# On Selective Unboundedness of VASS[1]

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**Abstract**

Numerous properties of vector addition systems with states amount to checking the (un)boundedness of some selective feature (e.g., number of reversals, counter values, run lengths). Some of these features can be checked in exponential space by using Rackoff's proof or its variants, combined with Savitch's Theorem. However, the question is still open for many others, e.g., regularity detection problem and reversal-boundedness detection problem. In the paper, we introduce the class of generalized unboundedness properties that can be verified in exponential space by extending Rackoff's technique, sometimes in an unorthodox way. We obtain new optimal upper bounds, for example for place boundedness problem, reversal-boundedness detection (several variants are present in the paper), strong promptness detection problem and regularity detection. Our analysis is sufficiently refined so as to obtain a polynomial-space bound when the dimension is fixed.

*Keywords:* vector addition systems with states, place boundedness problem, regularity detection problem, exponential space

## 1. Introduction

*Reversal-boundedness.* A standard approach to circumvent the undecidability of the reachability problem for counter automata [39] consists in designing subclasses with simpler decision problems. For instance, the reachability problem is decidable for vector addition systems with states (VASS) [38, 33, 36], for flat counter automata [11, 9, 15] or for lossy counter automata [1]. Among the other interesting subclasses of counter automata, reversal-bounded counter automata verify that any counter has a bounded number of reversals, alternations between a nonincreasing mode and a nondecreasing mode, and vice versa. Reversal-boundedness remains a standard concept that is introduced in [4] for multistack automata. A major property of such operational models is that reachability sets are effectively definable in Presburger arithmetic [28], which provides decision procedures for LTL existential model-checking and other related problems, see e.g. [12]. However, the class of reversal-bounded counter automata is not recursive [28] but a significant breakthrough is achieved in [20] by designing a procedure to determine when a VASS is reversal-bounded (or weakly reversal-bounded as defined later), even though the decision procedure can be nonprimitive recursive in the worst-case. This means that reversal-bounded VASS can benefit from the known techniques for Presburger arithmetic [44] in order to solve their verification

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problems. More precisely, we consider subclasses of counter systems for which the reachability sets of the form $\{\vec{x} \in \mathbb{N}^n : (q_0, \vec{x_0}) \xrightarrow{*} (q, \vec{x})\}$ are effectively Presburger-definable ($(q_0, \vec{x_0})$ and $q$ are fixed). By decidability of Presburger arithmetic, this allows us to solve problems restricted to such counter systems such as the reachability problem, the control state reachability problem, the boundedness problem or the covering problem. Indeed, suppose that given $(q_0, \vec{x_0})$ and $q$, one can effectively build a Presburger formula $\varphi$ such that $\varphi$ holds true exactly for the values in $\{\vec{x} \in \mathbb{N}^n : (q_0, \vec{x_0}) \xrightarrow{*} (q, \vec{x})\}$. One can then classically observe that $(q_0, \vec{x_0}) \xrightarrow{*} (q, \vec{z})$ iff the formula below is satisfiable:

$$\varphi(\mathsf{x}_1, \ldots, \mathsf{x}_n) \wedge \mathsf{x}_1 = \vec{z}(1) \wedge \cdots \wedge \mathsf{x}_n = \vec{z}(n).$$

Hence, there is an important gain to have effectively semilinear reachability sets, which can be witnessed by detecting reversal-boundedness or by detecting regularity (guaranteeing the semilinearity by Parikh's Theorem [40]).

*Selective unboundedness.* In order to characterize the complexity of detecting reversal-boundedness on VASS (the initial motivation for this work), we make a detour to selective unboundedness, as explained below. Numerous properties of vector addition systems with states amounts to checking the (un)boundedness of some selective feature. Some of these features can be verified in exponential space by using Rackoff's proof or its variants [45], whereas the question is still open for many of them. In the paper, we advocate that many properties can be decided as soon as we are able to decide selective unboundedness, which is a generalization of place unboundedness for Petri nets (a model known to be equivalent to VASS but of greater practical appeal). The boundedness problem was first considered in [31] and shown decidable by simply inspecting Karp and Miller trees: the presence of the infinity value $\infty$ (also denoted by $\omega$) is equivalent to unboundedness. So, unboundedness is equivalent to the existence of a witness run of the form $(q_0, \vec{x_0}) \xrightarrow{*} (q, \vec{x_1}) \xrightarrow{\pi} (q, \vec{x_2})$ such that $\vec{x_1} \prec \vec{x_2}$, assuming that the initial configuration is $(q_0, \vec{x_0})$ ($\prec$ is the standard strict ordering on tuples of natural numbers). In [45], it is shown that if there is such a run, there is one of length at most doubly exponential. This leads to the ExpSpace-completeness of the boundedness problem for VASS using the lower bound from [37] and Savitch's Theorem [50]. A variant problem consists in checking whether the $i$th component is bounded, i.e., is there a bound $B$ such that for every configuration reachable from $(q_0, \vec{x_0})$, its $i$th component is bounded by $B$? Again, inspecting Karp and Miller trees reveals the answer: the presence of the infinity value $\infty$ at the $i$th position of some extended configuration is equivalent to $i$-unboundedness. Surprisingly, the literature often mentions this alternative problem, see e.g. [46], but never specifies its complexity: ExpSpace-hardness can be obtained from [37] but as far as we know, no elementary complexity upper bound has been shown. A natural adaptation from boundedness is certainly that $i$-unboundedness could be witnessed by the existence of a run of the form $(q_0, \vec{x_0}) \xrightarrow{*} (q, \vec{x_1}) \xrightarrow{\pi} (q, \vec{x_2})$ such that $\vec{x_1} \prec \vec{x_2}$ with $\vec{x_1}(i) < \vec{x_2}(i)$. By inspecting the proof in [45], one can show that if there is such a run, then there is one of length at most doubly exponential. However, although existence of such a run is a sufficient condition for $i$-unboundedness (simply iterate $\pi$ infinitely), this is not a necessary condition. It might be explained by the fact that, if a VASS is unbounded, then there is a witness infinite run with an infinite number of distinct configurations. By contrast, it may happen that a VASS is $i$-unbounded but no infinite run has an infinite amount of distinct values at the $i$th position of the configurations of the run.

*A generalization.* In the paper, we present a generalization of place unboundedness by checking whether a set of components is simultaneously unbounded, possibly with some ordering (see Section 3.2). This amounts to specifying in the Karp and Miller trees, the ordering with which the value $\infty$ appears in the different components. Such a generalization is particularly useful since we show that many problems such as reversal-boundedness [28], strong reversal-boundedness [29], reversal-boundedness from [20] can be naturally reduced to simultaneous unboundedness. Moreover, this allows to extend the class of properties for which ExpSpace can be obtained, see e.g. standard results in [45, 24, 3].

*Our contribution.* In the paper we show the following results.

1. Detecting whether a VASS is reversal-bounded in the sense of [28] or in the sense of [20] is ExpSpace-complete by refining the decidability results from [20] (see Theorem 5.2).
2. To do so, we introduce the generalized unboundedness problem in which many problems can be captured such as the reversal-boundedness detection problems, the boundedness problem, the place boundedness problem, termination, strong promptness detection problem, regularity detection and many other decision problems on VASS. We show that this problem can be solved in exponential space by adapting [45] even though it does not fall into the class of *increasing path formulae* recently introduced in [3, 2] (see Theorem 4.6).
3. Consequently, we show that regularity and strong promptness detection problems for VASS are in ExpSpace. The ExpSpace upper bound has been left open in [3]. Even though most of our results essentially rest on the fact that the place boundedness problem can be solved in ExpSpace, our generalization is introduced to obtain new complexity upper bound for other related problems. On our way to this complexity result, we provide a witness run characterization for place unboundedness that can still be expressed in Yen's path logic [54, 3] but with a path formula of exponential size in the dimension.
4. As a by-product of our analysis and following a parameterized analysis initiated in [48, 27], for all the above-mentionned problems, we show that fixing the dimension of the VASS allows to get a PSpace upper bound.

The complexity of our witness run characterization for selective unboundedness partly explains why it has been ignored so far. It is clear that whenever the place boundedness problem is decidable, the boundedness problem is decidable too. However, the converse does not always hold true: for instance the boundedness problem for transfer nets is decidable unlike the place boundedness problem [17]. Place boundedness problem can be therefore intrinsically more difficult than the boundedness problem: there is always a simple way to be unbounded but if one looks for $i$-unboundedness, it might be much more difficult to detect it, if possible at all.
The paper has also original contributions as far as proof techniques are concerned. First, simultaneous unboundedness has a simple characterization in terms of Karp and Miller trees, but we provide in the paper a witness run characterization, which allows us to provide a complexity analysis along the lines of [45]. We also provide a witness pseudo-run characterization in which we sometimes admit negative component values. This turns out to be the right approach when a characterization from coverability graphs [31, 52] already exists. Apart from this unorthodox adaptation of [45], in the counterpart of Rackoff's proof about the induction on the dimension, we provide an induction on the dimension and on the length of the properties to be verified (see Lemma 4.4). The preliminary work [13] has already been used in [42] to obtain new complexity results. This is a genuine breakthrough comparable to [45, 48, 24, 3]. We believe this approach is still subject to extensions. Finally, a recent work [5] has also established similar ExpSpace upper

bounds for checking properties on VASS by introducing a temporal logic on coverability graphs.

*By the way, we pay a special attention to explain most of the technical developements, at the cost of repeating sometimes standard arguments (see e.g. the nondeterministic procedure in Section 4.1). We feel that this will considerably help the reader for understanding the chain of technical results.*

*Plan of the paper.* In Section 2, we present the vector addition systems with states as well as their decision problems including the simultaneous unboundedness problem and reversal-boundedness detection problem. Section 3 introduces the class of generalized unboundedness properties as well as the generalized unboundedness problem. We show how regularity detection, strong promptness and weak reversal-boundedness detection can be reduced to generalized unboundedness problem. In Section 4, we prove our main result: the generalized unboundedness problem for VASS is ExpSpace-complete. In Section 5, as a consequence of the main result, we show that the regularity detection problem and the strong promptness detection problem are in ExpSpace. Moreover, (weak) reversal-boundedness detection problem for VASS is also shown ExpSpace-complete.

## 2. Preliminaries

In this section, we recall the main definitions for vector addition systems with states (VASS), without states (VAS) as well as the notions of reversal-boundedness introduced in [28, 20]. We also present the simultaneous unboundedness problem, which generalizes the place unboundedness problem for Petri nets. First, we write $\mathbb{N}$ [resp. $\mathbb{Z}$] for the set of natural numbers [resp. integers] and $[m, m']$ with $m, m' \in \mathbb{Z}$ to denote the set $\{j \in \mathbb{Z} : m \leq j \leq m'\}$. Given a dimension $n \geq 1$ and $a \in \mathbb{Z}$, we write $\vec{a} \in \mathbb{Z}^n$ to denote the vector with all values equal to $a$. For $\vec{x} \in \mathbb{Z}^n$, we write $\vec{x}(1), \ldots, \vec{x}(n)$ for the entries of $\vec{x}$. For $\vec{x}, \vec{y} \in \mathbb{Z}^n$, $\vec{x} \preceq \vec{y} \overset{\text{def}}{\Leftrightarrow}$ for every $i \in [1, n]$, we have $\vec{x}(i) \leq \vec{y}(i)$. We also write $\vec{x} \prec \vec{y}$ when $\vec{x} \preceq \vec{y}$ and $\vec{x} \neq \vec{y}$.

### 2.1. Simultaneous unboundedness problem for VASS

*VASS.* A *vector addition system with states* [26] (VASS for short) is a finite-state automaton with transitions labelled by tuples of integers viewed as update functions. A *VASS* is a structure $\mathcal{V} = (Q, n, \delta)$ such that $Q$ is a nonempty finite set of *control states*, $n \geq 1$ is the *dimension*, and $\delta$ is the *transition relation* defined as a finite set of triples in $Q \times \mathbb{Z}^n \times Q$. Elements $t = (q, \vec{b}, q') \in \delta$ are called *transitions* and are often represented by $q \xrightarrow{\vec{b}} q'$. Moreover, a VASS has no initial control state and no final control state but in the sequel we introduce such control states on demand. Figure 1 presents a VASS of dimension 4 with two control states. VASS with a unique control state are called *vector addition systems* (VAS for short) [31]. In the sequel, a VAS $\mathcal{T}$ is represented by a finite nonempty subset of $\mathbb{Z}^n$, encoding naturally the transitions. VASS and VAS are equivalent to Petri nets, see e.g. [47]. In this paper, the decision problems are defined with the VASS model and the decision procedures are designed for VAS, assuming that we know how the problems can be reduced, see e.g. [26]. Indeed, we prefer to define problems with the help of the VASS model since when infinite-state transition systems arise in the modeling of computational processes, there is often a natural factoring of each system state into a control component and a memory component, where the set of control states is typically finite. In this paper, we use the reduction from VASS to VAS defined in [26] that allows to simulate a VASS of dimension $n$ by

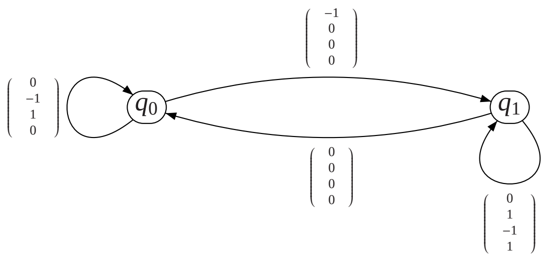



Figure 1: A simple VASS with two control states

a VAS of dimension $n + 3$, independently of its number of control states (formal definition is recalled in the proof of Lemma 2.5). Even though a simpler reduction exists that increments the dimension by the cardinal of the set of control states, the reduction from [26] is exactly what we need, since sometimes, at some intermediate stage, we may increase exponentially the number of control states.

*Runs.* A *configuration* of $\mathcal{V}$ is defined as a pair $(q, \vec{x}) \in Q \times \mathbb{N}^n$ (for VAS, we simply omit the control state). An *initialized VASS* is a pair made of a VASS and a configuration. Given two configurations $(q, \vec{x})$, $(q', \vec{x'})$ and a transition $t = q \xrightarrow{\vec{b}} q'$, we write $(q, \vec{x}) \xrightarrow{t} (q', \vec{x'})$ whenever $\vec{x'} = \vec{x} + \vec{b}$. We also write $(q, \vec{x}) \rightarrow (q', \vec{x'})$ when there is no need to specify the transition $t$. The operational semantics of VASS updates configurations; runs of such systems are essentially sequences of configurations. Every VASS induces a (possibly infinite) directed graph of configurations. Indeed, all the interesting problems on VASS can be formulated on its *transition system* $(Q \times \mathbb{N}^n, \rightarrow)$. Given a VASS $\mathcal{V} = (Q, n, \delta)$, a *run* $\rho$ is a nonempty (possibly infinite) sequence $\rho = (q_0, \vec{x_0}), \ldots, (q_k, \vec{x_k}), \ldots$ of configurations such that $(q_i, \vec{x_i}) \rightarrow (q_{i+1}, \vec{x_{i+1}})$ for all $i$. We set $Reach(\mathcal{V}, (q_0, \vec{x_0})) \stackrel{\text{def}}{=} \{(q_k, \vec{x_k}) :$ there is a finite run $(q_0, \vec{x_0}), \ldots, (q_k, \vec{x_k})\}$. Considering the VASS in Figure 1, one can show that

$$\{\begin{pmatrix} a \\ b \\ d \end{pmatrix} \in \mathbb{N}^3 : d \leq a \times b\} =$$

$$\{\begin{pmatrix} a \\ b \\ d \end{pmatrix} \in \mathbb{N}^3 : \exists \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \in \mathbb{N}^3,\ (q_0, \begin{pmatrix} a' \\ b' \\ c' \\ d \end{pmatrix}) \in Reach(\mathcal{V}, (q_0, \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}))\}$$

A run can be alternatively represented by an initial configuration and a sequence of transitions, assuming that no negative component values is obtained by applying the sequence of transitions. A *path* $\pi$ is a finite sequence of transitions whose successive control states respect $\delta$ (actually this notion is mainly used for VAS without control states). A *pseudo-configuration* is defined as an element of $Q \times \mathbb{Z}^n$. When $\pi = t_1 \cdots t_k$ is a path, the *pseudo-run* $\rho = (\pi, (q, \vec{x}))$ is defined as the sequence of pseudo-configurations $(q_0, \vec{x}_0) \cdots (q_k, \vec{x}_k)$ such that $(q_0, \vec{x}_0) = (q, \vec{x})$, and for every $i \in [1, k]$, there is $t = q_i \xrightarrow{\vec{b}} q_{i+1}$ such that $\vec{x}_i = \vec{x}_{i-1} + \vec{b}$. So, we deliberately distinguish the notion of path (sequence of transitions) from the notion of pseudo-run (sequence of elements in $Q \times \mathbb{Z}^n$ respecting the transition from $\mathcal{V}$). The pseudo-run $\rho$ is *induced* by the path $\pi$ and of *length* $k + 1$; the path $\pi$ is of *length* $k$. $(q_0, \vec{x}_0)$ is called the *initial* pseudo-configuration and $(q_k, \vec{x}_k)$ is called the *final* pseudo-configuration in the pseudo-run $\rho$. We also use the notation $(q, \vec{x}) \xrightarrow{t} (q', \vec{x'})$ with pseudo-configurations. Given a VASS $\mathcal{V}$ [resp. a pseudo-configuration $(q, \vec{x})$, etc.] of dimension $n$, we write $\mathcal{V}(I)$ [resp. $(q, \vec{x})(I)$, etc.] to denote the restriction of $\mathcal{V}$ [resp. $(q, \vec{x})$, etc.] to the components in $I \subseteq [1, n]$.

*Sizes.* Let us start by defining the size of some VAS $\mathcal{T}$ of dimension $n \geq 1$. Given $\vec{x} \in \mathbb{Z}^n$, we write $\mathrm{maxneg}(\vec{x})$ [resp. $\max(\vec{x})$] to denote the value $\max(\{\max(0, -\vec{x}(i)) : i \in [1, n]\})$. [resp. $\max(\{\vec{x}(i) : i \in [1, n]\})$]. By extension, we write $\mathrm{maxneg}(\mathcal{V})$ to denote $\max\{\mathrm{maxneg}(\vec{b}) : q \xrightarrow{\vec{b}} q' \in \delta\}$. Furthermore, we write $\mathrm{scale}(\mathcal{V})$ to denote the value $\max(\{|\vec{b}(i)| : q \xrightarrow{\vec{b}} q' \in \delta,\ i \in [1, n]\})$. For instance, $\mathrm{maxneg}((-2, 3)) = 2$ and $\mathrm{scale}(\{(-2, 3)\}) = 3$. The size of $\mathcal{T}$, written $|\mathcal{T}|$, is defined by the value below: $n \times \mathrm{card}(\mathcal{T}) \times (1 + \lceil log_2(1 + \mathrm{scale}(\mathcal{T}))\rceil)$. Given a finite subset $X$ of $\mathbb{Z}^n$, we also write $|X|$ to denote $n \times \mathrm{card}(X) \times (1 + \lceil log_2(1 + \mathrm{scale}(X))\rceil)$. We write $|\vec{x}|$ to denote the size of $\vec{x} \in \mathbb{Z}^n$ defined as the size of the singleton set $\{\vec{x}\}$. Given a VASS $\mathcal{V} = (Q, n, \delta)$, we write $|\mathcal{V}|$ to denote its size defined by

$$\mathrm{card}(Q) + n \times \mathrm{card}(\delta) \times (2 \times \mathrm{card}(Q) + (1 + \lceil log_2(1 + \mathrm{scale}(\mathcal{V}))\rceil))$$

Observe that $1 + \lceil log_2(1 + a)\rceil$ is a sufficient number of bits to encode integers in $[-a, a]$ for $a > 0$. Moreover $\mathrm{scale}(\mathcal{V}) \geq \mathrm{maxneg}(\mathcal{V})$, $\mathrm{scale}(\mathcal{V}) \leq 2^{|\mathcal{V}|}$ and $|\mathcal{V}| \geq 2$. In a few words, we adopt reasonably succint encodings for all the objects involved in decision problems, in particular the integers are encoded with a binary representation.

*Standard problems.* The reachability problem for VASS is decidable [38, 33, 47, 34, 36]. Nevertheless, the exact complexity of the reachability problem is open: we know it is ExpSpace-hard [37, 10, 18] and no primitive recursive upper bound exists. By contrast, the covering problem and boundedness problems seem easier since they are ExpSpace-complete [37, 45]. Decidability is established in [31] but with a worst-case nonprimitive recursive bound. The ExpSpace lower bound is due to Lipton and the upper bound to Rackoff. In order to be complete, one should make precise how vectors in $\mathbb{Z}^n$ are encoded. The upper bound holds true with a binary representation of integers whereas the lower bound holds true already with the values -1, 0 and 1. Consequently, the problem is ExpSpace-hard even with an unary encoding. The proof technique in [45] has been also used to establish that LTL model-checking problem for VASS is ExpSpace-complete [24]. By adding the possibility to reset counters in the system (providing the class of *reset VASS*), the boundedness and the reachability problems becomes undecidable, see e.g. [16]. By contrast, the covering problem for VASS with resets is decidable by using the theory of well-structured transition systems, see e.g. [22].

*Simultaneous unboundedness problem.* Let $(\mathcal{V}, (q_0, \vec{x}_0))$ be an initialized VASS of dimension $n$ and $X \subseteq [1, n]$. We say that $(\mathcal{V}, (q_0, \vec{x}_0))$ is *simultaneously $X$-unbounded* if for any $B \geq 0$, there is a run from $(q_0, \vec{x}_0)$ to some $(q, \vec{y})$ such that for every $i \in X$, we have $\vec{y}(i) \geq B$. When $X = \{j\}$, we say that $(\mathcal{V}, (q_0, \vec{x}_0))$ is *$j$-unbounded*. It is clear that $(\mathcal{V}, (q_0, \vec{x}_0))$ is bounded (i.e., the set $Reach(\mathcal{V}, (q_0, \vec{x}_0))$ is finite) iff for all $j \in [1, n]$, $(\mathcal{V}, (q_0, \vec{x}_0))$ is not $j$-unbounded. So, here is the simultaneous unboundedness problem.

Simultaneous unboundedness problem:

**Input:** Initialized VASS $(\mathcal{V}, (q_0, \vec{x}_0))$ of dimension $n$ and $X \subseteq [1, n]$.

**Question:** is $(\mathcal{V}, (q_0, \vec{x}_0))$ simultaneously $X$-unbounded?

Theorem 2.1. [31] Simultaneous unboundedness problem is decidable.

This follows from [31, 52]: $(\mathcal{V}, (q_0, \vec{x}_0))$ is simultaneously $X$-unbounded iff the coverability graph $CG(\mathcal{V}, (q_0, \vec{x}_0))$ (see e.g., [31, 52]) contains an extended configuration $(q, \vec{y})$ such that $\vec{y}(X) = \vec{\infty}$ (for $\alpha \in \mathbb{Z} \cup \{\infty\}$, we write $\vec{\alpha}$ to denote any vector of dimension $n \geq 1$ whose component values are $\alpha$). More properties about coverability graphs are recalled below but just note that in the sequel, we show that the simultaneous unboundedness problem is ExpSpace-complete too.

Before going any further, let us recall some properties about coverability graphs [31, 52], see complete definitions in [47]. Not only this will be useful to prove Lemma 3.1 but we will refer to it quite often.

A coverability graph approximates the set of reachable configurations from a given configuration and it is a finite structure that can be effectively computed. Let us start by preliminary definitions. Let us consider the structure $(\mathbb{N} \cup \{\infty\}, \leq)$ such that for all $k, k' \in \mathbb{N} \cup \{\infty\}$, $k \leq k' \overset{\text{def}}{\Leftrightarrow}$ either $k, k' \in \mathbb{N}$ and $k \leq k'$ or $k' = \infty$. We write $k < k'$ whenever $k \leq k'$ and $k \neq k'$. The ordering $\leq$ can be naturally extended to tuples in $(\mathbb{N} \cup \{\infty\})^n$ by defining it component-wise: for all $\vec{x}, \vec{x}' \in (\mathbb{N} \cup \{\infty\})^n$, $\vec{x} \leq \vec{x}' \overset{\text{def}}{\Leftrightarrow}$ for $i \in [1, n]$, either $\vec{x}(i), \vec{x}'(i) \in \mathbb{N}$ and $\vec{x}(i) \leq \vec{x}'(i)$ or $\vec{x}'(i) = \infty$. We also write $\vec{x} \prec \vec{x}'$ when $\vec{x} \leq \vec{x}'$ and $\vec{x} \neq \vec{x}'$. Given $\vec{x}, \vec{x}' \in (\mathbb{N} \cup \{\infty\})^n$ such that $\vec{x} < \vec{x}'$, we write $acc(\vec{x}, \vec{x}')$ to denote the element of $(\mathbb{N} \cup \{\infty\})^n$ such that for $i \in [1, n]$, if $\vec{x}(i) = \vec{x}'(i)$ then $acc(\vec{x}, \vec{x}')(i) \overset{\text{def}}{=} \vec{x}'(i)$, otherwise $acc(\vec{x}, \vec{x}')(i) \overset{\text{def}}{=} \infty$. Let us conclude this paragraph by a last definition. For all $\vec{x} \in (\mathbb{N} \cup \{\infty\})^n$ and for every $t \in \mathbb{Z}^n$, $\vec{x} + t$ is defined as an element of $(\mathbb{Z} \cup \{\infty\})^n$ such that for every $i \in [1, n]$, if $\vec{x}(i) \in \mathbb{N}$ then $(\vec{x} + t)(i) \overset{\text{def}}{=} \vec{x}(i) + t(i)$, otherwise $(\vec{x} + t)(i) \overset{\text{def}}{=} \infty$. Given a VASS $\mathcal{V} = (Q, n, \delta)$ and a configuration $(q_0, \vec{x}_0)$, we recall that the coverability graph $CG(\mathcal{V}, (q_0, \vec{x}_0))$ is a structure $(V, E)$ such that $V \subseteq Q \times (\mathbb{N} \cup \{\infty\})^n$ and $E \subseteq V \times \delta \times V$, see e.g. [31] or in [19] a generalization to well-structured transition systems. Here are essential properties of $CG(\mathcal{V}, (q_0, \vec{x}_0))$:

**(CG1)** $CG(\mathcal{V}, (q_0, \vec{x}_0))$ is a finite structure (consequence of König's Lemma and Dickson's Lemma).

**(CG2)** For any configuration $(q, \vec{y})$ reachable from $(q_0, \vec{x}_0)$ in $\mathcal{V}$, there is $(q, \vec{y}')$ in $CG(\mathcal{V}, (q_0, \vec{x}_0))$ such that $\vec{y} \leq \vec{y}'$. Otherwise said, any reachable configuration can be covered by an element of $CG(\mathcal{V}, (q_0, \vec{x}_0))$. Moreover, if $(q_0, \vec{x}_0) \xrightarrow{\pi} (q, \vec{y})$ is a run of $\mathcal{V}$, then $(q_0, \vec{x}_0) \xrightarrow{\pi} (q, \vec{y}')$ in $CG(\mathcal{V}, (q_0, \vec{x}_0))$.

**(CG3)** For every extended configuration $(q, \vec{y}')$ in $CG(\mathcal{V}, (q_0, \vec{x}_0))$ and for every bound $B \in \mathbb{N}$, there is a run $(q_0, \vec{x}_0) \xrightarrow{*} (q, \vec{y})$ in $\mathcal{V}$ such that for $i \in [1, n]$, if $\vec{y}'(i) = \infty$ then $\vec{y}(i) \geq B$ otherwise $\vec{y}(i) = \vec{y}'(i)$.

Unfortunately, even though $CG(\mathcal{T}, \vec{x}_0)$ is finite, in the worst-case its number of nodes can be nonprimitive recursive [52, 30]. Figure 2 presents a VASS of dimension 1 (on the left) and the corresponding coverability graph for the initial configuration $(q_0, 0)$.

### 2.2. Standard reversal-boundedness and its variant

A *reversal* for a counter occurs in a run when there is an alternation from nonincreasing mode to nondecreasing mode and vice-versa. For instance, in the sequence below, there are three reversals identified by an upper line:

$$0011223333444\overline{4}33222\overline{3}3344445555\overline{5}4$$

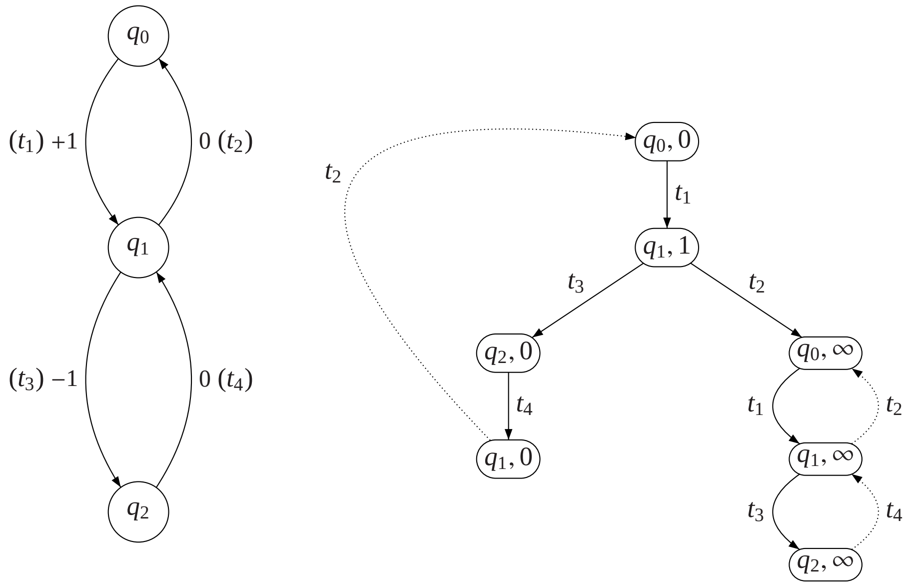



Figure 2: A VASS and its coverability graph from the initial configuration $(q_0, 0)$

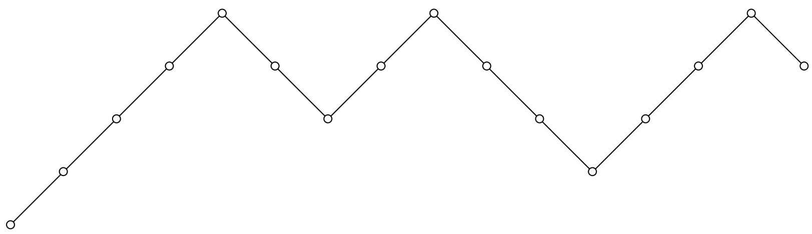


Figure 3: 5 reversals in a row

Similarly, the sequence 001112222233333334444 has no reversal. Figure 3 presents schematically the behavior of a counter with 5 reversals. A VASS is *reversal-bounded* whenever there is $r \geq 0$ such that for any run, every counter makes no more than $r$ reversals. This class of VASS has been introduced and studied in [28], partly inspired by similar restrictions on multistack automata [4]. A formal definition will follow, but before going any further, it is worth pointing out a few peculiarities of this subclass. Indeed, reversal-bounded VASS are augmented with an initial configuration so that existence of the bound $r$ is relative to the initial configuration. Secondly, this class is not defined from the class of VASS by imposing syntactic restrictions but rather semantical ones. In spite of the fact that the problem of deciding whether a counter automaton (VASS with zero-tests) is reversal-bounded is undecidable [28], we explain later why reversal-bounded counter automata have numerous fundamental properties. Moreover, a breakthrough has been achieved in [20] by establishing that checking whether a VASS is reversal-bounded is decidable. The decidability proof in [20] provides a decision procedure that requires nonprimitive recursive time in the worst case since Karp and Miller trees need to be built [31, 52]. In the sequel, we show that this can be checked with exponential space only, and this is optimal as far as worst-case complexity is concerned.

Let $\mathcal{V} = (Q, n, \delta)$ be a VASS. Let us define the auxiliary VASS $\mathcal{V}_{rb} = (Q', 2n, \delta')$ such that essentially, the $n$ new components in $\mathcal{V}_{rb}$ count the number of reversals for each component from $\mathcal{V}$. We set $Q' = Q \times \{\mathrm{DEC}, \mathrm{INC}\}^n$ and, for every $\vec{v} \in \{\mathrm{DEC}, \mathrm{INC}\}^n$ and every $i \in [1, n]$, $\vec{v}(i)$ encodes whether component $i$ is in a decreasing mode or in an increasing mode. Moreover, $(q, \vec{mode}) \xrightarrow{\vec{b'}} (q', \vec{mode}') \in \delta'$ (with $\vec{b'} \in \mathbb{Z}^{2n}$) $\stackrel{\text{def}}{\Leftrightarrow}$ there is $q \xrightarrow{\vec{b}} q' \in \delta$ such that $\vec{b'}([1, n]) = \vec{b}$ and for every $i \in [1, n]$, one of the conditions below is satisfied:

- $\vec{b}(i) < 0$, $\vec{mode}(i) = \vec{mode}'(i) = \mathrm{DEC}$ and $\vec{b'}(n + i) = 0$,
- $\vec{b}(i) < 0$, $\vec{mode}(i) = \mathrm{INC}$, $\vec{mode}'(i) = \mathrm{DEC}$ and $\vec{b'}(n + i) = 1$,
- $\vec{b}(i) > 0$, $\vec{mode}(i) = \vec{mode}'(i) = \mathrm{INC}$ and $\vec{b'}(n + i) = 0$,
- $\vec{b}(i) > 0$, $\vec{mode}(i) = \mathrm{DEC}$, $\vec{mode}'(i) = \mathrm{INC}$ and $\vec{b'}(n + i) = 1$,
- $\vec{b}(i) = 0$, $\vec{mode}(i) = \vec{mode}'(i)$ and $\vec{b'}(n + i) = 0$.

Initialized VASS $(\mathcal{V}, (q, \vec{x}))$ is *reversal-bounded* [28] $\stackrel{\text{def}}{\Leftrightarrow}$ for every $i \in [n + 1, 2n]$, $\{\vec{y}(i) : \exists \text{ run } (q_{rb}, \vec{x}_{rb}) \xrightarrow{*} (q', \vec{y}) \text{ in } \mathcal{V}_{rb}\}$ is finite with $q_{rb} = (q, \vec{\mathrm{INC}})$, $\vec{x}_{rb}$ restricted to the $n$ first components is $\vec{x}$ and $\vec{x}_{rb}$ restricted to the $n$ last components is $\vec{0}$. When $r \geq \max(\{\vec{y}(i) : \exists \text{ run } (q_{rb}, \vec{x}_{rb}) \xrightarrow{*} (q', \vec{y}) \text{ in } \mathcal{V}_{rb}\} : i \in [n + 1, 2n])$, $(\mathcal{V}, (q, \vec{x}))$ is said to be *r-reversal-bounded*. For a fixed $i \in [1, n]$, when $\{\vec{y}(n + i) : \exists \text{ run } (q_{rb}, \vec{x}_{rb}) \xrightarrow{*} (q', \vec{y}) \text{ in } \mathcal{V}_{rb}\}$ is finite, we say that $(\mathcal{V}, (q, \vec{x}))$ is *reversal-bounded with respect to i*. Reversal-boundedness for counter automata, and *a fortiori* for VASS, is very appealing because reachability sets are semilinear as recalled below.

Theorem 2.2. [28] Let $(\mathcal{V}, (q, \vec{x}))$ be an $r$-reversal-bounded VASS. For each control state $q'$, the set $\{\vec{y} \in \mathbb{N}^n : \exists \text{ run } (q, \vec{x}) \xrightarrow{*} (q', \vec{y})\}$ is effectively semilinear.

This means that one can compute effectively a Presburger formula that characterizes precisely the reachable configurations whose control state is $q'$. So, detecting reversal-boundedness for VASS, which can be easily reformulated as an unboundedness problem with the above reduction, is worth the effort since semilinearity follows and then decision procedures for Presburger arithmetic can be used. By a simple observation, boundedness and reversal-boundedness are related as follows.

Lemma 2.3. $(\mathcal{V}, (q, \vec{x}))$ is reversal-bounded with respect to $i$ iff $(\mathcal{V}_{rb}, (q_{rb}, \vec{x}_{rb}))$ is not $(n + i)$-unbounded.

An interesting extension of reversal-boundedness is introduced in [20, 49] for which we only count the number of reversals when their values occur for a counter value above a given bound $B$. For instance, finiteness of the reachability set implies reversal-boundedness in the sense of [20, 49], which we call *weak reversal-boundedness*.

Let $\mathcal{V} = (Q, n, \delta)$ be a VASS and a bound $B \in \mathbb{N}$. Instead of defining a counter automaton $\mathcal{V}_{rb}$ as done to characterize (standard) reversal-boundedness, we define directly an infinite directed graph that corresponds to a variant of the transition system of $\mathcal{V}_{rb}$: still, there are $n$ new counters that record the number of reversals but only if their values occur above a bound

$B$. That is why, the infinite directed graph $TS_B = (Q \times \{\text{DEC}, \text{INC}\}^n \times \mathbb{N}^{2n}, \rightarrow_B)$ is defined as follows: $(q, \vec{mode}, \vec{x}) \rightarrow_B (q', \vec{mode}', \vec{x}') \overset{\text{def}}{\Leftrightarrow}$ there is a transition $q \xrightarrow{\vec{b}} q' \in \delta$ such that $\vec{x}'([1,n]) = \vec{x}([1,n]) + \vec{b}$, and for every $i \in [1,n]$, one of the conditions below is satisfied:

- $\vec{b}(i) < 0$, $\vec{mode}(i) = \vec{mode}'(i) = \text{DEC}$ and $\vec{x}'(n+i) - \vec{x}(n+i) = 0$
- $\vec{b}(i) < 0$, $\vec{mode}(i) = \text{INC}$, $\vec{mode}'(i) = \text{DEC}$, $\vec{x}(i) \leq B$ and $\vec{x}'(n+i) - \vec{x}(n+i) = 0$,
- $\vec{b}(i) < 0$, $\vec{mode}(i) = \text{INC}$, $\vec{mode}'(i) = \text{DEC}$, $\vec{x}(i) > B$ and $\vec{x}'(n+i) - \vec{x}(n+i) = 1$,
- $\vec{b}(i) > 0$, $\vec{mode}(i) = \vec{mode}'(i) = \text{INC}$ and $\vec{x}'(n+i) - \vec{x}(n+i) = 0$,
- $\vec{b}(i) > 0$, $\vec{mode}(i) = \text{DEC}$, $\vec{mode}'(i) = \text{INC}$, $\vec{x}(i) > B$ and $\vec{x}'(n+i) - \vec{x}(n+i) = 1$,
- $\vec{b}(i) > 0$, $\vec{mode}(i) = \text{DEC}$, $\vec{mode}'(i) = \text{INC}$, $\vec{x}(i) \leq B$ and $\vec{x}'(n+i) - \vec{x}(n+i) = 0$,
- $\vec{b}(i) = 0$, $\vec{mode}(i) = \vec{mode}'(i)$ and $\vec{x}'(n+i) - \vec{x}(n+i) = 0$.

Given $B \geq 0$ and $r \geq 0$, the initialized VASS $(\mathcal{V}, (q, \vec{x}))$ is *r-reversal-B-bounded* $\overset{\text{def}}{\Leftrightarrow}$ for every $i \in [n+1, 2n]$, $\{\vec{y}(i) : (q_{rb}, \vec{x}_{rb}) \xrightarrow{*}_B (q', \vec{y})$ in $TS_B\}$ is finite and $r \geq \max(\{\vec{y}(i) : (q_{rb}, \vec{x}_{rb}) \xrightarrow{*}_B (q', \vec{y})$ in $TS_B\} : i \in [n+1, 2n])$. Initialized VASS $(\mathcal{V}, (q, \vec{x}))$ is *weakly reversal-bounded* [20] $\overset{\text{def}}{\Leftrightarrow}$ there is some $B \geq 0$ such that for every $i \in [n+1, 2n]$, $\{\vec{y}(i) : (q_{rb}, \vec{x}_{rb}) \xrightarrow{*}_B (q', \vec{y})$ in $TS_B\}$ is finite. Observe that whenever $(\mathcal{V}, (q, \vec{x}))$ is $r$-reversal-bounded, $(\mathcal{V}, (q, \vec{x}))$ is $r$-reversal-0-bounded. Reversal-boundedness for counter automata, and *a fortiori* for VASS, is again very appealing because reachability sets are semilinear as stated below.

Theorem 2.4. [28, 20] Let $(\mathcal{V}, (q, \vec{x}))$ be an initialized VASS that is (weakly) $r$-reversal-$B$-bounded for some $r, B \geq 0$. For each control state $q'$, the set $\{\vec{y} \in \mathbb{N}^n :$ run $(q, \vec{x}) \xrightarrow{*} (q', \vec{y})\}$ is effectively semilinear.

This means that one can compute effectively a Presburger formula that characterizes precisely the reachable configurations whose control state is $q'$. The original proof for reversal-boundedness can be found in [28] and its extension for weak reversal-boundedness is presented in [20]; whenever a counter value is below $B$, this information is encoded in the control state which provides a reduction to (standard) reversal-boundedness.

Reversal-boundedness detection problem

**Input:** Initialized VASS $(\mathcal{V}, (q, \vec{x}))$ of dimension $n$ and $i \in [1, n]$.

**Question:** Is $(\mathcal{V}, (q, \vec{x}))$ reversal-bounded with respect to the component $i$?

We also consider the variant with weak reversal-boundedness.

Let us conclude this section by Lemma 2.5 below. The proof is essentially based on [26, Lemma 2.1] and on the definition of the initialized VASS $(\mathcal{V}_{rb}, (q_{rb}, \vec{x}_{rb}))$. The key properties are that the dimension increases only linearly and the scale "only" exponentially in the dimension.

Lemma 2.5. Given a VASS $\mathcal{V} = (Q, n, \delta)$ and a configuration $(q, \vec{x})$, one can effectively build in polynomial space an initialized VAS $(\mathcal{T}, \vec{x}')$ of dimension $2n + 3$ such that $(\mathcal{V}, (q, \vec{x}))$ is reversal-bounded with respect to $i$ iff $(\mathcal{T}, \vec{x}')$ is not $(n+i)$-unbounded. Moreover, $\text{scale}(\mathcal{T}) = \max((\text{card}(Q) \times 2^n + 1)^2, \text{scale}(\mathcal{V}))$.

*Proof.* (Lemma 2.5) Let $\mathcal{V} = (Q, n, \delta)$ be a VASS and $(q, \vec{x}) \in Q \times \mathbb{N}^n$. Suppose that $Q$ has $m \geq 1$ control states with $Q = \{q_1, \ldots, q_m\}$. Let us recall the construction of an equivalent initialized VAS of dimension $n + 3$ from [26, Lemma 2.1], that we write $(((Q, n, \delta), (q, \vec{x})))^{\mathrm{HP}} = (\mathcal{T}, \vec{x}')$. We pose $a_i \stackrel{\text{def}}{=} i$ and $b_i \stackrel{\text{def}}{=} (m + 1)(m + 1 - i)$ for every $i \in [1, m]$. A configuration $(q_i, \vec{y})$ of $\mathcal{V}$ is encoded by the configuration $\vec{y}'$ in $\mathcal{T}$ such that $\vec{y}'([1, n]) = \vec{y}$ and $\vec{y}'([n + 1, n + 3]) = (a_i, b_i, 0)$. The initial configuration $\vec{x}'$ is computed from $(q, \vec{x})$ by using this encoding. It remains to define the transitions in $\mathcal{T}$.

- For each $t = q_i \xrightarrow{\vec{b}} q_j \in \delta$, we consider the transition $t' \in \mathcal{T}$ such that $t'([1, n]) = \vec{b}$ and $t'([n + 1, n + 3]) = (a_j - b_i, b_j, -a_i)$.
- For technical reasons, for every $i \in [1, m]$, we add two dummy transitions $t_i$ and $t_i'$ in $\mathcal{T}$ such that
  - $t_i([1, n]) = t_i'([1, n]) = \vec{0}$,
  - $t_i([n + 1, n + 3]) = (-a_i, a_{m+1-i} - b_i, b_{m+1-i})$,
  - $t_i'([n + 1, n + 3]) = (b_i, -a_{m+1-i}, -b_{m+1-i} + a_i)$.

Observe that for $t = q_i \xrightarrow{\vec{b}} q_j \in \delta$, $(t' + t_i + t_i')([1, n]) = \vec{b}$ and $(t' + t_i + t_i')([n + 1, n + 3]) = (a_j - a_i, b_j - b_i, 0)$. The proof of [26, Lemma 2.1] establishes that every run $(q_0', \vec{y}_0) \cdots (q_k', \vec{y}_k)$ in $\mathcal{V}$ leads to a run $\rho' = \vec{z}_0 \cdots \vec{z}_{3k}$ in $\mathcal{T}$ such that

- for every $i \in [0, k]$, $\vec{z}_{3i}([1, n]) = \vec{y}_i$ and $\vec{z}_{3i}$ is the standard encoding of $(q_i', \vec{y}_i)$. Moreover, each step $(q_i', \vec{y}_i) \xrightarrow{t} (q_{i+1}', \vec{y}_{i+1})$ corresponds to the three steps $\vec{z}_{3i} \xrightarrow{t_I t_I' t} \vec{z}_{3i+3}$ in $\rho'$ where $q_i'$ is the $I$th control state of $Q$.

An analogous property holds true in the converse direction (and this is the place where the dummy transitions play a crucial role). This implies that for every $i \in [1, n]$, $(\mathcal{V}, (q, \vec{x}))$ is $i$-unbounded iff $((\mathcal{V}, (q, \vec{x})))^{\mathrm{HP}}$ is $i$-unbounded.

Let us come back to our reduction. Let $\mathcal{V} = (Q, n, \delta)$, $(q, \vec{x})$ and $i$ be an instance of the reversal-boundedness detection problem. Using Lemma 2.3 and the properties of the construction in [26, Lemma 2.1], it is easy to show that

- $(\mathcal{V}, (q, \vec{x}))$ is reversal-bounded with respect to $i$ iff $((\mathcal{V}_{rb}, (q_{rb}, \vec{x}_{rb})))^{\mathrm{HP}}$ is not $(n+i)$-unbounded.
- The scale of the VAS $((\mathcal{V}_{rb}, (q_{rb}, \vec{x}_{rb})))^{\mathrm{HP}}$ is bounded by $\max((\mathrm{card}(Q) \times 2^n + 1)^2, \mathrm{scale}(\mathcal{V}))$ (as well as the scale of the target initial configuration).
- $((\mathcal{V}_{rb}, (q_{rb}, \vec{x}_{rb})))^{\mathrm{HP}}$ can be built in polynomial space.

It is worth noting that the cardinal of the set of control states of $\mathcal{V}_{rb}$ is $\mathrm{card}(Q) \times 2^n$ where $Q$ is the set of control states of $\mathcal{V}$. Hence, this excludes the possibility to construct $((\mathcal{V}_{rb}, (q_{rb}, \vec{x}_{rb})))^{\mathrm{HP}}$ in logarithmic space. □

Note that by using the simple reduction from VASS to VAS that increases the dimension by the number of control states, we would increase exponentially the dimension, which would disallow us to obtain forthcoming optimal complexity bounds. Indeed, the number of control states in $\mathcal{V}_{rb}$ is exponential in the number of control states in $\mathcal{V}$.

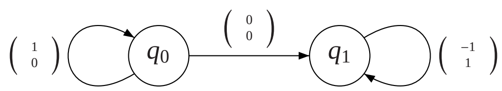
In Lemma 3.6, we explain how to reduce weak reversal-boundedness detection to a generalization of $(n + i)$-unboundedness.

## 3. Generalized Unboundedness Properties

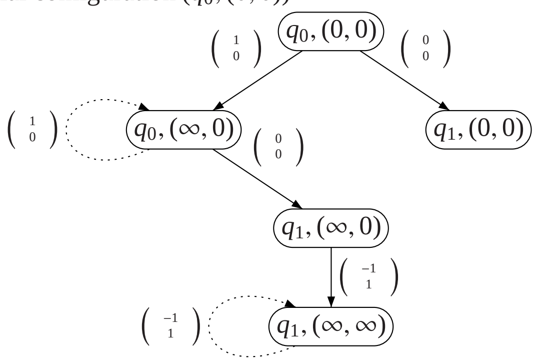
In this section, we essentially introduce the generalized unboundedness problem and we show how several detection problems can be naturally reduced to it.

### 3.1. Witness runs for simultaneous unboundedness

By [31, 52], we know that $(\mathcal{V}, (q_0, \vec{x}_0))$ is $i$-unbounded iff the coverability graph $CG(\mathcal{V}, (q_0, \vec{x}_0))$ contains an extended configuration with $\infty$ on the $i$th component. This is a simple characterization whose main disadvantage is to induce a nonprimitive recursive decision procedure in the worst case. By contrast, unboundedness of $(\mathcal{V}, (q_0, \vec{x}_0))$ (i.e. $i$-unboundedness for some $i \in [1, n]$) is equivalent to the existence of witness run of the form $(q_0, \vec{x}_0) \xrightarrow{*} (q_1, \vec{x}_1) \xrightarrow{+} (q_2, \vec{x}_2)$ such that $\vec{x}_1 \prec \vec{x}_2$ and $q_1 = q_2$. In [45], it is shown that if there is such a run, there is one of length at most doubly exponential. Given a component $i \in [1, n]$, a natural adaptation to $i$-unboundedness is to check the existence of a run of the form $(q_0, \vec{x}_0) \xrightarrow{*} (q_1, \vec{x}_1) \xrightarrow{\pi} (q_2, \vec{x}_2)$ such that $\vec{x}_1 \prec \vec{x}_2$, $q_1 = q_2$ and $\vec{x}_1(i) < \vec{x}_2(i)$. By inspecting the proof in [45], one can show that if there is such a run, then there is one of length at most doubly exponential. However, although existence of such a run is a sufficient condition for $i$-unboundedness (simply iterate $\pi$ infinitely), this is not necessary as shown on the VASS below:




The second component is unbounded from $(q_0, \vec{0})$ but no run $(q_0, \vec{0}) \xrightarrow{*} (q, \vec{x}_1) \xrightarrow{\pi} (q, \vec{x}_2)$ with $\vec{x}_1 \prec \vec{x}_2$, $\vec{x}_1(2) < \vec{x}_2(2)$ and $q \in \{q_0, q_1\}$ exists. Indeed, in order to increment the second component, the first component needs first to be incremented. Below, we present the coverability graph for this VASS with initial configuration $(q_0, (0, 0))$




Note that the only way to introduce $\infty$ in the second component is to introduce first $\infty$ on the first component. In general for VASS of dimension $n$, $i$-boundedness amounts to the existence of a run of the form

$$(q_0, \vec{x}_0) \xrightarrow{\pi'_0} (q_1, \vec{x}_1) \xrightarrow{\pi_1} (q_1, \vec{x}_2) \xrightarrow{\pi'_1} \cdots \xrightarrow{\pi'_{K-1}} (q_K, \vec{x}_{2K-1}) \xrightarrow{\pi_K} (q_K, \vec{x}_{2K})$$

where $\vec{x}_{2K}(i) > \vec{x}_{2K-1}(i)$. Moreover, for all $l \in [1, K]$ and for all $j \in [1, n]$, whenever $\vec{x}_{2l}(j) < \vec{x}_{2l-1}(j)$, there is $l' < l$ such that $\vec{x}_{2l'}(j) > \vec{x}_{2l'-1}(j)$ (this will be proved soon formally). This illustrates the idea that to be able to increment unboundedly the $i$th component, we may be able to increment earlier other components. Similarly, the ultimate condition for simultaneous unboundedness needs to specify the different ways to introduce the value $\infty$ along a given branch of the Karp and Miller coverability graphs. This is done thanks to the condition $\mathrm{PB}_\sigma$ defined below and further generalized in Section 3.2. A *disjointness sequence* is a nonempty sequence $\sigma = X_1 \cdot \cdots \cdot X_K$ of nonempty subsets of $[1, n]$ such that for $i \neq i'$, $X_i \cap X_{i'} = \emptyset$ (consequently $K \leq n$). A run of the form

$$(q_0, \vec{x}_0) \xrightarrow{\pi'_0} (q_1, \vec{x}_1) \xrightarrow{\pi_1} (q_2, \vec{x}_2) \xrightarrow{\pi'_1} \cdots \xrightarrow{\pi'_{K-1}} (q_{2K-1}, \vec{x}_{2K-1}) \xrightarrow{\pi_K} (q_{2K}, \vec{x}_{2K})$$

satisfies the *property* $\mathrm{PB}_\sigma$ (Place Boundedness with respect to a disjointness sequence $\sigma$) iff the conditions below hold true:

**(P0)** For every $l \in [1, K]$, $q_{2l-1} = q_{2l}$.

**(STRICT)** For all $l \in [1, K]$ and all $j \in X_l$, $\vec{x_{2l-1}}(j) < \vec{x_{2l}}(j)$.

**(NONSTRICT)** For all $l \in [1, K]$ and all $j \in ([1, n] \setminus X_l)$, $\vec{x_{2l}}(j) < \vec{x_{2l-1}}(j)$ implies $j \in \bigcup_{l' \in [1,l-1]} X_{l'}$.

Observe that when (STRICT) holds, the condition (NONSTRICT) is equivalent to: for all $l \in [1, K]$ and all $j \notin \bigcup_{l' \in [1,l-1]} X_{l'}$, we have $\vec{x}_{2l-1}(j) \leq \vec{x}_{2l}(j)$. Consequently, for all $l \in [1, K]$ and for all paths of the form $(\pi_l)^k$ for some $k \geq 1$, the effect on the $j$th component may be negative only if $j \in \bigcup_{l' \in [1,l-1]} X_{l'}$. Finally, note that the conditions on $X_1 \cdots X_K$ are reminiscent of chains in automata, see e.g. [41, Chapter 5].

It is now time to provide a witness run characterization for simultaneous $X$-unboundedness that is a direct consequence of the properties of the coverability graphs [52].

Lemma 3.1. Let $(\mathcal{V}, (q_0, \vec{x}_0))$ be an initialized VASS of dimension $n$ and $X \subseteq [1, n]$. Then, $(\mathcal{V}, (q_0, \vec{x}_0))$ is simultaneously $X$-unbounded iff there is a run $\rho$ starting at $(q_0, \vec{x}_0)$ satisfying $\mathrm{PB}_\sigma$ for some disjointness sequence $\sigma = X_1 \cdot \cdots \cdot X_K$ such that $X \subseteq (X_1 \cup \cdots \cup X_K)$ and $X \cap X_K \neq \emptyset$.

Consequently, $(\mathcal{V}, (q_0, \vec{x}_0))$ is $i$-unbounded iff there is a run $\rho$ starting at $(q_0, \vec{x}_0)$ satisfying $\mathrm{PB}_\sigma$ for some disjointness sequence $\sigma = X_1 \cdot \cdots \cdot X_K$ with $i \in X_K$.

*Proof.* As a consequence of the properties on the coverability graphs presented in Section 2.1, given $X \subseteq [1, n]$, $(\mathcal{V}, (q_0, \vec{x}_0))$ is simultaneously $X$-unbounded iff $CG(\mathcal{V}, (q_0, \vec{x}_0))$ contains some $(q, \vec{y})$ with $\vec{y}(X) = \vec{\infty}$ [31].
It is now time to show the statement.
($\leftarrow$) Let us consider the run $\rho$

$$(q_0, \vec{x_0}) \xrightarrow{\pi'_0} (q_1, \vec{x_1}) \xrightarrow{\pi_1} (q_2, \vec{x_2}) \xrightarrow{\pi'_1} (q_3, \vec{x_3}) \xrightarrow{\pi_2} \cdots (q_{2K-1}, \vec{x_{2K-1}}) \xrightarrow{\pi_K} (q_{2K}, \vec{x_{2K}})$$

of length $L$ satisfying the property $\mathrm{PB}_\sigma$. Let $B \geq 0$. We construct a run $\rho'$ satisfying $\mathrm{PB}_\sigma$ of the form $(\pi'_0(\pi_1)^{\beta_1}\pi'_1(\pi_2)^{\beta_2} \cdots (\pi_K)^{\beta_K}, (q_0, \vec{x}_0))$ for some $\beta_1, \ldots, \beta_K \geq 1$ such that $(B, \ldots, B) \leq \vec{x}_f(X)$ where $(q_f, \vec{x}_f)$ is the final configuration of $\rho'$. Now let us define $\beta_K, \ldots, \beta_1$ (in this very ordering):

- $\beta_K = B$.

- Now suppose that $\beta_{i+1}, \ldots, \beta_K$ are already defined and $i < K$. Let us define $\beta_i$ by

$$\beta_i \stackrel{\text{def}}{=} B + (K-i)(L-1)\text{maxneg}(\mathcal{V}) + \sum_{i' \in [i+1,K]} ((L-1)\text{maxneg}(\mathcal{V}))\beta_{i'}$$

For every $j \in [i+1, K]$, the path $\pi_j$ has at most $(L-1)$ transitions and each transition may decrease a component by at most maxneg$(\mathcal{V})$. The term $\sum_{i' \in [i+1,K]} ((L-1)\text{maxneg}(\mathcal{V}))\beta_{i'}$ guarantees that each component in $X_i$ is large enough to fire $\pi_j$ without reaching negative values. Similarly, each path $\pi'_j$ with $j \in [i, K-1]$, has at most $(L-1)$ transitions and each transition may decrease a component by at most maxneg$(\mathcal{V})$. The term $(K-i)(L-1)\text{maxneg}(\mathcal{V})$ guarantees that each component in $X_i$ is large enough to fire $\pi'_j$ without reaching negative values. Finally the term $B$ in $\beta_i$ guarantees that the final value of the component $i$ is greater than $B$. Consequently, the expression $(K-i)(L-1)\text{maxneg}(\mathcal{T})$ is related to the paths $\pi'_i, \ldots, \pi'_{K-1}$ whereas the expression $\Sigma_{i' \in [i+1,K]}((L-1)\text{maxneg}(\mathcal{V}))\beta_{i'}$ is related to the paths $\pi_{i+1}, \ldots, \pi_K$. It is not difficult to show that $(\pi'_0(\pi_1)^{\beta_1}\pi'_1(\pi_2)^{\beta_2} \cdots (\pi_K)^{\beta_K}, (q_0, \vec{x}_0))$ defines a run, it satisfies $\text{PB}_\sigma$ and $(B, \ldots, B) \preceq \vec{x}_f(X)$ where $\vec{x}_f$ is the final configuration of $\rho'$. Since the above construction can be performed for any $B$, we conclude that $(\mathcal{V}, (q_0, \vec{x}_0))$ is simultaneously $X$-unbounded.
($\rightarrow$) Now suppose that $(\mathcal{V}, (q_0, \vec{x}_0))$ is simultaneously $X$-unbounded. This means that $CG(\mathcal{V}, (q_0, \vec{x}_0))$ has an extended configuration $(q, \vec{y}) \in Q \times (\mathbb{N} \cup \{\infty\})^n$ such that $\vec{y}(X) = (\infty, \ldots, \infty)$. We can assume that $\vec{y}$ is the first extended configuration on that branch with $\vec{y}(X) = (\infty, \ldots, \infty)$. Let us consider the sequence below

$$(q_0, \vec{x}_0) \xrightarrow{\pi'_0} (q_1, \vec{y}_1) \xrightarrow{\pi_1} (q_2, \vec{y}_2) \xrightarrow{\pi'_1} (q_3, \vec{y}_3) \xrightarrow{\pi_2} \cdots \xrightarrow{\pi'_{K-1}} (q_{2K-1}, \vec{y}_{2K-1}) \xrightarrow{\pi_K} (q_{2K}, \vec{y}_{2K})$$

obtained from $CG(\mathcal{V}, (q_0, \vec{x}_0))$ such that

- For every $l \in [1, K]$, $q_{2l-1} = q_{2l}$, and $\vec{y}_{2K} = \vec{y}$.
- For every $l \in [1, K]$, $X_l \neq \emptyset$ with $X_l \stackrel{\text{def}}{=} \{j \in [1, n] : \vec{y}_{2l}(j) = \infty, \ \vec{y}_{2l-1}(j) \neq \infty\}$ and $\vec{y}_{2l-1} \prec \vec{y}_{2l}$.

Let us suppose that the above sequence in $CG(\mathcal{V}, (q_0, \vec{x}_0))$ has $L$ (extended) configurations and let us pose $\sigma = X_1 \cdots X_K$. It is easy to show that $\sigma$ is a disjointness sequence with $X \subseteq \bigcup_{l \in [1,K]} X_l$ and $X \cap X_K \neq \emptyset$. Again, we shall design a run $\rho$ satisfying $\text{PB}_\sigma$ of the form $(\pi'_0(\pi_1)^{\beta_1}\pi'_1(\pi_2)^{\beta_2} \cdots (\pi_K)^{\beta_K}, (q_0, \vec{x}_0))$ for some $\beta_1, \ldots, \beta_K \geq 1$. Now let us define $\beta_K, \ldots, \beta_1$ (in this very ordering):

- $\beta_K = 1$.
- Now suppose that $\beta_{i+1}, \ldots, \beta_K$ are already defined for $i < K$. Let us define $\beta_i$ by $\beta_i \stackrel{\text{def}}{=} 1 + (K-i)(L-1)\text{maxneg}(\mathcal{V}) + \sum_{i' \in [i+1,K]} ((L-1)\text{maxneg}(\mathcal{V}))\beta_{i'}$.

Now, it is not difficult to show that $\rho = (\pi'_0(\pi_1)^{\beta_1}\pi'_1(\pi_2)^{\beta_2} \cdots (\pi_K)^{\beta_K}, (q_0, \vec{x}_0))$ defines a run and it satisfies $\text{PB}_\sigma$. □

Existence of a run satisfying $\text{PB}_\sigma$ can be expressed in the logical formalisms from [54, 3] but this requires a formula of exponential size in the dimension because an exponential number of disjointness sequences needs to be taken into account. By contrast, each disjunct has a size only polynomial in $n$. The path formula looks like that (in order to fit exactly the syntax from [54, 3] we would need a bit more work since existential quantification cannot occur in the scope of disjunction):

$$\bigvee_{X_1\cdots X_K, i\in X_K} \exists\ \vec{x}_1,\ldots,\vec{x}_{2K}\ \bigwedge_{l=1}^{K} (\bigwedge_{j\in X_l} \vec{x}_{2l-1}(j) < \vec{x}_{2l}(j)) \wedge (\bigwedge_{j\notin(X_1\cup\cdots\cup X_{l-1})} \vec{x}_{2l-1}(j) \leq \vec{x}_{2l}(j))$$

It is worth noting that the satisfaction of $\mathrm{PB}_\sigma$ does not imply $\vec{x}_1 \preceq \vec{x}_{2K}$. This prevents us from defining this condition with an increasing path formula [3] and therefore the ExpSpace upper bound established in [3] does not apply straightforwardly to $i$-unboundedness.

### 3.2. A helpful generalization

We introduce below a slight generalization of the properties $\mathrm{PB}_\sigma$ in order to underline their essential features and to provide a future uniform treatment. Moreover, this allows us to express new properties, for instance those helpful to characterize nonregularity. The conditions (STRICT) and (NONSTRICT) specify inequality constraints between component values. We introduce intervals in place of such constraints. An *interval* is an expression of one of the forms $]-\infty,+\infty[$, $[a,+\infty[$, $]-\infty,b]$ or $[a,b]$ for some $a,b\in\mathbb{Z}$ interpreted as a subset of $\mathbb{Z}$ (with the obvious interpretation).

Definition 3.1. A *generalized unboundedness property* $\mathcal{P} = (\mathcal{I}_1,\ldots,\mathcal{I}_K)$ is a nonempty sequence of $n$-tuples of intervals. ∇

The *length* of $\mathcal{P}$ is $K$ and its *scale* is equal to the maximum between 1 and the maximal absolute value of integers occurring in the interval expressions of $\mathcal{P}$ (if any). A run of the form

$$(q_0,\vec{x}_0)\xrightarrow{\pi'_0}(q_1,\vec{x}_1)\xrightarrow{\pi_1}(q_2,\vec{x}_2)\xrightarrow{\pi'_1}(q_3,\vec{x}_3)\cdots\xrightarrow{\pi'_{K-1}}(q_{2K-1},\vec{x}_{2K-1})\xrightarrow{\pi_K}(q_{2K},\vec{x}_{2K})$$

satisfies the property $\mathcal{P}$ $\stackrel{\text{def}}{\Leftrightarrow}$ the conditions below hold true:

**(P0)** For every $l\in[1,K]$, $q_{2l-1}=q_{2l}$.

**(P1)** For every $l\in[1,K]$ and $j\in[1,n]$, we have $\vec{x}_{2l}(j)-\vec{x}_{2l-1}(j)\in\mathcal{I}_l(j)$.

**(P2)** For every $l\in[1,K]$ and $j\in[1,n]$, if $\vec{x_{2l}}(j)-\vec{x_{2l-1}}(j)<0$, then there is $l'<l$ such that $\vec{x}_{2l'}(j)-\vec{x}_{2l'-1}(j)>0$.

Given a run $\rho$, we say that $\rho$, *satisfies* $\mathcal{P}$ if $\rho$ admits a decomposition satisfying the conditions (P0)–(P2). By extension, $(\mathcal{V},(q_0,\vec{x}_0))$ *satisfies* $\mathcal{P}$ $\stackrel{\text{def}}{\Leftrightarrow}$ there is a finite run starting at $(q_0,\vec{x}_0)$ satisfying $\mathcal{P}$. It is easy to see that condition (P1) [resp. (P2)] is a quantitative counterpart for condition (STRICT) [resp. (NONSTRICT)] defined in Section 3.1.

Let us now introduce below our most general problem, especially tailored to capture selective unboundedness.

Generalized Unboundedness Problem

**Input:** Initialized VASS $(\mathcal{V},(q_0,\vec{x}_0))$ and generalized unboundedness property $\mathcal{P}$.

**Question:** Does $(\mathcal{V},(q_0,\vec{x}_0))$ satisfy $\mathcal{P}$?

Let us first forget about control states: we can safely restrict ourselves to VAS without any loss of generality, as it is already the case for many properties.

Lemma 3.2. There is a logarithmic-space many-one reduction from the generalized unboundedness problem for VASS to the generalized unboundedness problem for VAS. Moreover, an instance of the form $((\mathcal{V},(q,\vec{x})),\mathcal{P})$ is reduced to an instance of the form $((\mathcal{T},\vec{x}'),\mathcal{P}')$ such that

1. if $\mathcal{V}$ is of dimension $n$, then $\mathcal{T}$ is of dimension $n+3$,
2. $\mathcal{P}$ and $\mathcal{P}'$ have the same length and scale,
3. $\text{scale}(\mathcal{T}) = \max((\text{card}(Q)+1)^2, \text{scale}(\mathcal{V}))$ where $Q$ is the set of control states of $\mathcal{V}$.

The proof is essentially based on [26, Lemma 2.1].

*Proof.* Let $\mathcal{V} = (Q,n,\delta)$, $(q,\vec{x}) \in Q\times\mathbb{N}^n$ and $\mathcal{P} = (\mathcal{I}_1,\ldots,\mathcal{I}_K)$ be an instance of the generalized unboundedness problem for VASS. First, $(\mathcal{T},\vec{x}') = ((\mathcal{V},(q,\vec{x})))^{\text{HP}}$ following the construction from [26, Lemma 2.1] (see also the proof of Lemma 2.5). Let us now construct $\mathcal{P}'$.

- $\mathcal{P}' = (\mathcal{I}'_1,\ldots,\mathcal{I}'_K)$ with for every $l\in[1,K]$, $\mathcal{I}'_l([1,n]) = \mathcal{I}_l$ and $\mathcal{I}'_l([n+1,n+3]) = [0,\vec{0}]$.

We recall that every run $(q'_0,\vec{y}_0)\cdots(q'_k,\vec{y}_k)$ in $\mathcal{V}$ leads to a run $\rho' = \vec{z}_0\cdots\vec{z}_{3k}$ in the target VAS such that

- for every $i\in[0,k]$, $\vec{z}_{3i}([1,n]) = \vec{y}_i$ and $\vec{z}_{3i}$ is the standard encoding of $(q'_i,\vec{y}_i)$. Moreover, each step $(q'_i,\vec{y}_i)\xrightarrow{t}(q'_{i+1},\vec{y}_{i+1})$ corresponds to the steps $\vec{z}_{3i}\xrightarrow{t_I t'_I t}\vec{z}_{3i+3}$ in $\rho'$ where $q'_i$ is the $I$th control state of $Q$.

An analogous property holds true in the converse direction, which guarantees the correctness of the reduction. Observe that when $\vec{x}_{2l-1}([n+1,n+3]) = \vec{x}_{2l}([n+1,n+3])$ for some $l\in[1,K]$ with $\vec{x}_{2l-1}([n+1,n+3])$ not of the form $(a_i,b_i,0)$, we can always come back to such a situation since the dummy transitions are fired in a very controlled way. □

Generalized unboundedness properties can be expressed in more general formalisms for which decidability is known. However, in Section 4, we establish ExpSpace-completeness.

Theorem 3.3. [3, 2] The generalized unboundedness problem is decidable.

Given $(\mathcal{V},(q_0,\vec{x}_0))$, the existence of a run from $(q_0,\vec{x}_0)$ satisfying $\mathcal{P}$ can be easily expressed in Yen's path logic [54] and the generalized unboundedness problem is therefore decidable by [3, Theorem 3] and [38, 33]. We cannot rely on [54, Theorem 3.8] for decidability since [54, Lemma 3.7] contains a flaw, as observed in [3]. [3] precisely establishes that satisfiability in Yen's path logic is equivalent to the reachability problem for VASS. Moreover, it is worth noting that the reduction from the reachability problem to satisfiability [3, Theorem 2] uses path formulae that cannot be expressed as generalized unboundedness properties. Observe that the ExpSpace upper bound obtained for increasing path formulae in [3, Section 6] cannot be used herein since obviously generalized unboundedness properties are not necessarily increasing. That is why, we need directly to extend Rackoff's proof for boundedness [45].

### *3.3. From regularity to reversal-boundedness detection*

In this section, we explain how simultaneous unboundedness problem, regularity detection, strong promptness detection and weak reversal-boundedness detection can be reduced to generalized unboundedness problem. This will allow us to obtain ExpSpace upper bound for all these problems.

Lemma 3.4. Every property $PB_{\sigma}$ can be encoded as a generalized unboundedness property $\mathcal{P}_{\sigma}$ with length $K \leq n$ and $\text{scale}(\mathcal{P}_{\sigma}) = 1$.

*Proof.* From a disjointness sequence $\sigma = X_1 \cdots X_K$, we define the generalized unboundedness property $\mathcal{P}_{\sigma} = (\mathcal{I}_1, \ldots, \mathcal{I}_K)$ as follows. For every $l \in [1, K]$ and $j \in [1, n]$, if $j \in X_l$ then $\mathcal{I}_l(j) = [1, +\infty[$. Otherwise, if $j \in ([1, n] \setminus (\bigcup_{1 \leq l' \leq l} X_{l'}))$, then $\mathcal{I}_l(j) = [0, +\infty[$, otherwise $\mathcal{I}_l(j) = ]-\infty, +\infty[$. It is easy to check that $\mathcal{P}_{\sigma}$ and $PB_{\sigma}$ define the same set of runs. □

*Regularity detection.* Another example of properties that can be encoded by generalized unboundedness properties comes from the witness run characterization for nonregularity, see e.g. [52, 3]. Nonregularity of an initialized VASS $(\mathcal{V}, (q_0, \vec{x_0}))$ is equivalent to the existence of a run of the form

$$(q_0, \vec{x_0}) \xrightarrow{\pi'_0} (q_1, \vec{x_1}) \xrightarrow{\pi_1} (q_2, \vec{x_2}) \xrightarrow{\pi'_1} (q_3, \vec{x_3}) \xrightarrow{\pi_2} (q_4, \vec{x_4})$$

such that

1. $q_1 = q_2$,
2. $q_3 = q_4$,
3. $\vec{x_1} \prec \vec{x_2}$,
4. there is $i \in [1, n]$ such that $\vec{x_4}(i) < \vec{x_3}(i)$,
5. for all $j \in [1, n]$ such that $\vec{x_4}(j) < \vec{x_3}(j)$, we have $\vec{x_1}(j) < \vec{x_2}(j)$,

see e.g. [52, 3] and [47, Chapter 6]. Here, the language recognized by the initialized VASS is the set of finite sequences of transitions firable from the initial configuration (no final condition). Consequently, nonregularity condition can be viewed as a disjunction of generalized unboundedness properties of the form $(\mathcal{I}^i_1, \mathcal{I}^i_2)$ where $\mathcal{I}^i_1(i) = [1, +\infty[$, $\mathcal{I}^i_2(i) = ]-\infty, -1]$, and for $j \neq i$, we have $\mathcal{I}^i_1(j) = [0, +\infty[$ and $\mathcal{I}^i_2(j) = ]-\infty, +\infty[$. Condition (5.) above will be satisfied thanks to Condition (P1) in the definition of a generalized unboundedness property.

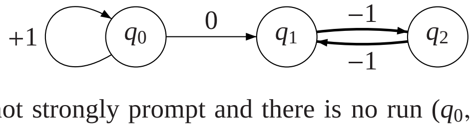
*Strong promptness detection.* We show below how the strong promptness detection problem can be reduced to the simultaneous unboundedness problem, leading to an ExpSpace upper bound. The *strong promptness detection problem* is defined as follows [51].

Strong promptness detection problem

**Input:** An initialized VASS $((Q, n, \delta), (q, \vec{x}))$ and a partition $(\delta_I, \delta_E)$ of $\delta$.

**Question:** Is there $k \in \mathbb{N}$ such that for every run $(q, \vec{x}) \xrightarrow{*} (q', \vec{x}')$, there is no run $(q', \vec{x}') \xrightarrow{\pi} (q'', \vec{x}'')$ using only transitions from $\delta_I$ and of length more than $k$ ($\pi \in \delta_I^*$)?

Let us consider below the VASS $\mathcal{V}$ of dimension 1 with $\delta_I$ made of the two transitions in bold.




$(\mathcal{V}, (q_0, 0))$ is not strongly prompt and there is no run $(q_0, 0) \xrightarrow{*} (q, \vec{x}) \xrightarrow{\pi} (q, \vec{y})$ for some $q \in \{q_0, q_1, q_2\}$ such that $\vec{x} \preceq \vec{y}$, $\pi$ is nonempty and contains only transitions in $\delta_I$.

Lemma 3.5. There is a logarithmic-space reduction from strong promptness detection problem to the complement of simultaneous unboundedness problem.

*Proof.* Let $(\mathcal{V}, (q, \vec{x}))$ be an initialized VASS with $\mathcal{V} = (Q, n, \delta)$ and equipped with the partition $(\delta_I, \delta_E)$. We construct the VASS $\mathcal{V}[\delta_I] = (Q \times \{0, 1\}, n + 1, \delta')$ made of two copies of $\mathcal{V}$. The 0-copy behaves exactly as $\mathcal{V}$ whereas the 1-copy contains only the transitions from $\delta_I$ and has an extra counter that is incremented for each transition. The transitions from the 0-copy to the 1-copy determines nondeterministically when the length of sequences of transitions in $\delta_I$ starts to be computed. $\mathcal{V}[\delta_I]$ is defined as follows: $(q, i) \xrightarrow{\vec{b}} (q', i') \in \delta'$ iff one of the conditions below holds true:

- $i = i' = 0$, $q \xrightarrow{\vec{b}([1,n])} q' \in \delta$, $\vec{b}(n + 1) = 0$,
- $i = 0$, $i' = 1$, $\vec{b} = \vec{0}$ and $q = q'$,
- $i = i = 1$, $q \xrightarrow{\vec{b}([1,n])} q' \in \delta_I$, $\vec{b}(n + 1) = +1$.

It is easy to show that $(\mathcal{V}, (q, \vec{x}))$ is strongly prompt with respect to the partition $(\delta_I, \delta_E)$ iff $(\mathcal{V}[\delta_I], (q, \vec{x}'))$ is not $(n + 1)$-unbounded for some $\vec{x}'$ with restriction to $[1, n]$ equal to $\vec{x}$. ☐

*Weak reversal-boundedness detection.* Complement of weak reversal-boundedness involves two universal quantifications (on $B$ and $r$) that can be understood as simultaneous unboundedness properties. Lemma 3.6 below is a key intermediate result in our investigation.

Lemma 3.6. Given a VASS $\mathcal{V} = (Q, n, \delta)$ and a configuration $(q, \vec{x})$, $(\mathcal{V}, (q, \vec{x}))$ is not weakly reversal-bounded with respect to $i$ iff $(\mathcal{V}_{rb}, (q_{rb}, \vec{x}_{rb}))$ has a run satisfying $\mathrm{PB}_\sigma$ for some disjointness sequence $\sigma = X_1 \cdots X_K$ with $n + i \in X_K$ and $i \in (X_1 \cup \cdots \cup X_{K-1})$.

*Proof.* (←) Let $\sigma = X_1 \cdots X_K$ be a disjointness sequence such that $n+i \in X_K$, $i \in (X_1 \cup \cdots \cup X_{K-1})$ and $(\mathcal{V}_{rb}, (q_{rb}, \vec{x}_{rb}))$ has a run $\rho$ satisfying $\mathrm{PB}_\sigma$. Suppose that $\rho$ is of the form below

$$(q_0, \vec{x}_0) \xrightarrow{\pi'_0} (q_1, \vec{x}_1) \xrightarrow{\pi_1} (q_2, \vec{x}_2) \xrightarrow{\pi'_1} \cdots \xrightarrow{\pi'_{K-1}} (q_{2K-1}, \vec{x}_{2K-1}) \xrightarrow{\pi_K} (q_{2K}, \vec{x}_{2K})$$

and of length $L$. By construction of $(\mathcal{V}_{rb}, (q_{rb}, \vec{x}_{rb}))$, a reversal for $i$ is operated on the path $\pi_K$, and the projection of $\rho$ on the $n$ first components and to $Q$ (for the control states from $Q \times \{\mathrm{INC}, \mathrm{DEC}\}^n$) corresponds to a run of $\mathcal{V}$. For all $B, B' \geq 1$, we define a run $\rho'$ that performs at least $B'$ reversals above $B$ for the component $i$, which guarantees that $(\mathcal{V}, (q, \vec{x}))$ is not weakly reversal-bounded with respect to $i$. The run $\rho'$ is of the form $(\pi'_0(\pi_1)^{\beta_1}\pi'_1(\pi_2)^{\beta_2} \cdots (\pi_K)^{\beta_K}, (q, \vec{x}))$. Let us define $\beta_K, \ldots, \beta_1 \geq 1$ as follows: first $\beta_K \stackrel{\text{def}}{=} B'$, then suppose that $\beta_{j+1}, \ldots, \beta_K$ are already defined and $j < K$. If $i \notin X_j$, then $\beta_j \stackrel{\text{def}}{=} \sum_{j' \in [j+1,K]} ((L - 1)\mathrm{maxneg}(\mathcal{V}))\beta_{j'}$, otherwise $\beta_j \stackrel{\text{def}}{=} (B + B' \times L \times \mathrm{maxneg}(\mathcal{V})) + (K - j)(L - 1)\mathrm{maxneg}(\mathcal{V}) + \sum_{j' \in [j+1,K]} ((L - 1)\mathrm{maxneg}(\mathcal{V}))\beta_{j'}$.
It is not difficult to show that $(\pi'_0(\pi_1)^{\beta_1}\pi'_1(\pi_2)^{\beta_2} \cdots (\pi_K)^{\beta_K}, (q, \vec{x}))$ defines a run and in the part of the run corresponding to the path $(\pi_K)^{\beta_K}$, at least $B'$ reversals above $B$ are observed for the $i$th component. Indeed, after firing $\pi'_0(\pi_1)^{\beta_1} \cdots \pi'_{K-1}$, the value for the component $i$ is greater than $B + B' \times L \times \mathrm{maxneg}(\mathcal{V})$. Moreover, after firing $\pi'_0(\pi_1)^{\beta_1} \cdots \pi'_{K-1}(\pi_K)^j$ with $j \in [1, B']$, the value for component $i$ is greater than $B + (B' - j) \times L \times \mathrm{maxneg}(\mathcal{V})$.
(→) Suppose that $(\mathcal{V}, (q, \vec{x}))$ is not weakly reversal-bounded. We use [20, Lemma 13] that characterizes weak reversal-boundedness on the coverability graph $CG(\mathcal{V}, (q, \vec{x}))$. First, let us recall [20, Lemma 13] formulated on the coverability graph $CG(\mathcal{V}, (q, \vec{x}))$: $(\mathcal{V}, (q, \vec{x}))$ is $r$-reversal-$B$-bounded with respect to $i$ for some $r$ and $B$ iff for every elementary loop in $CG(\mathcal{V}, (q, \vec{x}))$

that performs a reversal on the $i$th component, the $i$th component of every extended configuration on the loop is less than $B$. An elementary loop is a sequence of extended configurations respecting the edge relation $E$ of $CG(\mathcal{V},(q,\vec{x}))$ such that the two extremity (extended) configurations are identical and these are the only ones identical on the loop. Since $(\mathcal{V},(q,\vec{x}))$ is not weakly reversal-bounded and $CG(\mathcal{V},(q,\vec{x}))$ is a finite structure (with a finite amount of elementary loops), there is an elementary loop that performs a reversal on the $i$th component and such that one of its extended configuration has $\infty$ on the $i$th component (otherwise we would find a $B$ by finiteness). So, there is a sequence in $CG(\mathcal{V},(q,\vec{x}))$ of the form below

$$(q_0,\vec{x_0}) \xrightarrow{t_1} (q_1,\vec{x_1}) \xrightarrow{t_2} \cdots (q_{k'},\vec{x_{k'}}) \xrightarrow{t_{k'+1}} \cdots \xrightarrow{t_k} (q_k,\vec{x_k})$$

with $(q_0,\vec{x_0}) = (q,\vec{x})$, $k' < k$ and $(q_{k'},\vec{x_{k'}}) \xrightarrow{t_{k'+1}} \cdots \xrightarrow{t_k} (q_k,\vec{x_k})$ is an elementary loop. Remember that the $\vec{x_i}$'s are extended configurations. Since $(q_{k'},\vec{x_{k'}}) \xrightarrow{t_{k'+1}} \cdots \xrightarrow{t_k} (q_k,\vec{x_k})$ has an extended configuration with $\infty$ on the $i$th component, this entails that $\vec{x_{k'}}(i)$ is already equal to $\infty$. With a similar reasoning, all the extended configurations in $(q_{k'},\vec{x_{k'}}) \xrightarrow{t_{k'+1}} \cdots \xrightarrow{t_k} (q_k,\vec{x_k})$ have the same amount of components equal to $\infty$. Let $i_1,\ldots,i_K \leq k'$ be positions on which at least one component has been newly given the value $\infty$ and $\sigma = X_1 \cdots X_K$ be the disjointness sequence such that each $X_l$ is the set of components that have been newly given the value $\infty$ at the position $i_l$. It is then easy to see that $(t_1 \cdots t_k,(q_{rb},\vec{x}_{rb}))$ is a pseudo-run weakly satisfying $\mathcal{P}_{\sigma\cdot\{n+i\}}$ with $\mathcal{P}_{\sigma\cdot\{n+i\}}$ defined from $\sigma\cdot\{n+i\}$ as done in the beginning of Section 3.3 for dealing with simultaneous unboundedness. Weak satisfaction is introduced in Section 3.4. From Lemma 3.7, $(\mathcal{V}_{rb},(q_{rb},\vec{x}_{rb}))$ has a run $\rho'$ satisfying $\mathcal{P}_{\sigma\cdot\{n+i\}}$, which is equivalent to $\rho'$ satisfying $\mathrm{PB}_{\sigma\cdot\{n+i\}}$. Observe that $\sigma\cdot\{n+i\}$ is also of the appropriate form. ☐

As a corollary, we are in a position to present a witness run characterization for weak reversal-boundedness detection. $(\mathcal{V},(q_0,\vec{x_0}))$ is not weakly reversal-bounded with respect to $i$ iff there exist a disjointness sequence $\sigma = X_1 \cdots X_K$ and a run $(q_0,\vec{x_0}) \xrightarrow{\pi'_0} (q_1,\vec{x}_1) \xrightarrow{\pi_1} (q_2,\vec{x}_2) \xrightarrow{\pi'_1} \cdots \xrightarrow{\pi'_K} (q_{2K+1},\vec{x}_{2K+1}) \xrightarrow{\pi_{K+1}} (q_{2K+2},\vec{x}_{2K+2})$ such that

1. $\pi_{K+1}$ contains a reversal for the $i$th component,
2. the subrun $(q_0,\vec{x_0}) \xrightarrow{*} (q_{2K},\vec{x}_{2K})$ satisfies $\mathrm{PB}_\sigma$,
3. $i \in (X_1 \cup \cdots \cup X_K)$, and
4. for every $j \in [1,n]$, $\vec{x}_{2K+2}(j) < \vec{x}_{2K+1}(j)$ implies $j \in (X_1 \cup \cdots \cup X_K)$.

Based on Lemmas 2.3 and 3.1, a characterization for reversal-boundedness can be also defined.

### *3.4. A first relaxation*

Below, we relax the satisfaction of the property $\mathcal{P}$ by allowing negative component values in a controlled way. A pseudo-run of the form

$$(q_0,\vec{x_0}) \xrightarrow{\pi'_0} (q_1,\vec{x}_1) \xrightarrow{\pi_1} (q_2,\vec{x}_2) \xrightarrow{\pi'_1} (q_3,\vec{x}_3) \cdots \xrightarrow{\pi'_{K-1}} (q_{2K-1},\vec{x}_{2K-1}) \xrightarrow{\pi_K} (q_{2K},\vec{x}_{2K})$$

*weakly satisfies* $\mathcal{P}$ $\overset{\text{def}}{\Leftrightarrow}$ it satisfies (P0), (P1), (P2) (see Section 3.2) and (P3) defined below:

**(P3)** for every $j \in [1,n]$, every pseudo-configuration $\vec{x}$ such that $\vec{x}(j) < 0$ occurs after some $\vec{x}_{2l}$ for which $\vec{x}_{2l}(j) - \vec{x}_{2l-1}(j) > 0$.

If the run $\rho$ satisfies $\mathcal{P}$, then viewed as a pseudo-run, it also weakly satisfies $\mathcal{P}$. Lemma 3.7 below states that the existence of pseudo-runs weakly satisfying $\mathcal{P}$ is equivalent to the existence of runs satisfying $\mathcal{P}$ and their length can be compared. Later, we use the witness pseudo-run characterization.

Lemma 3.7. Let $\rho$ be a pseudo-run of length $L$ weakly satisfying $\mathcal{P}$ (of length $K$). Then, there is a run $\rho$ satisfying $\mathcal{P}$ of length at most $((L \times \mathrm{maxneg}(\mathcal{V}))^K \times (1 + K^2 \times L \times \mathrm{maxneg}(\mathcal{V})) + L$.

*Proof.* Let $\rho$ be a pseudo-run of the form below weakly satisfying the property $\mathcal{P} = (\mathcal{I}_1, \ldots, \mathcal{I}_K)$:

$$(q_0, \vec{x}_0) \xrightarrow{\pi'_0} (q_1, \vec{x}_1) \xrightarrow{\pi_1} (q_2, \vec{x}_2) \xrightarrow{\pi'_1} (q_3, \vec{x}_3) \xrightarrow{\pi_2} \cdots \xrightarrow{\pi'_{K-1}} (q_{2K-1}, \vec{x}_{2K-1}) \xrightarrow{\pi_K} (q_{2K}, \vec{x}_{2K})$$

We design a run $\rho$ satisfying $\mathcal{P}$ of the form

$$(\pi'_0(\pi_1)^{\beta_1}\pi'_1(\pi_2)^{\beta_2} \cdots (\pi_K)^{\beta_K}, (q_0, \vec{x}_0))$$

and of the appropriate length for some $\beta_1, \ldots, \beta_K \geq 1$. We use the same type of construction as in the proof of Lemma 3.1. First, let us define $X_1, \ldots, X_K \subseteq [1, n]$ that records when components are strictly increasing: for every $l \in [1, K]$, $X_l = \{j \in [1, n] : \vec{x}_{2l-1}(j) < \vec{x}_{2l}(j)\} \setminus (\bigcup_{l' < l} X_{l'})$. Observe that for $l \neq l'$, we have $X_l \cap X_{l'} = \emptyset$. Now let us define $\beta_K, \ldots, \beta_1$ (again, in this ordering):

- $\beta_K \stackrel{\text{def}}{=} 1$.
- Now suppose that $\beta_{i+1}, \ldots, \beta_K$ are already defined and $i < K$. Let us define $\beta_i$. If $X_i = \emptyset$, then $\beta_i \stackrel{\text{def}}{=} 1$. Otherwise $\beta_i \stackrel{\text{def}}{=} (K - i)(L - 1)\mathrm{maxneg}(\mathcal{V}) + \sum_{i' \in [i+1, K]} ((L - 1)\mathrm{maxneg}(\mathcal{V}))\beta_{i'}$.

The term $(K-i)(L-1)\mathrm{maxneg}(\mathcal{V})$ is related to the paths $\pi'_i, \ldots, \pi'_{K-1}$ whereas the term $\sum_{i' \in [i+1, K]} ((L-1)\mathrm{maxneg}(\mathcal{V}))\beta_{i'}$ is related to the paths $\pi_{i+1}, \ldots, \pi_K$. Again, it is worth noting that $L - 1$ transitions cannot decrease a component by more than $(L - 1)\mathrm{maxneg}(\mathcal{V})$. Now, it is not difficult to show that

$$(\pi'_0(\pi_1)^{\beta_1}\pi'_1(\pi_2)^{\beta_2} \cdots (\pi_K)^{\beta_K}, (q_0, \vec{x}_0))$$

defines a run (and not only a pseudo-run) and moreover it satisfies $\mathcal{P}$ which is witnessed by the decomposition below:

$$(q_0, \vec{x}_0) \xrightarrow{\pi'_0} (q_1, \vec{y}_1) \xrightarrow{\pi_1} (q_2, \vec{y}_2) \xrightarrow{(\pi_1)^{\beta_1 - 1}\pi'_1} (q_3, \vec{y}_3) \xrightarrow{\pi_2} \cdots$$

$$\cdots \xrightarrow{(\pi_{K-1})^{\beta_{K-1} - 1}\pi'_{K-1}} (q_{2K-1}, \vec{y}_{2K-1}) \xrightarrow{\pi_K} (q_{2K}, \vec{y}_{2K})$$

It remains to verify that this run is not too long. Let us define the sequence $\gamma_0, \ldots, \gamma_{K-1}$ with $\gamma_i = \Sigma_{i' \in [K-i, K]}\beta_{i'}$. So, $\gamma_0 = \beta_K = 1$ and $\gamma_{i+1} = \beta_{K-i-1} + \gamma_i$ with

$$\beta_{K-i-1} \leq (i + 1)(L - 1)\mathrm{maxneg}(\mathcal{V}) + ((L - 1)\mathrm{maxneg}(\mathcal{V}))\gamma_i$$

So $\gamma_{i+1} \leq (K \times L \times \mathrm{maxneg}(\mathcal{V})) + (L \times \mathrm{maxneg}(\mathcal{V}))\gamma_i$ for every $i \in [1, K-1]$. If $L \times \mathrm{maxneg}(\mathcal{V}) = 1$, then $\gamma_{K-1} \leq K(K \times L \times \mathrm{maxneg}(\mathcal{V}))$. Otherwise $\gamma_{K-1} \leq (L \times \mathrm{maxneg}(\mathcal{V}))^{K-1} \times (1 + K \times L \times \mathrm{maxneg}(\mathcal{V}))$. Finally, by using that the sum of the paths $\pi'_i$ is bounded by $L$, we get the desired bound. □

The principle of the proof of Lemma 3.7 (and part of the proof of Lemma 3.1) is identical to the idea of the proof of the following property of the coverability graph $CG(\mathcal{V}, (q_0, \vec{x_0}))$ (see e.g., details in [47]). For every extended configuration $(q, \vec{y'}) \in Q \times (\mathbb{N} \cup \{\infty\})^n$ in $CG(\mathcal{V}, (q_0, \vec{x_0}))$ and bound $B \in \mathbb{N}$, there is a run $(q_0, \vec{x_0}) \xrightarrow{*} (q, \vec{y})$ in $\mathcal{V}$ such that for $i \in [1, n]$, if $\vec{y'}(i) = \infty$ then $\vec{y}(i) \geq B$ otherwise $\vec{y}(i) = \vec{y'}(i)$. In the proof of Lemma 3.7, the paths $\pi_i$'s are repeated hierarchically in order to eliminate negative values.

Additionally, if $\rho$ is a pseudo-run of length $L$ weakly satisfying $\mathcal{P}$ and $L$ is at most doubly exponential in $N = |\mathcal{V}| + |(q_0, \vec{x_0})| + K + \text{scale}(\mathcal{P})$, then there is a run satisfying $\mathcal{P}$ and starting in $\vec{x_0}$ that is also of length at most doubly exponential in $N$.

So, standard unboundedness admits also a witness pseudo-run characterization with a disjunction of $n$ generalized unboundedness properties of length 1. But, if a pseudo-run $\rho$ weakly satisfies $\mathcal{P}$ of length 1, then $\rho$ is a run satisfying $\mathcal{P}$, explaining why only the witness run characterization is relevant for standard unboundedness.

## 4. ExpSpace Upper Bound

In this section, we deal with VAS only and we consider a current VAS $\mathcal{T}$ of dimension $n$ (see Lemma 3.2). Without any loss of generality, we can assume that $n > 1$, otherwise it is easy to show that the generalized unboundedness problem restricted to VAS of dimension 1 can be solved in polynomial space. Moreover, we assume that $\text{maxneg}(\mathcal{T}) \geq 1$.

### *4.1. Motivations for approximating properties*

Generalized unboundedness properties apply on runs but as it will be shown below, it would be more convenient to relax the conditions to pseudo-runs. A first step has been done in Section 3.4; we push further the idea in order to adapt Rackoff's proof. In forthcoming Section 4.2, we introduce approximations of generalized unboundedness properties and in Section 4.3, we explain how to shrink pseudo-runs satisfying such properties. To do so, we extend Rackoff's proof technique to obtain a small run property for runs witnessing (standard) unboundedness. In the rest of this section, first we recall main ingredients of Rackoff's proof (with references to forthcoming results about generalized unboundedness) and then, we motivate the main ingredients of our approximation properties.

*Ingredients in Rackoff's proof.* Let us briefly recall the structure of Rackoff's proof to show that the boundedness problem for VAS is in ExpSpace. Let $(\mathcal{T}, \vec{x_0})$ be an initialized VAS of dimension $n$. A witness run for unboundedness is of the form $\rho = \vec{x_0} \xrightarrow{*} \vec{y} \xrightarrow{+} \vec{y'}$ with $\vec{y} \prec \vec{y'}$. In [45], it is shown that $\rho$ can be of length at most doubly exponential. In order to get the ExpSpace upper bound, Savitch's Theorem is used. Let us be a bit more precise. A nondeterministic algorithm guessing such a run of length less than $L$ is defined as follows. Here is the algorithm with inputs $\mathcal{T}$, $\vec{x}$ and $L$:

1. guess $L'$ and $L''$ such that $L' < L'' \leq L$;
2. $i := 0$; $\vec{x_c} := \vec{x}$ (current configuration);
3. While $i < L'$ do
    (a) Guess a transition $t \in \mathcal{T}$; If $\vec{x_c} + t \notin \mathbb{N}^n$ then abort;
    (b) $i := i + 1$; $\vec{x_c} := \vec{x_c} + t$.
4. $\vec{y} := \vec{x_c}$;

5. While $i < L''$ do
   (a) Guess a transition $t \in \mathcal{T}$; If $\vec{x_c} + t \notin \mathbb{N}^n$ then abort;
   (b) $i := i + 1$; $\vec{x_c} := \vec{x_c} + t$.
6. Return $\vec{y} \prec \vec{x_c}$.

If the maximal absolute value in $\mathcal{T}$ and $\vec{x}$ is $2^N$ for some $N \geq 0$ and $L$ is doubly exponential in $N$, then the maximal absolute value appearing in the algorithm is doubly exponential in $N$ too. The decision procedure above guesses the small run and only requires exponential space thanks to the following additional arguments:

1. A counter with an exponential amount of bits can count until a double-exponential value.
2. Only two configurations need to be stored thanks to nondeterminism.
3. Comparing or adding two natural numbers requires logarithmic space only (if their values is doubly exponential in $N$, then their comparisons require only exponential space in $N$).
4. By Savitch's Theorem [50], a nondeterministic procedure for a given problem using space $f(N) \geq log(N)$ can be turned into a deterministic procedure using $f(N) \times f(N)$ space.
5. Exponential functions are closed under multiplication.

Rackoff's proof to establish the small run property goes as follows. First, a technical lemma shows that if there is some *i-B*-bounded pseudo-run (instance of the approximation property $\mathcal{A}$ introduced in forthcoming Section 4.2), then there is one of length at most $B^{|\mathcal{T}|^C}$ for some constant $C$. *i-B*-boundedness refers to the fact that the *i*th first components have values in $[0, B-1]$. The proof essentially shows that existence of such a pseudo-run amounts to solving an inequation system and by using [8], small solutions exist, whence the existence of a short *i-B*-bounded pseudo-run (the same technique is used in forthcoming Lemma 4.2). The idea of using small solutions of inequation system to solve problems on counter systems dates back from [45, 23] and nowadays, this is a standard proof technique, see e.g. [15]. This proof can be extended to numerous properties on pseudo-runs for which intermediate counter value differences can be expressed in Presburger arithmetic as done in [54, 3]. Then, a proof by induction on the dimension is performed by using this very technical lemma and the ability to repeat sequences of transitions; the proof can be extended when the first intermediate configuration is less or equal to the last configuration of the sequence (leading to the concept of increasing path formula in [3]). This condition allows to perform the induction on the dimension with a unique increasing formula. Unfortunately, generalized unboundedness properties are not increasing in the sense of [3]. Therefore, Rackoff's proof requires to be extended even though the essential ingredients remain, see the proof of Lemma 4.4. The generalization of the technical lemma corresponds to forthcoming Lemma 4.2; it is not surprising since generalized unboundedness properties are Presburger-definable properties. However, not only we need to refine the expression $B^{|\mathcal{T}|^C}$ in terms of various parameters (length of $\mathcal{P}$, scale($\mathcal{P}$), $n$, scale($\mathcal{T}$)) in order to get the final ExpSpace upper bound (or the PSpace upper bound with fixed dimension), but also we have to check that the new ingredients in the definition of the forthcoming approximation properties $\mathcal{A}$ do not prevent us from extending [45, Lemma 4.4]. Finally, it is important to specify the length of small pseudo-runs with respect to parameters from $\mathcal{P}$.

*What needs to be approximated.* Lemma 3.7 states that the existence of a run satisfying the generalized unboundedness property $\mathcal{P}$ is equivalent to the existence of a *pseudo-run weakly satisfying* $\mathcal{P} = (\mathcal{I}_1, \ldots, \mathcal{I}_K)$. Therefore, in the sequel, without any loss of generality, we can focus on weak satisfaction. Suppose that the pseudo-run $\rho = \vec{x}_0 \xrightarrow{\pi'_0} \vec{x}_1 \xrightarrow{\pi_1} \vec{x}_2 \cdots \vec{x}_{2K-1} \xrightarrow{\pi_K} \vec{x}_{2K}$ weakly satisfies $\mathcal{P}$. Note that

1. every element of $(\mathcal{I}_1, \ldots, \mathcal{I}_K)$ constraints $\rho$, as far as each path $\pi_j$ is concerned,
2. each pseudo-configuration $\vec{x}$ takes its values in $\mathbb{Z}^n$,
3. whenever a component value is negative, there is some earlier path $\pi_j$ that strictly increases that component (see Condition (P3)).

An approximation property $\mathcal{A}$ (parameterized by elements made explicit below) relaxes weak satisfaction in the following way (compare each condition $C$ above with $C'$).

1′. Only a suffix $(\mathcal{I}_l, \ldots, \mathcal{I}_K)$ of $\mathcal{P}$ is considered and a pseudo-run $\rho'$ satisfying $\mathcal{A}$ will be therefore of the form $\vec{y}_{2l-2} \xrightarrow{\pi'_{l-1}} \vec{y}_{2l-1} \xrightarrow{\pi_l} \vec{y}_{2l} \cdots \xrightarrow{\pi'_{K-1}} \vec{y}_{2K-1} \xrightarrow{\pi_K} \vec{y}_{2K}$. Hence $l \in [1, K]$ is a parameter of $\mathcal{A}$. Such a relaxation is useful when gluing pseudo-runs weakly satisfying distinct parts of $\mathcal{P}$.

2′. For each configuration $\vec{y}$ and for each component $i \in I$, $\vec{y}(i) \in [0, B-1]$ for some $I \subseteq [1, n]$ and $B \geq 0$ unless values on the $i$th component can be pumped (see 3′. below). Hence $I$ and $B$ are parameters of $\mathcal{A}$ too and such a relaxation will allow to provide a proof by induction on the dimension.

3′. Whenever a component value is negative in $\rho'$, either there is some earlier path $\pi_j$ in $\rho'$ that strictly increases that component or that component belongs to some set INCR of components whose values can be pumped. Hence, INCR $\subseteq [1, n]$ is a parameter of $\mathcal{A}$ and such a relaxation is useful when gluing pseudo-runs and pumping values can be performed thanks to paths occurring in other pseudo-runs (materialized by the fact that the component belongs to INCR).

*4.2. Approximating generalized unboundedness properties*

We are now in position to define the property $\mathcal{A}[\mathcal{P}, l, \text{INCR}, I, B]$. Given a generalized unboundedness property $\mathcal{P}$ of length $K$, $l \in [1, K]$, INCR $\subseteq [1, n]$, $I \subseteq [1, n]$ and $B \geq 0$, a pseudo-run of the form below

$$\vec{y}_{2l-2} \xrightarrow{\pi'_{l-1}} \vec{y}_{2l-1} \xrightarrow{\pi_l} \vec{y}_{2l} \cdots \xrightarrow{\pi'_{K-1}} \vec{y}_{2K-1} \xrightarrow{\pi_K} \vec{y}_{2K}$$

satisfies the *approximation property* $\mathcal{A}[\mathcal{P}, l, \text{INCR}, I, B]$ (also abbreviated by $\mathcal{A}$) $\overset{\text{def}}{\Leftrightarrow}$ the conditions below are verified:

**(P1′)** For every $l' \in [l, K]$ and for every $j \in [1, n]$, we have $\vec{y}_{2l'}(j) - \vec{y}_{2l'-1}(j) \in \mathcal{I}_{l'}(j)$ (only the suffix $(\mathcal{I}_l, \ldots, \mathcal{I}_K)$ is considered).

**(P2′)** For every $l' \in [l, K]$ and for every $j \in [1, n]$, if $\vec{y}_{2l'}(j) - \vec{y}_{2l'-1}(j) < 0$, then one of the conditions holds true:

- there is $l'' \in [l, l'-1]$ such that $\vec{y}_{2l''}(j) - \vec{y}_{2l''-1}(j) > 0$,
- $j \in$ INCR.

**(P3′)** For every pseudo-configuration $\vec{x}$ in $\rho$ occurring between $\vec{y}_{2l'}$ and strictly before $\vec{y}_{2l'+2}$ with $l' \geq l-1$, $\vec{x}(J) \in [0, B-1]^J$ with $J = I \setminus \text{PUMP}(l, l')$ where $\text{PUMP}(l, l') = (\text{INCR} \cup \{j : \exists\, l'' \in [l, l'], \vec{y}_{2l''}(j) - \vec{y}_{2l''-1}(j) > 0\})$.

Condition (P3′) reflects the intuition that only the values from components in $J$ need to be controlled. We also write $\mathcal{A}[\mathcal{P}, l, \text{INCR}, I, +\infty]$ to denote the property obtained from $\mathcal{A}[\mathcal{P}, l, \text{INCR}, I, B]$ by replacing $[0, B-1]^J$ by $\mathbb{N}^J$ in the condition (P3′). Observe that a pseudo-run satisfies $\mathcal{A}[\mathcal{P}, 1, \emptyset, [1, n], +\infty]$ iff it weakly satisfies $\mathcal{P}$ (see Section 3.4). The property $\mathcal{A}[\mathcal{P}, l, \text{INCR}, I, +\infty]$ is exactly the condition we need in the proof of Lemma 4.4 below thanks to the property stated below.

**Lemma 4.1.** If the pseudo-run $\rho = \vec{y}_{2l-2} \xrightarrow{\pi'_{l-1}} \vec{y}_{2l-1} \xrightarrow{\pi_l} \vec{y}_{2l} \cdots \xrightarrow{\pi_K} \vec{y}_{2K}$ satisfies the approximation property $\mathcal{A}[\mathcal{P}, l, \mathrm{INCR}, I, +\infty]$, then

$$(\pi'_{l-1}(\pi_l)^{n_l}\pi'_l(\pi_{l+1})^{n_{l+1}} \cdots (\pi_K)^{n_K}, \vec{y}_{2l-2})$$

also satisfies it, for all $n_l, \dots, n_K \geq 1$.

A similar statement does not hold for pseudo-runs satisfying $\mathcal{A}$ (values for components in $J$ might become out of $[0, B-1]$) and for runs satisfying $\mathcal{P}$ (component values might become negative).

*Proof.* (Lemma 4.1) Let $\rho'$ be the pseudo-run

$$\vec{y}_{2l-2} = \vec{z}_{2l-2} \xrightarrow{\pi'_{l-1}(\pi_l)^{n_l-1}} \vec{z}_{2l-1} \xrightarrow{\pi_l} \vec{z}_{2l} \cdots \xrightarrow{\pi'_{K-1}(\pi_K)^{n_K-1}} \vec{z}_{2K-1} \xrightarrow{\pi_K} \vec{z}_{2K}$$

obtained from $\rho$ by copying $n_i$ times the path $\pi_i$. For every $l' \in [l, K]$ and for every $j \in [1, n]$, $\vec{y}_{2l'}(j) - \vec{y}_{2l'-1}(j) = \vec{z}_{2l'}(j) - \vec{z}_{2l'-1}(j)$, whence $\rho'$ satisfies the conditions (P1′) and (P2′). Of course, we need also to take advantage that $\rho$ satisfies (P2′). Indeed, suppose that $\vec{z}_{2l'}(j) - \vec{z}_{2l'-1}(j) < 0$. So $\vec{y}_{2l'}(j) - \vec{y}_{2l'-1}(j) < 0$ and by satisfaction of (P2′) by $\rho$, we can also conclude that either $j \in \mathrm{INCR}$ or there is $l'' \in [l, l'-1]$ such that $\vec{y}_{2l''}(j) - \vec{y}_{2l''-1}(j) > 0$ (equivalent to $\vec{z}_{2l''}(j) - \vec{z}_{2l''-1}(j) > 0$).

Since $\rho$ satisfies condition (P3′), for every pseudo-configuration $\vec{x}$ in $\rho$ occurring between $\vec{y}_{2l'}$ and strictly before $\vec{y}_{2l'+2}$ with $l' \geq l-1$, $\vec{x}(J) \in \mathbb{N}^J$ with $J = I \setminus \mathrm{PUMP}(l, l')$. Now let $\vec{x}$ in $\rho'$ occurring between $\vec{z}_{2l'}$ and strictly before $\vec{z}_{2l'+2}$ with $l' \geq l-1$. Let $J = I \setminus \mathrm{PUMP}(l, l')$. For every $l'' \in [l, l']$ and for every $j \in J$, the path $\pi_{l''}$ has a positive effect on the component $j$. One can show that this entails that $\vec{x}(J) \in \mathbb{N}^J$ using the property (P3′) on $\rho$. □

Property $\mathcal{A}[\mathcal{P}, l, \mathrm{INCR}, I, B]$ can be viewed as a collection of *local* path increasing formulae in the sense of [3].

### *4.3. Bounding the length of pseudo-runs*

It is important to specify the length of small pseudo-runs with respect to parameters from $\mathcal{P}$ as done in Lemma 4.2 below.

**Lemma 4.2.** Let $\mathcal{T}$ be a VAS of dimension $n \geq 2$, $\mathcal{P}$ be a generalized unboundedness property of length $K$, $l \in [1, K]$, $B \geq 2$, $I, \mathrm{INCR} \subseteq [1, n]$ and $\rho$ be a pseudo-run satisfying $\mathcal{A}[\mathcal{P}, l, \mathrm{INCR}, I, B]$. Then, there exists a pseudo-run starting by the same pseudo-configuration, satisfying $\mathcal{A}[\mathcal{P}, l, \mathrm{INCR}, I, B]$ and of length at most $(1 + K) \times (\mathrm{scale}(\mathcal{T}) \times \mathrm{scale}(\mathcal{P}) \times B)^{n^{\mathsf{C}_1}}$ for some constant $\mathsf{C}_1$ independent of $K$, $\mathrm{scale}(\mathcal{P})$, $\mathrm{scale}(\mathcal{T})$, $B$ and $n$.

The length expression in Lemma 4.2 can be certainly refined in terms of card(INCR), card($I$) and $l$ but these values are anyhow bounded by $n$ and $K$ respectively, which is used in Lemma 4.2. The proof below is essentially a refinement of the proof of [45, Lemma 4.4].

*Proof.* Let $\mathcal{P} = (\mathcal{I}_1, \dots, \mathcal{I}_K)$, $l \in [1, K]$, $I, \mathrm{INCR} \subseteq [1, n]$ and $\rho$ be the pseudo-run described below satisfying $\mathcal{A}[\mathcal{P}, l, \mathrm{INCR}, I, B]$:

$$\rho = \vec{x}_{2l-2} \xrightarrow{\pi'_{l-1}} \vec{x}_{2l-1} \xrightarrow{\pi_l} \vec{x}_{2l} \cdots \xrightarrow{\pi'_{K-1}} \vec{x}_{2K-1} \xrightarrow{\pi_K} \vec{x}_{2K}$$

We pose $d_0 = \mathrm{card}(J_0)$ with $J_0 = I \setminus \mathrm{INCR}$. We suppose that the pseudo-run $\rho$ is induced by the path $t_1 \ldots t_k$ with $\rho = \vec{u}_0 \cdots \vec{u}_k$. Let $f : [2l-2, 2K] \to [0,k]$ be the map such that $\vec{x}_i = \vec{u}_{f(i)}$; consequently $f(2l-2) = 0$ and $f(2K) = k$. By the satisfaction of the condition (P3′) from $\mathcal{A}[\mathcal{P}, l, \mathrm{INCR}, I, B]$, for every pseudo-configuration $\vec{u_j}$ with $j \leq f(2l-1)$, we have $\vec{u_j}(J_0) \in [0, B-1]^{J_0}$. If the length of $\pi'_{l-1}$ is at least $B^{d_0}$, then there are two distinct positions $j < j' \leq f(2l-1)$ such that $\vec{u}_j(J_0) = \vec{u}_{j'}(J_0)$ (by the pigeonhole principle) and therefore $(t_1 \ldots t_j t_{j'+1} \ldots t_k, \vec{x}_{2l-2})$ also satisfies $\mathcal{A}[\mathcal{P}, l, \mathrm{INCR}, I, B]$. Observe that the values for components in $[1,n] \setminus J_0$ are allowed to be negative. By iterating this contraction process, without any loss of generality, we can assume that in $\rho$, we have $f(2l-1) - f(2l-2) < B^{d_0}$ and for every $l' \in [l-1, K-1]$, $f(2l'+1) - f(2l') < B^{\mathrm{card}(I)} \leq B^n$.

Now, for each $D \in [l, K]$ we shorten the pseudo-run $\vec{x}_{2D-1} \xrightarrow{\pi_D} \vec{x}_{2D}$. This is done by removing loops, as explained below, and by following the key steps of the proof of [45, Lemma 4.4]. We pose $d = \mathrm{card}(J)$ with

$$J = I \setminus (\mathrm{INCR} \cup \{j : \exists\, l' \in [l, D-1],\ \vec{x_{2l'}}(j) - \vec{x_{2l'-1}}(j) > 0\}) = I \setminus \mathrm{PUMP}(l, D-1).$$

A *simple loop with respect to J* is a pair $sl = (\vec{s}, \pi)$ such that $\vec{s} \in [0, B-1]^J$ and $\pi = t'_1 \ldots t'_\gamma$ is a path satisfying the conditions below:

**(SL1)** For every $j \in [1,\gamma]$, $\vec{s} + \sum_{i \in [1,j]} t'_i(J) \in [0, B-1]^J$ (the bound $B$ is never exceeded).

**(SL2)** $\sum_{i \in [1,\gamma]} t'_i(J) = 0$ (the total effect on the components in $J$ is zero),

**(SL3)** For $j < j' \in [1,\gamma]$ with $(j,j') \neq (1,\gamma)$, we have $\sum_{i \in [j,j']} t'_i(J) \neq 0$ (minimality of the path).

The *length* of $sl$ is defined as the length of its path $\pi$ and its *effect* is the value $\sum_{i \in [1,\gamma]} t'_i$ (remember that not all the components are in $J$). Consequently, let $\vec{y}_0 \cdots \vec{y}_\gamma$ be a pseudo-run induced by the simple loop $(\vec{y}_0(J), t'_1 \ldots t'_\gamma)$. Then,

1. $\vec{y}_0(J) = \vec{y}_\gamma(J)$ (by (SL2)).
2. For $j < j' \in [1,\gamma]$ such that $(j,j') \neq (1,\gamma)$, we have $\vec{y}_j(J) \neq \vec{y}_{j'}(J)$ (by (SL3)).

It is easy to show that the length of a simple loop with respect to $J$ is strictly below $B^d$ with $B^d \leq B^{d_0} \leq B^{\mathrm{card}(I)}$. Its effect is therefore in $[-\mathrm{scale}(\mathcal{T})B^d, \mathrm{scale}(\mathcal{T})B^d]^n$. Let $\vec{z}_1, \ldots, \vec{z}_\alpha$ be the effects of simple loops occurring in $\vec{x}_{2D-1} \xrightarrow{t_{2D-1}} \ldots \xrightarrow{t_{2D}} \vec{x}_{2D}$ as factors. Because the effects of simple loops are bounded (see above), we have

$$\alpha \leq (1 + 2 \times \mathrm{scale}(\mathcal{T})B^d)^n \leq (1 + 2 \times \mathrm{scale}(\mathcal{T}))^n B^{n^2}.$$

From the pseudo-run $\vec{x}_{2D-1} \xrightarrow{\pi_D} \vec{x}_{2D}$, we define a finite sequence of pairs made of a pseudo-run $\vec{y}^i_0 \cdots \vec{y}^i_{K_i}$ and a tuple $\vec{v}_i \in \mathbb{N}^\alpha$ such that

- $\vec{v}_0 = \vec{0}$ and $\vec{y}^0_0 \cdots \vec{y}^0_{K_0} = \vec{x}_{2D-1} \cdots \vec{x}_{2D}$.
- $\vec{y}^{i+1}_0 \cdots \vec{y}^{i+1}_{K_{i+1}}$ and $\vec{v}_{i+1}$ are computed from $\vec{y}^i_0 \cdots \vec{y}^i_{K_i}$ and $\vec{v}_i$ by removing a simple loop from $\vec{y}^i_0 \cdots \vec{y}^i_{K_i}$ with effect $\vec{z}_\beta$ and by computing $\vec{v}_{i+1}$ from $\vec{v}_i$ by only incrementing $\vec{v}_i(\beta)$, i.e. a simple loop is removed but we remember its effect by incrementing $\vec{v}_i(\beta)$.
- The length of the final pseudo-run $\vec{y}^N_0 \cdots \vec{y}^N_{K_N}$ (on which no simple loop can be removed) is less than $(1 + B^d)^2$. Explanations about this bound are provided below.

- $\{\vec{x}_{2D-1}(J), \ldots, \vec{x}_{2D}(J)\} = \{\vec{y}^i_0(J), \ldots, \vec{y}^i_{K_i}(J)\}$ for every $i \in [0, N]$. In words, the set of tuples restricted to components in $J$ remains even all over this process of removing simple loops. This will be useful to bring back simple loops.

Consequently, whenever $\vec{v}_i(j) > 0$, there is a simple loop $(\vec{s}, \pi)$ with effect some $\vec{z}_j$ such that $\vec{s} \in \{\vec{y}^i_0(J), \ldots, \vec{y}^i_{K_i}(J)\}$.

Let us explain how to compute $\vec{y}^{i+1}_0 \cdots \vec{y}^{i+1}_{K_{i+1}}$ and $\vec{v}_{i+1}$ from $\vec{y}^i_0 \cdots \vec{y}^i_{K_i}, \vec{v}_i$. Suppose that $\vec{y}^i_0 \cdots \vec{y}^i_{K_i}$ is induced by the path $\pi_i = t_1 \cdots t_{K_i}$. If $\pi_i$ has no simple loop $t_j \cdots t_{j'}$ as a factor such that

$$\{\vec{x}_{2D-1}(J), \ldots, \vec{x}_{2D}(J)\} = \{\vec{y}^i_0(J), \ldots, \vec{y}^i_{j-1}(J), \vec{y}^i_{j'}(J) \ldots, \vec{y}^i_{K_i}(J)\},$$

then $N = i$ (we stop the process). Otherwise, let $(\vec{y}^i_{j-1}(J), t_j \cdots t_{j'})$ be a simple loop with respect to $J$ such that

$$\{\vec{x}_{2D-1}(J), \ldots, \vec{x}_{2D}(J)\} = \{\vec{y}^i_0(J), \ldots, \vec{y}^i_{j-1}(J), \vec{y}^i_{j'}(J) \ldots, \vec{y}^i_{K_i}(J)\}.$$

Then $\vec{y}^{i+1}_0 \cdots \vec{y}^{i+1}_{K_{i+1}}$ is the pseudo-run $(t_1 \cdots t_{j-1}, t_{j'+1} \cdots t_{K_i}, \vec{y}^i_0)$ and $\vec{v}_{i+1}$ is equal to $\vec{v}_i$ except that $\vec{v}_{i+1}(\beta) = \vec{v}_i(\beta) + 1$ with $t_j, \ldots, t_{j'}$ having the effect $\vec{z}_\beta$. Since $\vec{x}_{2D-1} \xrightarrow{\pi_D} \vec{x}_{2D}$ is finite, it is clear that this process eventually stops and the above-mentioned conditions are clearly satisfied (except for the bound on the length of $\vec{y}^N_0 \cdots \vec{y}^N_{K_N}$).

Before going any further, let us briefly explain why eventually the length of $\vec{y}^N_0 \cdots \vec{y}^N_{K_N}$ is less than $(1 + B^d)^2$. Suppose that the pseudo-run $\vec{y}^i_0 \cdots \vec{y}^i_{K_i}$ has at least $(1 + B^d)^2$ pseudo-run configurations. First, observe that each block of $B^d + 1$ consecutive pseudo-configurations contains at least one simple loop. Moreover, we wish to preserve the set $\{\vec{x}_{2D-1}(J), \ldots, \vec{x}_{2D}(J)\}$, so we cannot remove any simple loop. The set $\{\vec{x}_{2D-1}(J), \ldots, \vec{x}_{2D}(J)\}$ has cardinal at most $B^d$. Consequently, there is a block of $B^d + 1$ successive pseudo-configurations so that all the restrictions to the components in $J$ have already appeared earlier.

Let $\vec{y}^N_0 \cdots \vec{y}^N_{K_N}$ be the final sequence induced by the path $t_1 \cdots t_{K_N}$ with final loop vector $\vec{v}_N \in \mathbb{N}^\alpha$.

Since the pseudo-run $\rho$ satisfies $\mathcal{A}[\mathcal{P}, l, \mathrm{INCR}, I, B]$, we have the following properties.

1. For every $j \in [1, n]$, we have $((\sum_{i \in [1,\alpha]} \vec{v}_N(i)\vec{z}_i) + \sum_{i \in [1,K_N]} t_i)(j) \in \mathcal{I}_D(j)$. Depending on the value of $\mathcal{I}_D(j)$, this can encoded by at most 2 inequality constraints of the form $\sum_{i \in [1,\alpha]} a_i \vec{v}_N(i)(j) \geq b_j$.
2. For every $j \in J$, $((\sum_{i \in [1,\alpha]} \vec{v}_N(i)\vec{z}_i) + \sum_{i \in [1,K_N]} t_i)(j) \geq 0$.

There is a bit of redundancy here for the components in $J$ since removing simple loops does not change the projection over $J$ of the first and last pseudo-configurations. Hence, we only need to bother about the components in $([1, n] \setminus J)$. The vector $\vec{v}_N$ is a solution to the following inequality system:

$$(\bigwedge_{j \in ([1,n] \setminus J)} ((\sum_{i \in [1,\alpha]} \vec{v}_N(i)\vec{z}_i) + \sum_{i \in [1,K_N]} t_i)(j) \in \mathcal{I}_D(j))$$

The number of inequalities can be bounded by $2n$, the number of variables is bounded by $(1 + 2 \times \mathrm{scale}(\mathcal{T}))^n B^{n^2}$ and all the absolute values of the components are bounded by $(1 + B^n)^2 \times \mathrm{scale}(\mathcal{T}) + \mathrm{scale}(\mathcal{P})$. It is time to apply [8] in order to obtain a small solution:

Theorem 4.3. [8] Let $A \in [-M, M]^{U\times V}$ and $\vec{b} \in [-M, M]^U$, where $U, V, M \in \mathbb{N}$. If there is $\vec{x} \in \mathbb{N}^V$ such that $A\vec{x} \geq \vec{b}$, then there is $\vec{y} \in [0, (\max\{V, M\})^{\mathsf{C}U}]^V$ such that $A\vec{y} \geq \vec{b}$, where $\mathsf{C}$ is some constant.

By application of Theorem 4.3 on the above system with the values below

1. $V = (1 + 2 \times \mathrm{scale}(\mathcal{T}))^n B^{n^2}$.
2. $M = (1 + B^n)^2 \times \mathrm{scale}(\mathcal{T}) + \mathrm{scale}(\mathcal{P})$.
3. $U = 2n$.

It has a solution $\mathcal{X} \in \mathbb{N}^V$ such that each value is indeed within the interval

$$[0, ((1 + 2 \times \mathrm{scale}(\mathcal{T}) \times \mathrm{scale}(\mathcal{P}))^n B^{2n^2})^{\mathsf{C}2n}]$$

Indeed, we have $\max(V, M) \leq ((1 + 2 \times \mathrm{scale}(\mathcal{T})\mathrm{scale}(\mathcal{P}))^n B^{2n^2})$. Now, it is time to re-inject in $\vec{y}_0^N \cdots \vec{y}_{K_N}^N$ the simple loops encoded by $\mathcal{X}$.

From $\vec{y}_0^N \cdots \vec{y}_{K_N}^N$ and $\vec{v}_N$, we define a finite sequence of pseudo-runs $\vec{u}_0^i \cdots \vec{u}_{L_i}^i = (t_1^i \cdots t_{L_i}^i, \vec{u}_0^i)$ such that

- $\vec{u}_0^0 \cdots \vec{u}_{L_0}^0 = \vec{y}_0^N \cdots \vec{y}_{K_N}^N$.
- The length of the sequence is exactly $\alpha + 1$ ($\alpha$ is the number of distinct effects).
- $\vec{u}_0^{j+1} \cdots \vec{u}_{L_{j+1}}^{j+1} = (t_1^{j+1} \cdots t_{L_{j+1}}^{j+1}, \vec{u}_0^{j+1})$ is computed from $(t_1^j \cdots t_{L_j}^j, \vec{u}_0^j)$ as follows. Let $(\vec{s}_{j+1}, \pi_{j+1})$ be a simple loop with effect $\vec{z}_{j+1}$. There exists $\beta$ such that $\vec{u}_\beta^j(J) = \vec{s}_{j+1}$. Then,

$$t_1^{j+1} \cdots t_{L_{j+1}}^{j+1} \stackrel{\text{def}}{=} t_1^j \cdots t_\beta^j \cdot (\pi_{j+1})^{\mathcal{X}(j+1)} \cdot t_{\beta+1}^j \cdots t_{L_j}^j$$

  and $\vec{u}_0^{j+1} \cdots \vec{u}_{L_{j+1}}^{j+1} = (t_1^{j+1} \cdots t_{L_{j+1}}^{j+1}, \vec{u}_0^j)$.

It is easy to check that $\vec{x}_{2D-1} = \vec{u}_0^\alpha$. By replacing $\vec{x}_{2D-1} \xrightarrow{\pi_D} \vec{x}_{2D}$ by $\vec{u}_0^\alpha \cdots \vec{u}_{L_\alpha}^\alpha$ for each $D \in [l, K]$, we obtain a pseudo-run satisfying $\mathcal{A}[\mathcal{P}, l, \mathrm{INCR}, I, B]$ whose length is bounded by the value below:

$$(K + 1)B^n + K[(B^n + 1)^2 + \overbrace{(1 + 2 \times \mathrm{scale}(\mathcal{T}))^n B^{n^2}}^{\text{number of effects}} \times \overbrace{[(1 + 2 \times \mathrm{scale}(\mathcal{T}) \times \mathrm{scale}(\mathcal{P}))^n B^{2n^2}]^{\mathsf{C}2n}}^{\text{maximal number of copies per effect}} \times \overbrace{(B^n + 1)}^{\text{bound on the length of simple loop}}]$$

This value is bounded by

$$(K + 1) \times \mathsf{C}' \times \mathrm{scale}(\mathcal{T})^{p_1(n)} \mathrm{scale}(\mathcal{P})^{p_2(n)} \times B^{p_3(n)}$$

where $\mathsf{C}'$ is a constant and $p_1(\cdot)$, $p_2(\cdot)$ and $p_3(\cdot)$ are polynomials. Since $n, B \geq 2$, this value is bounded by $(K + 1)(\mathrm{scale}(\mathcal{T}) \times \mathrm{scale}(\mathcal{P}) \times B)^{p(n)}$ for some polynomial $p(\cdot)$. Suppose that $p(n) = \Sigma_{i=0}^f a_i n^i$ (without any loss of generality, we can assume that the $a_i$'s are non-negative and $a_f \neq 0$). Let $f' \geq 0$ be such that $\Sigma_{i=0}^f a_i \leq 2^{f'}$. Since $n \geq 2$, $(\mathrm{scale}(\mathcal{T}) \times \mathrm{scale}(\mathcal{P}) \times B)^{p(n)}$ is bounded by $(\mathrm{scale}(\mathcal{T}) \times \mathrm{scale}(\mathcal{P}) \times B)^{n^{f+f'}}$. Hence, the length of the final pseudo-run satisfying $\mathcal{A}[\mathcal{P}, l, \mathrm{INCR}, I, B]$ and starting at $\vec{x}_0$ is bounded by $(K + 1) \times (\mathrm{scale}(\mathcal{T}) \times \mathrm{scale}(\mathcal{P}) \times B)^{n^{\mathsf{C}_1}}$ for some constant $\mathsf{C}_1$. □

For every $i \in [0, n]$, let us define the value $g(i)$ that serves to bound the length of pseudo-runs satisfying $\mathcal{A}$, not only the approximation:

$$g(i) \stackrel{\text{def}}{=} \begin{cases} (2\mu)^{n^{\mathrm{C}_1}} \text{ with } \mu = (1+K) \times \text{scale}(\mathcal{T}) \times \text{scale}(\mathcal{P}) & \text{if } i = 0, \\ (2\mu(\text{maxneg}(\mathcal{T}) \times g(i-1)))^{n^{\mathrm{C}_1}} + g(i-1) & \text{if } i > 0. \end{cases}$$

Lemma 4.4 below is an extension of [45, Lemmas 4.6 & 4.7], see also [3, Lemma 7].

Lemma 4.4. Let $I, \text{INCR} \subseteq [1, n]$, $l \in [1, K]$ and $\rho$ be a pseudo-run satisfying $\mathcal{A}[\mathcal{P}, l, \text{INCR}, I, +\infty]$. Then, there exists a pseudo-run $\rho'$ starting from the same pseudo-configuration, satisfying the property $\mathcal{A}[\mathcal{P}, l, \text{INCR}, I, +\infty]$ and of length at most $g(\text{card}(I))$.

*Proof.* Let $\rho = \vec{x}_{2l-2} \xrightarrow{\pi'_{l-1}} \vec{x}_{2l-1} \xrightarrow{\pi_l} \vec{x}_{2l} \cdots \xrightarrow{\pi'_{K-1}} \vec{x}_{2K-1} \xrightarrow{\pi_K} \vec{x}_{2K}$ be a pseudo-run satisfying the property $\mathcal{A}[\mathcal{P}, l, \text{INCR}, I, +\infty]$. We suppose that $\rho$ is induced by the path $t_1 \cdots t_k$ with $\rho = \vec{u}_0 \cdots \vec{u}_k$ and $f : [2l-2, 2K] \to [0, k]$ is the map such that $\vec{x}_i = \vec{u}_{f(i)}$. So $f(2l-2) = 0$ and $f(2K) = k$.

The proof is by induction on $i = \text{card}(I)$. If $i = 0$, then we apply Lemma 4.2 with $B = 2$ and we obtain a pseudo-run satisfying the approximation property $\mathcal{A}[\mathcal{P}, l, \text{INCR}, I, +\infty]$ leading to the bound $(\mu \times 2)^{n^{\mathrm{C}_1}}$.

Now suppose $\text{card}(I) = i + 1$ and $J = (I \setminus \text{INCR})$. We pose $B = \text{maxneg}(\mathcal{T}) \times g(i)$. We recall that $\mathcal{T}$ is the current VAS with $n \geq 2$. We perform a case analysis depending where in $\rho$ a value from a component in $J$ is strictly greater than $B - 1$ (if any).
**Case 1:** Every configuration $\vec{z}$ in $\rho$ satisfies $\vec{z}(J) \in [0, B-1]^J$, i.e., $\rho$ satisfies $\mathcal{A}[\mathcal{P}, l, \text{INCR}, I, B]$. Obviously, the case $J = \emptyset$ is captured here. By Lemma 4.2, there is a pseudo-run $\rho'$ starting at $\vec{x}_{2l-2}$ satisfying $\mathcal{A}[\mathcal{P}, l, \text{INCR}, I, B]$ of length at most $(1+K) \times (\text{scale}(\mathcal{T}) \times \text{scale}(\mathcal{P}) \times B)^{n^{\mathrm{C}_1}}$, which is bounded by $(\mu \times (\text{maxneg}(\mathcal{T}) \times g(i)))^{n^{\mathrm{C}_1}}$.
**Case 2:** A value for some component in $J$ is strictly greater than $B-1$ for the first time within the path $\pi'_D$ for some $D \in [l-1, K-1]$. Let $\alpha$ be the minimal position such that $\vec{u}_{\alpha+1}(J) \notin [0, B-1]^J$ and $\alpha + 1 \in [f(2D)+1, f(2D+1)]$, say $\vec{u}_{\alpha+1}(i_0) \geq B$ for some $i_0 \in J$. The pseudo-run $\rho$ can be decomposed as follows with $\pi'_D = \pi^1_D t_{\alpha+1} \pi^2_D$ (INCR$'$ is defined few lines below):

$$\underbrace{\vec{x}_{2l-2} \xrightarrow{\pi'_{l-1}} \vec{x}_{2l-1} \cdots \vec{x}_{2D}}_{\rho_1} = \underbrace{\vec{x}_{2D} \xrightarrow{\pi^1_D} \vec{u}_\alpha}_{\rho_2} \xrightarrow{t_{\alpha+1}} \overbrace{\underbrace{\vec{u}_{\alpha+1} \xrightarrow{\pi^2_D} \vec{x}_{2D+1} \cdots \vec{x}_{2K-1} \xrightarrow{\pi_K} \vec{x}_{2K}}_{\rho_3}}^{\text{satisfies } \mathcal{A}[\mathcal{P}, D+1, \text{INCR}', (I \setminus \{i_0\}), +\infty]}$$

We construct a pseudo-run of the form $\rho'_1\rho'_2\rho'_3$ such that each $\rho'_j$ is obtained by shortening $\rho_j$ and the length of $\rho'_1$ [resp. $\rho'_2, \rho'_3$] is bounded by $(\mu \times B)^{n^{\mathrm{C}_1}} + 1$ [resp. $B^{i+1} + 1$, $g(i) + 1$].

- If $D > l - 1$, then we introduce $\mathcal{P}^\star = (\mathcal{I}'_l, \ldots, \mathcal{I}'_D)$ with for every $l'' \in [l, D]$ and for every $j \in [1, n]$,
  - if $\vec{x}_{2l''}(j) - \vec{x}_{2l''-1}(j) > 0$ then $\mathcal{I}'_{l''}(j) = \mathcal{I}_{l''}(j) \cap [1, +\infty[$,
  - otherwise $\mathcal{I}'_{l''}(j) = \mathcal{I}_{l''}(j)$.

  The construction of $\mathcal{P}^\star$ allows us to preserve the set of elements in $[l, D]$ whose values can be arbitrarily increased. Moreover, above, by taking the intersection with $[1, +\infty[$, in $\rho'_1$, we preserve the set of components in which proper pumping is possible. By Lemma 4.2, there is a pseudo-run $\rho'_1 = (t^1_1 \cdots t^1_{\beta_1}, \vec{x}_{2l-2})$ satisfying $\mathcal{A}[\mathcal{P}^\star, 1, \text{INCR}, I, B]$ such that $\beta_1 \leq$

$(\mu \times B)^{n^{C_1}}$. Indeed, $\mathrm{scale}(\mathcal{P}^\star) \le \mathrm{scale}(\mathcal{P})$ and the length of $\mathcal{P}^\star$ is obviously bounded by $K$. Say $\rho'_1 = \vec{y}_{2l-2} \xrightarrow{*} \vec{y}_{2l-1} \xrightarrow{*} \vec{y}_{2l} \cdots \xrightarrow{*} \vec{y}_{2D-1} \xrightarrow{*} \vec{y}_{2D}$. Suppose that $\rho'_1 = \vec{u}^1_0 \cdots \vec{u}^1_{\beta_1}$ and $f_1 : [2l-2, 2D] \to [0, \beta_1]$ is the map such that $\vec{y}_i = \vec{u}^1_{f_1(i)}$ with $f_1(2l-2) = 0$ and $f_1(2D) = \beta_1$. If $D = l-1$, then $\rho_1 = (t_1 \cdots t_\alpha, \vec{x}_{2l-2})$ with an analogous decomposition in terms of $\vec{y}_i$'s.

So, whenever $D \ge l-1$, we have $\{j : \vec{y}_{2l'-1}(j) < \vec{y}_{2l'}(j),\ l' \in [l, D]\} = \{j : \vec{x}_{2l'-1}(j) < \vec{x}_{2l'}(j),\ l' \in [l, D]\}$ – partly by construction of $\mathcal{P}^\star$. We write $Z$ to denote the set $\{j : \vec{y}_{2l'-1}(j) < \vec{y}_{2l'}(j),\ l' \in [l, D]\}$.

- Now, by the pigeonhole principle, there is a pseudo-run

$$\rho'_2 = (t^2_1 \cdots t^2_{\beta_2}, \vec{y}_{2D})$$

such that $\vec{u}'_\alpha = \vec{y}_{2D} + t^2_1 + \cdots + t^2_{\beta_2}$, $\vec{u}'_\alpha(J) = \vec{u}_\alpha(J)$ and $\beta_2 < B^{\mathrm{card}(J)} \le B^{i+1}$. We pose $\vec{u}'_{\alpha+1} = \vec{u}'_\alpha + t_{\alpha+1}$.

- Finally, observe that $(t_{\alpha+2} \cdots t_k, \vec{u}'_{\alpha+1})$ satisfies $\mathcal{A}[\mathcal{P}, D+1, \mathrm{INCR}', (I \setminus \{i_0\}), +\infty]$ with $\mathrm{INCR}' \stackrel{\text{def}}{=} \mathrm{INCR} \cup Z$. By the induction hypothesis, there is a pseudo-run $\rho'_3 = (t^3_1 \cdots t^3_{\beta_3}, \vec{u}'_{\alpha+1})$ satisfying $\mathcal{A}[\mathcal{P}, D+1, \mathrm{INCR}', (I \setminus \{i_0\}), +\infty]$ and such that $\beta_3 \le g(i)$. Because $\vec{u}'_{\alpha+1}(i_0) \ge \mathrm{maxneg}(\mathcal{T}) \times g(i)$, $\rho'_3$ also satisfies $\mathcal{A}[\mathcal{P}, D+1, \mathrm{INCR}', I, +\infty]$.

Gluing the previous transitions, the pseudo-run

$$(t^1_1 \cdots t^1_{\beta_1} t^2_1 \cdots t^2_{\beta_2} t_{\alpha+1} t^3_1 \cdots t^3_{\beta_3}, \vec{x}_{2l-2})$$

satisfies the approximation property $\mathcal{A}[\mathcal{P}, l, \mathrm{INCR}, I, +\infty]$ and its length is bounded by $(\mu \times B)^{n^{C_1}} + B^{i+1} + g(i)$.

**Case 3:** A value for some component in $J$ is strictly greater than $B-1$ for the first time within the path $\pi_D$ for some $D \in [l, K]$.

The pseudo-run $\rho$ can be written as follows with $\pi_D = \pi^1_D \pi^2_D$ and $\pi^1_D \ne \varepsilon$

$$\vec{x}_{2l-2} \xrightarrow{\pi'_{l-1}} \vec{x}_{2l-1} \cdots \vec{x}_{2D-1} \xrightarrow{\pi^1_D} \vec{u}_{\alpha+1} \xrightarrow{\pi^2_D} \vec{x}_{2D} \cdots \vec{x}_{2K-1} \xrightarrow{\pi_K} \vec{x}_{2K}$$

By Lemma 4.1, the pseudo-run $\rho' = (\pi'_{l-1}\pi_l \cdots \pi'_{D-1}(\pi_D)^2 \pi'_D \cdots \pi_K, \vec{x}_{2l-2})$ also satisfies the approximation property $\mathcal{A}[\mathcal{P}, l, \mathrm{INCR}, I, +\infty]$ and can be written as $\vec{x}_{2l-2} \xrightarrow{\pi'_{l-1}} \vec{x}_{2l-1} \cdots \vec{x}_{2D-2} \xrightarrow{\pi'_{D-1}\pi_D} \vec{x}_{2D} = \vec{z}_{2D-1} \xrightarrow{\pi_D} \vec{z}_{2D} \xrightarrow{\pi'_{D+1}} \cdots \vec{z}_{2K-1} \xrightarrow{\pi_K} \vec{z}_{2K}$. We are therefore back to Case 2. □

We are now in position to bound the length of pseudo-runs weakly satisfying the generalized unboundedness property $\mathcal{P}$.

Lemma 4.5. If $\rho$ is a pseudo-run weakly satisfying $\mathcal{P}$, then there is a $\rho'$ starting from the same pseudo-configuration, weakly satisfying $\mathcal{P}$ and of length at most $(\mu \times 2 \times \mathrm{maxneg}(\mathcal{T}))^{n^{(2n+1)C}}$ for some $C > 1$ with $\mu = (1+K) \times \mathrm{scale}(\mathcal{T}) \times \mathrm{scale}(\mathcal{P})$.

*Proof.* Since $\mathcal{T}$ has a pseudo-run weakly satisfying $\mathcal{P}$ iff $\mathcal{T}$ has a pseudo-run satisfying $\mathcal{A}[\mathcal{P}, 1, \emptyset, [1, n], +\infty]$, by Lemma 4.4, it is sufficient to bound $g(n)$. By Lemma 4.4, for some constant $C_2 > C_1$ (for instance $C_2 = C_1 + 1$), we have

$$g(i) \le \begin{cases} (2\mu)^{n^{C_2}} & \text{if } i = 0, \\ (2\mu(\mathrm{maxneg}(\mathcal{T}) \times g(i-1)))^{n^{C_2}} & \text{if } i > 0. \end{cases}$$

By induction on $i$, we can show that $g(i) \leq (\nu^{i+1})^{n^{(2i+1)\mathsf{C}_2}}$ with $\nu = 2\mu \times \text{maxneg}(\mathcal{T})$. For $i = 0$ this is obvious. Otherwise,

$$g(i+1) \leq (2\mu \times \text{maxneg}(\mathcal{T}) \times g(i))^{n^{\mathsf{C}_2}} \leq (\nu(\nu^{i+1})^{n^{(2i+1)\mathsf{C}_2}})^{n^{\mathsf{C}_2}} \leq \ldots$$

$$\leq ((\nu^{i+2})^{n^{(2i+1)\mathsf{C}_2}})^{n^{\mathsf{C}_2}} \leq (\nu^{i+2})^{n^{(2i+2)\mathsf{C}_2}} < (\nu^{i+2})^{n^{(2i+3)\mathsf{C}_2}}$$

Hence, $g(n) \leq (\nu^{n+1})^{n^{(2n+1)\mathsf{C}_2}}$. As soon as $n \geq 2$, there is a constant $\mathsf{C}$ such that $g(n) \leq (2\mu \times \text{maxneg}(\mathcal{T}))^{n^{(2n+1)\mathsf{C}}}$. □

Let us conclude the section by the main result of the paper.

Theorem 4.6. (I) The generalized unboundedness problem for VASS is ExpSpace-complete. (II) For each $n \geq 1$, the generalized unboundedness problem restricted to VASS of dimension at most $n$ is in PSpace.

*Proof.* (I, upper bound) Let $(\mathcal{V}, (q, \vec{x}))$ be an initialized VASS of dimension $n$ and $\mathcal{P}$ be a generalized unboundedness property. By Lemma 3.2, one can compute in logarithmic space an initialized VAS $((\mathcal{T}, \vec{x}'), \mathcal{P}')$ such that $(\mathcal{V}, (q, \vec{x}))$ satisfies $\mathcal{P}$ iff $(\mathcal{T}, \vec{x}')$ satisfies $\mathcal{P}'$, $\mathcal{T}$ has dimension $n+3$, $\mathcal{P}$ and $\mathcal{P}'$ have the same length and $\text{scale}(\mathcal{T}) = \max((\text{card}(Q)+1)^2, \text{scale}(\mathcal{V}))$. The propositions below are equivalent:

1. $\mathcal{T}$ has a run satisfying $\mathcal{P}'$.
2. $\mathcal{T}$ has a pseudo-run weakly satisfying $\mathcal{P}'$ (see Lemma 3.7).
3. $\mathcal{T}$ has a pseudo-run satisfying $\mathcal{A}[\mathcal{P}', 1, \emptyset, [1, n+3], +\infty]$ (by definition of $\mathcal{A}$).
4. $\mathcal{T}$ has a pseudo-run weakly satisfying $\mathcal{P}'$ whose length is bounded by

$$((1+K) \times 2 \times \text{scale}(\mathcal{T}) \times \text{scale}(\mathcal{P}) \times \text{maxneg}(\mathcal{T}))^{(n+3)^{(2(n+3)+1)\mathsf{C}}}$$

(by Lemma 4.5).

Then, we guess a witness pseudo-run weakly satisfying $\mathcal{P}'$ whose length is bounded by

$$((1+K) \times 2 \times \text{scale}(\mathcal{T}) \times \text{scale}(\mathcal{P}) \times \text{maxneg}(\mathcal{T}))^{(n+3)^{(2(n+3)+1)\mathsf{C}}}$$

This can be done in exponential space in the combined size of $(\mathcal{V}, (q, \vec{x}))$ and $\mathcal{P}$. By Savitch's Theorem [50], we get the ExpSpace upper bound. It is indeed sufficient to adapt the nondeterministic algorithm designed at the beginning of Section 4.3 in order to consider the above-mentionned length. Actually, one needs to consider $K+1$ intermediate pseudo-configurations and a current set of components among $[1, n+3]$ in order to record which components can be strictly increased in preceeding loops.

(II) Easy consequence of the proof of (I, upper bound).

(I, lower bound) A first temptation is to state ExpSpace-hardness from ExpSpace-hardness of the unboundedness problem for VAS. However, we are looking for a logarithmic-space many-one reduction and an instance of unboundedness can be naturally reduced to $n$ instances of the generalized unboundedness problem with property of length 1 and scale 1. We shall directly adapt [37, 18] to obtain the lower bound.

By [39] (see also [53]), a deterministic Turing machine $\mathcal{M}$ of size $n$ running in space $K2^{n^K}$ can be simulated by a deterministic counter automaton $C$ of size $O(n)$ with 4 counters and that is $2^{2^{n^{K'}}}$-bounded (counter values are bounded by $2^{2^{n^{K'}}}$ when the initial configuration has zero counter values). Moreover, $\mathcal{M}$ can reach a halting state on the empty tape iff $C$ can reach a halting control state with a run starting with zero counter values. A deterministic counter automaton is understood as a simple machine with a finite set of control states equipped with counters and the only instructions on counters are increments, decrements and zero-tests. In [37, 18], it is shown that given a deterministic counter automaton $C$ of size $n$ with a halting control state, one can build a *net program* (equivalent to a Petri net) of size $O(n^2)$ simulating $C$. In particular, its dimension is also in $O(n^2)$. This net program can be easily shown equivalent to a VASS $\mathcal{V}$ of dimension $n'$ (in $O(n^2)$), with $m'$ control states (also in $O(n^2)$) and with two distinguished control states $q_0$, $q_h$ satisfying the following conditions:

- $C$ halts iff there is a run from $(q_0, \vec{0})$ reaching a configuration with control state $q_h$.
- Whenever the simulation of $C$ in a run in $\mathcal{V}$ is not faithful to $C$, then the run eventually terminates.
- $C$ does not halt iff there is an infinite run from $(q_0, \vec{0})$ that never reaches a configuration with control state $q_h$.

Consequently, when $C$ halts, all the runs from $(q_0, \vec{0})$ are finite and there is a finite number of runs from $(q_0, \vec{0})$. We define the VASS $\mathcal{V}'$ of dimension $n' + 1$ that behaves as $\mathcal{V}$ except that we add a self-loop transition to $q_h$ whose effect is to add one to the $(n' + 1)$st component. Then, we have $C$ halts iff there is a run in $\mathcal{V}'$ of the form $(q_0, \vec{0}) \xrightarrow{*} (q, \vec{x}) \xrightarrow{*} (q, \vec{x}')$ such that $\vec{x}([1, n']) = \vec{x}'([1, n'])$ and $\vec{x}(n' + 1) < \vec{x}'(n' + 1)$. This can be easily turned into an instance of the generalized unboundedness problem. The ExpSpace-hardness proof is therefore a simple adaptation of the ExpSpace-hardness result from [37, 18]. Reproducing the arguments would not add much apart from repeating arguments from [18]. More details about this standard reduction can be also found in the slides [14].

□

## 5. Other Applications

In this section, we draw conclusions from Theorem 4.6. First, as a by-product of Theorem 4.6 and using the reductions from Section 3.3, we can easily regain the ExpSpace upper bound mentioned below.

Corollary 5.1. The regularity detection problem and the strong promptness detection problem are in ExpSpace. The simultaneous unboundedness problem is ExpSpace-complete. For each fixed $n \geq 1$, their restriction to VASS of dimension at most $n$ are in PSpace.

*Proof.* The ExpSpace upper bound for regularity detection problem and strong promptness detection problem is a consequence of remarks from Section 3.3. Indeed, for both problems, one needs to guess a generalized unbounded property $\mathcal{P}$ of length at most $n$ (dimension of the input VASS) and of scale 1 and then check whether there is a run satisfying $\mathcal{P}$. In case of positive answer to this question, we answer negatively to the original instance of the original problem.

Let us establish the lower bound for the simultaneous unboundedness problem. Let $\mathcal{V}$ be the VASS from the lower bound proof for Theorem 4.6(I). We define the VASS $\mathcal{V}'$ of dimension $n'+1$

that behaves as $\mathcal{V}$ except that we add a self-loop transition to $q_h$ whose effect is to add one to the $(n'+1)$th component. Then, we have $C$ halts iff $(\mathcal{V}', (q_0, \vec{0}))$ is not $(n'+1)$-unbounded. Simultaneous unboundedness problem is therefore coExpSpace-hard but since coExpSpace= ExpSpace, the simultaneous unboundedness problem is ExpSpace-hard. Now, let us establish the upper bound for the simultaneous unboundedness problem. Let $(\mathcal{V}, (q, \vec{x}))$ be an initialized VASS of dimension $n$ and $X$ be a subset of $[1, n]$. We first guess a disjointness sequence $\sigma = X_1 \cdot \cdots \cdot X_K$ such that $X \subseteq \bigcup_{l \in [1,K]} X_l$ and $X \cap X_K \neq \emptyset$ (this requires only polynomial space). Let us now consider the generalized unboundedness property $\mathcal{P}_\sigma$ as defined in Section 3.3 for dealing with simultaneous unboundedness. Checking whether $(\mathcal{V}, (q, \vec{x}))$ satisfies $\mathcal{P}_\sigma$ can be reduced in logspace to an instance of the generalized unboundedness problem, that can be solved in exponential space in the size of $(\mathcal{V}, (q, \vec{x}))$: indeed the length of $\mathcal{P}_\sigma$ is bounded by $n$ and its scale is equal to one. □

The complexity upper bound for regularity detection problem has been left open in [3]. Decidability of the strong promptness detection problem is established in [51]. The ExpSpace upper bound has been already stated in [54, 3]. We cannot rely on [54] because of the flaw in [54, Lemma 7.7]. Condition 4. in [3, page 13] does not characterize strong promptness (but only promptness) as shown in Section 3.3. Finally, increasing path formulae from [3] cannot characterize strong promptness detection unlike generalized unboundedness properties. Therefore, the upper bound for strong promptness detection is also new. Below, we state how the previous results allow us to characterize the computational complexity of reversal-boundedness detection problem for VASS and its variant with weak reversal-boundedness.

Theorem 5.2.

**(I)** Reversal-boundedness detection problem for VASS is ExpSpace-complete.

**(II)** For each fixed $n \geq 1$, its restriction to VASS of dimension at most $n$ is in PSpace.

**(III)** (I) and (II) hold true for weak reversal-boundedness.

*Proof.* (I) Let us start by showing ExpSpace-hardness. Let $\mathcal{V}$ be the VASS from the lower bound proof for Theorem 4.6(I) obtained from [37, 18]. We define the VASS $\mathcal{V}'$ of dimension $n'+1$ that behaves as $\mathcal{V}$ except that we add two transitions $q_h \xrightarrow{e_{n'+1}} q_h$ and $q_h \xrightarrow{-e_{n'+1}} q_h$ where $e_i$ denotes the $i$th unit vector and $q_h$ is the halting control state of $\mathcal{V}$. Then, we have $C$ halts iff $(\mathcal{V}', (q_0, \vec{0}))$ is not reversal-bounded with respect to $n'+1$. Reversal-boundedness detection problem is therefore coExpSpace-hard but since coExpSpace= ExpSpace, the problem is ExpSpace-hard.

Now, let us show ExpSpace upper bound. Let $\mathcal{V} = (Q, n, \delta)$ be a VASS and $(q, \vec{x})$ be a configuration. By Lemma 2.5, $(\mathcal{V}, (q, \vec{x}))$ is not reversal-bounded with respect to $i$ iff $(\mathcal{T}, \vec{x}') = ((\mathcal{V}_{rb}, (q_{rb}, \vec{x}_{rb})))^{\mathrm{HP}}$ is $(n+i)$-unbounded. The operator $(\cdot)^{\mathrm{HP}}$ refers to the reduction from VASS to VAS in [26] (see also the proof of Lemma 2.5). scale$(\mathcal{T})$ is bounded by $\max((\mathrm{card}(Q) \times 2^n + 1)^2, \mathrm{scale}(\mathcal{V}))$ and $((\mathcal{V}_{rb}, (q_{rb}, \vec{x}_{rb})))^{\mathrm{HP}}$ can be built in polynomial space. Dimension of $\mathcal{T}$ is $2n+3$. First, we guess $\mathcal{P}$ of length at most $2n+3$ for characterizing $(n+i)$-unboundedness (this requires only polynomial space): its scale is equal to one. A witness pseudo-run weakly satisfying $\mathcal{P}$ (in $\mathcal{T}$) does not need to be longer than

$$((1 + 2n + 3) \times 2 \times \max((\mathrm{card}(Q) \times 2^n + 1)^2, \mathrm{scale}(\mathcal{V}))^2)^{(2n+3)^{(2(2n+3)+1)C}},$$

which is doubly exponential in the size of $\mathcal{V}$ and $(q, \vec{x})$ (our initial instance). This comes from Lemma 4.5. A nondeterministic algorithm guessing such a pseudo-run requires only exponential

space.
(II) When $n$ is fixed, the above expression is only exponential in the size of $\mathcal{V}$ and $(q, \vec{x})$.
(III) This part is similar to (I) and (II). By combining Lemmas 3.6 and 3.2, we build in polynomial space an initialized VAS $(\mathcal{T}, \vec{x}')$ such that $(\mathcal{V}, (q, \vec{x}))$ is not weakly reversal-bounded iff $(\mathcal{T}, \vec{x}')$ satisfies $\mathcal{P}'_\sigma$ for some disjointness sequence $\sigma = X_1 \cdots X_K$ with $n + i \in X_K$, $i \in (X_1 \cup \cdots \cup X_{K-1})$ and such that

- $\mathcal{P}'_\sigma$ is defined from $\mathcal{P}_\sigma$ as done in the proof of Lemma 3.2 (length bounded by $n$ and scale equal to 1),
- the dimension of $\mathcal{T}$ is $2n + 3$,
- $\mathrm{scale}(\mathcal{T}) \leq \max((\mathrm{card}(Q) \times 2^n + 1)^2, \mathrm{scale}(\mathcal{V}))$.

Again, a witness pseudo-run weakly satisfying $\mathcal{P}'_\sigma$ (in $(\mathcal{T}, \vec{x}')$) does not need to be longer than

$$((1 + 2n + 3) \times 2 \times \max((\mathrm{card}(Q) \times 2^n + 1)^2, \mathrm{scale}(\mathcal{V}))^2)^{(2n+3)^{(2(2n+3)+1)C}}$$

which is doubly exponential in the size of $\mathcal{V}$ and $(q, \vec{x})$ (our initial instance). A nondeterministic algorithm guessing such a pseudo-run requires only exponential space.

Let us establish the ExpSpace-hardness. Let $\mathcal{V}$ be the VASS from the lower bound proof for Theorem 4.6(I). We define the VASS $\mathcal{V}'$ of dimension $n'+1$ that behaves as $\mathcal{V}$ except that we add two transitions $q_h \xrightarrow{2 \times e_{n'+1}} q'_h \xrightarrow{-e_{n'+1}} q_h$ Then, $\mathcal{C}$ halts iff $(\mathcal{V}', (q_0, \vec{0}))$ is not weakly reversal-bounded with respect to $n' + 1$. Weak reversal-boundedness detection problem is therefore coExpSpace-hard, whence ExpSpace-hard. □

By Theorem 5.2(I), once an initialized VASS is shown to be reversal-bounded, one can compute effectively semilinear sets corresponding to reachability sets, for instance one by control state, see recent developments in [32]. The size of the representation of such sets is at least polynomial in the maximal number of reversals. However, we know that an initialized VASS can be bounded but still the cardinality of its reachability set may be nonprimitive recursive, see e.g. [52]. A similar phenomenon occurs with reversal-boundedness, as briefly explained below. Not only we wonder what is the computational complexity of the problem of determining whether a VASS is reversal-bounded but also in case of reversal-boundedness, it is important to evaluate the size of the maximal reversal $r$ in terms of the size of the VASS, see e.g. the recent work [32] following [27] that uses in an essential way the value $r$. In case of reversal-boundedness, the maximal reversal can be nonprimitive recursive in the size of the initialized VASS in the worst case, which, we admit, is not an idyllic situation for analyzing reversal-bounded VASS. Indeed, given $n \geq 0$, one can compute in polynomial time in $n$ an initialized VASS $(\mathcal{V}_n, (q_0, \vec{x}_n))$ that generates a finite reachability set of cardinal $O(A(n))$ for some nonprimitive recursive map $A(\cdot)$ similar to Ackermann function, see e.g., the construction in [30]. Let us precise what this means by recalling a variant of Ackermann function:

- $A_0(m) = 2m + 1$, $A_{n+1}(0) = 1$.
- $A_{n+1}(m + 1) = A_n(A_{n+1}(m))$.
- $A(n) = A_n(2)$.

The function $A(n)$ majorizes the primitive recursive functions.

Moreover, $(\mathcal{V}_n, (q_0, \vec{x}_n))$ can be shown to admit only finite runs, see details in [30]. It is then easy to compute a variant VASS $\mathcal{V}'_n$ by adding a component and such that each transition of $\mathcal{V}_n$

is replaced by itself followed by incrementating the new component and then decrementing it (creating a reversal). Still $\mathcal{V}'_n$ has no infinite computation, $(\mathcal{V}'_n, (q_0, \vec{x}'_n))$ is reversal-bounded ($\vec{x}'_n$ restricted to the components of $\mathcal{V}_n$ is equal to $\vec{x}_n$) and its maximal reversal is in $O(A(n))$.

## 6. Concluding Remarks

We have proved the ExpSpace upper bound for the generalized unboundedness problem (both the initialized VASS and the generalized unboundedness property are part of the inputs). For example, this allows us to show, for the first time (apart from the preliminary version [13]), that the following problems on VASS can be solved in exponential space:

- the place boundedness problem,
- the reversal-boundedness detection problem,
- the regularity detection problem,
- the strong promptness detection problem.

We have shown that these problems can be solved in polynomial space when the dimension is fixed. Even though our proof technique is clearly tailored along the lines of [45], we had to provide a series of adaptations in order to get the final ExpSpace upper bound (and the PSpace upper bound for fixed dimension). In particular, we advocate the use of witness pseudo-run characterizations (instead of using runs) when there exist decision procedures using coverability graphs.

Let us conclude by possible continuations. First, our ExpSpace proof can be obviously extended for example by admitting covering constraints, to replace intervals in properties by more complex sets of integers or to combine our proof technique with the one from [3], see also [5]. The robustness of our proof technique still deserves to be determined. A challenging question is to determine the complexity of checking when a reachability set obtained by an initialized VASS is semilinear. Indeed, it was proved independently by Hauschildt and Lambert that the class of semilinear VASS is recursive: checking whether a given VASS has a semilinear reachability set is decidable [see the unpublished works by 25, 35]. Moreover, the reachability set is effectively computable when it is semilinear. Observe that regularity, boundedness or reversal-boundedness imply semilinearity.

Another direction consists in considering a richer class of models. It is shown in [21] that checking whether an initialized VASS with one zero-test is reversal-bounded is decidable, but with a nonprimitive recursive worst-case complexity, the existence of an ExpSpace upper bound being open; see also recent results on VASS with one zero-test [6, 7].

Besides, various subclasses of VASS exist for which decision problems are of lower complexity. For instance, in [43], the boundedness problem is shown to be in PSpace for a class of VASS with so-called bounded *benefit depth*. It is unclear for which subclasses of VASS, the generalized unboundedness problem can be solved in polynomial space too.

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