rangle = \sum_{n \geq 1} Ap^*(\gamma q \beta q^*)^{n-1} \gamma q \beta_0^* \langle \beta_0 \rangle = \sum_{n \geq 1} Ap^*(\gamma q \beta q^*)^{n-1} \gamma q \beta_0^* = A_q$, so $A_q \leq \langle \beta_0 \rangle$. Since $\langle \beta_0 \rangle = A_q \beta^* \beta A_q$ by *(viii)*, $\langle \beta_0 \rangle \leq A_q$, so **(d)** $\langle \beta_0 \rangle = A_q$.

[Consider again an arbitrary token $n \in A_p$ traveling from the top left A of (2, left). It travels down along a to $A \otimes A$ as $p(n)$, then rightwards along $p \beta p^*$ to $p(\beta_0(n))$. Since the range of β_0 is A_q and n is arbitrary in A_p, $p(\beta_0(n))$ is arbitrary in $pA_q p^*$. Since every such n eventually exits at the bottom left A, $p(\beta_0(n))$ cannot be thrown away by γ, so the domain of γ contains pA_qp^*. Similarly, using (2, right), the domain of γ also contains qA_pq^*. No element in the domain of γ can be in pA_pp^* or in qA_qq^*, otherwise it would also be in the domain of a^*, and would travel up at this point. So the domain of γ is exactly $pA_qp^* + qA_pq^* = A_q \otimes A_p$. Since $\gamma^* = \gamma$, $pA_qp^* + qA_pq^*$ is also the range of γ.]

[As we have seen above, $p(\beta_0(n))$ (at the rightmost $A \otimes A$ of the left-hand side) cannot travel up along a^*, so it must travel leftwards along γ to $\gamma(p(\beta_0(n)))$. If $\gamma(p(\beta_0(n)))$ was in pA_qp^*, i.e., if it was of the form $p(m)$ with $m \in A_q$, then it would travel again rightwards along $p\beta p^*$, to $p(\beta_0^*(m)) \in pA_pp^*$, then upwards along a^* to $\beta_0^*(m)$, which is impossible. So $\gamma(p(\beta_0(n)))$ is in qA_pq^*, i.e., it is of the form $q(m)$ with $m \in A_p$, and exits as m at the bottom left A. But it can only exit as n, so $m = n$, and therefore $\gamma(p(\beta_0(n))) = q(n)$. Since β_0 is total from A_p to A_q, $p(\beta_0(n))$ is arbitrary in pA_qp^*, therefore γ maps every $m \in pA_qp^*$ to $q(\beta_0^*(p^*(m)))$. Since $\gamma = \gamma^*$, γ also maps every $m \in qA_pq^*$ to $p(\beta_0(q(m)))$. In short, $\gamma = q\beta_0^*p^* + p\beta_0 q$, i.e., **(e)** holds.]

By Condition **c** of the definition of morphisms in $\mathcal{DR}(M)$, $\gamma a = 0$, so by **(a)** $\gamma pA_p + \gamma qA_q = 0$, whence $\gamma pA_p = 0$ and $\gamma qA_q = 0$. Since $A = A_p + A_q$, we get $A \otimes A = pAp^* + qAq^* = pA_pp^* + pA_qp^* + qA_pq^* + qA_qq^*$. Since $\gamma = \gamma(A \otimes A)$, *(ix)* $\gamma = \gamma(pA_qp^* + qA_pq^*)$. Qua idempotent, $pA_qp^* + qA_pq^*$ is less than 1, so *(x)* $\langle \gamma^* \rangle \leq pA_qp^* + qA_pq^*$.

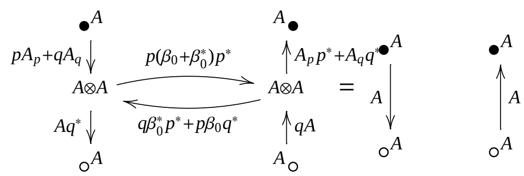
Note that, since the codomain $\langle \beta_0{}^* \rangle$ of β_0^* is A_p by **(c)**, $\left\langle \gamma p \beta_0^* \right\rangle = \gamma p \langle \beta_0{}^* \rangle p^* \gamma^* = \gamma pA_pp^*\gamma^* = \gamma(pA_qp^* + qA_pq^*)pA_pp^*\gamma^* = 0$, since $A_qA_p = 0$; so $\gamma p\beta_0^* = 0$. In particular, $\gamma p\beta p^* = \gamma p(\beta_0 + \beta_0^*)p^* = \gamma p\beta_0 p^*$, using **(b)**. The domain of $\gamma p\beta p^*$ is then $p\beta_0^*p^*\gamma^*\gamma p\beta_0 p^* \leq p\beta_0^*p^*(pA_qp^* + qA_pq^*)p\beta_0 p^*$ (by *(x)*) $= p\beta_0^*A_q\beta_0 p^* = p\beta_0^*\beta_0 p^*$ (by **(d)**) $= p \langle \beta_0{}^* \rangle p^*$; by **(c)**, it follows that *(xi)* $\langle (\gamma p\beta p^*)^* \rangle \leq pA_pp^*$. Similarly, $\gamma q\beta_0 = 0$, $\gamma q\beta q^* = \gamma q\beta_0^*q^*$, so *(xii)* $\langle (\gamma q\beta q^*)^* \rangle \leq qA_qq^*$.

Compute $(\gamma p\beta p^*)\gamma p$. First, $\langle ((\gamma p\beta p^*)\gamma p)^* \rangle = p^*\gamma^* \langle (\gamma p\beta p^*)^* \rangle \gamma p \leq p^*\gamma^* pA_pp^* \gamma p$ by *(xi)*, and $\gamma^* pA_p = \gamma pA_p$ since γ is self-inverse. But $\gamma pA_p = 0$, as we have noticed above (or using *(ix)*), so $\langle ((\gamma p\beta p^*)\gamma p)^* \rangle = 0$. So $(\gamma p\beta p^*)\gamma p = 0$. It follows that *(v)* $A_p = \sum_{n \geq 1} Aq^*(\gamma p\beta p^*)^{n-1}\gamma p\beta_0$ simplifies to *(v')* $A_p = Aq^*\gamma p\beta_0$, since all summands vanish except for $n = 1$. Similarly, using *(xii)* we obtain $(\gamma q\beta q^*)\gamma q = 0$, so *(vi)* $A_q = \sum_{n \geq 1} Ap^*(\gamma q\beta q^*)^{n-1}\gamma q\beta A_q$ simplifies to $A_q = Ap^*\gamma q\beta A_q = Ap^*\gamma q\beta_0^*$ (by *(viii)*); that is, *(vi')* $A_q = Ap^*\gamma q\beta_0^*$.

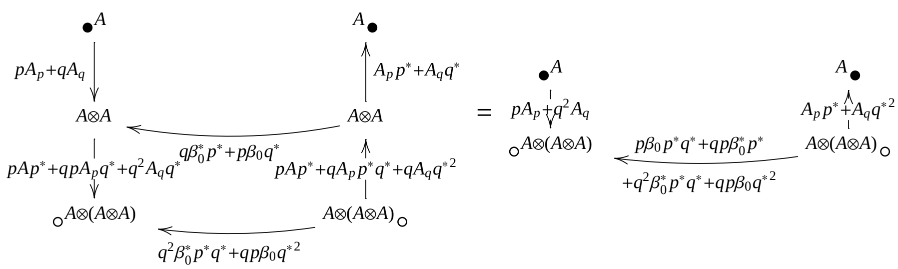
Multiply *(v')* by $\beta_0^*p^*$ on the right and qA_p on the left. Using **(d)**, $qA_p\beta_0^*p^* = qA_pq^*\gamma p \langle \beta_0 \rangle p^* = qA_pq^*\gamma pA_qp^*$. By **(c)**, $A_p\beta_0^* = \langle \beta_0{}^* \rangle \beta_0^* = \beta_0^*$, so *(v")* $q\beta_0^*p^* = qA_pq^*\gamma pA_qp^*$. Taking inverses, and since $\gamma^* = \gamma$, *(vi")* $p\beta_0 q^* = pA_qp^*\gamma qA_pq^*$.

Look at the $n = 1$ summand in *(vii)*: $A_p\beta p^*\gamma p\beta A_p = 0$. Since $\beta_0 = \beta A_p$, and $A_p\beta = A_p\beta^* = (\beta A_p)^* = \beta_0^*$, we obtain $\beta_0^*p^*\gamma p\beta_0 = 0$. Multiplying by $p\beta_0$ on the left, by $\beta_0^*p^*$ on the right, and using **(d)**, *(v''')* $pA_qp^*\gamma pA_qp^* = 0$. Similarly, looking at the $n = 1$ summand in the path from the top left A to the top right A in (2, right), we get *(vi''')* $qA_pq^*\gamma qA_pq^* = 0$. By *(ix)* and $\gamma^* = \gamma$, $\gamma = (pA_qp^* + qA_pq^*)\gamma$, so by *(ix)* again, $\gamma = (pA_qp^* + qA_pq^*)\gamma(pA_qp^* + qA_pq^*) = pA_qp^*\gamma pA_qp^* + pA_qp^*\gamma qA_pq^* + qA_pq^*\gamma pA_qp^* + qA_pq^*\gamma qA_pq^* = 0 + p\beta_0 q^* + q\beta_0^*p^* + 0$ by *(v''')*, *(vi")*, *(v")*, *(vi''')*: **(e)** obtains.

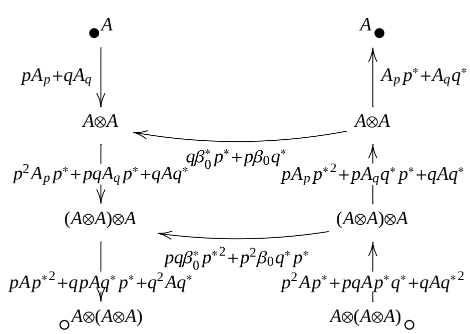
Conversely, assume that A_p, A_q form a partitition of A, and that **(a)–(e)** hold. The left counit law (2, left) is then shown on the right. It is straightforward to check this equality. In particular, the A down arrow is obtained from the left-hand side as one A_q going straight down, plus one A_p obtained by going once through the loop; no contribution arises from looping twice or more.



The verification of the right counit laws proceeds by similar means. It remains to establish coassociativity. On the one hand, $(\mathrm{id}_A \otimes d_A) \circ d_A$ equals:



where we have used $A_pA_q = 0$, $\beta_0 A_q = 0$, $\beta_0 A_p = \beta_0$, $A_p\beta_0 = 0$, $A_q\beta_0 = \beta_0$ several times to simplify the sums on arrows; while $\alpha_{A,A,A} \circ (d_A \otimes \mathrm{id}_A) \circ d_A$ equals the following morphism:

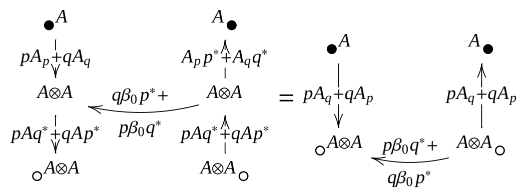


It is easy to check that this simplifies to the same value, using the same equations. $\square$

Curiously, note that coassociativity is for free once (A, d_A, e_A) obeys the left and the right counit laws. Cocommutativity is an entirely different matter:

Theorem 7.2. *Let M be weakly Cantorian. The only cocommutative comonoid in $\mathcal{DR}(M)$ is $(I, \ell_I^{-1}, \mathrm{id}_I)$, or explicitly $(0, d_0, e_0)$, where $d_0 : 0 \to 0 \otimes 0$ and $e_0 : 0 \to 0$ are the all zero morphisms.*

Proof. Let (A, d_A, e_A) be some cocommutative comonoid in $\mathcal{DR}(M)$, written as in Theorem 7.1. $c_{A,A} \circ d_A$ is given by the morphism shown on the right. While the bottom arrow always coincides with that of d_A, the vertical arrows only coincide provided that $pA_q + qA_p = pA_p + qA_q$. Multiply by A_p on the right: since $A_q A_p = 0$, $qA_p = pA_p$. Multiply by A_q on the right: $pA_q = qA_q$. So $qA = q(A_p + A_q) = qA_p + qA_q = pA_p + pA_q = p(A_p + A_q) = pA$. Multiply by p^* on the left: $p^* qA = 0$, while $p^* pA = A$, so $A = 0$. $\square$



Corollary 7.3 ($\mathcal{DR}(M)$ Is Not New-Lafont). *$\mathcal{DR}(M)$ is (the $\mathcal{C}$ component of) a new-Lafont category iff M is the trivial semigroup* $\{0\}$.

Proof. Assume $\mathcal{DR}(M)$ is new-Lafont. Since $U \dashv F$, for any object A of $\mathcal{DR}(M)$, there is a bijection between morphisms from $(0, d_0, e_0)$ to $F(A)$ in $\mathcal{M}$ and morphisms from $U(0, d_0, e_0) = 0$ to A in $\mathcal{DR}(M)$. But there is only one morphism of the first kind, namely $0 : (0, d_0, e_0) \to (0, d_0, e_0)$, by Theorem 7.2. So there is exactly one morphism from 0 to A in $\mathcal{DR}(M)$. As in Proposition 6.1, this implies $A = 0$ for every idempotent A, so M is trivial. $\square$

We won't recall the definitions of linear category [7] or that of an LNL category [6]. The deep connections between these and new-Lafont categories [27] then allow us to conclude that $\mathcal{DR}(M)$ is a linear category, resp. an LNL category, iff $M = \{0\}$ [18, Theorem 6.2.15, Theorem 6.2.16]. Again, this is definitive: if M is non-trivial, then $\mathcal{DR}(M)$ *cannot* be a categorical model of linear logic. This includes any attempt to invent boxes, dereliction, weakening and promotion constants in M. In particular, there is no way to turn the constructions of e.g. [11, 10, 25, 16] into categorical models of linear logic, of any kind.

8. Coherence Completions

However, we can build a category *on top* of $\mathcal{DR}(M)$, preserving the existing multiplicative structure $(\otimes, I)$, while adding all exponentials and additives.

Following Melliès [27, Definition 7], let a (classical) *Lafont category* be a ($*$-) autonomous category with finite products where for each object A there is a free cocommutative comonoid $(!A, \underline{d}_A, \underline{e}_A)$. Lafont categories are probably the strongest form of categorical model of linear logic: any Lafont category is new-Lafont, linear, and LNL in particular. An important example is the category $\mathcal{C}oh$ of coherence spaces. This has coherence spaces as objects (Section 2), and *linear maps* $f : X \to Y$ as morphisms. Letting $X = (|X|, \frown_X)$ and $Y = (|Y|, \frown_Y)$, this is a binary relation between the *webs* $|X|$ and $|Y|$, such that whenever $(x, y), (x', y') \in f$ and $x \frown_X x'$ then [$y \frown_Y y'$, and $y = y'$ implies $x = x'$]. (Brackets added for precision.) $\mathcal{C}oh$ is Lafont, provided we take the multiclique interpretation of $!A$, not the clique interpretation used e.g. in [17].

The main construction we use now is the *coherence completion* $\mathcal{COH}(\mathcal{C})$ of a $*$-autonomous category $\mathcal{C}$, due to Hu and Joyal [21]. Interestingly, this is the second place in this work where coherence plays a crucial role, after the definition of linear inverse semigroups. While the original notion of coherence completion only preserves existing exponentials, we show that a simple modification of the construction *creates* them. To obtain a comonad $(!, \epsilon, \delta)$ on $\mathcal{COH}(\mathcal{C})$ giving meaning to the exponential connectives, Hu and Joyal assume a comonad $(!_{\mathcal{C}}, \epsilon_{\mathcal{C}}, \delta_{\mathcal{C}})$ so that for each object A of $\mathcal{C}$, $!_{\mathcal{C}}A$ is a cocommutative comonoid and $!_{\mathcal{C}}(A \times B) = !_{\mathcal{C}}A \otimes !_{\mathcal{C}}B$. (Hu and Joyal assume finite products in $\mathcal{C}$ at this point.) If $\mathcal{C} = \mathcal{DR}(M)$, we will have none of that... in a very strong sense, as we have seen. Instead, we just take $!_{\mathcal{C}}$ to be the *identity* comonad.

Concrete Coherence Spaces. To alleviate a slight ambiguity in Hu and Joyal's original construction, we consider a subcategory $\mathcal{CCOH}$ of $\mathcal{COH}$, the full subcategory of so-called *concrete* coherence spaces. Let $\Sigma_{\mathcal{C}oh}$ be the signature $\{\mathbf{1}/0, \langle\_,\_\rangle/2, \mathbf{i}_1/1, \mathbf{i}_2/1, \mathbf{nil}/0, ::/2\}$. Write $\langle s, t\rangle$ for $\langle\_,\_\rangle$ applied to s and t, $s::t$ for $::$ applied to s and t, and $[s_1, s_2, \ldots, s_n]$ for $s_1::(s_2:: \ldots (s_n::\mathbf{nil}) \ldots)$. Fix, once and for all, a total ordering $\preceq$ on ground terms built on $\Sigma_{\mathcal{C}oh}$.

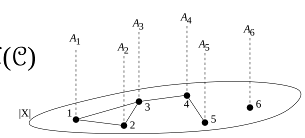
A *concrete coherence space* $X = (|X|, \frown_X)$ is any coherence space whose web $|X|$ is a set of ground terms over $\Sigma_{\mathcal{C}oh}$. Let $\mathcal{CCOH}$ be the full subcategory of $\mathcal{COH}$ consisting of concrete coherence spaces: $\mathcal{CCOH}$ is a Lafont category, and all structures are inherited from $\mathcal{COH}$ [18, Section 6.4.1]. Its only purpose is to endow $|X|$ with a canonical total ordering for all X, inherited from $\preceq$. This is needed to define ! unambiguously in $\mathcal{CCOH}(\mathcal{C})$ below.

Paraphrasing definitions for $\mathcal{COH}$, the morphisms from $X = (|X|, \frown_X)$ to $Y = (|Y|, \frown_Y)$ in $\mathcal{CCOH}$ are linear maps. Identity on X is $\{(x, x) \mid x \in |X|\}$, composition is relation composition: $g \circ f = \{(x, z) \mid \exists y \cdot (x, y) \in f, (y, z) \in g\}$. The tensor unit is $\mathbb{I} = (\{\mathbf{1}\}, \frown_{\mathbb{I}})$ where $\frown_{\mathbb{I}}$ relates $\mathbf{1}$ to itself. Tensor product of $X = (|X|, \frown_X)$ and $Y = (|Y|, \frown_Y)$ is $X \otimes Y = (|X \otimes Y|, \frown_{X \otimes Y})$, where $|X \otimes Y| = |X| \times |Y| = \{\langle i, j\rangle \mid i \in |X|, j \in |Y|\}$, and coherence on $X \otimes Y$ is given by $\langle i, j\rangle \frown_{X \otimes Y} \langle i', j'\rangle$ iff $i \frown_X i'$ and $j \frown_Y j'$. The associativity $\alpha_{X,Y,Z}$ is $\{(\langle\langle i, j\rangle, k\rangle, \langle i, \langle j, k\rangle\rangle) \mid i \in |X|, j \in |Y|, k \in |Z|\} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, the commutativity $c_{X,Y}$ is $\{(\langle i, j\rangle, \langle j, i\rangle \mid i \in |X|, j \in |Y|\} : X \otimes Y \to Y \otimes X$, the left neutral is $\ell_X = \{(\langle \mathbf{1}, i\rangle, i) \mid i \in |X|\} : \mathbb{I} \otimes X \to X$, and the right neutral is $r_X = \{(\langle i, \mathbf{1}\rangle, i) \mid i \in |X|\} : X \otimes \mathbb{I} \to X$. Given this symmetric monoidal structure, $\mathcal{CCOH}$ is autonomous: the linear function space $X \multimap Y$ of $X = (|X|, \frown_X)$ and $Y = (|Y|, \frown_Y)$ is given by $|X \multimap Y| = |X| \times |Y|$, and coherence on $X \multimap Y$ is given by $\langle i, j\rangle \frown_{X \multimap Y} \langle i', j'\rangle$ iff, when $i \frown_X i'$ then $[j \frown_Y j'$, and if $j = j'$ then $i = i']$. Next, $\mathcal{CCOH}$ is $*$-autonomous: the dual, a.k.a. the linear negation of $X = (|X|, \frown_X)$ is $X^\perp = (|X|, \asymp_X)$, where $i \asymp_X i'$ iff, when $i \frown_X i'$ then $i = i'$; equivalently, if $i \not\frown_X i'$ or $i = i'$. The natural transformation $C_X : \sim\sim X \to X$ (linear control operator), where $\sim X = X \multimap \perp \cong X^\perp$ is the linear trace $\{(\langle\langle i, \mathbf{1}\rangle, \mathbf{1}\rangle, i) \mid i \in |X|\}$. $\mathcal{CCOH}$ also has finite products and coproducts. The terminal object $\top$ is $(\emptyset, \frown_\top)$ where $\frown_\top$ is the empty relation. This is also the initial object. The binary product $X \times Y$ of $X = (|X|, \frown_X)$ and $Y = (|Y|, \frown_Y)$ is defined as $(|X \times Y|, \frown_{X \times Y})$, where $|X \times Y| = \{\mathbf{i}_1(i) \mid i \in |X|\} \cup \{\mathbf{i}_2(j) \mid j \in |Y|\}$, and coherence is given by: $\mathbf{i}_1(i) \frown_{X \times Y} \mathbf{i}_1(i')$ if and only if $i \frown_X i'$, $\mathbf{i}_2(j) \frown_{X \times Y} \mathbf{i}_2(j')$ iff $j \frown_Y j'$, and $\mathbf{i}_1(i) \frown_{X \times Y} \mathbf{i}_2(j)$ for every i, j. The first projection is $\pi_1 : X \times Y \to X = \{(\mathbf{i}_1(i), i) \mid i \in |X|\}$, the second projection is $\pi_2 : X \times Y \to Y = \{(\mathbf{i}_2(j), j) \mid j \in |Y|\}$, and pairing of $f : Z \to X$ and $g : Z \to Y$ is $\langle f, g\rangle : Z \to X \times Y = \{(k, \mathbf{i}_1(i)) \mid (k, i) \in f\} \cup \{(k, \mathbf{i}_2(j)) \mid (k, j) \in g\}$.

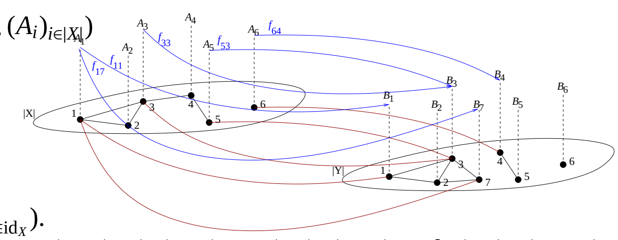
Binary coproducts are defined through duality.

There are several choices here for the ! functor, but only one that makes our version of concrete coherence completion work: take $|!X|$ to be the set of all multicliques of X, where a *multiclique* is a finite multiset of pairwise coherent elements. In concrete coherence spaces, multicliques will be encoded as sorted lists of pairwise coherent elements. So, let a *concrete multiclique* of $X = (|X|, \coh_X)$ be any sorted list $[i_1, i_2, \ldots, i_k]$, where by sorted we mean $i_1 \leq i_2 \leq \ldots \leq i_k$, and $i_1, i_2, \ldots, i_k$ form a clique in $|X|$. We shall abuse notation: if e is a concrete multiclique $[i_1, i_2, \ldots, i_k]$, we shall understand e ambiguously as the multiset $\{\!|i_1, i_2, \ldots, i_k|\!\}$; we retrieve the concrete multiclique from the multiset by sorting. In particular, multiset union $\uplus$ makes sense on concrete multicliques. The functor ! of $\mathcal{CCOH}$ maps $X = (|X|, \coh_X)$ to $!X = (|!X|, \coh_{!X})$, where $|!X|$ is the set of concrete multicliques of X, and $e \coh_{!X} e'$ iff $e \uplus e'$ is again a concrete multiclique. Given any morphism $f : X \to Y$, where $X = (|X|, \coh_X)$ and $Y = (|Y|, \coh_Y)$, and two multicliques $e = [i_1, i_2, \ldots, i_k]$ in X and $e' = [j_1, j_2, \ldots, j_k]$ in Y, of the same *length* k, let an f*-matching* of e with e' be any permutation π of $\{1, 2, \ldots, k\}$ such that $(i_\ell, j_{\pi(\ell)}) \in f$ for every ℓ, $1 \leq \ell \leq k$. Then $!f : !X \to !Y$ is defined by $!f = \{(e, e') \in |!X| \times |!Y| \mid$ there is an f-matching of e and $e'\}$. Such an f-matching π need not be unique, but the multiset $\{e\}f\{e'\}$ defined as $\{\!|(i_\ell, j_{\pi(\ell)}) \mid 1 \leq \ell \leq k|\!\}$ is independent of π, and only depends on f, e and e'. This is by linearity of f.

We get a comonad $(!, \delta, \epsilon)$ by letting $\delta_X : !X \to !!X$ be the linear trace $\{(e, \{\!|e_1, \ldots, e_n|\!\}) \mid e \in |!X|, e = e_1 \uplus \ldots \uplus e_n\}$, and $\epsilon_X : !X \to X$ be $\{(\{\!|x|\!\}, x) \mid x \in |X|\}$. And we get a cocommutative comonoid $(!X, \underline{d}_X, \underline{e}_X)$ by letting $\underline{d}_X : !X \to !X \otimes !X$ be the linear trace $\{(e_1 \uplus e_2, \langle e_1, e_2 \rangle) \mid e_1, e_2 \in |!X|, e_1 \coh_{!X} e_2\}$, and $\underline{e}_X : !X \to \mathbb{I}$ be the linear trace $\{(\{\!|\,|\!\}, \mathbf{1})\}$. $(!X, \underline{d}_X, \underline{e}_X)$ is the free cocommutative comonoid over X, as shown by van de Wiele [27]. That is, the functor U mapping each cocommutative comonoid (X, d, e) to X in $\mathcal{CCOH}$ has a right adjoint. The non-trivial part of the proof is in building the unit η of the adjunction: for any cocommutative comonoid (X, d_X, e_X) in $\mathcal{CCOH}$, η_X is the set of all pairs $(a, \{\!|a_1, \ldots, a_n|\!\}) \in |X| \times |!X|$, for every $n \in \mathbb{N}$ such that $(a, \langle a_1, \langle a_2, \ldots, \langle a_n, \mathbf{1} \rangle \ldots \rangle\rangle) \in d_X^n$, where *n-fold comultiplication* $d_X^n : X \to X \otimes (X \otimes \ldots \otimes (X \otimes \mathbb{I}) \ldots)$ is defined by: $d_X^0 = e_X$, $d_X^{n+1} = (\mathrm{id}_X \otimes d_X^n) \circ d_X$.



The Concrete Coherence Completion of $\mathcal{C}$. For any category $\mathcal{C}$, the *concrete coherence completion* $\mathcal{CCOH}(\mathcal{C})$ has as objects all pairs $(X, (A_i)_{i \in |X|})$ where the *base* X is a concrete coherence space $X = (|X|, \coh_X)$, and the *fiber* $(A_i)_{i \in |X|}$ is a family of objects of $\mathcal{C}$, indexed, by the web $|X|$. An object $(X, (A_i)_{i \in |X|})$ is conveniently seen as on the picture shown next.



The morphisms from $(X, (A_i)_{i \in |X|})$ to $(Y, (B_j)_{j \in |Y|})$ are all pairs $(f, (a_{ij})_{(i,j) \in f})$ where f is a linear map from X to Y, and a_{ij} is a morphism from A_i to B_j in $\mathcal{C}$. The identity on $(X, (A_i)_{i \in |X|})$ is $(\mathrm{id}_X, (\mathrm{id}_{A_i})_{(i,i) \in \mathrm{id}_X})$. The composition of $(f, (a_{ij})_{(i,j) \in f}) : (X, (A_i)_{i \in |X|}) \to (Y, (B_j)_{j \in |Y|})$ with $(g, (a_{jk})_{(j,k) \in g}) : (Y, (B_j)_{j \in |Y|}) \to (Z, (C_k)_{k \in |Z|})$ is $(g \circ f, (c_{ik})_{(i,k) \in g \circ f})$, where $c_{ik} = b_{jk} \circ a_{ij}$ with j such that

$$
\begin{array}{rcl}
(X, (A_i)_{i\in|X|}) \multimap (Y, (B_j)_{j\in|Y|}) & = & (X \multimap Y, (A_i \multimap B_j)_{\langle i,j\rangle\in|X\multimap Y|}) \\
\mathsf{app}_{(X,(A_i)_{i\in|X|}),(Y,(B_j)_{j\in|Y|})} & = & (\mathsf{app}_{X,Y}, (\mathsf{app}_{A_i,B_j})_{(\langle\langle i,j\rangle,i\rangle,j)} \in \mathsf{app}_{X,Y}) \\
\lambda^{(X,(A_i)_{i\in|X|})}_{(Y,(B_j)_{j\in|Y|}),(Z,(C_k)_{k\in|Z|})}(f, (a_{\langle i,j\rangle\ k})_{(\langle i,j\rangle,k)\in f}) & = & (\lambda^X_{Y,Z}(f), (\lambda^{A_i}_{B_j,C_k}(a_{\langle i,j\rangle\ k}))_{(i,\langle j,k\rangle)\in\lambda^X_{Y,Z}(f)}) \\
\bot = (\bot, (\bot)) & & C_{(X,(A_i)_{i\in|X|})} = (C_X, (C_{A_i})_{(\langle\langle i,\mathbf{1}\rangle,\mathbf{1}\rangle,i)\in C_X})
\end{array}
$$

Figure 3: $*$-Autonomous structure on $\mathcal{CCOH}(\mathcal{C})$

$(i, j) \in f$ and $(j, k) \in g$—j is unique by linearity of f and g, and this is the crucial point [21].

Then $\mathcal{CCOH}(\mathcal{C})$ is ($*$-)autonomous as soon as $\mathcal{C}$ is. In detail, tensor unit is $(\mathbb{I}, (I))$ (where (I) denotes the family of just one object, I) and tensor product $(X, (A_i)_{i\in|X|}) \otimes (Y, (B_j)_{j\in|Y|})$ defined as $(X \otimes Y, (A_i \otimes B_j)_{(i,j)\in|X\otimes Y|})$. Associativity, commutativity, neutrals are defined in the obvious way. The $*$-autonomous structure is given in Figure 3. Also, $\mathcal{CCOH}(\mathcal{C})$ has all finite products and coproducts, *whatever* $\mathcal{C}$ is. E.g., the binary product ("with") $(X, (A_i)_{i\in|X|}) \times (Y, (B_j)_{j\in|Y|})$ is $(X \times Y, (A_i)_{\mathbf{i}_1(i)\in|X\times Y|} \cup (B_j)_{\mathbf{i}_2(j)\in|X\times Y|})$, where $|X \times Y|$ is the disjoint sum of $|X|$ and $|Y|$. The first projection, from $(X, (A_i)_{i\in|X|}) \times (Y, (B_j)_{j\in|Y|})$ to $(X, (A_i)_{i\in|X|})$, is $(\pi_1, (\mathrm{id}_{A_i})_{(\mathbf{i}_1(i),i)\in\pi_1})$, where π_1 denotes first projection in $\mathcal{CCOH}$. Note that morphisms in the fibers are just identities id_{A_i}—this is why we don't need any structure from $\mathcal{C}$ for products to exist in $\mathcal{CCOH}(\mathcal{C})$, a fortunate state when $\mathcal{C} = \mathcal{DR}(M)$ indeed.

The ! comonad is slightly more complex. Recall that in $\mathcal{CCOH}$ all webs $|X|$ are totally ordered by $\preceq$, so we can represent any multiclique $e = \{i_1, \ldots, i_k\}$ as a sorted list $[i_1, \ldots, i_k]$. We may then define $\bigotimes_{i\in e} A_i$ as $A_{i_1} \otimes (A_{i_2} \otimes \ldots (A_{i_k} \otimes \mathbb{I}) \ldots)$. For any object $(X, (A_i)_{i\in|X|})$ of $\mathcal{CCOH}(\mathcal{C})$, let $!(X, (A_i)_{i\in|X|}) = \left(!X, \left(\bigotimes_{i\in e} A_i\right)_{e\in|!X|}\right)$. For any morphism $(f, (a_{ij})_{(i,j)\in f}) : (X, (A_i)_{i\in|X|}) \to (Y, (B_j)_{j\in|Y|})$, define $!(f, (a_{ij})_{(i,j)\in f}) :!(X, (A_i)_{i\in|X|}) \to !(Y, (B_j)_{j\in|Y|})$ as $\left(!f, \left(\bigotimes_{(i,j)\in\{e_1\}f\{e_2\}} a_{ij}\right)_{(e_1,e_2)\in!f}\right)$. Note that this again assumes no extra structure from $\mathcal{C}$, contrarily to [21]. Then ! is an endofunctor of $\mathcal{CCOH}(\mathcal{C})$. Turn it into a comonad by letting comultiplication $\delta_{(X,(A_i)_{i\in|X|})} :!(X, (A_i)_{i\in|X|}) \to !!(X, (A_i)_{i\in|X|})$ be $\left(\delta_X, (\cong_{e_1,\ldots,e_n})_{e\in|!X|, e=e_1\uplus\ldots\uplus e_n}\right)$, where $\cong_{e_1,\ldots,e_n}$ denotes the obvious natural iso from $\bigotimes_{i_1\in e_1} A_{i_1} \otimes \ldots \otimes \bigotimes_{i_n\in e_n} A_{i_n} \otimes \mathbb{I}$ to $\bigotimes_{i\in e_1\uplus\ldots\uplus e_n} A_i$, defined from associativity, commutativity and the neutrals of the tensor product $\otimes$; and counit $\epsilon_{(X,(A_i)_{i\in|X|})} :!(X, (A_i)_{i\in|X|}) \to (X, (A_i)_{i\in|X|})$ as $\left(\epsilon_X, (r_{A_i})_{i\in|X|}\right)$, where $r_{A_i} : A_i \otimes I \to A_i$ is right neutral in $\mathcal{C}$.

Observe here the role of the total ordering $\preceq$. Although any choice of $\preceq$ would produce an isomorphic object $\bigotimes_{i\in e} A_i$, the isomorphism need not be monoidal. Another way to model the ! comonad, without a $\preceq$ ordering, would be to equip the set of indices with a group action, in such a way that all constructions above commute with group actions. Similar ideas arise elsewhere [3, 28]. In concrete coherence completions, we use $\preceq$ to pick a distinguished element of each orbit instead.

The cocommutative comonoid structure on $!(X, (A_i)_{i\in|X|})$ is as follows. Comultiplication $\underline{d}_{(X,(A_i)_{i\in|X|})} :!(X,(A_i)_{i\in|X|}) \to !(X,(A_i)_{i\in|X|}) \otimes !(X,(A_i)_{i\in|X|})$ is $\left(\underline{d}_X, (\cong_{e_1,e_2})_{e_1,e_2\in|!X|, e_1 \frown_{!X} e_2}\right)$, where $\cong_{e_1,e_2}$ denotes the obvious natural iso from $\bigotimes_{i\in e_1 \uplus e_2} A_i$ to $\bigotimes_{i\in e_1} A_i \otimes \bigotimes_{i\in e_2} A_i$, defined from associativity, commutativity and the neutrals of $\otimes$. Then, $\underline{e}_{(X,(A_i)_{i\in|X|})}$: $!(X,(A_i)_{i\in|X|}) \to (\mathbb{I},(I))$ is $\left(\underline{e}_X, (\mathrm{id}_I)\right)$, where (id_I) is the family only containing id_I : $\bigotimes_{i\in\{\}} A_i \to I$.

To show that this defines a Lafont category, we must show that ! is the free cocommutative comonoid comonad on $\mathcal{CCOH}(\mathcal{C})$. The counit of the adjunction is ϵ, and the unit η is defined from the corresponding unit η in $\mathcal{CCOH}$. Precisely, the forgetful functor $U : co\mathcal{M}on(\mathcal{CCOH}(\mathcal{C})) \to \mathcal{CCOH}(\mathcal{C})$ maps each cocommutative comonoid $((X,(A_i)_{i\in|X|}), \widetilde{d}, \widetilde{e})$ to $(X,(A_i)_{i\in|X|})$. Its right adjoint $F : \mathcal{CCOH}(\mathcal{C}) \to co\mathcal{M}on(\mathcal{CCOH}(\mathcal{C}))$ maps each object $(X,(A_i)_{i\in|X|})$ to $(!(X,(A_i)_{i\in|X|}), \underline{d}_{(X,(A_i)_{i\in|X|})}, \underline{e}_{(X,(A_i)_{i\in|X|})})$, and each morphism $(f,(a_{ij})_{(i,j)\in f}) : (X,(A_i)_{i\in|X|}) \to (Y,(B_j)_{j\in|Y|})$ to $!(f,(a_{ij})_{(i,j)\in f})$. The counit of the adjunction $U \dashv F$ is $\epsilon_{(X,(A_i)_{i\in|X|})} :!(X,(A_i)_{i\in|X|}) \to (X,(A_i)_{i\in|X|})$. The unit is the most challenging construct. For each object $((X,(A_i)_{i\in|X|}), \widetilde{d}, \widetilde{e})$ of $co\mathcal{M}on(\mathcal{CCOH}(\mathcal{C}))$, let $\eta_{((X,(A_i)_{i\in|X|}),\widetilde{d},\widetilde{e})} : ((X,(A_i)_{i\in|X|}), \widetilde{d}, \widetilde{e}) \to (!(X,(A_i)_{i\in|X|}), \underline{d}_{(X,(A_i)_{i\in|X|})}, \underline{e}_{(X,(A_i)_{i\in|X|})})$ be the morphism $\left(\eta_X, \left(\cong \circ\, a^n_{i,i_1,\ldots,i_n}\right)_{(i,\{i_1,\ldots,i_n\})\in|X|\times|!X|, (i,\langle i_1,\langle i_2,\ldots,\langle i_n,\mathbf{1}\rangle\ldots\rangle\rangle)\in d^n_X}\right)$ where $\cong$ is the obvious natural iso from $A_{i_1} \otimes (A_{i_2} \otimes \ldots (A_{i_n} \otimes I)\ldots)$ to $\bigotimes_{j\in\{i_1,i_2,\ldots,i_n\}} A_j$ defined from associativity, commutativity and the neutrals of $\otimes$, $\widetilde{d} = (d_X, (a_{ijk} : A_i \to A_j \otimes A_k)_{(i,\langle j,k\rangle)\in d_X})$, and $\widetilde{e} = (e_X, (b_i : A_i \to I)_{(i,\mathbf{1})\in e_X})$; and finally $a^k_{i,i_{n-k+1},\ldots,i_n} : A_i \to A_{i_{n-k+1}} \otimes (A_{i_{n-k+2}} \otimes \ldots (A_{i_n} \otimes I)\ldots)$ is defined whenever $0 \le k \le n$ and $(i, \langle i_{n-k+1}, \ldots, \langle .i_n, \mathbf{1}\rangle \ldots\rangle) \in d^k_X$, by: $a^0_i = b_i$, and $a^{k+1}_{i,i_{n-k},\ldots,i_n} = (\mathrm{id}_{A_{i_{n-k}}} \otimes a^k_{j,i_{n-k+1},\ldots,i_n}) \circ a_{ii_{n-k}j}$ for some j such that $(i,\langle i_{n-k}, j\rangle) \in d_X$ and $(j, \langle i_{n-k+1}, \ldots, \langle .i_n, \mathbf{1}\rangle \ldots\rangle) \in d^k_X$.

Theorem 8.1. *Let $\mathcal{C}$ be any ($*$-)autonomous category. Then $\mathcal{CCOH}(\mathcal{C})$ is a (classical) Lafont category, hence a (classical) new-Lafont category.*

Proof. First, j in the definition of $a^n_{i,i_1,\ldots,i_n}$ exists and is unique. It exists because we assume $(i, \langle i_{n-k}, \langle i_{n-k+1}, \ldots, \langle .i_n, \mathbf{1}\rangle \ldots\rangle\rangle) \in d^{k+1}_X$ in defining $a^{k+1}_{i,i_{n-k},\ldots,i_n}$, and because $d^{n+1}_X = (\mathrm{id}_X \otimes d^n_X) \circ d_X$, so there is a j such that $(i,\langle i_{n-k}, j\rangle) \in d_X$ and $(j, \langle i_{n-k+1}, \ldots, \langle .i_n, \mathbf{1}\rangle \ldots\rangle) \in d^k_X$. If there was another, say j', then since d_X is linear we would have $j \frown j'$, and since j and j' would be mapped to the same element by d^k_X, necessarily $j = j'$.

Checking all equations is tedious, and in fact uninformative. Instead, here is an intuitive argument. First, think of $a_{ijk} : A_i \to A_j \otimes A_k$ as some multiplication written in the reverse direction, where we multiply an item of type A_j by one of type A_k to get one of type A_i; we can do this whenever $(i,\langle j,k\rangle)$ is in d_X. On the other hand, b_i is the unit of type A_i, when $(i,\mathbf{1}) \in e_X$. (This i is unique, since e_X is linear.) Because $((X,(A_i)_{i\in|X|}), \widetilde{d}, \widetilde{e})$ is a cocommutative comonoid, it does not matter in which order we multiply elements from $A_{i_1}, A_{i_2}, \ldots, A_{i_n}$ (one from each), getting one in A_i: $\cong \circ\, a^n_{i,i_1,\ldots,i_n}$: $A_i \to \bigotimes_{j\in\{i_1,i_2,\ldots,i_n\}} A_j$ is precisely this multiplication operation.

Naturality of η on fibers (η is natural on bases, by the structure of $\mathcal{CCOH}$) then means that given any cocommutative comonoid morphism $(f,(c_{ii'})_{(i,i')\in f})$ from the object $((X,(A_i)_{i\in|X|}), \widetilde{d}, \widetilde{e})$ to $((X',(A'_{i'})_{i'\in|X'|}), \widetilde{d'}, \widetilde{e'})$, i.e., given that the maps $c_{ii'}$ commute with multiplication and send units to units, then they also commute with n-fold multiplications of the form $\cong \circ a^n_{i,i_1,\ldots,i_n}$. This is clear.

We must check that $F\epsilon_{(X,(A_i)_{i\in|X|})} \circ \eta_{F(X,(A_i)_{i\in|X|})}$ is identity on $F(X,(A_i)_{i\in|X|})$. Equality on bases is clear. Now $F(X,(A_i)_{i\in|X|}) = (!(X,(A_i)_{i\in|X|}), \underline{d}_{(X,(A_i)_{i\in|X|})}, \underline{e}_{(X,(A_i)_{i\in|X|})})$. The fiber part of $F\epsilon_{(X,(A_i)_{i\in|X|})} \circ \eta_{F(X,(A_i)_{i\in|X|})}$ is then a collection of morphisms in $\mathcal{C}$ (which also commute with various a_{ijk}'s and b_i's as above) indexed by $(e,e) \in \mathrm{id}_{!X}$. For each such e, (looking at bases) there is a unique multiset $\{e_1,\ldots,e_n\}$ such that $(e,\{e_1,\ldots,e_n\}) \in \eta_{!X}$ and $(\{e_1,\ldots,e_n\},e) \in !\epsilon_X$. By definition of ! and ϵ_X in $\mathcal{CCOH}$, and letting $e = \{i_1,\ldots,i_n\}$, this multiset must be exactly $\{\{i_1\},\ldots,\{i_n\}\}$. Looking at the fiber above (e,e) (passing through $\{\{i_1\},\ldots,\{i_n\}\}$), we find the composite (from right to left) that does all 1-fold multiplications from A_{i_1} to A_{i_1}, ..., from A_{i_n} to A_{i_n}. A 1-fold multiplication is the composite of a binary multiplication $\underline{d}$ with a unit $\underline{e}$ on the right, so 1-fold multiplications are just identities. So the fiber above (e,e) is $\mathrm{id}_{A_{i_1}} \otimes (\mathrm{id}_{A_{i_2}} \otimes \ldots (\mathrm{id}_{A_{i_n}} \otimes \mathrm{id}_I)\ldots)$, i.e., identity.

We must finally check that the composition $\epsilon_{U((X,(A_i)_{i\in|X|}),\widetilde{d},\widetilde{e})} \circ U\eta_{((X,(A_i)_{i\in|X|}),\widetilde{d},\widetilde{e})}$ is the identity on $U((X,(A_i)_{i\in|X|}),\widetilde{d},\widetilde{e}) = (X,(A_i)_{i\in|X|})$. The fibers are indexed by those $(i,i) \in \mathrm{id}_X$, passing through $\{i\}$. The corresponding morphism in this fiber is the composite of an identity (coming from the ϵ part) and a 1-fold multiplication (defined using $\widetilde{d},\widetilde{e}$), so is identity again. $\square$

Corollary 8.2. *Let M be an arbitrary weakly Cantorian linear inverse semigroup. Then $\mathcal{CCOH}(\mathcal{DR}(M))$ is a classical Lafont category, hence also a classical new-Lafont category.*

In other words, $\mathcal{CCOH}(\mathcal{DR}(M))$ is a categorical model of full classical linear logic, in the strongest known sense.

Discussion. The $\mathcal{CCOH}$ construction seems orthogonal to $\mathcal{DR}$, and it is time to explain how we came to the idea this was the right construction. The key to understanding Theorem 7.2 and the failure of cocommutativity is to realize that d_A has vertical arrows of the form $pA_p + qA_q$, while $c_{A,A} \circ d_A$ has $pA_q + qA_p$ instead. Intuitively, the only way $\mathcal{DR}(M)$ is able to talk about two occurrences of the same variable x in a λ-term, i.e., to duplicate x through d_A, is to declare there will be a p occurrence of x (in A_p) and a q occurrence of x (in A_q). With more than two occurrences, iterating the process amounts to *numbering* all occurrences of the same variable with words over $\{p,q\}$. The law that fails, $c_{A,A} \circ d_A = d_A$, says that exchanging two occurrences of the same variable x in a term does not change the term; but exchanging the two occurrences x_p and x_q of x in $\lambda y, x \cdot y x_p x_q$ produces $\lambda y, x \cdot y x_q x_p$, which is different, and the geometry of interaction constructions cannot escape it. Instead, to build a categorical model we used the fact that the $\mathcal{DR}$ construction is able to talk about *partial* λ-terms, i.e., collections of paths that do not exhaust all paths of a given λ-term, and define the actual λ-term as a *superposition* of these partial λ-terms, as though they were drawn on tracing paper. In the example of $\lambda y, x \cdot yxx$, draw (the interpretation in $\mathcal{DR}(M)$ of) $\lambda y, x \cdot yx\textvisiblespace$ and of $\lambda y, x \cdot y\textvisiblespace x$, which we see as the down-closed clique of (weights of) paths that do not go through the positions shown as $\textvisiblespace$. The intended λ-term $\lambda y, x \cdot yxx$ is obtained by superposing the two—which are coherent in the sense that no two distinct letters ever get on top of each other, unless one is the blank $\textvisiblespace$. The $\mathcal{CCOH}$ construction does this, or at least half of it: what it does not do is check that the two partial terms are coherent in the sense just given.

That $\mathcal{CCOH}$ really builds superposed sheets of tracing paper (the fibers) on which we draw partial λ-terms (in $\mathcal{C} = \mathcal{DR}(M)$) is probably more visible in the interpretation of additives. (Note how the idea of layering additive strata looks like Hughes and van Glabbeek's idea [22], although the technical developments differ.) Take the example of two closed MLL λ-terms, or proof nets, t_1 and t_2, interpreted as morphisms $(0, 0, \gamma_1)$ and $(0, 0, \gamma_2)$ from 0 to some idempotent A in $\mathcal{DR}(M)$. In $\mathcal{CCOH}(\mathcal{DR}(M))$, these are interpreted as morphisms from $(\mathbb{I}, (0))$ to $(\mathbb{I}, (A))$, namely just one morphism from the unique fiber 0 over $\mathbb{I}$ to the unique fiber A over $\mathbb{I}$. The additive pair $\langle t_1, t_2\rangle$ will then be just drawn as two morphisms ($(0, 0, \gamma_1)$ and $(0, 0, \gamma_2)$ again) from $(\mathbb{I}, (0))$ to the two fibers above $(\mathbb{I} \times \mathbb{I}, (A, A))$. In the general case, where multiplicatives, additives, and exponentials are intermingled, the construction will be more complex, but coherence completions help organize the clutter in an elegant way.

9. Conclusion

To conclude, we would like, first, to stress the importance of *coherence*, both in the coherence completion construction, where its role is obvious, and in geometry of interaction, where it is really at the core of the $\mathcal{DR}$ construction. This may have remained hidden until now. However, the two notions of coherence are completely independent, and whether this is needed or whether one could define a refinement of the construction where the two coherences would interact, is still a mystery.

Second, we would like to suggest that our results that $\mathcal{DR}(M)$ has no additive and no exponential are in fact good news: none of the products, coproducts, comonads that the $\mathcal{CCOH}$ construction creates have a chance of conflicting with any similar preexisting construction in $\mathcal{DR}(M)$—there just isn't any. We conjecture that the semantics of linear logic proofs inside $\mathcal{CCOH}(\mathcal{DR}(M))$, for suitable M, is both equationally complete (any two proofs that have the same denotation are equal) and fully complete. For this, we need proof nets for full linear logic, where cut elimination implements all expected categorical equalities. A candidate is given in [18, Chapter 4], but its theory has not been worked out yet.

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