Cyclic Ordering through Partial Orders

To the honor of Maurice Pouzet

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Abstract. The orientation problem for ternary cyclic order relations has been attacked in the literature from combinatorial perspectives, through rotations, and by connection with Petri nets. We propose here a two-fold characterization of orientable cyclic orders in terms of symmetries of partial orders as well as in terms of separating sets (cuts). The results are inspired by properties of non-sequential discrete processeses, but also apply to dense structures of any cardinality.

1 Introduction

In girum imus nocte et consumimur igni[1].

Partial orders can be seen as the canonical way of describing or specifying distributed and interacting processes in all technical areas. Their axiomatization is simple, and their theory is rich in results and algorithms. On the other hand, systems that repeat the same actions and states periodically, suggest an intuitive way of ordering in *cycles*: On the left hand side of Figure 1, event b always occurs between a and c, c always between b and d, etc. It is obvious that under the cyclic symmetry, an axiomatization of this relation with binary transitive relations will not be able to express the orientation of the cycle: since every event "precedes" every other, precedence is an equivalence here. The axiomatizations existing in the literature use either

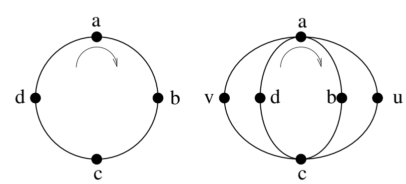

Fig. 1. Cyclic orders

- *ternary* relations ([ANP91,ChN83,Gen71,IC00,Jak94,Meg76,Nov83,Qui89,Qui91] and the present article),
- *pairs* of binary relations ([ER94]),
- or tuples/words of variable length ≥ 2 ([Hun16,Hun24a,Hun24b,Hun38,Ste98]).

We will focus here on a canonical ternary relation framework.

The distinction between total and partial orders carries over from the acyclic to the cyclic case. While the left hand side of Figure 1 gives a *total* cyclic arrangement of four elements, the right hand side illustrates a truly *partial* cyclic order: b and u are both ordered w.r.t. all other elements, since they are between a and c, yet there is no ordering between the two. This article deals precisely with the connection between partial *cyclic* orders and partial *acyclic* orders . – The following Section 2 introduces or recalls key concepts ; Section 3 proves the main results, and Section 4 concludes.

[1] "We enter the circle after dark and are consumed by fire"; Latin palindrome said to describe the movement of moths around (and into) a flame; author unknown

2 Problem Statement

Partial Orders and Szpilrajn's theorem. $\Pi = (\mathcal{X}, <)$ with $\mathcal{X}$ a non-empty set and $<$ a binary relation over $\mathcal{X}$ is a **partial order** (*PO*) or **poset** iff $<$ is *i)* transitive: $x < y$ and $y < z$ imply $x < z$; and *ii)* irreflexive: $x \not< y$.
Let $\mathsf{li} \triangleq (< \cup <^{-1})$ denote **comparability** and $co \triangleq (\mathcal{X} \times \mathcal{X}) \backslash (\mathsf{id}_{\mathcal{X}} \cup \mathsf{li})$ **incomparability** of pairs of nodes; here, $\mathsf{id}_{\mathcal{X}} \triangleq \{(x, x) \mid x \in \mathcal{X}\}$ is the identity relation. If $\mathcal{X}^2 = \mathsf{li} \cup \mathsf{id}_{\mathcal{X}}$, then $<$ is a **total order** (*TO*). According to Szpilrajn's Theorem [Szp30], every *PO* has an embedding into some *TO*, called its *linearization*.

We will examine which axioms are meaningful for cyclic ordering; the counterpart of Szpilrajn's theorem will turn out to hold only in a non-trivial important subclass for cyclic orders, for which we give a novel characterization in terms of partial orders.

Stehr [Ste98] shows that for discrete cyclic orders, global orientation is equivalent to (i) having a Petri net representation and (ii) existence of a true cut (called cycle separator below), i.e. an set of pairwise independent nodes that separates all cycles. Here, we prove a generalization of this result, for general cyclic orders and only requiring existence of some superstructure that contains a separator, and also showing a close connection between acyclic partial orders, i.e. posets, and the orientable cyclic orders.

We first fix some notations and definitions. An **n-ary relation** over $\mathcal{X}$ is a non-empty subset $\mathcal{R} \subseteq \mathcal{X}^n$; the important cases here will be $n = 2$ **(binary)** and $n = 3$ **(ternary)**. Write $x_1 \ \mathcal{R} \ x_2$ to express that $(x_1, x_2) \in \mathcal{R}$ for a binary relation $\mathcal{R}$; if $\mathcal{R}$ is ternary, we write $\mathcal{R}(x_1, x_2, x_3)$ iff $(x_1, x_2, x_3) \in \mathcal{R}$. For $\mathcal{Y} \subseteq \mathcal{X}$, $\mathcal{Y}$ non-empty, and $\mathcal{R}$ an n-ary relation over $\mathcal{X}$, denote by $\mathcal{R}_{|\mathcal{Y}}$ the **restriction** of $\mathcal{R}$ to $\mathcal{Y}$. If $\mathcal{X}_1 \subseteq \mathcal{X}_2$ and $\mathcal{R}_1 = \mathcal{R}_{2|\mathcal{X}_1}$, call $\Theta_1 = (\mathcal{X}_1, \mathcal{R}_1)$ a **substructure** of $\Theta_2 = (\mathcal{X}_2, \mathcal{R}_2)$, and Θ_2 a **superstructure** of Θ_1. If for $\mathcal{R}_1$ and $\mathcal{R}_2$ of same arity and over the same set $\mathcal{X}$ it holds that $\mathcal{R}_2 \subseteq \mathcal{R}_1$, say that $\mathcal{R}_1$ **embeds** $\mathcal{R}_2$ (or: $\mathcal{R}_1$ **is an embedding for** $\mathcal{R}_2$).

If Π_1, Π_2 are *PO*s and Π_1 is a substructure of Π_2, call Π_1 a *SubPO* of Π_2 and Π_2 a *SuperPO* of Π_1.

An **equivalence** is a transitive, symmetric and reflexive binary relation. A non-empty set $\mathcal{E} \subseteq \mathcal{X}$ is an $\mathcal{R}$**-clique** iff $x \ \mathcal{R} \ y$ for all $x, y \in \mathcal{E}$ such that $x \neq y$.

2.1 Cyclic Orders and Orientability

We represent cyclic orders as **ternary** rather than binary relations. This is not an arbitrary choice: it requires a ternary structure to discern **senses of rotation,** i.e. tell **"clockwise"** from **"counterclockwise"**. In artificial intelligence, some recent work on qualitative spatial reasoning uses ternary cyclic ordering, see [IC00]; the situation there, however, is simplified by the absence of *co* (only **total** cyclic orders are used, see below). The following is the usual[2] definition of **partial** cyclic orders (see [Qui91,Qui89,Jak94,Ste98]).

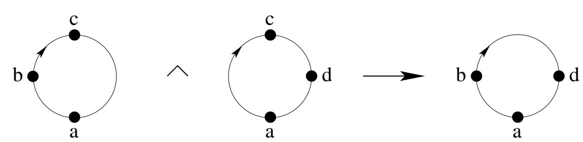

Fig. 2. Ternary Transitivity

Definition 21 (Cyclic Orders) *Let $\lessdot$ be a ternary relation over the set $\mathcal{X}$. Then $\Gamma = (\mathcal{X}, \lessdot)$ is a* **cyclic order (***CyO***)** *iff it satisfies, for $a, b, c, d \in \mathcal{X}$:*

1. **inversion asymmetry***: If $\lessdot(a, b, c)$, then $\lessdot(b, a, c)$ does* **not** *hold;*
2. **rotational symmetry***: If $\lessdot(a, b, c)$, then $\lessdot(c, a, b)$;*
3. **ternary transitivity***: If $[\lessdot(a, b, c) \ \wedge \lessdot(a, c, d)]$, then $\lessdot(a, b, d)$.*

Definition 22 *Call a ternary relation $\mathcal{R}$* **simple** *iff $\mathcal{R}(x, y, z)$ implies that $x \neq y \neq z \neq x$.*

From Definition 21, one obtains:

Lemma 23 *If $(\mathcal{X}, \lessdot)$ is a CyO, then $\lessdot$ is simple.*

[2] The non-ternary approach of Stehr [KS97,Ste98]) provides an equivalent representation of cyclic orders; the results here carry over after careful translation.

Proof: Assume $\lhd$ is not simple. Then rotational symmetry implies that there exist $x, y \in \mathcal{X}$ such that $\lhd(x, x, y)$; hence inversion asymmetry is violated. □

Note that ternary transitivity *resembles* binary transitivity; compare Figure 2.

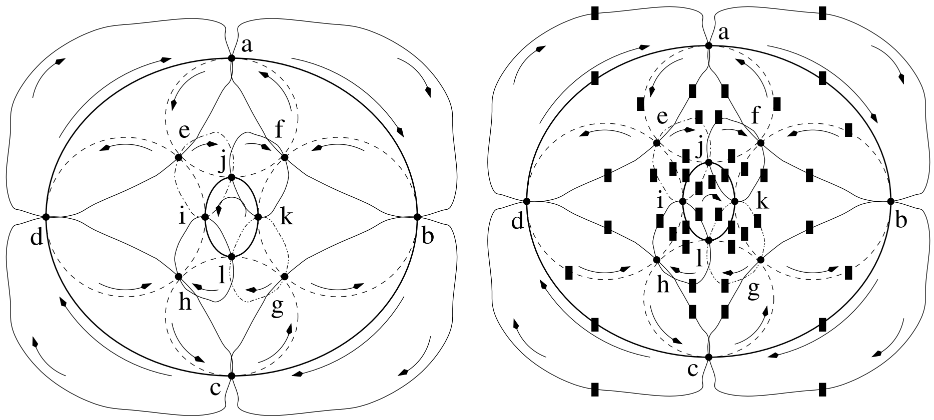

Fig. 3. Top: a cyclic order ; bottom: one of its li-oriented extensions. Neither version is orientable.

Definition 24 *A CyO* $\Gamma_{tot} = (\mathcal{X}, \lhd_{tot})$ *is called* **total** *or a TCO if for all* $a, b, c \in \mathcal{X}$,

$$(x \neq y \neq z \neq x) \Rightarrow \lhd(a, b, c) \ \vee \ \lhd(b, a, c).$$

Note that there are only two different ways to orient a given triple in a cyclic order, since all other arrangements are rotations of either (a, b, c) or (b, a, c).

Orientations of cyclic orders. Let $\Gamma = (\mathcal{X}, \lhd)$ be a *CyO*. If there exists a total *CyO* Γ_{tot} on $\mathcal{X}$ such that Γ_{tot} **embeds** Γ, then Γ is called **orientable**, and Γ_{tot} an **orientation** of Γ. The existence of an orientation for Γ is equivalent to Γ having a graphical representation by **clock cycles**, i.e. as a collection of directed loops in the two-dimensional plane such that the origin is avoided and such that all loops run clockwise around the origin[3]. Orientable *CyO*s are therefore also called **globally oriented** ([Ste98]). They are characterized by the fact that a counterpart to Szpilrajn's Theorem [Szp30] hold. The cyclic order Γ of Figure 5 is orientable. In fact, it is already given in clock cycles, an orientation is found by projecting, from the center, to some cycle surrounding N, and then ordering in an arbitrary way those transitions or places that may happen to be mapped to the same point (as could be the case, say, for α and β): in Γ, such nodes were necessarily in *co*.

Is every *CyO* orientable ? The answer is *negative*: additional properties are needed to ensure orientability. Consider the example on top of Fig. 3; there,

$$\begin{array}{llll}
\lhd(a,b,c), & \lhd(a,c,d), & \lhd(l,k,j), & \lhd(l,j,i), \\
\lhd(a,e,j), & \lhd(a,j,f), & \lhd(a,e,d), & \lhd(d,h,i), \\
\lhd(d,i,e), & \lhd(c,g,l), & \lhd(c,l,h), & \lhd(b,f,k), \\
\lhd(b,k,g), & \lhd(d,h,c) & \lhd(c,g,b), & \lhd(b,f,a), \\
\lhd(e,j,i), & \lhd(i,l,h) & \lhd(g,l,k), & \lhd(f,k,j),
\end{array}$$

plus the triples obtained using transitivity and rotation. This structure[4] can be shown to have no orientation and no clock cycle representation at all. Informally speaking, any total order that contains (a, b, c) and

[3] cf. the **arc orders** in Alles, Nešetřil, Poljak [ANP91].

[4] the example is due to Genrich [Gen71], with completion by Stehr[Ste98]; for a different example of a non-orientable cyclic order, cf. Megiddo [Meg76]

(a, c, d) as well as (i, l, k) and (i, k, j), will violate one of the other triples; the readers are invited to attempt this totalization themselves ! Hence the example cannot be extended into an orientable superstructure, since any *CyO* is orientable iff all its *SubCyO*s are.

Thus, the **orientability problem** consists in finding minimal additional properties that, together with the above axioms, ensure orientability. Some further formal preparations next:

Definition 25 *Let $\Gamma = (\mathcal{X}, \lhd)$ be a CyO. Set*

$$\mathsf{li} \triangleq \{(x, y) \mid \exists z : \lhd(x, y, z) \vee \lhd(x, z, y)\} \text{ and } co \triangleq \mathcal{X}^2 - (\mathsf{id}_{\mathcal{X}} \cup \mathsf{li}),$$

and denote the maximal cliques of li *as* **rounds** *and those of co as* **cuts**. *We will denote rounds by O and the set of all rounds of Γ as $\mathcal{O}(\Gamma)$; for cuts, we use* **c** *and $\mathcal{C}(\Gamma)$ as in the acyclic case.*

The names li and co are a tribute to the axiomatic *Concurrency Theory* initiated by C. A. Petri, see [Pet96,Pet96,KS97]. The graph of li is the *Gaifman graph* of Γ; a total *CyO* satisfies $co = \emptyset$. Note that all rounds have at least three elements. We shall require a strong link between them $\lhd$:

Definition 26 *A cyclic order $\Gamma = (\mathcal{X}, \lhd)$ is* **round-oriented** *or a ROCO iff for any round $\{a, b, c\}$ of* li*, either $\lhd(a, b, c)$ or $\lhd(b, a, c)$.*

Figure 4 shows that round-orientability is independent of global orientation. The *CyO* represented is not round-oriented (consider $\{a, b, c\}$). Yet there exists a round-oriented extension: add (b, a, c) and its rotations. This extension is also globally oriented, since the *CyO* given by the single cycle $\langle b, a, u, v, c, w\rangle$ provides an orientation. Comparison with Figure 3 shows that round-orientation is (necessary, but) insufficient for global orientation. Round-orientation is a stronger condition than the requirements in Quilliot [Qui91,Qui89], Jakubík [Jak94] etc.; cf. Stehr [Ste98]. Some CyO's that are not round-oriented may be completed to a*ROCO*; not all CyOs will allow this. It is obvious, however, that only those CyO's extensible to a *ROCO* can be globally oriented, and that they are globally oriented iff one of their round-oriented super-CyO's is; therefore, we restrict our attention to *ROCO*s.

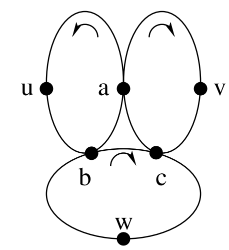

Fig. 4. Global vs round-orientability

2.2 Windings of posets

We focus on cyclic orders that can be obtained from periodic partial orders. For this, we will now introduce **windings**, and show how they lead to a two-fold characterization of orientable *ROCO*s; the main result on orientability is Theorem 313 below.

Observe that $\overline{N}$ in Figure 5 and its associated poset are **periodic**: they display translational symmetries, corresponding to particular **order automorphisms**, that is, bijections $\mathbf{G} : \mathcal{X} \to \mathcal{X}$ of a poset $\Pi = (\mathcal{X}, <)$ such that for all $x, y \in \mathcal{X}$, one has $x < y \Leftrightarrow \mathbf{G}x < \mathbf{G}y$. We define:

Definition 27 *For any mapping $\phi : \mathcal{X} \to \mathcal{X}$ and subset $A \subseteq \mathcal{X}$, we say that ϕ* **contracts** *A iff $\phi(A) \subseteq A$, and that A is ϕ-***invariant** *iff $\phi(A) = A$. Further, let $\Pi = (\mathcal{X}, <)$ be a poset and $\mathbf{G}$ an automorphism of Π. Then $\mathbf{G}$ is called a* **shift** *if $x < \mathbf{G}x$ for all $x \in \mathcal{X}$.*

Let $\Pi = (\overline{\mathcal{X}}, <)$ be a poset with a shift $\mathbf{G}$, and $\mathcal{G}$ the group of Π-automorphisms generated by $\mathbf{G}$; $\mathcal{G}$ is isomorphic to $(\mathbb{Z}, +)$. Write $\overline{x} \sim_{\mathbf{G}} \overline{y}$ iff there exists $k \in \mathbb{Z}$ such that $\mathbf{G}^k\overline{x} = \overline{y}$; then $\sim_{\mathbf{G}}$ is an equivalence relation on $\overline{\mathcal{X}}$. The equivalence class $[\overline{x}] \triangleq [x]_{\sim_{\mathbf{G}}}$ of $\overline{x}$ is the $\mathcal{G}$-**orbit** of $\overline{x}$; the associated **winding map** is $\beta_{\mathbf{G}} : \overline{\mathcal{X}} \to \mathcal{X}$, $\overline{x} \mapsto [\overline{x}]_{\sim_{\mathbf{G}}}$. We will note orbits by simple lower case letters $x, y, ...$, and their elements as overlined lower case letters $\overline{x}, \overline{y}, ...$ Orbits are obviously shift-invariant. We define:

Definition 28 *Let $\Pi = (\overline{\mathcal{X}}, <)$ a poset with shift $\mathbf{G}$, and $\mathcal{X} \triangleq \overline{\mathcal{X}}_{\sim_{\mathbf{G}}}$. Define*

$$\lhd(x, y, z) \text{ iff } \exists \left\{ \begin{array}{c} \overline{x} \in x \\ \overline{y} \in y \\ \overline{z} \in z \end{array} \right\} : \overline{x} < \overline{y} < \overline{z} < \mathbf{G}\overline{x},$$

where we identify an element $x \in \mathcal{X}$ with the orbit $[\overline{x}]$ in $\overline{\mathcal{X}}$. We say that $\Gamma = (\mathcal{X}, \lhd)$ is **wound** *from Π using $\mathbf{G}$ (or, equivalently, $\beta_{\mathbf{G}}$).*

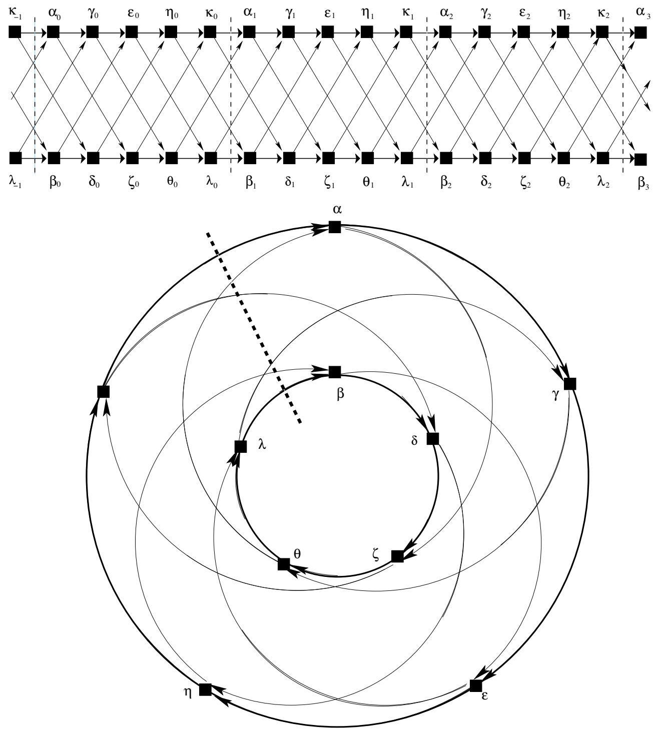

Fig. 5. A cyclic order (bottom) wound from the acyclic order on top

For an example, consider Figure 5: $\lhd(\alpha, c, \gamma)$ holds since $\alpha_0 < c_0 < \gamma_0 < \alpha_1 = \mathbf{G}\alpha_0$, where $\mathbf{G}$ is the shift that takes each dashed vertical line in Figure 5 to its right neighbor. In this way, $\mathbf{G}$ winds the partial order on $\overline{N}$ to a cyclic order on N.

Lemma 29 *Let $\varGamma_1 = (\mathcal{X}_1, \lhd_1)$ be a ROCO and $\varGamma_2 = (\mathcal{X}_2, \lhd_2)$ a sub-ROCO of $\varGamma_1$, i.e. $\mathcal{X}_2 \subseteq \mathcal{X}_1$ and $\lhd_2 = \lhd_{1|\mathcal{X}_2}$. If $\varGamma_1$ is obtained by winding, then so is $\varGamma_2$.*

Proof: Let $\varPi_1 = (\overline{\mathcal{X}}_1, <_1)$ be wound to $\varGamma_1$ using $\mathbf{G}$, and set $\overline{\mathcal{X}}_2 \triangleq \mathbf{G}^{-1}(\mathcal{X}_2) \subseteq \overline{\mathcal{X}}_1$. With $<_2 \triangleq <_{1|\overline{\mathcal{X}}_2}$, one checks that $\varPi_2 \triangleq (\overline{\mathcal{X}}_2, <_2)$ is wound to $\varGamma_2$ using $\mathbf{G}_2 \triangleq \mathbf{G}_{1|\overline{\mathcal{X}}}$. □

We say that a winding is **loop-free (LF)** iff for all $\overline{x} \in \overline{\mathcal{X}}$, there exist $\overline{y}, \overline{z} \in \overline{\mathcal{X}}$ s.th. $\overline{x} < \overline{y} < \overline{z} < \mathbf{G}\overline{x}$. Under loop-freeness, all nodes of $\mathcal{X}$ are contained in some triple of $\lhd$, and there is no pair $\overline{x}, \overline{y} \in \overline{\mathcal{X}}$ such that $\overline{x} < \overline{y} < \mathbf{G}\overline{x}$ but $\neg([\overline{x}] \mathrel{\text{li}} [\overline{y}])$ in $\lhd$. If a winding has a loop, the result may not be a cyclic order; consider the total order $(\overline{\mathcal{X}}, <)$ with $\overline{\mathcal{X}} = \{x_i, y_i \mid \imath \in \mathbb{Z}\}$ and $<$ given by $x_i < y_i < x_{i+1}$ for all $i \in \mathbb{Z}$ with the obvious shift $x_i \mapsto x_{i+1}$, $y_i \mapsto y_{i+1}$. Then $\mathcal{X} = \{x, y\}$, which allows no triple, and therefore yields an empty $\lhd$-relation. On the other hand, we have:

Theorem 210 *LF windings generate ROCOs.*

Proof: Let $\varPi = (\overline{\mathcal{X}}, <)$ be wound to $\varGamma = (\mathcal{X}, \lhd)$ using $\mathbf{G}$, and let $\beta \triangleq \beta_{\mathbf{G}}$.
To show **inversion asymmetry**, suppose $\lhd(x, y, z)$ and $\lhd(y, x, z)$. Then there exist $\overline{x} \in \beta^{-1}(x)$, $\overline{y} \in \beta^{-1}(y)$, and $\overline{z} \in \beta^{-1}(z)$ such that $\overline{x} < \overline{y} < \overline{z} < \mathbf{G}\overline{x}$, but also some $k \in \mathbb{Z}$ such that $\mathbf{G}^k(\overline{y}) < \mathbf{G}^k\overline{x} < \mathbf{G}^k\overline{z} < \mathbf{G}^{k+1}\overline{y}$. Applying $\mathbf{G}^{-k}$ yields $\overline{y} < \overline{x}$, contradicting the acyclicity of $<$.
For **rotational symmetry**, suppose $\lhd(x, y, z)$; then there exist $\overline{x} \in \beta^{-1}(x)$, $\overline{y} \in \beta^{-1}(y)$, and $\overline{z} \in \beta^{-1}(z)$ such that $\overline{x} < \overline{y} < \overline{z} < \mathbf{G}\overline{x}$, hence $\overline{z} < \mathbf{G}\overline{x} < \mathbf{G}\overline{y} < \mathbf{G}\overline{z}$. Since $\beta(\mathbf{G}\overline{x}) = \beta(\overline{x}) = x$ and $\beta(\mathbf{G}\overline{y}) = \beta(\overline{y}) = y$, we thus obtain $\lhd(z, x, y)$.

For **ternary transitivity**, assume $\lhd(x,y,z)$ and $\lhd(x,z,u)$. Then there exist $\overline{x} \in \beta^{-1}(x)$, $\overline{y} \in \beta^{-1}(y)$, $\overline{z} \in \beta^{-1}(z)$ and $\overline{u} \in \beta^{-1}(u)$ such that

$$\overline{x} < \overline{y} < \overline{z} < \mathbf{G}\overline{x} \tag{1}$$

$$\exists\, k \in \mathbb{Z} : \mathbf{G}^k\overline{x} < \mathbf{G}^k\overline{z} < \mathbf{G}^k\overline{u} < \mathbf{G}^{k+1}\overline{x}. \tag{2}$$

But since $\mathbf{G}$ is an automorphism, it follows that $\overline{x} < \overline{z} < \overline{u} < \mathbf{G}\overline{x}$, thus $\overline{x} < \overline{y} < \overline{u} < \mathbf{G}\overline{x}$ by transitivity of $<$, and therefore $\lhd(x,y,u)$.
For **round-orientation**, assume there exist three distinct elements $x,y,z \in \mathcal{X}$ such that $x \parallel y$, $y \parallel z$, and $x \parallel z$. Then, by definition of $\lhd$, and the fact than $\mathbf{G}$ is an order automorphism, one has that for every $\overline{x} \in \beta_{\mathbf{G}}^{-1}x$, there exists $\overline{y} \in \beta_{\mathbf{G}}^{-1}y$ such that

$$\overline{x} \; < \; \overline{y} \; < \; \mathbf{G}\overline{x} \; < \; \mathbf{G}\overline{y}. \tag{3}$$

Using the same arguments, one finds that there exists $\overline{z} \in \beta_{\mathbf{G}}^{-1}z$ such that

$$\overline{y} \; < \; \overline{z} \; < \; \mathbf{G}\overline{y} \; < \; \mathbf{G}\overline{z}. \tag{4}$$

Since $x \parallel z$, we have $\mathbf{G}\overline{x} \parallel \overline{z}$. Now, (3) and (4) imply that $\overline{x} < \overline{z} < \mathbf{G}^2\overline{y}$; it remains to determine the ordering of $\mathbf{G}\overline{x}$ and $\overline{z}$. Assume first that $\mathbf{G}\overline{x} < \overline{z}$; then also $\mathbf{G}^2\overline{x} < \mathbf{G}\overline{z}$. Combining this with (3) and (4), we obtain

$$\mathbf{G}\overline{x} \; < \; \overline{z} \; < \; \mathbf{G}\overline{y} \; < \; \mathbf{G}^2\overline{x} \; < \; \mathbf{G}\overline{z} < \; \mathbf{G}^2\overline{y}.$$

This yields $\lhd(x,z,y)$ since $\beta_{\mathbf{G}}(\mathbf{G}\overline{x}) = \beta_{\mathbf{G}}(\overline{x}) = x$ and $\beta_{\mathbf{G}}(\mathbf{G}\overline{y}) = \beta_{\mathbf{G}}(\overline{y}) = y$. Now, if $\overline{z} < \mathbf{G}\overline{x}$, then $\overline{x} < \overline{y} < \overline{z} < \mathbf{G}\overline{x}$ and thus $\lhd(x,y,z)$; in either case, the set $\{x,y,z\}$ is ordered by $\lhd$. □

3 Characterizing Orientability

In the light of Theorem 210, two questions arise:
1) Is it also true that any loop-free winding will preserve successor relations ?
2) Which properties characterize those ROCOs that have a representation as a winding ?

3.1 Mind the gap !

Not all loop-free windings preserve successor relations; a study of this issue will reveal the dangers of *gaps* (compare [BF88]). We first need some supplementary relations for both acyclic and cyclic orders:

Definition 31 *Let $(\overline{\mathcal{X}},<)$ be aposet and $(\mathcal{X},\lhd)$ a cyclic order. Define the successor relations $<\!\cdot$ for an acyclic order, and $\lessdot$ for a cyclic order, by*

- $\overline{x} <\!\cdot\, \overline{y}$ *iff*
  1. $\overline{x} < \overline{y}$ *and*
  2. *for all* $\overline{z} \in \overline{\mathcal{X}}$, $\overline{x} < \overline{z} < \overline{y}$ *implies* $\overline{z} \in \{\overline{x},\overline{y}\}$*;*
- $x \lessdot y$ *iff* (a) $x \parallel y$, *and* (b) *for all* $z \in \mathcal{X} - \{x,y\}$, $x \parallel z$ *and* $z \parallel y$ *imply that* $\lhd(x,y,z)$.

1. $\overline{x}$ **covers** $\overline{y}$ **from below**, *written* $x \veebar y$, *iff (a)* $\overline{x} < \overline{y}$, *and (b) for all* $\overline{z} \in \mathcal{X}$, $\overline{z} < \overline{y}$ *implies* $\overline{z} \leqslant \overline{x}$.
2. $\overline{y}$ **covers** $\overline{x}$ **from above**, *written* $y \barwedge x$, *iff (a)* $\overline{x} < \overline{y}$, *and (b) for all* $\overline{z} \in \mathcal{X}$, $\overline{x} < \overline{z}$ *implies* $\overline{y} \geqslant \overline{z}$.
3. *In* $(\mathcal{X},\lhd)$, x **covers** y, *written* $x \sqsupset y$, *iff (a)* $x \parallel y$, *and (b) for all* $z,u \in \mathcal{X}$, $\lhd(z,u,y)$ *implies* $\lhd(z,u,x)$.

Using this terminology, we define gaps to be successor pairs without covering:

Definition 32 *1. A* **gap** *in* $(\overline{\mathcal{X}},<)$ *is a pair* x,y *such that* $x <\!\cdot y$ *holds, but neither* $x \barwedge y$ *nor* $y \veebar x$.
2. *A* **gap** *in* $(\mathcal{X},\lhd)$ *is a pair* x,y *such that* $x \lessdot y$ *holds, and* $x \sqsupset y$ *does not hold.*

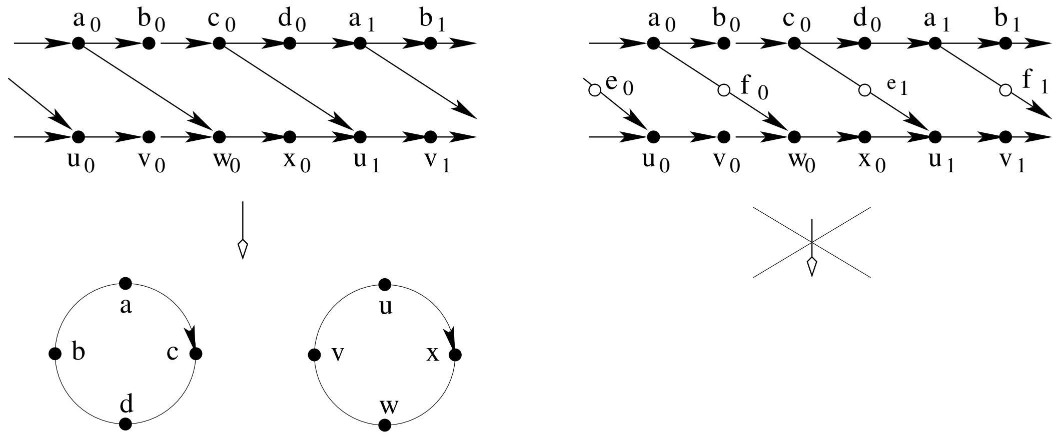

Fig. 6. Left: A partial order with a winding that destroys the successor relation; right: a gap-free version of the partial order from the left hand side. It admits no shift symmetry and therfore no winding.

Note that $y \barwedge x$ does **not** imply $x \veebar y$, nor the converse: on the right hand side of Figure 6, $w_0 \barwedge f_0$ and $a_0 \veebar f_0$, but neither $f_0 \barwedge a_0$ nor $f_0 \veebar w_0$ hold. The right hand side of Figure 6 is gap-free; on the left hand side, all pairs a_n, w_n and c_n, u_{n+1} are gaps for $n \in \mathbb{Z}$.

Lemma 33 *Suppose* $(\overline{\mathcal{X}}, <)$ *is gap-free and* $\beta_{\mathbf{G}}$ *winds* $(\overline{\mathcal{X}}, <)$ *to* $(\mathcal{X}, \triangleleft)$*. If* $\beta_{\mathbf{G}}$ *is loop-free, then* $\beta_{\mathbf{G}}$ *maps* $\lessdot$ *surjectively to* $\triangleleft\!\cdot$ *.*

> **Proof:** Let $\overline{x} \lessdot \overline{y}$; then either a) $\overline{x} \veebar \overline{y}$ or b) $\overline{y} \barwedge \overline{x}$. Consider case a); b) is analogous. We then have $\mathbf{G}\overline{x} \lessdot \mathbf{G}\overline{y}$; since $\overline{y} < \mathbf{G}\overline{y}$, the assumptions imply $\overline{y} < \mathbf{G}\overline{x} \lessdot \mathbf{G}\overline{y}$. Since $\beta_{\mathbf{G}}$ is loop-free, there exists $\overline{z} \in \overline{\mathcal{X}}$ such that $\overline{y} < \overline{z} < \mathbf{G}\overline{x} \lessdot \mathbf{G}\overline{y}$, and hence $\triangleleft(x, y, z)$, where $\beta_{\mathbf{G}}(\overline{x}) = x, \beta_{\mathbf{G}}(\overline{y}) = y$, and $\beta_{\mathbf{G}}(\overline{z}) = z$ as usual. Now, suppose there exists $u \in \mathcal{X}$ such that $\triangleleft(x, u, y)$; then there must be a $\overline{u} \in \beta_{\mathbf{G}}^{-1}(\{u\})$ such that $\overline{x} < \overline{u} < \overline{y}$, contradicting the assumption $\overline{x} \lessdot \overline{y}$. Hence we have $x \triangleleft\!\cdot\, y$. □

Figure 6 shows that gaps may be responsible for loss of successor relations in a winding: one has successor relation $a_0 \lessdot w_0$ but no $a \triangleleft\!\cdot\, w$, not even $a \parallel w$ in fact; the reason is that there is no $n > 0$ such that $w_0 < a_n$. The cyclic order obtained under the winding degenerates into two separate components, all links between the subsets are lost. The right hand side shows an extension of the partial order in which all gaps have been filled by new elements e_k, f_k, $k \in \mathbb{Z}$. But now there is no winding at all anymore: since $e_k \mathsf{co} e_l$ for all $k \neq l$, there exists only the trivial shift for this partial order, and no winding. That is, filling the gaps helped detect an intrinsic lack of symmetry of the partial order. Note that dense orders (i.e. where $\lessdot$ is empty) are gap-free.

3.2 Separators in Partial Orders

Definition 34 *In a partial order* Π*, a maximal clique of* $\parallel$ *is called a* **line***, and the set of lines is denoted* $\mathcal{L}(\Pi)$*. Dually, let* $\mathcal{C}(\Pi)$ *be the set of* **cuts** *of* Π*, i.e. its maximal co-cliques.*

A cut $\mathbf{c}$ can be viewed as a **global** state of the set of local processes that are represented by lines.

The intersection of $\mathbf{c}$ with line L then yields the state of L, seen as a local process, on the "snapshot" $\mathbf{c}$. This leads to the question whether $\mathbf{c}$ *does* intersect every L. We define:

Definition 35 *Let* $\Pi = (\mathcal{X}, <)$ *be a partial order. Then we say that* $\mathbf{c} \in \mathcal{C}(\Pi)$ *is a* **separator**[5] *iff* $\mathbf{c} \cap L \neq \emptyset$ *for all* $L \in \mathcal{L}(\Pi)$*.* Π *is* **weakly separable** *iff it has a separator, and* **(strongly) separable** *iff every cut of* Π *is a separator.*

[5] called a **true cut** in [Ste98]

Strong separability of partial orders has been extensively studied, see for instance [BF88] (where it is called K-density). It should be noted that separability can be destroyed by gaps in the sense introduced below, or by the presence of infinite lines; [BF88] gives an extensive tableau on strong separability. We add the following result on **weak** separability:

Lemma 36 *Let $\mathcal{X}$ be a non-empty set and $\Pi = (\mathcal{X}, <)$ a partial order. Then there exist a weakly separable super-poset Π of Π.*

Proof: Consider any total extension $\Pi_{tot} = (\mathcal{X}, <_{tot})$ of Π, and fix $x \in \mathcal{X}$; we will insert "x-witnesses" into all lines of Π. Let $\mathcal{Y}$ be a set such that $\mathcal{X} \cap \mathcal{Y} = \emptyset$, and $\psi : \mathcal{L}(\Pi) \to \mathcal{Y}$, $L \mapsto y_L$ injective. Set

$$\begin{aligned} \mathcal{X}^\sharp &\triangleq \mathcal{X} \cup \psi(\mathcal{L}(\Pi)) \\ <^\sharp &\triangleq < \cup \{(u, y_L) \mid u \in L \wedge u <_{tot} x\} \cup \{(y_L, u) \mid u \in L \wedge x <_{tot} u\} \\ &\quad \cup \{(u, v) \mid u < v \wedge u <_{tot} x <_{tot} v\}. \end{aligned}$$

Then $\Pi^\sharp \triangleq (\mathcal{X}^\sharp, <^\sharp)$ is a *SuperPO* of Π. By construction, $\mathbf{c}_x \triangleq \{y_L \mid L \in \mathcal{L}(\Pi)\}$ is a separator of $\Pi^\sharp$. □

Note that Lemma 36. does not carry over to cyclic orders, as we will see below. Theorem 313 shows that this property is intrinsically linked to orientability, as indicated by the results in [Ste98] in finitary settings. In order to generalize the notion of cycle from graph theory, we need first the following auxiliary notions:

Definition 37 *Let $\Pi = (\mathcal{X}, <)$ be a poset, $x, y \in \mathcal{X}$, and $x < y$. The* **intervals** *spanned by x and y are*

$$\begin{aligned}]x, y[&\triangleq \{z \mid x < z < y\} \\ [x, y[&\triangleq]x, y[\cup \{x\}, \\ ]x, y] &\triangleq]x, y[\cup \{y\} \\ [x, y] &\triangleq]x, y] \cup \{x\}. \end{aligned}$$

For $x \not< y$, $[x, y] =]x, y] = [x, y[=]x, y[\triangleq \emptyset$. An **edge** *of Π is a li-clique $\mathcal{E}$ such that there exist $a, b \in \mathcal{X}$ satisfying $a \ \mathsf{li} \ b$ and $\mathcal{E} \subseteq [a, b]$, and $\mathcal{E}$ is maximal relative $[a, b]$, i.e. for any $u \in [a, b]$ such that $v \ \mathsf{li} \ u$ for all $v \in \mathcal{E}$, one has $u \in \mathcal{E}$ (observe that also $a, b \in \mathcal{E}$). Let $start(\mathcal{E}) \triangleq a$ and $end(\mathcal{E}) \triangleq b$ be the start and end elements of $\mathcal{E}$, respectively.*

Moving from acyclic to cyclic orders, we have to consider separately **rounds** and **cycles**, respectively. Intuitively, cycles are arbitrary closed paths, while rounds are special cycles that 'wrap around the structure only once'.A **cycle** is composed of a sequence of **edges**, i.e. segments of a total cyclic sub-order (compare Def. 37).

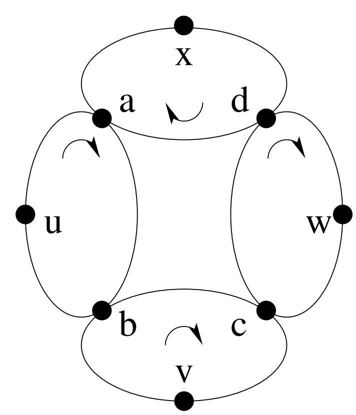

Fig. 7. A *ROCO.*

Definition 38 *For a CyO $\Gamma = (\mathcal{X}, \lhd)$ and $a \ \mathsf{li} \ b$, define :*

$$\begin{aligned}]a, b[&\triangleq \{x \in \mathcal{X} \mid \lhd(a, x, b)\} \\ [a, b[&\triangleq]a, b[\cup \{a\}, \\ ]a, b] &\triangleq]a, b[\cup \{b\} \\ [a, b] &\triangleq]a, b] \cup \{a\}. \end{aligned}$$

Definition 39 *An* **edge** *of Γ is a li-clique $\mathcal{E}$ such that there exist $a, b \in \mathcal{X}$ with $a \ \mathsf{li} \ b$ and $\mathcal{E} \subseteq [a, b]$, and $\mathcal{E}$ is* ***maximal*** *relative $[a, b]$: for any $u \in [a, b]$ such that $\forall \ v \in \mathcal{E} : \ v \ \mathsf{li} \ u$, one has $u \in \mathcal{E}$.*

If $\mathcal{E}$ is an edge, set $start(\mathcal{E}) \triangleq a$ and $end(\mathcal{E}) \triangleq b$. Note that, as in the acyclic case, $start(\mathcal{E}) \in \mathcal{E}$ and $end(\mathcal{E}) \in \mathcal{E}$, and every edge $\mathcal{E}$ can be represented as the intersection of an appropriate round $O_{\mathcal{E}}$ with $[start(\mathcal{E}), end(\mathcal{E})]$. So we are ready to define:

Definition 310 (Cycles of a *ROCO***)** *Let $\Gamma = (\mathcal{X}, \lhd)$ be a ROCO. $\mathcal{C} \subseteq \mathcal{X}$ is a* **cycle** *of Γ iff there exist edges $\mathcal{E}_1, \ldots, \mathcal{E}_n$ such that $start(\mathcal{E}_1) = end(\mathcal{E}_n)$, $start(\mathcal{E}_{i+1}) = end(\mathcal{E}_i)$ for $1 \leqslant i \leqslant n-1$, and $\mathcal{C} = \bigcup_{i=1}^{n-1} \mathcal{E}_i$. Denote as $\mathcal{D}(\Gamma)$ the set of cycles of Γ.*

So every round in Γ is a cycle, but the converse is not true:: in Figure 5, the cycle through transitions $\alpha, \zeta, \theta, \lambda, \beta, \delta, \eta, \kappa$ is not a round since α *co* β, etc.

Definition 311 *Let Γ be a ROCO. A cut* **c** *is called a* **separator** *iff $\mathbf{c} \cap O \neq \emptyset$ for all $O \in \mathcal{O}(\Gamma)$, and a* **cycle separator** *iff $\mathbf{c} \cap \gamma \neq \emptyset$ for all $\gamma \in \mathcal{D}(\Gamma)$; Γ is called* **weakly (cycle) separable** [6] *iff there exists a (cycle) separator $\mathbf{c} \in \mathcal{C}(\Gamma)$, and* **strongly (cycle) separable** *iff all its cuts are (cycle) separators. If there is a superstructure cyclic order Γ' of Γ such that Γ' is (strongly) cycle separable, then Γ is called* **(strongly) saturable.**

Thus cycle separation implies round separation etc., but the converse is not true: Figure 7 shows a cyclic order with a cut $\{u, v, w, x\}$ that is a separator since it intersects each round $\{b, c, v\}$, $\{c, dw\}$, $\{d, a, x\}$, and $\{u, a, b\}$, but is not a cycle separator since it fails to intersect the cycle $\langle a, b, c\rangle$. Note: that structure is nonetheless weakly cycle separable since $\{a, c\}$ is a cycle separator.

To conclude, consider the following special case: Since a total *CyO* contains only one cycle, every singleton is a cycle separator; hence, since all cuts are singletons in total cyclic orders, we obtain:

Lemma 312 *Every total CyO is strongly separable.*

Before turning to the orientability of general *ROCO*ss, observe that every total *CyO* has a winding representation : in fact, let $\{x\}$ be a strong separator of $(\mathcal{X}, \lhd)$ (which must exist according to Lemma312). Then one obtains a winding representation by taking copies $(x_k)_{k\in\mathbb{Z}}$ of x, "gluing" successive copies $\mathcal{X}_k$ of $\mathcal{X}$ "between" x_k and x_{k+1}. More formally, set $\overline{\mathcal{X}} \triangleq \mathcal{X} \times \mathbb{Z}$, and let $<$ be the smallest transitive binary relation on $\overline{\mathcal{X}}$ such that for all $k \in \mathbb{Z}$ and $y \in \mathcal{X}\backslash\{x\}$, one has $x_k < y_k < y_{k+1}$; then one checks that $(\overline{\mathcal{X}}, <)$ is a winding representation of $(\mathcal{X}, \lhd)$. Looking at Lemma312 once again, it appears that winding representations and separability might be linked in a more general way : in fact, the following theorem 313 establishes exactly this, using a construction that extends the informal "unwinding" sketched above, from total *CyO* s to the general case.

3.3 Characterization of Orientable *ROCO*s

We have now completed the preparations for our central theorem. The result shows the connection between cycle separability, winding representability, and orientability; it characterizes *all* orientable *ROCO*s, regardless of their cardinality.

Theorem 313 *Let $\mathcal{X} \neq \emptyset$, and $\Gamma = (\mathcal{X}, \lhd)$ a ROCO. Then the following are equivalent:*

1. *Γ is strongly saturable;*
2. *there exists a winding representation for Γ, i.e. a partial order $\Pi = (\overline{\mathcal{X}}, <)$ with shift* **G** *such that $\mathcal{X} = \overline{\mathcal{X}}_{/\mathbf{G}}$, and $\phi_{\mathbf{G}}$ winds Π to Γ ;*
3. *Γ is orientable.*

Proof: (1) $\Rightarrow$ (2): By Lemma 29, we only have to consider the case where Γ is itself weakly cycle separable. So let $\mathbf{c} \in \mathcal{C}(\Gamma)$ be a cycle separator; we have to construct a partial order wound to Γ. Set $\overline{\mathcal{X}} \triangleq \mathcal{X} \times \mathbb{Z}$; we write x_k for (x, k). Let $\mathbf{G} : \overline{\mathcal{X}} \to \overline{\mathcal{X}}$ be given by $\mathbf{G}x_k = x_{k+1}$ for $k \in \mathbb{Z}$. Define relation $\prec$ on $\overline{\mathcal{X}}$ by:

$$\begin{aligned} \mathcal{R}_0 \triangleq \quad & \{(a_0, b_0) \mid a \in \mathbf{c} \ \wedge \ a \ \text{li} \ b\} \\ & \cup \ \{(a_0, b_0) \mid \exists\, x \in \mathbf{c} : \ \lhd(a, b, x)\} \\ & \cup \{(a_k, a_{k+l}) \mid a \in \mathcal{X}, \ k \in \mathbb{Z}, \ l \in \mathbb{N}\} \\ \prec \triangleq & \{(a_{l+k}, b_{m+k}) \mid a_l \ \mathcal{R}_0 \ b_m, \ k \in \mathbb{Z}\}\,. \end{aligned}$$

[6] In a setting that corresponds to discrete cyclic orders, cycle separability has been introduced as "F-density" in [KS97,Ste98]

By construction, *(i)* $\forall\, u,v \in \mathcal{X}\ \forall\, k,m \in \mathbb{Z}:\ u_k \prec\ v_m \Rightarrow\ k \leqslant m$, and *(ii)* the set of minimal elements with index 0 is $\mathbf{c} \times \{0\}$, i.e.

$$\{x \in \mathcal{X} \times \{0\} \mid \forall\, y \in (\mathcal{X}\backslash\{x\}) \times \{0\} : \neg\,(y \prec x)\} \\ = (\mathbf{c} \times \{0\});$$

Now, let $<$ be the transitive closure of $\prec$. We claim that $<$ is a partial order. It suffices to show that $<$ is irreflexive; so assume $u_k < u_k$ for some $u \in \mathcal{X}$, $k \in \mathbb{Z}$. Without loss of generality, $k = 0$. Then there exist $n \in \mathbb{N}$, elements $y^1, \ldots, y^n \in \mathcal{X}$, and indices $k_1, \ldots, k_n \in \mathbb{N}$ such that (i) $u_0\ \prec\ y^1_{k_1}$, (ii) $y^i_{k_i}\ \mathcal{R}_1\ y^{i+1}_{k_{i+1}}$ for $i \in \{1, \ldots, n-1\}$, and (iii) $y^n_{k_n}\ \mathcal{R}_1\ u_0$. If $u \in \mathbf{c}$, this is impossible for any value of ν since it contradicts *(i),(ii)*. So assume $u \notin \mathbf{c}$, and let n be minimal with the above properties; then by 3.3.), $k_i = 0$ for all $1 \leqslant i \leqslant n$. Now, since $y^i \notin \mathbf{c}$ for all i by the choice of the y^i, there exist $n{+}1$ elements $x^i \in \mathbf{c}$, $1 \leqslant i \leqslant n+1$, that satisfy $\lhd(u, y^1, x^1)$ and $\lhd(y^n, u, x^{n+1})$, and $\lhd(y^{i-1}, y^i, x^i)$ for $2 \leqslant i \leqslant n$. So one can choose edges $\mathcal{E}_j$, $1 \leqslant j \leqslant n-1$, such that $start(\mathcal{E}_1) = end(\mathcal{E}_n) = u$, and $end(\mathcal{E}_j) = start(\mathcal{E}_{j+1}) = y^j$, and such that the cycle $\mathcal{C}_u \triangleq \bigcup_{j=1}^n \mathcal{E}_j$ does not intersect $\mathbf{c}$ (since no $\mathcal{E}_j$ does); this contradicts the assumption that $\mathbf{c}$ is cycle separating. Hence $\mathit{\Pi} = (\overline{\mathcal{X}}, <)$ is a poset; moreover, $\mathbf{G}$ is a shift for $\mathit{\Pi}$, and by construction, the mapping $\beta_{\mathbf{G}} : (\mathcal{X} \times \mathbb{Z}) \to \mathcal{X}$, $(x,z) \mapsto x$, winds $\mathit{\Pi}$ to $\mathit{\Gamma}$.
(2) ⇒ (3): Let $\mathit{\Pi}_\sharp$ be a weakly separable *SuperPO* of $\mathit{\Pi}$, and $\mathbf{c}_0$ a separator of $\mathit{\Pi}_\sharp$. Let $\mathbf{c}_k$ be the cut $\mathbf{c}_k \triangleq \mathbf{G}^k \mathbf{c}_0$, and define $\mathcal{U}_k \triangleq \mathcal{U}^\sharp_k \cap \overline{\mathcal{X}}$, where

$$\mathcal{U}^\sharp_k \triangleq \bigcup_{\substack{\overline{y}_k \in \mathbf{c}_k \\ \overline{y}_{k+1} \in \mathbf{c}_{k+1}}} [\overline{y}_k, \overline{y}_{k+1}[.$$

Then the $\mathcal{U}_k$ are pairwise disjoint and cover $\overline{\mathcal{X}}$. Moreover, $<$ induces a partial order $<_k$ on $\mathcal{U}_k$. Now, set $\mathit{\Pi}_k \triangleq (\mathcal{U}_k, <_k)$; then $\mathbf{G}$ induces, for every $n,m \in \mathbb{Z}$, an order isomorphism $\mathbf{G}_{n,m} : \mathcal{U}_n \to \mathcal{U}_m$ from $\mathit{\Pi}_n$ to $\mathit{\Pi}_m$. By Szpilrajn's Theorem, there exists a total ordering $\mathit{\Pi}^{tot}_0$ on $\mathcal{U}_0$ embedding $<_0$. Now, the mapping $\sigma : \overline{\mathcal{X}} \to \overline{\mathcal{X}}$, given by $\sigma_{|\mathcal{U}_z} \triangleq \mathbf{G}_{0,z} \circ \mathsf{id}_0 \circ \mathbf{G}_{z,0}$ for $z \in \mathbb{Z}$, is a well-defined order isomorphism; σ embeds $\mathit{\Pi}$ into a total order $\mathit{\Pi}^{tot}$ on $\overline{\mathcal{X}}$, whose restriction to $\mathcal{U}_z$ is $\mathit{\Pi}$. Then, by construction, $\mathbf{G} \circ \sigma = \sigma \circ \mathbf{G}$, and $\mathit{\Pi}^{tot}$ under the winding $\beta_{\mathbf{G}\circ\sigma}$ induces an orientation of $\mathit{\Gamma}$.
(3) ⇒ (1): If $\mathit{\Gamma}$ is total, we are done by Lemma 312. So assume $\mathit{\Gamma}$ is not total, and let $\mathit{\Gamma}_{tot} = (\mathcal{X}, \lhd_{tot})$ be an orientation of $\mathit{\Gamma}$. As in the proof of Lemma 36, fix $x \in \mathcal{X}$, let $\mathcal{Y}$ be a set disjoint from $\mathcal{X}$, and $\psi : \mathcal{O} \to \mathcal{Y}$ injective; then, set $A_x \triangleq \{O \in \mathcal{O}(\mathit{\Gamma}) \mid x \notin O\}$ and $\mathcal{X}_x \triangleq \mathcal{X} \cup \psi(A_x)$, and let $\iota_x : \mathcal{X} \to \mathcal{X}_x$ be the insertion of $\mathcal{X}$ into $\mathcal{X}_x$. For every cycle $O \in A_x$, set $x_O \triangleq \psi(O)$; further, for every edge $[s_i, e_i]$ of O, let $\lhd_x(s_i, x_O, e_i)$ if $\lhd_{tot}(s_i, x, e_i)$, and $\lhd_x(s_i, e_i, x_O)$ otherwise. $\mathit{\Gamma}_x = (\mathcal{X}_x, \lhd_x)$ is a superstructure of $\mathit{\Gamma}$. We have $\mathcal{O}(\mathit{\Gamma}_x) = (\mathcal{O}(\mathit{\Gamma})\backslash A_x) \cup \{O \cup \{x_O\} \mid O \in A_x\}$. Moreover, $\mathbf{c}_x \triangleq \psi(\mathcal{O}(\mathit{\Gamma}))$ is a *co*-clique by construction. $\mathbf{c}_x$ is also maximal with this property since, for every round $O \in \mathcal{O}(\mathit{\Gamma})\backslash A_x$, one has $\mathbf{c}_x \cap O = \{x\}$, and for all $O \in A_x$, $\mathbf{c}_x \cap O \cup \{x_O\} \ni \{x_O\}$; this also shows that $\mathbf{c}_x$ is a separator. We claim that $\mathbf{c}_x$ is also a cycle separator for $\mathcal{X}_x$: Let $\mathcal{C} = \bigcup_{i=1}^k \mathcal{E}_i$ be a cycle; we have to show $\mathbf{c}_x\ \cap\ \mathcal{C}\ \neq\ \emptyset$. If there is an index $1 \leqslant j \leqslant k$ such that $start(\mathcal{E}_j) \in \mathbf{c}_x$ or $end(\mathcal{E}_j) \in \mathbf{c}_x$, we are done. Otherwise, we will show that there exists at least one index $1 \leqslant \nu \leqslant k$ such that $\lhd_{tot}(start(\mathcal{E}_\nu), x, end(\mathcal{E}_\nu))$. In fact, suppose this is not true. Denote, for all i, $s_i \triangleq start(\mathcal{E}_i)$ and $e_i \triangleq end(\mathcal{E}_i)$. Then we have $\mathit{\Gamma}_{tot}(x, s_i, s_{i+1})$ for all $1 \leqslant i \leqslant n-1$; by transitivity, this implies $\mathit{\Gamma}_{tot}(x, s_1, s_n)$. But since $e_n = s_1$, our assumption also implies that $\mathit{\Gamma}_{tot}(x, s_n, s_1)$, a contradiction. For the ν thus found, let $O \in \mathcal{O}(\mathit{\Gamma})$ such that $\mathcal{E}_\nu = O \cap [start(\mathcal{E}_\nu), end(\mathcal{E}_\nu)]$; then $\lhd_x(start(\mathcal{E}_\nu), x_O, end(\mathcal{E}_\nu))$ by construction, so $x_O \in \mathcal{E}_\nu$, and hence $(\mathcal{C}_x \cap \mathcal{C}) \neq \emptyset$. □

We close by some remarks on the results:

Remark 1. Inspection of Part "**(1) ⇒ (2)**" of the proof of Theorem 313 shows that for a given $\mathit{\Gamma}$ and fixed cycle separator $\mathbf{c}$ for $\mathit{\Gamma}$, there is a unique unwinding $\mathit{\Pi}_{\mathbf{c}}(\overline{\mathcal{X}}_{\mathbf{c}}, <)$ and associated shift $\mathbf{G}$ obtained from the above construction; denote this automorphism as $\mathbf{G}(\mathit{\Gamma}, \mathbf{c})$. In this, any separator $\overline{\mathbf{c}}'$ of $\mathit{\Pi}$ will be wound to cycle separator $\mathbf{c}'$ of $\mathit{\Gamma}$; all cycle separators $\tilde{\mathbf{c}}$ obtained in this way are equivalent to $\mathbf{c}$ in the sense that there exists an isomorphism $\mathit{\Psi}_{\mathbf{c},\tilde{\mathbf{c}}}$ from $\mathit{\Pi}_{\mathbf{c}}$ to $\mathit{\Pi}_{\tilde{\mathbf{c}}}$ such that

$$\mathit{\Psi}_{\mathbf{c},\tilde{\mathbf{c}}} \circ \mathbf{G}(\mathit{\Gamma}, \mathbf{c}) = \mathbf{G}(\mathit{\Gamma}, \tilde{\mathbf{c}})$$
$$\mathit{\Psi}_{\mathbf{c},\tilde{\mathbf{c}}} \circ \beta_{\mathbf{G}(\mathit{\Gamma},\mathbf{c})} = \beta_{\mathbf{G}(\mathit{\Gamma},\tilde{\mathbf{c}})}.$$

Remark 2. In Alles, Nešetřil, Poljak [ANP91], a CyO $\varGamma = (\mathcal{X}, \lhd)$ is generated from a poset $\varPi = (\mathcal{X}, <)$ on the same set $\mathcal{X}$ by simply taking the *rotational (symmetric) closure*; that is, set

$$\lhd^\circ := \{(a, b, c) \mid a \; < \; b \; < \; c\},$$

and let $\lhd$ be the smallest superset of $\lhd^\circ$ that is rotationally symmetric, i.e., $(x, y, z) \in \lhd$ implies $(y, z, x) \in \lhd$. This is **not at all** equivalent to windings. Obviously, the rotational closure acts injectively, so the cyclic order has as many elements as its generating poset, whereas all pre-images under windings are infinite. But even the restriction to one section of the wound poset does not yield an isomorphic cyclic order: in Figure 5, consider only the elements with index 0. Then the cyclic order generated by rotational closure contains the triple $(\alpha_0, \gamma_0, \lambda_0)$, but $(\alpha, \gamma, \lambda)$ does not belong to the cyclic order winding since $co(\lambda_0, \alpha_1)$. More generally, one has from the construction that, for any $a < b$ in a poset $\varPi = (\mathcal{X}, <)$, the structure $\varGamma = (\mathcal{X}, \lhd)$ generated from $\varPi$ by rotational closure satisfies $\neg(a \; co \; b)$. This means also that the orientable cyclic order in Figure 5 cannot be obtained as a rotational closure ! As a consequence, we also see that orientability cannot be characterized by rotational closure.

Gaps revisited. Recall that in the presence of gaps in a partial order, winding may lead to cyclic orders that do not reflect all successor relations. However, the reverse is not possible:

Lemma 314 *Let $\varGamma = (\mathcal{X}, \lhd)$ a ROCO, and let $\varPi = (\overline{\mathcal{X}}, <)$ and* $\mathbf{G}$ *be the poset and shift, respectively, constructed in the first part of the proof of Theorem 313. Then $\beta_{\mathbf{G}}$ preserves $<\!\cdot$, i.e. $\overline{x} <\!\cdot \overline{y}$ implies $x \lhd\!\cdot\, y$.*

Proof: By construction of $\varPi = (\overline{\mathcal{X}}, <)$, $\overline{x} <\!\cdot \overline{y}$ implies $\overline{x} \prec \overline{y}$, and therefore $x \parallel y$. Now suppose $\lhd(x, z, y)$; then there is $\overline{z} \in \overline{\mathcal{X}}$ such that $\overline{x} \prec \overline{z} \prec \overline{y}$, contradicting the assumption that $\overline{x} <\!\cdot \overline{y}$. □

4 Conclusion

In this article, we have studied the connection between partial orders with shifts and cyclic ordering; a central result was the equivalence - under mild saturation conditions - between oriented cyclic ordering and the existing of a winding.

This connection allows to reduce problems concerning cyclic orders to known ones for partial orders. Supposing that a separator $\mathbf{c}$ is known for $\varGamma = (\mathcal{X}, \lhd)$, an unwinding prefix $\varPi_1$ from $\mathbf{c}_0$ to $\mathbf{c}_1$ is sufficient, e.g., for constructing a total cyclic order that embeds $\varGamma$, by computing a totalization of $\varPi_1$ that respects the winding morphism on $\mathbf{c}_0$ and $\mathbf{c}_1$.

Finally, note that the saturability properties required in Theorem 313 resemble Dedekind cuts; see [PeSm86] for a discussion of Dedekind continuity in the context of partial orders.

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