# A polynomial space construction of tree-like models for logics with local chains of modal connectives

Stéphane Demri [1]

*Lab. Spécification et Vérification*
*École Norm. Supér. de Cachan and CNRS UMR 8643*
*61 Av. Président. Wilson, 94235 Cachan Cedex, France* [2]

**Abstract**

The LA-logics ("logics with Local Agreement") are polymodal logics defined semantically such that at any world of a model, the sets of successors for the different accessibility relations can be linearly ordered and the accessibility relations are equivalence relations. In a previous work, we have shown that every LA-logic defined with a finite set of modal indices has an **NP**-complete satisfiability problem. In this paper, we introduce a class of LA-logics with a countably infinite set of modal indices and we show that the satisfiability problem is **PSPACE**-complete for every logic of such a class. The upper bound is shown by exhibiting a tree structure of the models. This allows us to establish a surprising correspondence between the modal depth of formulae and the number of occurrences of distinct modal connectives. More importantly, as a consequence, we can show the **PSPACE**-completeness of Gargov's logic DALLA and Nakamura's logic LGM restricted to modal indices that are rational numbers, for which the computational complexity characterization has been open until now. These logics are known to belong to the class of information logics and fuzzy modal logics, respectively.

*Key words:* computational complexity, modal logic, Ladner-like algorithm, local agreement.

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[2] Email: `demri@lsv.ens-cachan.fr`.

## 1 Introduction

**Complexity of modal logics.** The worst-case complexity of modal logics is a flourishing research activity not only because of the ever growing number of new modal logics (program logics, temporal logics, description logics, information logics, . . . ) but also because known techniques in theoretical computer science can be applied to problems for such logics. The best illustration of this phenomenon has been the design of automata-theoretical decision procedures for dynamic and temporal logics (see e.g. (VW86; VW94; EJS01; Var98)) to quote two types of logics. Such an automata-based approach has been fruitful for characterizing **EXPTIME**-complete logics but it does not adapt easily to **PSPACE**-complete logics as far as satisfiability problems are concerned. For model-checking problems the situation differs essentially (VW94), e.g. for PLTL model-checking. In order to establish a **PSPACE** upper bound for the satisfiability problem of modal logics, one of the best known methods is due to Ladner (Lad77) in which trees with branches of polynomial length are explored. Such a method admits numerous technical variants either based on decision procedures from analytic proof systems (see e.g. (HM92; Vig00; Dem01)) or based on semantics-based algorithms (see e.g. (Lad77; Spa93b; Dem00; Bal01; BS01)). When the models admit a tree-like structure, in order to establish the **PSPACE** upper bound, the main difficulty is to show that the path depth into the tree is polynomially bounded (a polynomially bounded branching width is easier to obtain). Unlike the **EXPTIME** decision procedure based on automata machinery, no path of exponential length needs to be constructed. In the present paper, we consider a class of multimodal logics that are shown to be **PSPACE**-complete. The lower bound is established by a reduction from QBF whereas the upper bound is shown via a Ladner-like algorithm.

**Chains of S5 modal connectives.** A standard result for multimodal logics due to (HM92) states that the multimodal logic with $n$ independent S5 connectives has a **PSPACE**-complete satisfiability problem as soon as $n$ is greater than two. This upper bound is preserved if we consider a countably infinite set of S5 modal connectives instead of a finite one. Surprisingly, in the finite case, the problem becomes **NP**-complete if the equivalence relations are ordered locally (Dem98), that is to say, in the models for every world $w \in W$, for all relation indices $i, j \in \{1, \ldots, n\}$, either $R_i(w) \subseteq R_j(w)$ or $R_j(w) \subseteq R_i(w)$. In other words, for every world $w \in W$, there is a permutation $\sigma$ on $\{1, \ldots, n\}$ such that

$$R_{\sigma(1)}(w) \subseteq \ldots \subseteq R_{\sigma(n)}(w).$$

The binary relations $R_i$ and $R_j$ are said to be in *local agreement* (Gar86). The **NP**-completeness is preserved if we consider chains of relations of the form

$$R_1 \subseteq R_2 \subseteq \ldots \subseteq R_n,$$

that is to say we enforce the permutation $\sigma$ to be identity. However, it is open whether the problem remains in **NP** if a countably infinite set of connectives is considered.

**Our contribution.** We introduce a class of multimodal logics with a countably infinite set of modal connectives characterized by models in which the local agreement condition holds between any two equivalence relations of the models. The logics in the class, called *nice unbounded LA-logics*, differ by the admitted permutations $\sigma$ that can locally occur. For instance, such a class contains the logic for which the relations of the models satisfy

$$R_0 \subseteq R_1 \subseteq \ldots \subseteq R_n \subseteq \ldots.$$

We show that every logic in that class has a **PSPACE**-complete satisfiability problem. The lower bound is obtained by reducing QBF, a well-known **PSPACE**-complete problem (Sto77). The **PSPACE** upper bound is shown with a sophisticated Ladner-like procedure. Hence, the **NP** upper bound of the finite case does not extend to the infinite case (unless **PSPACE** = **NP**). This is in sharp contrast with the situation for independent S5 modal connectives and is reminiscent of the complexity of the basic modal logic K restricted to formulae of fixed modal depth: K satisfiability is **PSPACE**-complete (Lad77) whereas for every fixed $k \geq 0$, K satisfiability restricted to formulae of modal depth at most $k$ is **NP**-complete (Hal95). So, in nice unbounded LA-logics the number of distinct modal connectives occurring in a formula can be viewed as a measure of the modal depth as far as complexity issues are concerned. Actually, such a statement can be made even more precise for certain nice unbounded LA-logics (see e.g. Corollary 2.3). As applications of our main result, the logic DALLA introduced in (Gar86) (see also (BO99) for an equivalent logic defined with relative accessibility relations) is shown to be **PSPACE**-complete. The best known upper bound was **NEXPTIME** and the best known lower bound was **NP**. Moreover, Nakamura's logic LGM (Nak93) restricted to modal indices in the set of rational numbers is also shown to be **PSPACE**-complete.

The reason why DALLA is of interest to model reasoning in presence of incomplete information is the following. The Data Analysis Logic DAL (FdCO85) is the paradigm logic for reasoning about indiscernibility relations derived from information systems (Paw91). Unfortunately, very few results are known for DAL (its decidability status for example). That is why, variants of DAL have been proposed for which more results have been established while preserving some important features of DAL (see e.g. (Gar86; AT89; Bal96)). One of such logics is the logic DALLA introduced in (Gar86). An axiomatization is proposed in (Gar86) and the first decidability proof appeared in (Dem96). More about DAL and DALLA can be found in the forthcoming (DO) (see also Section 5).

It is possible to adapt our results to logics for which the relations in the models are not necessarily equivalence relations but the **PSPACE**-completeness result is far less interesting since the standard modal logics K, T, B, and S4 are already known to be **PSPACE**-complete (Lad77; CL94).

**Related work.** The paper uses the presentation of Ladner-like algorithms from (Spa93b) as it is also done in (Dem00). More generally, the **PSPACE** procedure designed in the paper is closely related in spirit to algorithms presented in (Lad77; HM92; Spa93b; BS01).

**Plan of the paper.** The rest of the paper is structured as follows. In Section 2, we introduce the class of LA-logics, remarkable elements in it and subclasses that are of particular interest in our computational complexity investigations. In Section 3, we show that every nice unbounded LA-logic has a **PSPACE**-hard satisfiability problem by taking advantage of the tree-like structure of the models. A Ladner-like algorithm for nice unbounded LA-logics is studied in Section 4, which allows us to conclude that every nice unbounded LA-logic has a satisfiability problem in **PSPACE**. In Section 5, we show how the **PSPACE**-completeness for the auxiliary logics DALLA′ and LGM′ can be lifted to Gargov's logic DALLA and Nakamura's logic LGM, respectively. Section 6 contains concluding remarks.

Some of the proofs are relegated to the appendix in order to facilitate the reading.

## 2 Logics with local agreement

In this section, we introduce the class of *nice unbounded LA-logics* which are modal logics defined semantically with a countable number of modal connectives. Some of the logics in this class are of special interest since they can be related to the logics DALLA (Gar86; DO98) and LGM (Nak93). Studying a class of modal logics in which some members are distinguished is a natural approach in modal logic theory, see e.g. (Sah75; Spa93a; Kra96) to quote only a few examples.

### *2.1 Language*

Given the set $\mathrm{PRP} = \{p_i, r_i, d_i : i \geq 0\}$ of propositional variables, the set FOR of modal formulae $\phi$ is inductively defined as follows:

$$\phi \ ::= \ p_i \ | \ r_i \ | \ d_i \ | \ \neg\phi \ | \ \phi \wedge \phi' \ | \ \phi \Rightarrow \phi' \ | \ \phi \Leftrightarrow \phi' \ | \ \phi \vee \phi' \ | \ [i]\phi,$$

where $i \in \mathbb{N}$. For every $i \geq 0$, we write $\mathrm{FOR}_i$ to denote the fragment of FOR restricted to modal connectives in $\{[0], \ldots, [i]\}$. All the natural numbers occurring in formulae are encoded in binary writing as a bit-string. For the sake of simplicity, we always write $n$ in decimal representation. Standard abbreviations include $\langle i \rangle$, $\top$, $\perp$. The set $\mathrm{sub}(\phi)$ of *subformulae* of the formula $\phi$ is defined in the standard way. We write $\mathbb{N}(\phi)$ to denote the finite subset of $\mathbb{N}$ of modal indices occurring in $\phi$. For instance, for every $i \geq 0$, for every $\phi \in \mathrm{FOR}_i$, $\mathbb{N}(\phi) \subseteq \{0, \ldots, i\}$. The *modal depth* of an occurrence of a formula $\psi$ in $\phi$ is the number of occurrences of elements of the form $[i]$ in $\phi$ such that $\psi$ is in their scope. We write $\mathrm{md}(\phi)$ to denote the *modal depth* of the formula $\phi$, that is the maximum of the modal depths of the subformulae of $\phi$.

### *2.2 Semantics*

A *frame* is a structure $\mathcal{F} = \langle W, (R_i)_{i \in \mathbb{N}} \rangle$ such that $W$ is a non-empty set and for every $i \in \mathbb{N}$, $R_i$ is a binary relation on $W$. Similarly, an *i-frame* is a structure of the form $\langle W, R_0, R_1, \ldots, R_i \rangle$. A *model* [resp. *i-model*] is a structure $\mathcal{M} = \langle W, (R_i)_{i \in \mathbb{N}}, V \rangle$ [resp. $\mathcal{M} = \langle W, R_0, \ldots, R_i, V \rangle$] such that $\mathcal{F} = \langle W, (R_i)_{i \in \mathbb{N}} \rangle$ [resp. $\mathcal{F} = \langle W, R_0, \ldots, R_i \rangle$] is a frame [resp. an $i$-frame] and $V$ is a valuation $V : \mathrm{PRP} \rightarrow 2^W$. The satisfiability relation $\models$ is defined inductively in the usual way:

- $\mathcal{M}, x \models p \overset{\text{def}}{\Leftrightarrow} x \in V(p)$ for every $p \in \mathrm{PRP}$;
- $\mathcal{M}, x \models \phi_1 \wedge \phi_2 \overset{\text{def}}{\Leftrightarrow} \mathcal{M}, x \models \phi_1$ and $\mathcal{M}, x \models \phi_2$;
- $\mathcal{M}, x \models \neg\phi \overset{\text{def}}{\Leftrightarrow}$ not $\mathcal{M}, x \models \phi$;
- $\mathcal{M}, x \models [i]\phi \overset{\text{def}}{\Leftrightarrow}$ for every $x' \in R_i(x)$, we have $\mathcal{M}, x' \models \phi$, where $R_i(x) \overset{\text{def}}{=} \{x' \in W : \langle x, x' \rangle \in R_i\}$.

We omit the standard clauses for the other connectives. A modal formula $\phi$ is said to be *true* in the model $\mathcal{M}$ (written $\mathcal{M} \models \phi$) $\overset{\text{def}}{\Leftrightarrow}$ for every $x \in W$, $\mathcal{M}, x \models \phi$. A modal formula $\phi$ is said to be *true* in the frame $\mathcal{F}$ (written $\mathcal{F} \models \phi$) $\overset{\text{def}}{\Leftrightarrow}$ $\phi$ is true in every model based on $\mathcal{F}$.

### *2.3 LA-logics*

The definitions in Section 2.1 and 2.2 are quite standard. In the sequel, we introduce the class of unbounded LA-logics (as a subclass of logics introduced in (Dem98)) in which each logic is characterized by a set of linear orders over $\mathbb{N}$.

An *unbounded LA-logic* [resp. *bounded LA-logic*] $\mathcal{L}$ is a pair $\langle \mathrm{FOR}, \mathcal{S} \rangle$ [resp. $\langle \mathrm{FOR}_i, \mathcal{S} \rangle$ for some $i \geq 0$] where $\mathcal{S}$ is a non-empty class of frames [resp. $i$-

frames] such that there exists a non-empty class $lo(\mathcal{L})$ of linear orders [3] on $\mathbb{N}$ [resp. on $\{0, \ldots, i\}$] such that for every frame $\mathcal{F} = \langle W, (R_i)_{i \in \mathbb{N}} \rangle$ [resp. $i$-frame $\mathcal{F} = \langle W, R_0, \ldots, R_i \rangle$], $\mathcal{F} \in \mathcal{S} \stackrel{\text{def}}{\Leftrightarrow}$

**(EQUIV)** for every $j \in \mathbb{N}$ [resp. $j \in \{0, \ldots, i\}$], $R_j$ is an equivalence relation;
**(LA)** for every $w \in W$, there is $\preceq \in lo(\mathcal{L})$ such that for all $j, j' \in \mathbb{N}$ [resp. $j, j' \in \{0, \ldots, i\}$], $j \preceq j'$ implies $R_j(w) \subseteq R_{j'}(w)$.

Condition (LA) is the local agreement condition and more generally we say that the relations $R$ and $R'$ on $W$ are in *local agreement* $\stackrel{\text{def}}{\Leftrightarrow}$ for every $w \in W$, either $R(w) \subseteq R'(w)$ or $R'(w) \subseteq R(w)$.

A modal formula is said to be $\mathcal{L}$-*satisfiable* $\stackrel{\text{def}}{\Leftrightarrow}$ there exist a model $\mathcal{M}$ based on some frame from $\mathcal{S}$ and $x \in W$ such that $\mathcal{M}, x \models \phi$. A formula $\phi$ is said to be $\mathcal{L}$-*valid* $\stackrel{\text{def}}{\Leftrightarrow}$ for every frame $\mathcal{F} \in \mathcal{S}$, we have $\mathcal{F} \models \phi$.

EXAMPLE 2.1. The standard modal logic S5 is a bounded LA-logic as well as the bimodal logic with two S5 modal connectives [1] and [2] such that $[2]p \Rightarrow [1]p$ is valid (semantically equivalent to $R_1 \subseteq R_2$ in the models). Similarly, the logics with a countably infinite set of modal connectives characterized by the frames $\langle W, (R_i)_{i \in \mathbb{N}} \rangle$ such that for every $i \in \mathbb{N}$, $R_i$ is an equivalence relation and $R_0 \subseteq R_1 \subseteq R_2 \subseteq R_3 \subseteq \ldots$ is an unbounded LA-logic.

The LA-logics satisfy the finite model property as stated below.

THEOREM 2.1. (Dem98, Proposition 4.5) Let $\mathcal{L}$ be an LA-logic either bounded or unbounded.

**(I)** A formula $\phi$ is $\mathcal{L}$-satisfiable iff $\phi$ is satisfied in an $\mathcal{L}$-model of cardinality less than $1 + n \times |\phi|^n$ where $n$ is the cardinality of $\mathbb{N}(\phi)$.
**(II)** Every bounded LA-logic has an **NP**-complete satisfiability problem.

Theorem 2.1(I) is proved by using an extension of the construction done for S5 in (Lad77). In (Dem98), it is not shown whether the exponential size of the models is unavoidable in the worst case. Remember that $n$ depends on $|\phi|$ and $n < |\phi|$. The rest of the paper is dedicated to show that this exponential size of the models is inescapable and that nevertheless many unbounded LA-logics admit a **PSPACE** satisfiability problem.

[3] A linear order is a binary relation that is reflexive, transitive, totally connected and antisymmetric.

The class of linear orders on $\mathbb{N}$ is uncountable and therefore one can expect that there exist undecidable unbounded LA-logics. In this section, we introduce a class of unbounded LA-logics that shall be shown to be decidable in polynomial space. An unbounded LA-logic is said to be *nice* $\overset{\text{def}}{\Leftrightarrow}$

**(NICE1)** there is a map $f : \mathbb{N} \rightarrow \mathbb{N}^*$ in logarithmic space such that for every $n \geq 1$, $f(n)$ is a string of $n+1$ elements, say $f(n) = i_1 \cdot \ldots \cdot i_{n+1}$, for which there is $\preceq \in lo(\mathcal{L})$ verifying $i_1 \preceq i_2 \preceq \ldots \preceq i_{n+1}$.

**(NICE2)** for every non-empty string $i_1 \cdot \ldots \cdot i_n$ of natural numbers (encoded in binary writing), deciding whether there is some $\preceq \in lo(\mathcal{L})$ such that $i_1 \preceq \ldots \preceq i_n$ can be done in polynomial space in the size of $\Sigma_{j=1}^{n} log(i_j)$. If such a linear order $\preceq$ exists for $i_1 \cdot \ldots \cdot i_n$, then we say that $i_1 \cdot \ldots \cdot i_n$ is $\mathcal{L}$-*extendable*.

**(NICE3)** for every non-empty string $i_1 \cdot \ldots \cdot i_n$, $n \geq 1$, for all equivalence relations $R'_{i_1}, \ldots, R'_{i_n}$, on a finite set $W$ that are pairwise in local agreement and for every $w \in W$, there is a bijection $\sigma : \{1, \ldots, n\} \rightarrow \{i_1, \ldots, i_n\}$ such that $R'_{\sigma(1)}(w) \subseteq \ldots \subseteq R'_{\sigma(n)}(w)$ and $\sigma(1) \cdot \ldots \cdot \sigma(n)$ is $\mathcal{L}$-extendable, there is an $\mathcal{L}$-frame $\langle W, (R_k)_{k \in \mathbb{N}} \rangle$ such that for every $j \in \{1, \ldots, n\}$, $R'_{i_j} = R_{i_j}$.

Condition (NICE1) almost states that there is a simple distinguished element in $lo(\mathcal{L})$ except that in (NICE1) the linear ordering $\preceq$ depends on $n$. Simplicity is reflected by the requirement that $f$ is in logarithmic space. For instance, if the usual ordering $\leq$ on natural numbers is in $lo(\mathcal{L})$, then $f(n)$ can take the value $0 \cdot 1 \cdot \ldots \cdot (n-1) \cdot n$. In the sequel, without any loss of generality, we can assume that $\leq \in lo(\mathcal{L})$ so that $f(n) = 0 \cdot 1 \cdot \ldots \cdot (n-1) \cdot n$. Condition (NICE1) is mainly used to show that every nice unbounded LA-logic has a **PSPACE**-hard satisfiability problem (see the proof of Lemma 3.1). Condition (NICE2) states that from a finite string one can decide in polynomial space whereas this chain can be extended as an infinite one viewed as a linear ordering from $lo(\mathcal{L})$ (see the proof of Theorem 4.6). This condition has to do with the fact that we want to define a class of logics in **PSPACE**. Condition (NICE3) guarantees that every finite structure satisfying certain local conditions can be extended as an $\mathcal{L}$-model (see the proof of Lemma 4.7).

Below we give examples of nice unbounded LA-logics.

**DALLA′** is the nice unbounded LA-logic such that $lo$(DALLA′) is the set of all the linear orders on $\mathbb{N}$ (Dem98) (see Section 5 for understanding the relationship between DALLA′ and Gargov's logic DALLA (Gar86)).

**LGM′** is the nice unbounded LA-logic such that $lo$(LGM′) is the singleton set $\{\geq\}$ (Dem98) (see also (Nak93)). The map $f$ can be defined as $f(n) = n \cdot (n-1) \cdot \ldots \cdot 1 \cdot 0$.

**LGM$'_k$** : for every $k \geq 0$, let LGM$'_k$ [resp. LGM$''_k$] be the nice unbounded

LA-logic such that $lo(\mathrm{LGM}'_k)$ is the class of linear orders $\preceq$ such that $\preceq$ restricted to $\{k+1, k+2, \ldots\}$ is $\geq$ [resp. $\leq$] restricted to $\{k+1, k+2, \ldots\}$ and for every $i \in \{0, \ldots, k\}$, $k+1 \preceq i$ [resp. $i \preceq k+1$]. Obviously, LGM′ is $\mathrm{LGM}'_0$ and $\mathrm{card}(lo(\mathrm{LGM}'_k)) = (k+1)!$ for every $k \geq 0$.

### *2.5 A few properties*

For the modal logic S5 it is known that every modal formula $\phi$ is equivalent to a formula $\phi'$ of modal depth at most one. $\phi'$ can be effectively built from $\phi$ (see e.g. (HC68)). Lemma 2.2 below generalizes this result to LGM′, providing an analogy between the modal depth and the number of distinct modal connectives.

LEMMA 2.2. Every LGM′-formula $\phi$ has an LGM′-equivalent formula $\psi$ such that if $[i]\varphi_1$ occurs in the scope of $[i']\varphi_2$ in $\psi$, then $i > i'$.

The proof in the appendix is based on results for the modal logic S5 (HC68) and on results for modal logics augmented with a universal modal connective (GP92). This allows us to view the number of distinct modal connectives in local agreement as a modal depth.

COROLLARY 2.3. Every LGM′-formula $\phi$ has an LGM′-equivalent formula, say $\psi$, such that the modal depth of $\psi$ is equal the number of different modal indices occurring in $\phi$.

Each class of linear orders over $\mathbb{N}$ determines a unique unbounded LA-logic. Since the class of linear orders on $\mathbb{N}$ is uncountable, there exist unbounded LA-logics that are undecidable. However, the conditions (NICE1)-(NICE3) guarantee decidability.

THEOREM 2.4. For every nice unbounded LA-logic $\mathcal{L}$, the $\mathcal{L}$-satisfiability problem is decidable.

PROOF: Let $\phi$ be a formula for which we want to know whether $\phi$ is $\mathcal{L}$-satisfiable. By Theorem 2.1, $\phi$ is $\mathcal{L}$-satisfiable iff $\phi$ is satisfied in an $\mathcal{L}$-model of cardinality less than $1 + n \times |\phi|^n$ where $n$ is the cardinality of $\mathbb{N}(\phi)$. In order to check whether $\phi$ is $\mathcal{L}$-satisfiable, enumerate all the structures $\mathcal{M} = \langle W, (R'_i)_{i \in \mathbb{N}(\phi)}, V \rangle$ (modulo the isomorphic copies) such that $\mathrm{card}(W) \leq 1 + n \times |\phi|^n$, the $R'_i$s are binary relations on $W$ and $V$ is a valuation restricted to the propositional variables occurring in $\phi$. Check the following properties:

**(i)** Is $\mathcal{M}, w \models \phi$ for some $w \in W$? This model-checking instance can be done in time $\mathcal{O}(\mathrm{card}(W)^2 \times |\phi|)$.

**(ii)** Is it the case that for all $i \in \mathbb{N}(\phi)$, the $R'_i$'s are equivalence relations and they are in local agreement? This can be checked in polynomial time in $\mathrm{card}(W) + |\phi|$.

**(iii)** Is it the case that for every $w \in W$, there is $\preceq \in lo(\mathcal{L})$ such that for all $i, j \in \mathbb{N}(\phi)$, $i \preceq j$ implies $R_i(w) \subseteq R_j(w)$? By satisfaction of the condition (NICE2), this is a decidable question.

By satisfaction of the condition (NICE3), one can easily show that $\phi$ is $\mathcal{L}$-satisfiable iff there is some structure $\langle W, (R'_i)_{i \in \mathbb{N}(\phi)}, V \rangle$ satisfying the above conditions (i)-(iii). Q.E.D.

The rest of the paper is mainly dedicated to show that one can refine this decidability result to obtain **PSPACE**-completeness of the satisfiability problem of any nice unbounded LA-logic.

## 3 Tree-like models and the PSPACE lower bound

In the rest of this section, $\mathcal{L}$ is a nice unbounded LA-logic such that $\leq \in lo(\mathcal{L})$. Lemma 3.1 below states that there exist $\mathcal{L}$-satisfiable formulae with exponential size models. Such an exponential bound can be obtained for the standard modal logics K, T and S4 (see e.g. (HM92)) but not for the bounded LA-logics (Dem98).

LEMMA 3.1. There is a family $(\psi_n)_{n \geq 1}$ of formulae such that for every $n \geq 1$, $|\psi_n|$ is in $\mathcal{O}(n)$, $\psi_n$ is $\mathcal{L}$-satisfiable and every $\mathcal{L}$-model for $\psi_n$ has cardinality at least $2^n$.

PROOF: We define recursively the formulae $renam_j$ and $tree_j$ for every $j \geq 1$.

- $renam_1 \stackrel{\text{def}}{=} \top$;
- $tree_1 \stackrel{\text{def}}{=} \langle 1 \rangle [0] p_1 \ \wedge \ \langle 1 \rangle [0] \neg p_1$;
- $renam_{j+1} \stackrel{\text{def}}{=} [j+1] renam_j \wedge [j+1](tree_j \Leftrightarrow r_j)$;
- $tree_{j+1} \stackrel{\text{def}}{=} \langle j+1 \rangle (([j] p_{j+1}) \wedge r_j) \ \wedge \ \langle j+1 \rangle (([j] \neg p_{j+1}) \wedge r_j)$.

The propositional variable $r_j$ is a renaming variable for the formula $tree_j$. We define $\psi_n$ as the conjunction $renam_n \wedge tree_n$. By induction on $n$, one can show that if $\mathcal{M}, w_0 \models \psi_n$ for some $\mathcal{L}$-model $\mathcal{M}$, then $\mathrm{card}(R_n(w_0)) \geq 2^n$ and $\bigcup \{R_j(w_0) : 1 \leq j \leq n-1\} \subseteq R_n(w_0)$. So, an alternative definition for $\psi_n$, $n \geq 2$, is

$$\psi_n \stackrel{\text{def}}{=} \bigwedge_{j=1}^{n-1} [n](tree_j \Leftrightarrow r_j) \wedge tree_n.$$

This is due to the fact that $\bigcup \{R_j(w_0) : 1 \leq j \leq n-1\} \subseteq R_n(w_0)$ implies $\mathcal{M}, w_0 \models [n][n_1] \ldots [n_k]\psi \Leftrightarrow [n]\psi$ for every formula $\psi$ and for every finite sequence $n_1 \cdot \ldots \cdot n_k \in \{1, \ldots n\}^*$.

The formula $\psi_n$ can be defined in a way that makes clear its logarithmic space construction in $n$. Indeed, the formula $tree_j$ is not defined inductively. For every $n \geq 2$, let $\psi_n$ be defined as follows:

$$\psi_n \stackrel{\text{def}}{=} \bigwedge_{j=1}^{n-1} [n]((\overbrace{\langle j \rangle([j-1]p_j \wedge r_{j-1}) \ \wedge \ \langle j \rangle([j-1]\neg p_j \wedge r_{j-1})}^{tree_j}) \Leftrightarrow r_j) \wedge$$

$$\overbrace{\langle n \rangle([n-1]p_n \wedge r_{n-1}) \ \wedge \ \langle n \rangle([n-1]\neg p_n \wedge r_{n-1})}^{tree_n}$$

In Figure 1, we present a skeleton of an $\mathcal{L}$-model for $\psi_2$. The family $(R_i)_{i \in \mathbb{N}}$ defined in Figure 1 is the smallest family of equivalence relations containing the pairs of worlds from the skeleton such that $R_0 \subset R_1 \subset R_2$ and for every $j > 2$, $R_j = R_2$. Obviously $R_0$ is the identity relation.

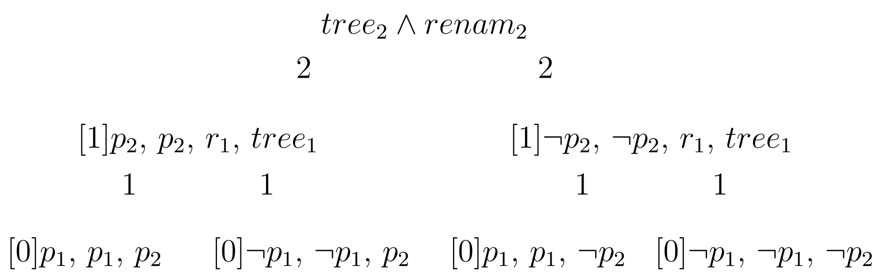


Fig. 1. $\mathcal{L}$-model skeleton for $\psi_2$

Q.E.D.

As shown in the proof of Lemma 3.1, the $\mathcal{L}$-models for the formula $\psi_n$ have a tree-like structure and this shall be exploited for reducing QBF into $\mathcal{L}$-satisfiability. If the usual ordering $\leq$ is not in $lo(\mathcal{L})$, then in the proof of Lemma 3.1, for every $j \in \{0, \ldots, n\}$, we replace $[j]$ by $[j']$ where $j'$ is the $j+1$th element of $f(n)$ and $f$ is the map from the condition (NICE1). We invite the reader to check that $\psi_n$ can be built in logarithmic space in $n$ because the composition of logarithmic space reductions is still a logarithmic space reduction (see e.g. (Pap94, Proposition 8.3)). Hence, Lemma 3.1 entails that the exponential size of models in Theorem 2.1(I) is unavoidable.

Given a Quantified Boolean formula $\phi = Q_n p_n Q_{n-1} p_{n-1} \ldots Q_1 p_1 \ \phi'$ where $\phi'$ is a propositional formula built over $\{p_1, \ldots, p_n\}$ (without modal connectives), and $\{Q_1, \ldots, Q_n\} \subseteq \{\exists, \forall\}$, deciding whether $\phi$ is true is **PSPACE**-complete (Sto77).

Lemma 3.2. There is a logarithmic space many-one reduction from QBF into $\mathcal{L}$-satisfiability.

PROOF: Let $\phi = Q_n p_n Q_{n-1} p_{n-1} \ldots Q_1 p_1 \; \phi'$ be a quantified Boolean formula in prenex form. Let us define an $\mathcal{L}$-formula $\psi$ such that $\phi$ is true iff $\psi$ is $\mathcal{L}$-satisfiable by taking advantage of the construction of Lemma 3.1. For every $j \geq 1$, we use the following abbreviation: $l_j \stackrel{\text{def}}{=} \neg d_1 \wedge \ldots \wedge \neg d_{j-1} \wedge d_j$ ("level $j$"). The propositional variable $d_j$ stands for "depth $j$". For instance $l_1 = d_1$. We define recursively the formulae $renam_j$, $tree_j$, $quanti_j$ for every $j \geq 1$.

- $renam_1 \stackrel{\text{def}}{=} \top$;
- $tree_1 \stackrel{\text{def}}{=} \langle 1 \rangle([0]p_1 \wedge l_1) \wedge \; \langle 1 \rangle([0]\neg p_1 \wedge l_1) \wedge \; [1](l_1 \Rightarrow ([0]p_1 \vee [0]\neg p_1))$;
- $quanti_1 \stackrel{\text{def}}{=} [1](l_1 \Rightarrow \phi')$ if $Q_1 = \forall$, otherwise $quanti_1 \stackrel{\text{def}}{=} \langle 1 \rangle(l_1 \wedge \phi')$;
- $renam_{j+1} \stackrel{\text{def}}{=} [j+1]renam_j \wedge [j+1](tree_j \Leftrightarrow r_j)$;
- $tree_{j+1} \stackrel{\text{def}}{=} \langle j+1 \rangle(([j]p_{j+1} \wedge l_{j+1})) \wedge \langle j+1 \rangle(([j]\neg p_{j+1} \wedge l_{j+1})) \wedge [j+1](l_{j+1} \Rightarrow (([j]p_{j+1} \vee [j]\neg p_{j+1}) \wedge r_j))$;
- $quanti_{j+1} \stackrel{\text{def}}{=} [j+1](l_{j+1} \Rightarrow quanti_j)$ if $Q_{j+1} = \forall$, otherwise $quanti_{j+1} \stackrel{\text{def}}{=} \langle j+1 \rangle(l_{j+1} \wedge quanti_j)$.

Finally, $\psi \stackrel{\text{def}}{=} quanti_n \wedge renam_n \wedge tree_n$. One can check that this many-one reduction is in logarithmic space. This is simpler to observe than in the proof of Lemma 3.1: to define formulae at stage $j+1$, one needs at most one copy of each formula of the stage $j$. The main differences with the formulae introduced in the proof of Lemma 3.1 are the following.

(1) Propositional variables for depths are introduced. As usual, they allow to distinguish the different quantification depths in the tree-like structure.
(2) In the definition of the formulae $(tree_j)_{1 \leq j \leq n}$, one cannot express that internal nodes have exactly two children. Instead, as usual, we can enforce that at most two children of internal nodes behave differently as far as the modal language is concerned.
(3) Once the tree structure is encoded by the satisfaction of the formula $renam_n \wedge tree_n$, the formulae $(quanti_j)_{1 \leq j \leq n}$ allow to quantify over the leaves of the tree, mimicking the quantifications in the QBF formula $\phi$.

Let us show that $\phi$ is true iff $\psi$ is $\mathcal{L}$-satisfiable.

Assume that $\phi$ is true. Then, $\psi$ is satisfied in the $\mathcal{L}$-model $\mathcal{M} = \langle W, (R_i)_{i \in \mathbb{N}}, V \rangle$ defined as follows:

- $W \stackrel{\text{def}}{=} \{u \in \{0,1\}^* : |u| \leq n\}$;
- for every $j \in \{0, \ldots, n\}$, we define the auxiliary relation $R'_j$ as follows: $\langle u, v \rangle \in R'_j \stackrel{\text{def}}{\Leftrightarrow} v = u \cdot i$ for some $i \in \{0,1\}$ and $|u| + j = n$;
- for every $j \in \{0, \ldots, n\}$, $R_j \stackrel{\text{def}}{=} (\bigcup_{0 \leq k \leq j}(R'_k \cup R'^{-1}_k))^*$;
- for every $j > n$, $R_j \stackrel{\text{def}}{=} R_n$;
- for every $1 \leq j \leq n$,
  - $V(d_j) \stackrel{\text{def}}{=} \{u \in W : |u| = n+1-j\}$;
  - $V(p_j) \stackrel{\text{def}}{=} \{u \in W : |u| \geq n+1-j, \; (n+1-j)\text{th bit of } u \text{ is } 1\}$;

· $V(r_j) \stackrel{\text{def}}{=} \{u \in W : |u| = n - j\}$ if $j \leq n - 1$.

$\mathcal{M}$ is an $\mathcal{L}$-model since $R_0 \subseteq R_1 \subseteq R_2 \subseteq R_3 \subseteq \ldots$ and $\leq \in lo(\mathcal{L})$. For every $j \in \{0, \ldots, n\}$, for every $u \in W$ such that $|u| = j$, we can show that $\mathcal{M}, u \models tree_{n-j} \wedge renam_{n-j}$. Hence, $\mathcal{M}, \epsilon \models tree_n \wedge renam_n$, where $\epsilon$ is the empty string.

A valuation over the propositional variables $\{p_1, \ldots, p_n\}$ can be represented by a string $u$ of length $n$ in $\{0, 1\}^*$ such that $p_i$ is true iff the $i$th bit of $u$ is 1. Here the ordering of bits is from the right to the left. It is a standard characterization of QBF that the QBF formula $\phi$ is true iff there is a non-empty set $X$ of valuations over $\{p_1, \ldots, p_n\}$ such that:

**(QBF1)** for every string $u \in X$, $u$ satisfies $\phi'$ in the propositional sense;
**(QBF2)** for every string $u \in X$, for every $r$ such that $Q_r = \forall$, there is $u' \in X$ such that the $r$th bit of $u'$ is different from the $r$th bit of $u$ and for all $r < r' \leq n$, the $r'$th bit of $u'$ is equal to the $r'$th bit of $u$.

One can show that $\{u \in W : |u| = n, \ \mathcal{M}, u \models \phi'\}$ is such a set and therefore $\mathcal{M}, \epsilon \models quanti_n$. Consequently, $\mathcal{M}, \epsilon \models \psi$.

Conversely, one can show that if $\mathcal{M}, w \models \psi_n$ for some $\mathcal{L}$-model $\mathcal{M}$, then one can easily extract from $\mathcal{M}$ a set $X$ of valuations over $\{p_1, \ldots, p_n\}$ satisfying the above conditions (QBF1) and (QBF2). Q.E.D.

If the usual ordering $\leq$ is not in $lo(\mathcal{L})$, one can easily adapt the proof in Lemma 3.2 to get a logarithmic space many-one reduction as done for the proof of Lemma 3.1.

Other translations from QBF into modal and temporal logics can be found in the papers (Lad77; SSS91; HM92; DS01). As usual, our reduction in the proof of Lemma 3.2 builds a tree-like models in which the leaves encode propositional valuations and the quantification in QBF are encoded by modal connectives in $\mathcal{L}$. The peculiarity of our reduction is in the fact that each quantifier in the QBF formula corresponds to a modal operator with different index, depending on the depth of this quantifier. This allows us to enforce the local linear orders.

## 4 PSPACE complexity upper bound

In this section, we show that every nice unbounded LA-logic has a satisfiability problem in **PSPACE** using a sophisticated Ladner-like algorithm (Lad77).

- In Section 4.1 we present preliminary definitions and results. Since the algorithm explores the branches of the tree-like potential $\mathcal{L}$-models, we need to fix what are the nodes of the trees (sets of formulae) and how they are related to each other.

- In Section 4.2, we present the definition of the main algorithm.
- In Section 4.3, we show that the main algorithm terminates by emphasizing what is the measure that strictly decreases depending on the history of the recursive calls.
- In Section 4.4, we finally show that the algorithm solves the $\mathcal{L}$-satisfiability problem.

### *4.1 Preliminary results*

In Definition 4.1 below, we introduce a closure operator for sets of modal formulae as it is done for Propositional Dynamic Logic in (FL79).

DEFINITION 4.1. Let $X$ be a set of formulae. $\mathrm{cl}(X)$ is the smallest set of formulae such that:

- $X \subseteq \mathrm{cl}(X)$;
- $\mathrm{cl}(X)$ is closed under subformulae;
- if $[i]\phi, [j]\phi' \in \mathrm{cl}(X)$, then $[i]\phi' \in \mathrm{cl}(X)$.

$\nabla$

A set $X$ of formulae is said to be *closed* $\stackrel{\text{def}}{\Leftrightarrow}$ $\mathrm{cl}(X) = X$. Observe that for every finite set $X$ of formulae, $\mathrm{md}(\mathrm{cl}(X)) = \mathrm{md}(X)$ and $\mathrm{card}(\mathrm{cl}(\{\phi\})) \leq \mathrm{card}(\mathbb{N}(\phi)) \times \mathrm{card}(\mathrm{sub}(\phi)) < |\phi|^2$. Indeed, one can consider that each subformula of $\phi$ generates at most $\mathbb{N}(\phi)$ formulae in $\mathrm{cl}(\{\phi\})$. This is a crucial property since in order to establish the **PSPACE** upper bound, $\mathrm{card}(\mathrm{cl}(\{\phi\}))$ is bounded by a polynomial in $|\phi|$.

The forthcoming algorithm defined in Section 4.2 explores branches of a tree and for each node one can abstract its path from the root by a finite word in $\mathbb{N}^*$. For each path, we define the admitted formulae that have to be taken into account for the nodes reachable from the root with such a path. Definition 4.2 states how such sets of formulae are defined.

DEFINITION 4.2. Let $\phi$ be a formula. For every $u \in \mathbb{N}^*$, $\mathrm{cl}(u, \phi)$ is the smallest set such that:

(1) $\mathrm{cl}(\epsilon, \phi) = \mathrm{cl}(\{\phi\})$;
(2) for every $v$, $\mathrm{cl}(v, \phi)$ is closed;
(3) for every $v$, for every $i \in \mathbb{N}$, if $[i]\psi \in \mathrm{cl}(v, \phi)$, then $[i]\psi \in \mathrm{cl}(v \cdot i, \phi)$.

$\nabla$

For instance,

$$\text{cl}(1, [1][2]p \vee [2]q) = \{[1][2]p, [2]p, p, [2][2]p, [1]p, [2]q, q, [1]q\}.$$

Lemma 4.1 contains some basic properties about the sets $\text{cl}(u, \phi)$.

LEMMA 4.1. Let $\phi$ be a formula and $u, u' \in \mathbb{N}^*$ be such that $u$ is a prefix of $u'$. Then,

(I) $\text{cl}(u', \phi) \subseteq \text{cl}(u, \phi)$;
(II) if $\text{md}(\text{cl}(u, \phi)) = 0$, then $\text{cl}(u \cdot i, \phi) = \emptyset$ for every $i \in \mathbb{N}(\phi)$.

In Definition 4.3 below, we define relations between sets of formulae that will be the basis to build binary relations in the $\mathcal{L}$-models obtained from the algorithm defined in Section 4.2 (see the proof of Lemma 4.7).

DEFINITION 4.3. Let $X, Y$ be sets of formulae. The binary relation $\approx_i$ is defined as follows: $X \approx_i Y \overset{\text{def}}{\Leftrightarrow}$ for every $[i]\psi \in X$, we have $[i]\psi \in Y$ and for every $[i]\psi \in Y$, we have $[i]\psi \in X$. $\nabla$

Let `Refl` be the set of subsets $Y$ of $\text{cl}(\{\phi\})$ such that $[i]\psi \in Y$ implies $\psi \in Y$. The relation $\approx_i$ is an equivalence relation on `Refl`. As maximally consistent sets for building canonical models from Hilbert-style proof systems for modal logics (see e.g. (BRV01)), we define below a notion of consistency adapted to the forthcoming tableaux-like method defined in Section 4.2.

DEFINITION 4.4. Let $\phi$ be a formula such that $\mathbb{N}(\phi) \geq 1$. Let $X$ be a subset of $\text{cl}(u, \phi)$ for some $u \in \mathbb{N}^*$ and $\sigma$ be a bijection $\sigma : \{1, \ldots, \text{card}(\mathbb{N}(\phi))\} \rightarrow \mathbb{N}(\phi)$. The set $X$ is said to be $\langle u, \sigma \rangle$-*consistent* $\overset{\text{def}}{\Leftrightarrow}$ for every $\psi \in \text{cl}(u, \phi)$:

(1) if $\psi = \neg\varphi$, then $\varphi \in X$ iff not $\psi \in X$;
(2) if $\psi = \varphi_1 \wedge \varphi_2$, then $\{\varphi_1, \varphi_2\} \subseteq X$ iff $\psi \in X$ (and similar conditions for $\vee, \Rightarrow, \Leftrightarrow$);
(3) if $\psi = [i]\varphi$ and $\psi \in X$, then $\varphi \in X$;
(4) if $\psi = [\sigma(j)]\varphi$ for some $j \in \{2, \ldots, \text{card}(\mathbb{N}(\phi))\}$ and $\psi \in X$, then $[\sigma(j-1)]\varphi \in X$.

$\nabla$

Roughly speaking, the $\langle u, \sigma \rangle$-consistency entails the maximal propositional consistency with respect to the set $\text{cl}(u, \phi)$ of formulae. Furthermore, the conditions (3)-(4) in Definition 4.4 are added in order to take into account the reflexivity of $R_i$ and the series of inclusions $R_{\sigma(1)} \subseteq \ldots \subseteq R_{\sigma(n)}$ we wish to enforce, where $n = \text{card}(\mathbb{N}(\phi))$. Lemma 4.2 below states the natural relationships between the relation $\approx_i$ and the relation $R_i$.

function LA-WORLD$(\Sigma, u, \sigma, index, Z, \phi)$
**(Consistency)** if $last(\Sigma)$ is not $\langle u, \sigma\rangle$-consistent, then return false;
($last(\Sigma)$ denotes the last element of the sequence $\Sigma$)
**(Witnesses)** for every $[i]\psi \in \mathrm{cl}(u, \phi) \setminus last(\Sigma)$ with $i \in Z$ do
if one of the two conditions below holds,
**(NewIndex)** $index \neq \sigma^{-1}(i)$;
**(NoExistingWitness)** there is no $X \in \Sigma$ such that
**(i)** $\Sigma = \Sigma_1 X \Sigma_2$ and $u = u' \cdot i^{|\Sigma_2|}$ for some sequences $\Sigma_1, \Sigma_2$ such that $\Sigma_2$ is non-empty;
**(ii)** $\psi \notin X$;
**(iii)** $last(\Sigma) \approx_{\sigma(j)} X$ for every $j \in \{index, \ldots, \mathrm{card}(\mathbb{N}(\phi))\}$;
then
**(SearchWitness)** for every $X_\psi \subseteq \mathrm{cl}(u \cdot i, \phi) \setminus \{\psi\}$, for every bijection $\sigma' : \{1, \ldots, \mathrm{card}(\mathbb{N}(\phi))\} \rightarrow \mathbb{N}(\phi)$ such that
**(iv)** $\sigma'(j) = \sigma(j)$ for every $j \in \{\sigma^{-1}(i), \ldots, \mathrm{card}(\mathbb{N}(\phi))\}$;
**(v)** $last(\Sigma) \approx_{\sigma'(j)} X_\psi$ for every $j \in \{\sigma^{-1}(i), \ldots, \mathrm{card}(\mathbb{N}(\phi))\}$;
**(vi)** $\sigma'(1), \ldots, \sigma'(\mathrm{card}(\mathbb{N}(\phi)))$ is $\mathcal{L}$-extendable;
call LA-WORLD$(\Sigma \cdot X_\psi, u \cdot i, \sigma', \sigma'^{-1}(i), Z^i_{\sigma'}, \phi)$ with $Z^i_{\sigma'} \stackrel{\text{def}}{=} \{\sigma'(j) : j \in \{1, \ldots, \sigma'^{-1}(i)\}\}$.
**(NoWitnessFound)** If all these calls return false, then return false;
**(AllWitnessesFound)** Return true.

Fig. 2. Algorithm LA-WORLD

LEMMA 4.2. For every $\mathcal{L}$-model $\mathcal{M} = \langle W, (R_i)_{i\in\mathbb{N}}, V\rangle$, for every $u \in \mathbb{N}^*$, for every formula $\phi$ such that $\mathbb{N}(\phi) \neq \emptyset$, for all $w, w' \in W$, for every bijection $\sigma : \{1, \ldots, \mathrm{card}(\mathbb{N}(\phi))\} \rightarrow \mathbb{N}(\phi)$ such that $R_{\sigma(j-1)}(w) \subseteq R_{\sigma(j)}(w)$ for every $j \in \{2, \ldots, \mathrm{card}(\mathbb{N}(\phi))\}$, for every $k \in \mathbb{N}(\phi)$, the following conditions are satisfied:

**(I)** $X_{w,u} \stackrel{\text{def}}{=} \{\psi \in \mathrm{cl}(u, \phi) : \mathcal{M}, w \models \psi\}$ is $\langle u, \sigma\rangle$-consistent;
**(II)** if $\langle w, w'\rangle \in R_k$, then $X_{w,u} \approx_{\sigma(j)} X_{w',u\cdot k}$ for every $j \in \{\sigma^{-1}(k), \ldots, n\}$.

The proof of the above lemma is by an easy verification using that $\mathrm{cl}(u, \phi)$ is closed. Lemma 4.2(II) is a counterpart of (Dem98, Proposition 4.2(2)).

### *4.2 The algorithm*

In Figure 2, the function LA-WORLD$(\Sigma, u, \sigma, index, Z, \phi)$ returning a Boolean value is defined.

The arguments of LA-WORLD are of the following form:

- $\Sigma$ is a finite non-empty sequence of subsets of $\mathrm{cl}(\{\phi\})$;

- $u \in \mathbb{N}(\phi)^*$;
- $\sigma$ is a 1-1 mapping $\{1, \ldots, \mathrm{card}(\mathbb{N}(\phi))\} \rightarrow \mathbb{N}(\phi)$;
- $index \in \{1, \ldots, \mathrm{card}(\mathbb{N}(\phi))\}$;
- $Z \subseteq \mathbb{N}(\phi)$.

`LA-WORLD` is defined on the model of the function `K-WORLD` in (Lad77) (see also (Spa93b; DLNN97; Dem00; Mas00)). $\Sigma$ and $u$ are historical information about the parent calls to `LA-WORLD`. For this reason, we shall have $|\Sigma| = |u| + 1$. We shall establish that if `LA-WORLD`$(\Sigma, u, \sigma, index, Z, \phi)$ returns true, then there is an $\mathcal{L}$-model $\mathcal{M} = \langle W, (R_i)_{i \in \mathbb{N}}, V \rangle$ and $w \in W$ such that

**(Prop1)** for every $\psi \in last(\Sigma)$, we have $\mathcal{M}, w \models \psi$;
**(Prop2)** $R_{\sigma(1)}(w) \subseteq \ldots \subseteq R_{\sigma(n)}(w)$.

The inputs $\Sigma$, *index* and $Z$ guarantee termination of the algorithm. For instance, no recursive call is performed when the argument $Z$ is empty. Similarly, every sequence of more than $|\phi|^2 + 1$ successive recursive calls strictly decreases the cardinality of the argument $Z$ (see Section 4.3 for details).

In Figure 2, the condition (Consistency) is just a consistency check whereas the condition (Witnesses) is the part searching for witnesses for negations of formulae of the form $[i]\psi$. If a new witness is really needed (by satisfaction of either the condition (NewIndex) or the condition (NoExistingWitness)), then one tests potential new witnesses by checking local conditions ((iv), (v), and (v)) and global one via the recursive call. The arguments need to be appropriately updated in such a call in order to guarantee termination. As other tableaux-like procedures, `LA-WORLD`$(\Sigma, u, \sigma, index, Z, \phi)$ tries to build a quasi $\mathcal{L}$-model for $last(\Sigma)$ having a tree-like structure. Then, proving the correctness of our algorithm partly consists in showing that from this quasi $\mathcal{L}$-model, it is possible to complete it providing a standard $\mathcal{L}$-model satisfying the conditions (Prop1) and (Prop2). More technical details are provided in the sequel.

We say that `LA-WORLD`$(\Sigma, u, \sigma, index, Z, \phi)$ *directly calls*
`LA-WORLD`$(\Sigma', u', \sigma', index', Z', \phi) \stackrel{\text{def}}{\Leftrightarrow}$ `LA-WORLD`$(\Sigma', u', \sigma', index', Z, \phi)$ is called at depth one in the computation tree of `LA-WORLD` calls from the execution of `LA-WORLD`$(\Sigma, u, \sigma, index, Z, \phi)$. If the call is at some depth (not necessarily one) in the computation tree of `LA-WORLD` calls from the execution of
`LA-WORLD`$(\Sigma, u, \sigma, index, Z, \phi)$, we simply say that
`LA-WORLD`$(\Sigma', u', \sigma', index', Z', \phi)$ is called *in* `LA-WORLD`$(\Sigma, u, \sigma, index, Z, \phi)$.

Given an $\mathcal{L}$-formula $\phi$ such that $\mathbb{N}(\phi)$ is empty, it is easy to show that $\phi$ is $\mathcal{L}$-satisfiable iff there is $Y \subseteq \mathrm{cl}(\{\phi\})$ such that `LA-WORLD`$(Y, \epsilon, \sigma, 0, \emptyset, \phi)$ returns true. In the sequel, we treat the case $\mathbb{N}(\phi) \neq \emptyset$.

In this section, we shall show that termination of `LA-WORLD` is guaranteed for calls of the form `LA-WORLD`$(Y, \epsilon, \sigma, n, \mathbb{N}(\phi), \phi)$ where

(1) $\phi$ is an $\mathcal{L}$-formula, $Y \subseteq \text{cl}(\{\phi\})$ and $n = \text{card}(\mathbb{N}(\phi))$;
(2) $\sigma : \{1, \ldots, \text{card}(\mathbb{N}(\phi))\} \to \mathbb{N}(\phi)$ is a bijection.

For every call `LA-WORLD`$(\Sigma, u, \sigma, index, Z, \phi)$ in `LA-WORLD`$(Y, \epsilon, \sigma, n, \mathbb{N}(\phi), \phi)$ the following invariant conditions can be easily checked:

**(INV1)** $last(\Sigma) \subseteq \text{cl}(u, \phi)$;
**(INV2)** $Z_\sigma^{index} = Z$ (see the definition of $Z_\sigma^{index}$ from the point (vi) of Figure 2);
**(INV3)** $last(u) = \sigma(index)$;
**(INV4)** $\sigma^{-1}(i) > index$ for every $i \in \mathbb{N}(\phi) \setminus Z$.

Clearly, the number of arguments in `LA-WORLD` can be reduced but minimality is not our main purpose here.

Lemma 4.3. If `LA-WORLD`$(\Sigma, u, \sigma, index, Z, \phi)$ is called in `LA-WORLD`$(Y, \epsilon, \sigma, n, \mathbb{N}(\phi), \phi)$, then there is no substring of the form $i^{|\phi|^2+1}$ for some $i \in \mathbb{N}(\phi)$ occurring in $u$.

The proof of Lemma 4.3 can be found in the appendix. It can be viewed as a refined variant of the proof showing that the modal logic S5 has the linear size model property (Lad77). A corollary of Lemma 4.3 is that there is no sequence of successive calls of length greater than $|\phi|^2$ such that the value of the argument *index* does not change. Lemma 4.4 below states that when the value of the argument *index* changes, the number of elements of the argument $Z$ strictly decreases. This is particularly interesting since whenever the argument $Z$ is empty no recursive call `LA-WORLD` is possible.

Lemma 4.4.
Let `LA-WORLD`$(\Sigma, u, \sigma, index, Z, \phi)$ and `LA-WORLD`$(\Sigma', u', \sigma', index', Z', \phi)$ be calls in `LA-WORLD`$(Y, \epsilon, \sigma, n, \mathbb{N}(\phi), \phi)$ such that `LA-WORLD`$(\Sigma, u, \sigma, index, Z, \phi)$ directly calls `LA-WORLD`$(\Sigma', u', \sigma', index', Z', \phi)$. If $index \neq index'$, then $Z' \subset Z$ (proper inclusion).

Proof: The proof is by an easy verification using the condition (INV2), the condition (iv) in Figure 2 and the fact that only bijections of the form $\{1, \ldots, n\} \to \mathbb{N}(\phi)$ with $n = \text{card}(\mathbb{N}(\phi))$ are involved in the algorithm defined in Figure 2. Q.E.D.

Consequently, we obtain the following result.

LEMMA 4.5. Let $\texttt{LA-WORLD}(\Sigma, u, \sigma, index, Z, \phi)$ be a call in $\texttt{LA-WORLD}(Y, \epsilon, \sigma, n, \mathbb{N}(\phi), \phi)$. Then, $|u| = |\Sigma| - 1 \leq |\phi|^3$.

Since the logic $\mathcal{L}$ satisfies the condition (NICE2), the condition (vi) in Figure 2 can be checked in polynomial space in $|\phi|$. For DALLA′, this can be done in constant space and for LGM′ in linear space. Consequently, since the depth of the recursion is polynomial in $|\phi|$ from $\texttt{LA-WORLD}(Y, \epsilon, \sigma, n, \mathbb{N}(\phi), \phi)$ and since at each level of the recursion, we need only polynomial space in $|\phi|$, we can deduce the following result.

THEOREM 4.6. For every $\mathcal{L}$-formula $\phi$, for every $Y \subseteq \text{cl}(\{\phi\})$, for every bijection $\sigma : \{1, \ldots, \text{card}(\mathbb{N}(\phi))\} \rightarrow \mathbb{N}(\phi)$, $\texttt{LA-WORLD}(Y, \epsilon, \sigma, \text{card}(\mathbb{N}(\phi)), \mathbb{N}(\phi), \phi)$ terminates and requires polynomial space in $\phi$.

The subsets of $\text{cl}(\{\phi\})$ can be implemented as bit-strings of polynomial length in $|\phi|$ and the sequences $\Sigma$ and $u$ can be implemented as global stacks. Refinements are possible but they are omitted here since they do not essentially decrease the space bounds. For instance, it is easy to design a variant of LA-WORLD from Figure 2 for which $|\Sigma| \leq |\phi|^2$ in the conditions of Lemma 4.5. Indeed, we need to recall the history of the recursive calls (we assume that past is linear and finite) only when the argument *index* does not change.

### 4.4 Correctness

Lemmas 4.7 and 4.8 below state that not only LA-WORLD terminates but also it solves the $\mathcal{L}$-satisfiability problem.

LEMMA 4.7. Let $\phi$ be an $\mathcal{L}$-formula such that $\text{card}(\mathbb{N}(\phi)) = n \geq 1$, $Y \subseteq \text{cl}(\{\phi\})$, and $\sigma$ be a bijection $\{1, \ldots, n\} \rightarrow \mathbb{N}(\phi)$ such that $\phi \in Y$.
If $\texttt{LA-WORLD}(Y, \epsilon, \sigma, n, \mathbb{N}(\phi), \phi)$ returns true, then $\phi$ is $\mathcal{L}$-satisfiable.

PROOF: Assume that $\texttt{LA-WORLD}(Y, \epsilon, \sigma, n, \mathbb{N}(\phi), \phi)$ returns true. Let us build an $\mathcal{L}$-model $\mathcal{M} = \langle W, (R_i)_{i \in \mathbb{N}}, V \rangle$ such that for some $w \in W$ for every $\psi \in \text{cl}(\{\phi\})$, we have $\mathcal{M}, w \models \psi$ iff $\psi \in Y$.

We define $W$ as the set of quintuples $\langle \Sigma, u, \sigma, index, Z \rangle$ such that there is a finite sequence $\langle \Sigma_1, u_1, \sigma_1, index_1, Z_1 \rangle, \ldots, \langle \Sigma_k, u_k, \sigma_k, index_k, Z_k \rangle$, $k \geq 1$, such that

(1) $\langle \Sigma_1, u_1, \sigma_1, index_1, Z_1 \rangle = \langle Y, \epsilon, \sigma, n, \mathbb{N}(\phi) \rangle$;
(2) $\langle \Sigma_k, u_k, \sigma_k, index_k, Z_k \rangle = \langle \Sigma, u, \sigma, index, Z \rangle$;
(3) for every $i \in \{1, \ldots, k\}$, $\texttt{LA-WORLD}(\Sigma_i, u_i, \sigma_i, index_i, Z_i)$ returns true;
(4) for every $i \in \{1, \ldots, k-1\}$, $\texttt{LA-WORLD}(\Sigma_i, u_i, \sigma_i, index_i, Z_i, \phi)$ directly calls $\texttt{LA-WORLD}(\Sigma_{i+1}, u_{i+1}, \sigma_{i+1}, index_{i+1}, Z_{i+1}, \phi)$.

By assumption, $\langle Y, \epsilon, \sigma, n, \mathbb{N}(\phi)\rangle \in W$ and $\langle \Sigma, u, \sigma, index, Z\rangle \in W$ implies that $last(\Sigma)$ is $\langle u, \sigma\rangle$-consistent. In order to define $(R_i)_{i\in\mathbb{N}}$, we introduce the auxiliary families of relations $(S_i)_{i\in\mathbb{N}}$ and $(T_i)_{i\in\mathbb{N}}$. Whereas each relation $S_i$ can be viewed as a "single step relation", $T_i$ is its extension by considering the local inclusions with the permutations. For every $i \in \mathbb{N}(\phi)$, let us define $S_i$ on $W$ as follows: $\langle \Sigma, u, \sigma, index, Z\rangle S_i \langle \Sigma', u', \sigma', index', Z'\rangle \overset{\text{def}}{\Leftrightarrow}$ (1) $\texttt{LA-WORLD}(\Sigma, u, \sigma, index, Z, \phi)$ calls directly $\texttt{LA-WORLD}(\Sigma', u', \sigma', index', Z', \phi)$ and (2) $u' = u \cdot i$. For every $i \in \mathbb{N}(\phi)$, we define the auxiliary relation $T_i$ as follows:

$$T_i \overset{\text{def}}{=} \{\langle w, w'\rangle \in W^2 : \langle w, w'\rangle \in S_j,\ j \in \mathbb{N}(\phi) \text{ such that } \sigma^{-1}(j) \leq \sigma^{-1}(i)\}.$$

In the definition of $T_i$, the map $\sigma^{-1}$ is from $w = \langle \ldots, \sigma, \ldots\rangle$. The definition of $\mathcal{M}$ can be now completed.

- For every $i \in \mathbb{N}(\phi)$, $R_i \overset{\text{def}}{=} (T_i \cup T_i^{-1})^*$ (reflexive, symmetric, and transitive closure of $T_i$).
- By construction of the family $(R_i)_{i\in\mathbb{N}(\phi)}$, one can show that for every $w = \langle \Sigma, u, \sigma, index, Z\rangle \in W$, $R_{\sigma(1)}(w) \subseteq \ldots \subseteq R_{\sigma(n)}(w)$. Moreover every relation in $(R_i)_{i\in\mathbb{N}(\phi)}$ is an equivalence relation. Hence, by the condition (NICE3), it is possible to define the relations $R_i$ with $i$ ranging over $\mathbb{N} \setminus \mathbb{N}(\phi)$, such that the resulting structure $\langle W, (R_i)_{i\in\mathbb{N}}\rangle$ is an $\mathcal{L}$-frame.
- For every $p \in$ PRP, $V(p) \overset{\text{def}}{=} \{\langle \Sigma, u, \sigma, index, Z\rangle \in W : p \in last(\Sigma)\}$.

Hence, $\mathcal{M}$ is an $\mathcal{L}$-model. One can show that for every $i \in \mathbb{N}(\phi)$,

- $S_i \subseteq T_i$ and $S_i^* \subseteq T_i^* \subseteq R_i$;
- $\langle\langle \Sigma, u, \sigma, index, Z\rangle, \langle \Sigma', u', \sigma', index', Z'\rangle\rangle \in R_i$ implies $last(\Sigma) \approx_i last(\Sigma')$ ($\approx_i$ is an equivalence relation on `Refl`).

By induction on the structure on $\psi$, we shall show that for every quintuple $w = \langle \Sigma, u, \sigma, index, Z\rangle$ in $W$, for every $\psi \in \text{cl}(u, \phi)$, $\psi \in last(\Sigma)$ iff $\mathcal{M}, w \models \psi$. The case when $\psi$ is a propositional variable, say $p$, holds by the definition of $V(p)$.
*Induction hypothesis*: for every $\psi \in \text{cl}(\{\phi\})$ such that $|\psi| \leq n$, for every $w = \langle \Sigma, u, \sigma, index, Z\rangle \in W$, if $\psi \in \text{cl}(u, \phi)$, then $\psi \in last(\Sigma)$ iff $\mathcal{M}, w \models \psi$.
Let $\psi$ be a formula in $\text{cl}(\{\phi\})$ such that $|\psi| \leq n + 1$. The cases when the outermost connective of $\psi$ is Boolean are consequences of the $\langle u, \sigma\rangle$-consistency of $last(\Sigma)$ and the induction hypothesis. Let us treat the other case.
Let $\psi = [i]\psi' \in \text{cl}(\{\phi\})$ and $w = \langle \Sigma, u, \sigma, index, Z\rangle$ be an element of $W$ such that $\psi \in \text{cl}(u, \phi)$.

Assume that $[i]\psi' \in last(\Sigma)$. Since, for every $w' = \langle \Sigma', u', \sigma', index', Z'\rangle \in R_i(w)$, we have $last(\Sigma) \approx_i last(\Sigma')$, we can conclude that $\psi' \in last(\Sigma')$. By the induction hypothesis, for every $w' \in R_i(w)$, $\mathcal{M}, w' \models \psi'$. Consequently, $\mathcal{M}, w \models [i]\psi'$.

Now assume that $[i]\psi' \not\in last(\Sigma)$.
*Case 1*: $\psi = [i]\psi'$ for some $i \in Z$.
*Case 1.1*: $\sigma^{-1}(i) = index$ and there is $X$ in $\Sigma$ satisfying the conditions (i)-(iii) in Figure 2.
We use notations from Figure 2 and this case corresponds to the situation when the conditions (NewIndex) and (NoExistingWitness) from Figure 2 do not hold. By definition of the set $W$, $\texttt{LA-WORLD}(\Sigma_1 \cdot X, u', \sigma', index', Z', \phi)$ returns true for some $u'$, $\sigma'$, $index'$ and $Z'$. By definition of the binary relation $S_i$, $\langle w', w\rangle \in S_i^*$ and $\langle w, w'\rangle \in R_i$ with $w'$ taking the value $\langle \Sigma_1 \cdot X, u', \sigma', index', Z'\rangle$. By the induction hypothesis, $\mathcal{M}, w' \not\models \psi'$ and therefore $\mathcal{M}, w \not\models [i]\psi'$.
*Case 1.2*: the condition for the Case 1.1 does not hold.
Since $w \in W$, $\texttt{LA-WORLD}(\Sigma, u, \sigma, index, Z, \phi)$ returns true and therefore there exist $X_{\psi'} \subseteq \mathrm{cl}(u \cdot i, \phi) \setminus \{\psi'\}$ and a bijection $\sigma' : \{1, \ldots, \mathrm{card}(\mathbb{N}(\phi))\} \rightarrow \mathbb{N}(\phi)$ such that the conditions (iv)-(vi) from Figure 2 hold true and $\texttt{LA-WORLD}(\Sigma \cdot X_{\psi'}, u \cdot i, \sigma', {\sigma'}^{-1}(i), Z^i_{\sigma'}, \phi)$ returns true. By definition of $S_i$, $\langle w, \langle \Sigma \cdot X_{\psi'}, u \cdot i, \sigma', {\sigma'}^{-1}(i), Z^i_{\sigma'}\rangle\rangle \in S_i \subseteq R_i$. By the induction hypothesis, $\mathcal{M}, \langle \Sigma \cdot X_{\psi'}, u \cdot i, \sigma', {\sigma'}^{-1}(i), Z^i_{\sigma'}\rangle \not\models \psi'$ and therefore $\mathcal{M}, w \not\models [i]\psi'$.
*Case 2*: $\psi = [i]\psi'$ and $i \not\in Z$.
One can show that there is a finite sequence

$$\langle \Sigma_1, u_1, \sigma_1, index_1, Z_1\rangle, \ldots, \langle \Sigma_j, u_j, \sigma_j, index_j, Z_j\rangle, \; j \geq 2,$$

in $W$ such that

(1) $\langle \Sigma_j, u_j, \sigma_j, index_j, Z_j\rangle = \langle \Sigma, u, \sigma, index, Z\rangle$;
(2) for every $k \in \{1, \ldots, j-1\}$, $\texttt{LA-WORLD}(\Sigma_k, u_k, \sigma_k, index_k, Z_k, \phi)$ directly calls $\texttt{LA-WORLD}(\Sigma_{k+1}, u_{k+1}, \sigma_{k+1}, index_{k+1}, Z_{k+1}, \phi)$;
(3) $[i]\psi' \in \mathrm{cl}(s, \phi) \setminus last(\Sigma_1)$, $i \in Z_1$ and $i \not\in Z_2 \cup \ldots \cup Z_j$;
(4) there is $w' = \langle \Sigma', u', \sigma', index', Z'\rangle \in W$ such that $\langle\langle \Sigma_1, u_1, \sigma_1, index_1, Z_1\rangle, \langle \Sigma', u', \sigma', index', Z'\rangle\rangle \in S_i^*$ and $\psi' \not\in last(\Sigma')$.

Since the condition (INV4) is satisfied, one can show that for every $k \in \{1, \ldots, j-1\}$, $\langle \Sigma_k, u_k, \sigma_k, index_k, Z_k\rangle \; T_i \; \langle \Sigma_{k+1}, u_{k+1}, \sigma_{k+1}, index_{k+1}, Z_{k+1}\rangle$. For every $k \in \{1, \ldots, j-1\}$, we have

$$\langle \Sigma_k, u_k, \sigma_k, index_k, Z_k\rangle (S_i \cup T_i)^* \langle \Sigma_{k+1}, u_{k+1}, \sigma_{k+1}, index_{k+1}, Z_{k+1}\rangle.$$

Therefore, $\langle w, w'\rangle \in R_i$. By the induction hypothesis, we get $\mathcal{M}, w' \not\models \psi'$. Hence, $\mathcal{M}, w \not\models \psi$. This part of the proof can be viewed as an algorithmic explanation for the most interesting case in the proof of (Dem98, Proposition 4.4). Q.E.D.

The model $\mathcal{M}$ built in the proof of Lemma 4.7 is finite and it has a tree structure obtained from the tree of successful recursive calls to $\texttt{LA-WORLD}$. Combined with Lemma 4.8 below this provides a tree model property for $\mathcal{L}$.

This is a property known to be essential for decidability of modal logics (see e.g. (Grä99)).

Lemma 4.8. For every $\mathcal{L}$-satisfiable formula $\phi$ such that $\mathrm{card}(\mathbb{N}(\phi)) = n \geq 1$, there exist $Y \subseteq \mathrm{cl}(\{\phi\})$ and a bijection $\sigma : \{1, \ldots, n\} \rightarrow \mathbb{N}(\phi)$ such that $\phi \in Y$ and $\texttt{LA-WORLD}(Y, \epsilon, \sigma, n, \mathbb{N}(\phi), \phi)$ returns true.

The proof of Lemma 4.8 can be found in the appendix. From Lemmas 3.2, 4.7, 4.8 and Theorem 4.6, we can conclude the main result of the paper.

Theorem 4.9. Every nice unbounded LA-logic has a **PSPACE**-complete satisfiability problem.

Corollary 4.10.
DALLA′-satisfiability and LGM′-satisfiability are **PSPACE**-complete.

## 5 Applications to information and fuzzy modal logics

In this section, we just explain how the **PSPACE**-completeness for DALLA′ and LGM′ can be lifted to DALLA (Gar86) and LGM (Nak93), respectively.

### *5.1 DALLA*

The logic DALLA defined in (Gar86) is a polymodal logic with a countably infinite set of modal connectives of the form $[e]$ where $\Pi_D \ni e ::= c_i \mid e \cup^* e' \mid e \cap e'$. Each $c_i$, $i \geq 0$, is a modal constant. DALLA is a variant of the Data Analysis Logic DAL introduced in (FdCO85) (see also another variant in (AT89)) and designed to model reasoning on Pawlak's information systems (see e.g. (Paw91)). The first decidability proof for DALLA can be found in (Dem96) (see a generalization in (Dem98)) but its complexity characterization has been open until now. The DALLA-models are the Kripke style structures of the form $\langle W, (R_e)_{e \in \Pi_D}, V \rangle$ where

**(equiv)** every relation $R_e$ is an equivalence relation;
**($\cap + \cup^*$)** $\cap$ and $\cup^*$ are interpreted as intersection and transitive and reflexive closure of the union, respectively;
**(la)** $R_e$ and $R_{e'}$ are in local agreement for all $e, e' \in \Pi_D$.

It is not difficult to see that DALLA′ can be viewed as a fragment of DALLA by identifying $[i]$ and $[c_i]$. Indeed, one can replace in the definition of DALLA-models the condition (la) by the condition (la′) defined below and still get the same class of models (see e.g. (DO98, Proposition 10.7)):

**(la′)** $R_c$ and $R_{c'}$ are in local agreement for all the modal constants $c$ and $c'$.

As a consequence, DALLA-satisfiability is **PSPACE**-hard. The **PSPACE** upper bound for DALLA satisfiability problem can be proved by using the renaming techniques (see e.g. (Min88)) and results for DALLA′ as shown in the proof of Lemma 5.1 below.

LEMMA 5.1. There is polynomial time many-one reduction from DALLA into DALLA′.

PROOF: Let $f$ be the map from the set of DALLA-formulae into the set of DALLA′-formulae such that $f(\phi) \stackrel{\text{def}}{=} \phi$ if neither $\cap$ nor $\cup^*$ occurs in $\phi$, otherwise

$$f(\phi) \stackrel{\text{def}}{=} \phi' \wedge \bigwedge_{\alpha=1}^{l} \bigwedge_{i=1}^{\beta} [c_\alpha](p^i_{new} \Leftrightarrow \psi^i)$$

such that

- $\{\phi^1, \ldots, \phi^\beta\}$, $\beta \geq 1$, are subformulae of $\phi$ such that for every $i \in \{1, \ldots, \beta\}$, $\phi^i$ is of the form $[e \oplus e']\psi^i$ for some $\oplus \in \{\cap, \cup^*\}$ and neither $\cap$ nor $\cup^*$ occurs in $\psi^i$;
- the modal constants in $\phi$ are $\{c_1, \ldots c_l\}$, $l \geq 1$;
- $p^1_{new}, \ldots, p^\beta_{new}$ are distinct propositional variables that do not occur in $\phi$;
- $\phi'$ is obtained from $\phi$ by substituting every occurrence of $[e \oplus e']\psi^i$ by $[e]p^i_{new} \wedge [e']p^i_{new}$ if $\oplus = \cup^*$ and by $[e]p^i_{new} \vee [e']p^i_{new}$ if $\oplus = \cap$.

Obviously, every propositional variable $p^i_{new}$ behaves as a renaming variable similarly to every propositional variable $r_j$ in the proof of Lemma 3.1. Observe also that for every formula $\psi$ built on the modal constants $\{c_1, \ldots c_l\}$, the formula $[c_1]\psi \wedge \ldots \wedge [c_l]\psi$ behaves as the formula $\psi$ prefixed by a universal modal connective. $f(\phi)$ can be computed in polynomial-time in $|\phi|$ and $\phi$ is DALLA-satisfiable iff $f(\phi)$ is DALLA-satisfiable. Hence $\phi$ is DALLA-satisfiable iff $f^{|\phi|}(\phi)$ is DALLA-satisfiable. $f^{|\phi|}(\phi)$ can be computed in polynomial-time in $|\phi|$ and it is a DALLA′-formula by identifying $[c_i]$ and $[i]$. Q.E.D.

Hence, an important consequence of Theorem 4.9 is the following result.

THEOREM 5.2. DALLA-satisfiability is **PSPACE**-complete.

As a corollary, the logic $S5^S_{L3}$ introduced in (BO99) has a **PSPACE**-complete satisfiability problem, since DALLA and $S5^S_{L3}$ can be viewed as syntactic variants.

*5.2 LGM*

The logic of graded modalities LGM introduced in (Nak93) is based on the graded equivalence relations, i.e., the graded similarity in Zadeh's meaning. Here, by LGM we mean the restriction of LGM defined in (Nak93) where only rational numbers [4] can occur in modal connectives. Following (Dem98, Section 5.2), one can show that LGM-satisfiability and LGM′-satisfiability are of the same computational complexity. Consequently,

THEOREM 5.3. LGM-satisfiability is **PSPACE**-complete.

## 6 Concluding remarks

In the paper we have introduced a class of nice unbounded LA-logics and we have shown that every nice unbounded LA-logic has a **PSPACE**-complete satisfiability problem whereas for bounded LA-logics, the problem is "only" **NP**-complete. The large class of nice unbounded LA-logics includes DALLA′ and LGM′ and consequently we were able to show that the logics LGM and DALLA are **PSPACE**-complete (see Section 5). The best known complexity bounds for DALLA were **NP**-hardness and **NEXPTIME**-easiness. A by-product of our technical developments is the observation that for LA-logics, bounding the number of distinct modal connectives is similar to the effect of bounding the modal depth for the standard modal logic K (see e.g. (Hal95)). We conjecture that Corollary 2.3 can be extended to other unbounded LA-logics.

An interesting open problem is the decidability (and computational complexity if meaningful) of the satisfiability problem for the Data Analysis Logic DAL defined in (FdCO85). DAL has the same language as DALLA and the DAL models are the structures satisfying the conditions (equiv) and ($\cap + \cup^*$). What is known so far is that DAL satisfiability is **EXPTIME**-hard by (Hem96, Theorem 5.1) and therefore it is highly probable that DAL behaves differently from DALLA as far as complexity issues are concerned. The largest fragment of DAL we are aware of for which the satisfiability problem is in **EXPTIME** is studied in (HM97; HM99).

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[4] In (Dem98) the argument for the decidability of LGM only applies when the set of modal connectives is countable, as it is the case for instance when we consider the set of rational numbers.

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## A Proof of Lemma 2.2

The proof is based on results from (HC68; GP92) and is done by induction on $\mathrm{card}(\mathbb{N}(\phi))$.

For every $X \subseteq \mathbb{N}$, an *LGM′(X)-formula* $\psi$ is an LGM′-formula such that $\mathbb{N}(\psi) \subseteq X$.
*Base case*: $\mathrm{card}(\mathbb{N}(\phi)) \leq 1$.
Immediate from the results for the modal logic S5 (HC68, Chapter 3).
*Induction step*: assume that for every LGM′-formula $\phi$ such that $\mathrm{card}(\mathbb{N}(\phi)) \leq n$, $n \geq 1$, there is an LGM′-equivalent formula $\psi$ such that if $[i]\varphi_1$ occurs in the scope of $[i']\varphi_2$ in $\psi$, then $i > i'$.
Let $\emptyset \neq X \subseteq \mathbb{N}$ be such that $\mathrm{card}(X) = n+1$ and $i_1 = min(X)$. By adapting definitions from (GP92), an elementary $X$-disjunction is an LGM′($X$)-formula of the form

$$\phi_{-1} \vee \langle i_1 \rangle \phi_0 \vee [i_1]\phi_1 \vee \ldots \vee [i_1]\phi_N,$$

where $\phi_{-1}, \phi_0, \ldots, \phi_N$ are LGM′($X \setminus \{i_1\}$)-formulae. For every LGM′($X$)-formula $\phi$, if $\phi \Leftrightarrow \psi$ is LGM′-valid and $\psi$ is a conjunction of elementary $X$-disjunctions, then $\psi$ is said to be a *$X$-conjunctive form* of $\phi$. We show that for every LGM′($X$)-formula $\phi$, there exists an $X$-conjunctive form that can be effectively computed. Let $i \in X$ and $\varphi, \varphi'$ be LGM′($X$)-formulae such that $\varphi'$ is a Boolean combination of formulae prefixed by $[i_1]$ or $\langle i_1 \rangle$. The LGM′($X$)-formulae below are LGM′-valid:

**(i)** $[i](\varphi \vee \varphi') \Leftrightarrow ([i]\varphi) \vee \varphi'$;
**(ii)** $[i_1](\varphi \vee \varphi') \Leftrightarrow ([i_1]\varphi) \vee \varphi'$.

By the induction on the structure of $\phi$ one can show that $\phi$ is equivalent to a conjunction of elementary $X$-disjunctions. The base case ($\phi$ is a propositional variable) and the cases in the induction step when the outermost connective of $\phi$ is Boolean are standard and they are omitted here. Let $\phi = [i]\phi_1$ with $i \in X \setminus \{i_1\}$. By the induction hypothesis, there is a finite set $\{\kappa_1, \ldots, \kappa_M\}$ of elementary $X$-disjunctions such that $\kappa_1 \wedge \ldots \wedge \kappa_M$ is a $X$-conjunctive form of $\phi_1$. Hence, $\phi \Leftrightarrow [i]\kappa_1 \wedge \ldots \wedge [i]\kappa_M$ is LGM′-valid. The condition (i) above guarantees that for every $j \in \{1, \ldots, M\}$, $[i]\kappa_j$, has a $X$-conjunctive form. So $\phi$ has a $X$-conjunctive form. When $\phi = [i_1]\phi_1$, the proof is similar to the previous case except that the condition (ii) above is used instead of (i).

Consequently, for every LGM′($X$)-formula $\phi$, there is a finite set $\{\kappa_1, \ldots, \kappa_M\}$ of elementary $X$-disjunctions such that $\phi \Leftrightarrow \kappa_1 \wedge \ldots \wedge \kappa_M$ is LGM′-valid. Each $\kappa_i$ is of the form $\phi^i_{-1} \vee \langle i_1 \rangle \phi^i_0 \vee [i_1]\phi^i_1 \vee \ldots \vee [i_1]\phi^i_N$, where $\phi^i_{-1}, \phi^i_0, \ldots, \phi^i_N$ are LGM′($X \setminus \{i_1\}$)-formulae. By the induction hypothesis, each $\phi^i_j$ has an LGM′-formula $\psi^i_j$ equivalent to $\phi^i_j$ such that $[i]\varphi_1$ occurs in the scope of $[i']\varphi_2$ in $\psi$, then $i > i'$. The replacement of LGM′ equivalent formulae preserves validity, which allows us to complete the proof easily.

## B Proof of Lemma 4.3

Let $\texttt{LA-WORLD}(\Sigma, u, \sigma, index, Z, \phi)$ be a call in $\texttt{LA-WORLD}(Y, \epsilon, \sigma, n, \mathbb{N}(\phi), \phi)$. Suppose that $\Sigma$ is of the form

$$\Sigma' \cdot X_{i_0} \cdot X_{i_0+1} \cdot \ldots \cdot X_{i_0+|\phi|^2+1} \cdot \Sigma''$$

such that for every $j \in \{1, \ldots, |\phi|^2 + 1\}$, the $(i_0 + j)$th element of $u$ is $i$. By definition of LA-WORLD (condition (v)), for every $j \in \{0, \ldots, |\phi|^2\}$, we have $X_{i_0+j} \approx_i X_{i_0+j+1}$. Suppose that $[i]\psi_{new} \in \mathrm{cl}(u, \phi) \setminus X_{i_0+|\phi|^2+1}$ and that there is no $X \in \Sigma$ verifying the conditions (i)-(iii) from Figure 2. By the definition of $\approx_i$, for all $j, j' \in \{0, \ldots, |\phi|^2 + 1\}$, $\{[i]\varphi : [i]\varphi \in X_{i_0+j}\} = \{[i]\varphi : [i]\varphi \in X_{i_0+j'}\}$. Since for every $\alpha \in \{1, \ldots, |\phi|^2\}$, $\texttt{LA-WORLD}(\Sigma' \cdot X_{i_0} \cdot X_{i_0+1} \cdot \ldots \cdot X_{i_0+\alpha-1}, u' \cdot i^{\alpha-1}, \sigma_{\alpha-1}, i, Z, \phi)$ calls directly $\texttt{LA-WORLD}(\Sigma' \cdot X_{i_0} \ldots X_{i_0+\alpha}, u' \cdot i^{\alpha}, \sigma_\alpha, i, Z, \phi)$, there are formulae $\psi_1, \ldots, \psi_{|\phi|^2+1}$ in $\mathrm{cl}(\{\phi\})$ such that for every $\alpha \in \{1, \ldots, |\phi|^2+1\}$. $\psi_\alpha \notin X_{i_0+\alpha}$ and for every $\alpha' \in \{0, \ldots, \alpha-1\}$, $\psi_\alpha \in X_{i_0+\alpha'}$. Hence $\psi_1, \ldots, \psi_{|\phi|^2+1}$ are $|\phi|^2 + 1$ distinct formulae in $\mathrm{cl}(\{\phi\})$, which is in contradiction with $\mathrm{card}(\mathrm{cl}(\{\phi\})) \leq |\phi|^2$.

## C Proof of Lemma 4.8

For every $u \in \mathbb{N}^*$, for every $j \in \{1, \ldots, |u|\}$, we write $u(j)$ [resp. $u[j]$] to denote the $j$th element of $u$ [resp. to denote the prefix of $u$ of length $j$]. By convention $u[0] = \epsilon$. Assume that $\phi$ is $\mathcal{L}$-satisfiable with $\mathrm{card}(\mathbb{N}(\phi)) = n$. There is an $\mathcal{L}$-model $\mathcal{M} = \langle W^0, (R_i^0)_{i\in\mathbb{N}}, V\rangle$ and $w^0 \in W^0$ such that $\mathcal{M}^0, w^0 \models \phi$. We show that

**(i)** for every $u \in \mathbb{N}^*$, for every sequence $\sigma_0, \ldots, \sigma_{|u|}$ of bijections of the form $\{1, \ldots, n\} \to \mathbb{N}(\phi)$, for every sequence $\Sigma = X_0, \ldots, X_{|u|}$ such that for every $j \in \{0, \ldots, |u|\}$,

- (1) $X_j \subseteq \mathrm{cl}(u[j], \phi)$ is $\langle u[j], \sigma_j\rangle$-consistent;
- (2) for every $k \in \{\sigma_j^{-1}(u(j)), \ldots, \mathrm{card}(\mathbb{N}(\phi))\}$, $\sigma_{j-1}(k) = \sigma_j(k)$;
- (3) if $j < |u|$, then $X_j \approx_{\sigma_j(u(k))} X_{j+1}$ for every $k \in \{\sigma_j^{-1}(u(j+1)), \ldots, n\}$;
- (4) if $j < |u|$, then there is $[u(j)]\psi \in \mathrm{cl}(u[j], \phi) \setminus X_j$ such that
  - (a) $\psi \notin X_{j+1}$;
  - (b) there is no $k' \in \{0, \ldots, j\}$ such that for every $k \in \{\sigma_j^{-1}(u(j)), \ldots, n\}$, $X_j \approx_{\sigma_j(k)} X_{k'}$ and $u[j] = u[k'] \cdot u(j)^{j-k'}$;
  - (c) $u(j+1) \in Z_{\sigma_j}^{u(j)}$;

if there is an $\mathcal{L}$-model $\mathcal{M}$ and $w \in W$ such that
(5) for every $\psi \in \text{cl}(u, \phi)$, $\mathcal{M}, w \models \psi$ iff $\psi \in X_{|u|}$,
(6) $R_{\sigma_{|u|}(1)}(w) \subseteq \ldots \subseteq R_{\sigma_{|u|}(n)}(w)$,
then $\texttt{LA-WORLD}(\Sigma, u, \sigma_{|u|}, \sigma_{|u|}^{-1}(last(u)), Z_{\sigma_{|u|}}^{last(u)}, \phi)$ returns true.

The conditions (1)-(4) state conditions between arguments of successive calls to LA-WORLD whereas the conditions (5)-(6) are similar to the conditions (Prop1) and (Prop2) from Section 4.2.

By convention, if $u = \epsilon$, then $\sigma_{|u|}^{-1}(last(u)) = n$ and $Z_{\sigma_{|u|}}^{last(u)} = \mathbb{N}(\phi)$. Consequently, by taking $Y = \{\psi \in \text{cl}(\epsilon, \phi) : \mathcal{M}^0, w^0 \models \psi\}$, $\sigma = \sigma_0$ for some bijection $\sigma_0$ such that $R_{\sigma_0(1)}(w^0) \subseteq \ldots \subseteq R_{\sigma_0(n)}(w^0)$, we obtain that $\texttt{LA-WORLD}(Y, \epsilon, \sigma, n, \mathbb{N}(\phi), \phi)$ returns true.
The proof of (i) is by the induction on the length of $u$.
*Base case*: $|u| > |\phi|^3$. By a reasoning similar to the proof of Lemma 4.5, one can show that no sequence $\Sigma = X_0, \ldots, X_{|u|}$ satisfies the conditions above.
*Induction step*: for every $u \in \mathbb{N}^*$ of length $|u| \geq m$, (i) holds true.
Now assume that $u \in \mathbb{N}^*$ is of length $m-1$. Let $\sigma_0, \ldots, \sigma_{m-1}$ and $\Sigma = X_0, \ldots, X_{m-1}$ satisfy the conditions (1)-(4). Let $\mathcal{M} = \langle W, (R_i)_{i \in \mathbb{N}}, V \rangle$ be an $\mathcal{L}$-model and $w \in W$ satisfying the conditions (5)-(6). Since $X_{m-1}$ is $\langle u, \sigma_{m-1} \rangle$-consistent, $\texttt{LA-WORLD}(\Sigma, u, \sigma_{m-1}, \sigma_{m-1}^{-1}(last(u)), Z_{\sigma_{m-1}}^{last(u)}, \phi)$ returns false only if there is $[i]\psi \in \text{cl}(u, \phi) \setminus X_{m-1}$ with $i \in Z_{\sigma_{m-1}}^{last(u)}$ such that for every $X_\psi \subseteq \text{cl}(u \cdot i, \phi)$, for every bijection $\sigma' : \{1, \ldots, n\} \to \mathbb{N}(\phi)$ such that the conditions (iv)-(vi) from Figure 2 hold true, $\texttt{LA-WORLD}(\Sigma \cdot X_\psi, u \cdot i, \sigma', \sigma'^{-1}(i), Z_{\sigma'}^{i}, \phi)$ returns false. By satisfaction of the condition (5), we obtain $\mathcal{M}, w \not\models [i]\psi$. So there is $w' \in W$ such that $\langle w, w' \rangle \in R_i$ and $\mathcal{M}, w' \models \psi$. Let $\sigma_m$ be a bijection $\{1, \ldots, n\} \to \mathbb{N}(\phi)$ such that $R_{\sigma_m(1)}(w') \subseteq \ldots \subseteq R_{\sigma_m(n)}(w')$. Such a bijection exists because $\mathcal{M}$ is an $\mathcal{L}$-model. Let $Y \subseteq \text{cl}(u \cdot i, \phi)$ be defined as follows: for every $\varphi \in \text{cl}(u \cdot i, \phi)$, $\varphi \in Y \overset{\text{def}}{\Leftrightarrow} \mathcal{M}, w' \models \varphi$. So $\psi \notin Y$ and for every $j \in \{\sigma_{m-1}^{-1}(i), \ldots, n\}$, $X_{m-1} \approx_{\sigma_{m-1}(j)} Y$. If either $\sigma_{m-1}^{-1}(last(u)) \neq \sigma_{m-1}^{-1}(i)$ or there is no $l \in \{1, \ldots, m-1\}$ such that $\psi \notin X_l$, $u = u' \cdot i^{l-1}$ and for every $j \in \{\sigma_{m-1}^{-1}(last(u)), \ldots, n\}$, $X_{m-1} \approx_{\sigma_{m-1}(j)} X_l$, by the induction hypothesis, $\texttt{LA-WORLD}(\Sigma \cdot Y, u \cdot i, \sigma_m, \sigma_m^{-1}(i), Z_{\sigma_m}^{i}, \phi)$ returns true, a contradiction.