

COMPLEXITY HIERARCHIES BEYOND ELEMENTARY

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ABSTRACT. We introduce a hierarchy of fast-growing complexity classes and show its suitability for completeness statements of many non elementary problems. This hierarchy allows the classification of many decision problems with a non-elementary complexity, which occur naturally in logic, combinatorics, formal languages, verification, etc., with complexities ranging from simple towers of exponentials to Ackermannian and beyond.

1. INTRODUCTION

Complexity classes, along with the associated notions of reductions and completeness, provide our best theoretical tools to classify and compare computational problems. The richness and liveness of this field can be experienced by taking a guided tour of the *Complexity Zoo*,¹ which presents succinctly most of the known specimen. The visitor will find there a wealth of classes at the frontier between tractability and intractability, starring the classes P and NP, as they help in understanding what can be solved efficiently by algorithmic means.

From this tractability point of view, it is not so surprising to find much less space devoted to the “truly intractable” classes, in the exponential hierarchy and beyond. Such classes are nevertheless quite useful for classifying problems, and employed routinely in logic, combinatorics, formal languages, verification, etc. since the 70’s and the exponential lower bounds proven by Meyer and Stockmeyer [60, 79].

Non Elementary Problems. Actually, these two seminal articles go quite further than mere exponential lower bounds: they show respectively that satisfiability of the weak monadic theory of one successor (WS1S) and equivalence of star-free expressions (SFEq) are both *non-elementary*, as they require space bounded above and below by towers of exponentials of height depending (elementarily) on the size of the input. Those are just two examples among many others of problems with non-elementary complexities [see e.g. 61, 36, 84], but they are actually good representatives of problems with a tower of exponentials as complexity, i.e. one would expect them to be *complete* for some suitable complexity class.

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¹<https://complexityzoo.uwaterloo.ca>

What might then come as a surprise is the fact that the Zoo does not provide any intermediate stops where classical problems like **WS1S** and **SFEq** would fit adequately: they are not in **ELEMENTARY** (henceforth **ELEM**), but the next class is **PRIMITIVE-RECURSIVE** (aka **PR**), which is way too big: **WS1S** and **SFEq** are not *hard* for **PR** under any reasonable notion of reduction. In other words, we seem to be missing a “TOWER” complexity class, which ought to sit somewhere between **ELEM** and **PR**. Going higher, we find a similar uncharted area between **PR** and **RECURSIVE** (aka **R**); these absences are not specific to the Complexity Zoo: they seem on the contrary universal in textbooks on complexity theory—which seldom even mention **ELEM** or **PR**. Somewhat oddly, the complexities above **R** are better explored and can rely on the arithmetical and analytical hierarchies.

Drawing distinctions based on complexity characterizations can guide the search for practically relevant restrictions to the problems. In addition, non-elementary problems are much more pervasive now than in the 70’s, and they are also considered for practical applications, motivating the implementation of tools, e.g. **MONA** for **WS1S** [25]. It is therefore high time for the definition of hierarchies suited for their classification.

Our Contribution. In this paper, we propose an ordinal-indexed hierarchy $(\mathbf{F}_\alpha)_\alpha$ of *fast growing* complexity classes for non-elementary complexities. Besides the already mentioned **TOWER** $\stackrel{\text{def}}{=} \mathbf{F}_3$, for which **WS1S** and **SFEq** are examples of complete problems, this hierarchy includes non primitive-recursive classes, for which quite a few complete problems have arisen in the recent years, e.g. \mathbf{F}_ω in [58, 43, 83, 75, 29, 14], $\mathbf{F}_{\omega^\omega}$ in [18, 66, 51, 8, 16, 9], $\mathbf{F}_{\omega^{\omega^\omega}}$ in [40], and $\mathbf{F}_{\varepsilon_0}$ in [38].

The classes \mathbf{F}_α are related to the Grzegorzcyk $(\mathcal{E}^k)_k$ [37] and extended Grzegorzcyk $(\mathcal{F}_\alpha)_\alpha$ [56] hierarchies, which have been used in complexity statements for non-elementary bounds. The $(\mathcal{F}_\alpha)_\alpha$ classes are extremely well-suited for characterizing various classes of functions, for instance computed by forms of **for** programs [62] or terminating **while** programs [26], or provably total in fragments of Peano arithmetic [27, 77], and they characterize some important milestones like **ELEM** or **PR**. They are however too large to classify our decision problems and do not lead to *completeness* statements—in fact, one can show that there are *no* “**ELEM**-complete” nor “**PR**-complete” problems—; see Section 2. Our \mathbf{F}_α share however several nice properties with the \mathcal{F}_α classes: for instance, they form a strict hierarchy (Section 5) and are *robust* to slight changes in their generative functions and to changes in the underlying model of computation (Section 4).

In order to argue for the suitability of the classes \mathbf{F}_α for the classification of high-complexity problems, we sketch two completeness proofs in Section 3, and present an already long list of complete problems for \mathbf{F}_ω and beyond in Section 6. A general rule of thumb seems to be that statements of the form “ L is in \mathcal{F}_α but not in \mathcal{F}_β for any $\beta < \alpha$ ” found in the literature can often be replaced by the much more precise “ L is \mathbf{F}_α -complete.”

There are of course essential limitations to our approach: there is no hope of defining such ordinal-indexed hierarchies that would exhaust \mathbb{R} using sensible ordinal notations [28]; this is called the “subrecursive stumbling block” in [77, Section 5.1]. Our aim here is more modestly to provide suitable definitions “from below” for naturally-occurring complexity classes above ELEM.

In an attempt not to drown the reader in the details of subrecursive functions and their properties, most of the technical contents appears in Appendix A at the end of the paper.

2. FAST-GROWING COMPLEXITY CLASSES

We define in this section the complexity classes \mathbf{F}_α . The hierarchies of functions, function classes, and complexity classes we employ in order to deal with non-elementary complexities are all indexed using ordinals, and we reuse the very rich literature on subrecursion [e.g. 70, 64, 77]. We strive to employ notations compatible with those of Schwichtenberg and Wainer [77, Chapter 4], and refer the interested reader to their monograph for proofs and additional material.

2.1. Cantor Normal Forms and Fundamental Sequences. In this paper, we only deal with ordinals below ε_0 , i.e. ordinals that can be denoted as terms in *Cantor Normal Form*:

$$\alpha = \omega^{\alpha_1} \cdot c_1 + \cdots + \omega^{\alpha_n} \cdot c_n \text{ where } \alpha > \alpha_1 > \cdots > \alpha_n \text{ and } \omega > c_1, \dots, c_n > 0. \quad (\text{CNF})$$

In this representation, $\alpha = 0$ if and only if $n = 0$. An ordinal α with CNF of form $\alpha' + 1$ is called a *successor* ordinal—it has $n > 0$ and $\alpha_n = 0$ —, and otherwise if $\alpha > 0$ it is called a *limit* ordinal, and can be written as $\gamma + \omega^\beta$ by setting $\gamma = \omega^{\alpha_1} \cdot c_1 + \cdots + \omega^{\alpha_n} \cdot (c_n - 1)$ and $\beta = \alpha_n$. We usually employ ‘ λ ’ to denote limit ordinals.

A *fundamental sequence* for a limit ordinal λ is a sequence $(\lambda(x))_{x < \omega}$ of ordinals with supremum λ . We consider a standard assignment of fundamental sequences for limit ordinals, defined inductively by

$$(\gamma + \omega^{\beta+1})(x) \stackrel{\text{def}}{=} \gamma + \omega^\beta \cdot (x + 1), \quad (\gamma + \omega^\lambda)(x) \stackrel{\text{def}}{=} \gamma + \omega^{\lambda(x)}. \quad (2.1)$$

This is one particular choice of a fundamental sequence, which verifies e.g. $0 < \lambda(x) < \lambda(y)$ for all $x < y$ and limit ordinals λ . For instance, $\omega(x) = x + 1$, $(\omega^{\omega^4} + \omega^{\omega^3 + \omega^2})(x) = \omega^{\omega^4} + \omega^{\omega^3 + \omega \cdot (x+1)}$. We also consider ε_0 , which is the supremum of all the ordinals writable in CNF, as a limit ordinal with fundamental sequence defined by $\varepsilon_0(0) \stackrel{\text{def}}{=} \omega$ and $\varepsilon_0(x + 1) \stackrel{\text{def}}{=} \omega^{\varepsilon_0(x)}$, i.e. a tower of ω ’s of height $x + 1$.

2.2. The Extended Grzegorzczk Hierarchy $(\mathcal{F}_\alpha)_{\alpha < \varepsilon_0}$ is an ordinal-indexed infinite hierarchy of classes of functions f with argument(s) and images in \mathbb{N} [56]. The extended Grzegorzczk hierarchy has multiple natural

characterizations: for instance as `loop` programs for $\alpha < \omega$ [62], as ordinal-recursive functions with bounded growth [85], as functions computable with restricted resources as we will see in (2.5), as functions that can be proven total in fragments of Peano arithmetic [27], etc.

2.2.1. *Fast-Growing Functions.* At the heart of each \mathcal{F}_α lies the α th *fast-growing function* $F_\alpha: \mathbb{N} \rightarrow \mathbb{N}$, which is defined inductively on the ordinal index: as the successor function at index 0

$$F_0(x) \stackrel{\text{def}}{=} x + 1, \quad (2.2)$$

by iteration at successor indices $\alpha + 1$

$$F_{\alpha+1}(x) \stackrel{\text{def}}{=} F_\alpha^{\omega(x)}(x) = \overbrace{F_\alpha(\cdots(F_\alpha(x))\cdots)}^{\omega(x) \text{ times}}, \quad (2.3)$$

and by diagonalization on the fundamental sequence at limit indices λ

$$F_\lambda(x) \stackrel{\text{def}}{=} F_{\lambda(x)}(x). \quad (2.4)$$

For instance, $F_1(x) = 2x + 1$, $F_2(x) = 2^{x+1}(x + 1) - 1$, F_3 is a non elementary function that grows faster than $\text{tower}(x) \stackrel{\text{def}}{=} 2^{\cdot^{\cdot^2}}\}_{x \text{ times}}$, F_ω a non primitive-recursive ‘‘Ackermannian’’ function, F_{ω^ω} a non multiply-recursive ‘‘hyper-Ackermannian’’ function, and $F_{\varepsilon_0}(x)$ cannot be proven total in Peano arithmetic. For every α , the F_α function is strictly *monotone* in its argument, i.e. $x < y$ implies $F_\alpha(x) < F_\alpha(y)$. As $F_\alpha(0) = 1$, it is therefore also strictly *expansive*, i.e. $F_\alpha(x) > x$ for all x .

2.2.2. *Computational Characterization.* The extended Grzegorzcyk hierarchy itself is defined by means of recursion schemes with the $(F_\alpha)_\alpha$ as generators (see § 5.3.1). Nevertheless, for $\alpha \geq 2$, each of its levels \mathcal{F}_α is also characterized as a class of functions computable with bounded resources [85]. More precisely, for $\alpha \geq 2$, it is the class of functions computable by deterministic Turing machines in time bounded by $O(F_\alpha^c(n))$ for some constant c , when given an input of size n :

$$\mathcal{F}_\alpha = \bigcup_{c < \omega} \text{FDTIME}(F_\alpha^c(n)). \quad (2.5)$$

Note that the choice between deterministic and nondeterministic, or between time-bounded and space-bounded computations in (2.5) is irrelevant, because $\alpha \geq 2$ and F_2 is already a function of exponential growth.

2.2.3. *Main Properties.* Each class \mathcal{F}_α is closed under (finite) composition. Every function f in \mathcal{F}_α is *honest*, i.e. can be computed in time bounded by some function also in \mathcal{F}_α [85, 27]—this is a relaxation of the *time constructible* condition, which asks instead for computability in time $O(f(n))$. Since each f in \mathcal{F}_α is also bounded by F_α^c for some c [56, Theorem 2.10], this means that

$$\mathcal{F}_\alpha = \bigcup_{f \in \mathcal{F}_\alpha} \text{FDTIME}(f(n)). \quad (2.6)$$

In particular, the function F_α belongs to \mathcal{F}_α for every α , and therefore F_α^c also belongs to \mathcal{F}_α .

Every f in \mathcal{F}_β is also eventually bounded by F_α if $\beta < \alpha$ [56], i.e. there exists a rank x_0 such that, for all x_1, \dots, x_n , if $\max_i x_i \geq x_0$, then $f(x_1, \dots, x_n) \leq F_\alpha(\max_i x_i)$ —a fact that we will use copiously. However, for all $\alpha > \beta > 0$, $F_\alpha \notin \mathcal{F}_\beta$, and the hierarchy $(\mathcal{F}_\alpha)_{\alpha < \varepsilon_0}$ is therefore strict for $\alpha > 0$.

2.2.4. Milestones. At the lower levels, $\mathcal{F}_0 = \mathcal{F}_1$ contains (among others) all the linear functions (see § 5.3.2). We focus however in this paper on the non-elementary classes by restricting ourselves to $\alpha \geq 2$. Writing

$$\mathcal{F}_{<\alpha} \stackrel{\text{def}}{=} \bigcup_{\beta < \alpha} \mathcal{F}_\beta, \quad (2.7)$$

we find for instance $\mathcal{F}_2 = \mathcal{F}_{<3} = \text{FELEM}$ the set of Kalmar-elementary functions, $\mathcal{F}_{<\omega} = \text{FPR}$ the set of primitive-recursive functions, $\mathcal{F}_{<\omega^\omega} = \text{FMR}$ the set of multiply-recursive functions, and $\mathcal{F}_{<\varepsilon_0} = \text{FOR}$ the set of ordinal-recursive functions (up to ε_0). We are dealing here with classes of functions, but writing \mathcal{F}_α^* for the restriction of \mathcal{F}_α to $\{0, 1\}$ -valued functions, i.e.

$$\mathcal{F}_\alpha^* = \bigcup_{c < \omega} \text{DTIME}(F_\alpha^c(n)), \quad \mathcal{F}_{<\alpha}^* \stackrel{\text{def}}{=} \bigcup_{\beta < \alpha} \mathcal{F}_\beta^*, \quad (2.8)$$

we obtain the corresponding classes for decision problems $\mathcal{F}_{<3}^* = \text{ELEM}$, $\mathcal{F}_{<\omega}^* = \text{PR}$, $\mathcal{F}_{<\omega^\omega}^* = \text{MR}$, and $\mathcal{F}_{<\varepsilon_0}^* = \text{OR}$.

2.3. Fast-Growing Complexity Classes. Unfortunately, the classes in the extended Grzegorzcyk hierarchy are not quite satisfying for some interesting problems, which are non elementary (or non primitive-recursive, or non multiply-recursive, ...), but only *barely* so. The issue is that complexity classes like e.g. \mathcal{F}_3^* , which is the first class that contains non-elementary problems, are very large: \mathcal{F}_3^* contains for instance problems that require space $F_3^{100}(n)$, more than a hundred-fold compositions of towers of exponentials. As a result, hardness for \mathcal{F}_3^* cannot be obtained for the classical examples of non-elementary problems.

We therefore introduce smaller classes of problems:

$$\mathbf{F}_\alpha \stackrel{\text{def}}{=} \bigcup_{p \in \mathcal{F}_{<\alpha}} \text{DTIME}(F_\alpha(p(n))). \quad (2.9)$$

In contrast with \mathcal{F}_α^* in (2.8), only a single application of F_α is possible, composed with some “lower” *reduction* function p from $\mathcal{F}_{<\alpha}$. As previously, the choice of DTIME rather than NTIME or SPACE is irrelevant for $\alpha \geq 3$ (see Lemma 4.5).

This definition yields for instance the desired class $\text{TOWER} \stackrel{\text{def}}{=} \mathbf{F}_3$, closed under elementary reductions (i.e., reductions in \mathcal{F}_2), but also a class $\text{ACK} \stackrel{\text{def}}{=} \mathbf{F}_\omega$ of Ackermannian problems closed under primitive-recursive reductions, a

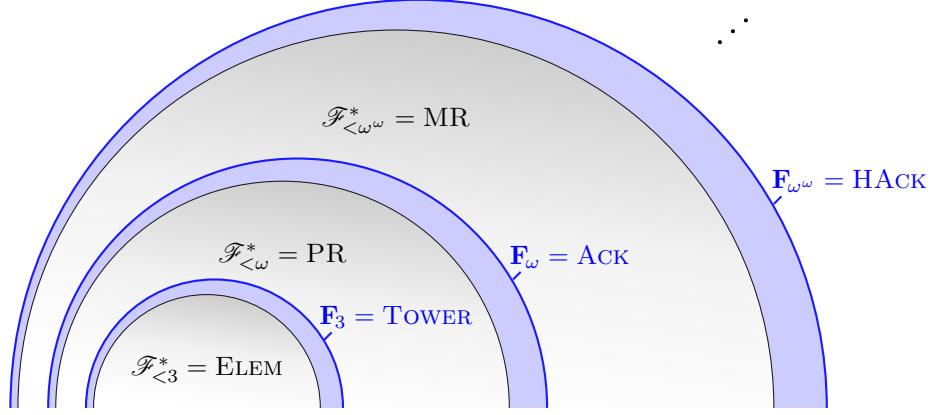


FIGURE 1. Some complexity classes beyond ELEM.

class $\text{HACK} \stackrel{\text{def}}{=} \mathbf{F}_{\omega^\omega}$ of hyper-Ackermannian problems closed under multiply-recursive reductions, etc. In each case, we can think of \mathbf{F}_α as the class of problems not solvable with resources in $\mathcal{F}_{<\alpha}$, but barely so: non-elementary problems for \mathbf{F}_3 , non primitive-recursive ones for \mathbf{F}_ω , non multiply-recursive ones for $\mathbf{F}_{\omega^\omega}$, and so on. See Figure 1 for the first main steps of the hierarchy.

2.3.1. *Reduction Classes.* Of course, we could replace in (2.9) the class of reductions $\mathcal{F}_{<\alpha}$ by a more traditional one, like logarithmic space (FL) or polynomial time (FP) functions. We feel however that our definition in (2.9) better captures the intuition we have of a problem being “complete for F_α .” Moreover, using at least \mathcal{F}_2 as our class of reductions allows to effectively compute the F_α function in the functional version \mathbf{FF}_α of \mathbf{F}_α (see Section 5.1), leading to interesting combinatorial algorithms (see §3.2.3 for an example).

Unless stated differently, we always assume many-one $\mathcal{F}_{<\alpha}$ reductions when discussing hardness for \mathbf{F}_α in the remainder of this paper, but we could just as easily consider Turing reductions (see §4.2.3).

2.3.2. *Basic \mathbf{F}_α -Complete Problems.* By (2.9), \mathbf{F}_α -hardness proofs can reduce from the acceptance problem of some input string x by some deterministic Turing machine M working in time $F_\alpha(p(n))$ for some p in $\mathcal{F}_{<\alpha}$. This can be simplified to a machine working in time $F_\alpha(n)$. Because p in $\mathcal{F}_{<\alpha}$ is honest, $p(n)$ can be computed in $\mathcal{F}_{<\alpha}$. Thus the acceptance of x by M can be reduced to the acceptance problem of a #-padded input string $x' \stackrel{\text{def}}{=} x\#^{p(|x|)-|x|}$ of length $p(|x|)$ by a machine M' that simulates M , and treats # as a blank symbol—now M' works in time $F_\alpha(n)$. Another similarly basic \mathbf{F}_α -hard problem is the halting problem for Minsky machines with the sum of counters bounded by $F_\alpha(n)$ [see 35]

To sum up, we have by definition of the $(\mathbf{F}_\alpha)_\alpha$ classes the following two \mathbf{F}_α -complete problems—which incidentally have been used in most of the master

reductions in the literature in order to prove non primitive-recursiveness, non multiple-recursiveness, and other hardness results [43, 83, 75, 18, 40, 38]:

F_α -Bounded Turing Machine Acceptance (F_α -TM)

instance: A deterministic Turing machine M working in time F_α and an input x .

question: Does M accept x ?

F_α -Bounded Minsky Machine Halting (F_α -MM)

instance: A deterministic Minsky machine M with sum of counters bounded by $F_\alpha(|M|)$.

question: Does M halt?

See Section 6 for a catalog of natural complete problems, which should be easier to employ in reductions.

3. FAST-GROWING COMPLEXITIES IN ACTION

We present now two short tutorials for the use of fast-growing complexities, namely for **SFEq** (Section 3.1) and reachability in lossy counter systems (Section 3.2), pointing to the relevant technical results from later sections. We also briefly discuss in each case the palliatives employed this far in the literature for expressing such complexities.

3.1. A TOWER-Complete Example can be found in the seminal paper of Stockmeyer and Meyer [79], and is quite likely already known by many readers. Define a *star-free expression* over some alphabet Σ as a term e with abstract syntax

$$e ::= a \mid \varepsilon \mid \emptyset \mid e + e \mid ee \mid \neg e$$

where ‘ a ’ ranges over Σ and ‘ ε ’ denotes the empty string. Such expressions are inductively interpreted as languages included in Σ^* by:

$$\begin{aligned} \llbracket a \rrbracket &\stackrel{\text{def}}{=} \{a\} & \llbracket \varepsilon \rrbracket &\stackrel{\text{def}}{=} \{\varepsilon\} & \llbracket \emptyset \rrbracket &\stackrel{\text{def}}{=} \emptyset \\ \llbracket e_1 + e_2 \rrbracket &\stackrel{\text{def}}{=} \llbracket e_1 \rrbracket \cup \llbracket e_2 \rrbracket & \llbracket e_1 e_2 \rrbracket &\stackrel{\text{def}}{=} \llbracket e_1 \rrbracket \cdot \llbracket e_2 \rrbracket & \llbracket \neg e \rrbracket &\stackrel{\text{def}}{=} \Sigma^* \setminus \llbracket e \rrbracket . \end{aligned}$$

The decision problem **SFEq** asks, given two such expressions e_1, e_2 , whether they are *equivalent*, i.e. whether $\llbracket e_1 \rrbracket = \llbracket e_2 \rrbracket$. Stockmeyer and Meyer [79] show that this problem is hard for $\text{tower}(\log n)$ space under FL reductions if $|\Sigma| \geq 2$. The problem **WS1S** can be shown similarly hard thanks to a reduction from **SFEq**.

3.1.1. Completeness. Recall that **TOWER** is defined as \mathbf{F}_3 , i.e. by the instantiation of (2.9) for $\alpha = 3$, as the problems decidable by a Turing machine working in time F_3 of some elementary function of the input size:

$$\text{TOWER} \stackrel{\text{def}}{=} \mathbf{F}_3 = \bigcup_{p \in \text{FELEM}} \text{DTIME}(F_3(p(n))) . \quad (3.1)$$

Once hardness for $\text{tower}(\log n)$ is established, hardness for TOWER under elementary reductions is immediate; a detailed proof can apply Theorem 4.1 and (4.7) to show that

$$\text{TOWER} = \bigcup_{p \in \text{FELEM}} \text{SPACE}(\text{tower}(p(n))) \quad (3.2)$$

and use a padding argument as in §2.3.2 to conclude.

That SFEq is in TOWER can be checked using an automaton-based algorithm: construct automata recognizing $\llbracket e_1 \rrbracket$ and $\llbracket e_2 \rrbracket$ respectively, using determinization to handle each complement operator at the expense of an exponential blowup, and check equivalence of the obtained automata in PSPACE—the overall procedure is in space polynomial in $\text{tower}(n)$, thus in \mathbf{F}_3 . A similar automata-based procedure yields the upper bound for WS1S.

3.1.2. Discussion. Regarding upper bounds, there was a natural candidate in the literature for the missing class TOWER: Grzegorzczuk [37] defines an infinite hierarchy of function classes $(\mathcal{E}^k)_{k \in \mathbb{N}}$ inside FPR with $\mathcal{E}^{k+1} = \mathcal{F}_k$ for $k \geq 2$. This yields $\text{FELEM} = \mathcal{E}^3$, and the tower function is in $\mathcal{E}^4 \setminus \mathcal{E}^3$. Thus WS1S and SFEq are in “time \mathcal{E}^4 ,” and such a notation has occasionally been employed, for instance for β -Eq the β equivalence of simply typed λ -terms [78, 76, 10]. Again, we face the issue that \mathcal{E}^4 is much too large a resource bound, as it contains for instance all the finite iterates of the tower function, and there is therefore no hope of proving the hardness for \mathcal{E}^4 of WS1S, SFEq, or indeed β -Eq, at least if using a meaningful class of reductions.

Regarding non-elementary lower bounds, recent papers typically establish hardness for k -EXPTIME (or k -EXPSPACE) for *infinitely many* k (possibly through a suitable parametrization of the problem at hand), for instance by reducing from the acceptance of an input of size n by a $\underbrace{2^{\cdot^{2^n}}}_{k \text{ times}}$ time-bounded Turing machine. Provided that such a lower bound argument is *uniform* for those infinitely many k , it immediately yields a TOWER-hardness proof, by choosing $k \geq n$. On a related topic, note that, in contrast with e.g. the relationship between PH and PSPACE, because the exponential hierarchy is known to be strict, we know for certain that

- for all k , $k\text{-EXPTIME} \subsetneq \text{ELEM} = \bigcup_k k\text{-EXPTIME}$,
- there are no “ELEM-complete problems,” and
- $\text{ELEM} \subsetneq \text{TOWER}$.

3.2. An ACK-Complete Example. Possibly the most popular ACK-complete problem in use in reductions, LCM Reachability asks whether a given configuration is reachable in a *lossy counter machine* (LCM) [75]. Such counter machines are syntactically defined like Minsky machines $\langle Q, \mathcal{C}, \delta, q_0 \rangle$, where transitions $\delta \subseteq Q \times \mathcal{C} \times \{=0?, ++, --\} \times Q$ operate on a set \mathcal{C} of counters

through *zero-tests* $\mathbf{c}=0?$, *increments* $\mathbf{c}++$ and *decrements* $\mathbf{c}--$. The semantics of an LCM differ however from the usual, “reliable” semantics of a counter machine in that the counter values can decrease in an uncontrolled manner at any point of the execution. These unreliable behaviors make several problems decidable on LCMs, contrasting with the situation with Minsky machines.

Formally, a configuration $\sigma = (q, \vec{v})$ associates a control location q in Q with a counter valuation \vec{v} in $\mathbb{N}^{\mathcal{C}}$, i.e. counter values can never go negative. Let the initial configuration be $(q_0, \vec{0})$. For two counter valuations \vec{v} and \vec{v}' , write $\vec{v} \leq_{\times} \vec{v}'$ if $\vec{v}(c) \leq_{\times} \vec{v}'(c)$ for all c in \mathcal{C} , i.e. for the product ordering over $\mathbb{N}^{\mathcal{C}}$. Then a zero-test $(q, \mathbf{c}=0?, q')$ updates (q, \vec{v}) into (q', \vec{v}') , written $(q, \vec{v}) \rightarrow (q', \vec{v}')$, if $\vec{v}' \leq_{\times} \vec{v}$ and $\vec{v}'(c) = 0$; an increment $(q, \mathbf{c}++, q')$ if $\vec{v}' \leq_{\times} \vec{v} + \vec{c}$ where $\vec{v} + \vec{c}(c) = \vec{v}(c) + 1$ and $\vec{v} + \vec{c}(c') = \vec{v}(c')$ for all $c' \neq c$; finally, a decrement $(q, \mathbf{c}--, q')$ if $\vec{v}' \leq_{\times} \vec{v} - \vec{c}$ where $\vec{v} - \vec{c}(c) = \vec{v}(c) - 1$ (thus $\vec{v}(c) > 0$) and $\vec{v} - \vec{c}(c') = \vec{v}(c')$ for all $c' \neq c$.

The reachability problem for such a system asks whether a given configuration τ can be reached, i.e. whether $(q_0, \vec{0}) \rightarrow^* \tau$. The hardness proof of Schnoebelen [75] immediately yields that this problem is ACK-hard [see also 83, 74], where ACK is defined as an instance of (2.9): it is the class of problems decidable with F_{ω} resources of some primitive-recursive function of the input size:

$$\text{ACK} \stackrel{\text{def}}{=} \mathbf{F}_{\omega} = \bigcup_{p \in \text{FPR}} \text{DTIME}(F_{\omega}(p(n))). \quad (3.3)$$

3.2.1. Decidability of LCM. Lossy counter machines define *well-structured* transition systems over the set of configurations $Q \times \mathbb{N}^{\mathcal{C}}$, for which generic algorithms have been designed [4, 34], which rely on the existence of a well-quasi-ordering [wqo, see 49] over the set of configurations. The particular variant of the algorithm we present here is well-suited for a complexity analysis, and is taken from [73].

Call a sequence of configurations $\sigma_0, \sigma_1, \dots, \sigma_n$ a *witness* if $\sigma_0 = \tau$ is the target configuration, $\sigma_n = (q_0, \vec{0})$ is the initial configuration, and $\sigma_{i+1} \rightarrow \sigma_i$ for all $0 \leq i < n$. An instance of LCM is positive if and only if there exists a witness, which we will search for backwards, starting from τ and attempting to reach the initial configuration $(q_0, \vec{0})$.

Consider the ordering over configurations defined by $(q, \vec{v}) \leq (q', \vec{v}')$ if and only if $q = q'$ and $\vec{v} \leq_{\times} \vec{v}'$, and observe that, if $\sigma \rightarrow \sigma'$ and $\sigma \leq \tau$, then $\tau \rightarrow \sigma'$. This means that, if there is a witness, then there is a *minimal* one, i.e. one where for all $0 < i < n$, $\sigma_{i+1} \in \text{MinPre}(\sigma_i)$ where $\text{MinPre}(\sigma) \stackrel{\text{def}}{=} \min_{\leq} \{\sigma' \mid \sigma' \rightarrow \sigma\}$. Observe furthermore that, if $\sigma_0, \sigma_1, \dots, \sigma_n$ is a *shortest* minimal witness, then for all $i < j$, $\sigma_i \not\leq \sigma_j$, i.e. is a *bad* sequence for \leq , or we could have picked σ_j at step i and obtained a shorter minimal witness. Hence, if there is a minimal witness, then there is one which is a bad sequence.

Now, because $(\leq, Q \times \mathbb{N}^{\mathbb{C}})$ is a well-quasi-order by Dickson's Lemma,

- (1) for all i , the set $\text{MinPre}(\sigma_i)$ is finite, and
- (2) any bad sequence, i.e. any sequence $\sigma_0, \sigma_1, \dots$ where $\sigma_i \not\leq \sigma_j$ for all $i < j$, is finite.

Therefore, an algorithm for LCM can proceed by exploring a tree of prefixes of potential minimal witnesses, which has finite degree by (1) and finite height by (2), hence by König's Lemma is finite.

3.2.2. Length Function Theorems. A nondeterministic version of this search for a witness for LCM will see its complexity depend essentially on the height of the tree, i.e. on the length of bad sequences. Define the size of a configuration as its infinite norm $|(q, \vec{v})| = \max_{c \in \mathbb{C}} \vec{v}(c)$, and note that any σ in $\text{MinPre}(\sigma_i)$ is of size $|\sigma| \leq |\sigma_i| + 1$. This means that in any sequence $\tau = \sigma_0, \sigma_1, \dots$ where $\sigma_{i+1} \in \text{MinPre}(\sigma_i)$ for all i , $|\sigma_i| \leq |\tau| + i = g^i(|\tau|)$ the i th iterate of the successor function $g(x) = x + 1$. We call such a sequence *controlled* by g .

What a *length function theorem* provides is an upper bound on the length of controlled bad sequences over a wqo, depending on the control function g —here the successor function—and the maximal order type of the wqo—here $\omega^{|\mathbb{C}|} \cdot |Q|$. In our case, the theorems in [71, 72] provide an

$$F_{h,|\mathbb{C}|}^{|\mathbb{C}|}(|\tau|) \leq F_{h,\omega}(\max\{|\mathbb{C}|, |Q|, |\tau|\}) \stackrel{\text{def}}{=} \ell \quad (3.4)$$

upper bound on both this length and the maximal size of any configuration in the sequence, where

- $h: \mathbb{N} \rightarrow \mathbb{N}$ is an increasing polynomial function (which depends on g) and
- for any increasing $h: \mathbb{N} \rightarrow \mathbb{N}$, $(F_{h,\alpha})_\alpha$ is a *relativized* fast-growing hierarchy that uses h instead of the successor function as base function with index 0:

$$F_{h,0}(x) \stackrel{\text{def}}{=} h(x), \quad F_{h,\alpha+1}(x) \stackrel{\text{def}}{=} F_{h,\alpha}^{\omega(x)}(x), \quad F_{h,\lambda}(x) \stackrel{\text{def}}{=} F_{h,\lambda_x}(x). \quad (3.5)$$

3.2.3. A Combinatorial Algorithm. We have now established an upper bound on the length of a shortest minimal witness, entailing that if a witness exists, then it is of length bounded by ℓ defined in (3.4). This bound can be exploited by a nondeterministic *forward* algorithm, which

- (1) computes ℓ in a first phase: by Theorem 5.1, this can be performed in time $F_{h,\omega}(e(n))$ for some elementary function e ,
- (2) then nondeterministically explores the reachable configurations, starting from the initial configuration $(q_0, \vec{0})$ and attempting to reach the target configuration τ —but aborts if the upper bound on the length is reached. This second phase uses at most ℓ steps, and each step can be performed in time polynomial in the size of the current configuration, itself bounded by ℓ . The whole phase can thus be performed

in time polynomial in ℓ , which is bounded by $F_{h,\omega}(f(n))$ for some primitive-recursive f by Lemma 4.5.

Thus the overall complexity of this algorithm can be bounded by $F_{h,\omega}(p(n))$ where h and p are primitive-recursive. Because by Corollary 4.3 and (4.7), for any primitive-recursive strictly increasing h ,

$$\text{ACK} = \bigcup_{p \in \text{FPR}} \text{NTIME}(F_{h,\omega}(p(n))), \quad (3.6)$$

this means that LCM is in ACK.

3.2.4. Discussion. The oldest statement of ACK-completeness (under polynomial time Turing reductions) we are aware of is due to Clote [20] for FCP, the finite containment problem for Petri nets; see §6.1.1. As observed by Clote, his definition of ACK as $\text{DTIME}(F_\omega(n))$ is somewhat problematic, since the class is not robust under changes in the model of computation, for instance RAM vs. multitape Turing machines. A similar issue arises with the definition $\bigcup_{c < \omega} \text{DTIME}(F_\omega(n+c))$ employed in [40]: though robust under changes in the model of computation, it is not closed under reductions. Those classes are too tight to be convenient.

Conversely, stating that a problem is “in \mathcal{F}_ω^* but not in \mathcal{F}_k^* for any k ” [e.g. 31] is much less informative than stating that it is \mathbf{F}_ω -complete: $\mathcal{F}_\omega^* \setminus \mathcal{F}_{<\omega}^*$ is too large to allow for completeness statements, see Section 5.

4. ROBUSTNESS

In the applications of fast-growing classes we discussed in sections 3.1 and 3.2, we relied on both counts on their “robustness” to minor changes in their definition. More precisely, we employed space or time hierarchies indifferently, and alternative generating functions: first for the lower bound of SFEq and WS1S, when we used the tower function instead of F_3 in the reduction, and later for the upper bound of LCM, where we relied on a relativized version of F_ω . In this section, we prove these and other small changes to be innocuous.

4.1. Generating Functions. There are many variants for the definition of the fast-growing functions $(F_\alpha)_\alpha$, but they are all known to generate essentially the same hierarchy $(\mathcal{F}_\alpha)_\alpha$.² Nevertheless, because the fast-growing complexity classes \mathbf{F}_α we defined are smaller, there is no guarantee for these classical results to hold for them.

²See [69] and [56, pp. 48–51] for such results—and the works of Weiermann et al. on phase transitions for investigations of when changes do have an impact [e.g. 65].

4.1.1. *Ackermann Hierarchy.* We start here with one particular variant, which is rather common in the literature: define $A_\alpha: \mathbb{N} \rightarrow \mathbb{N}$ for $\alpha > 0$ by:

$$A_1(x) \stackrel{\text{def}}{=} 2x, \quad A_{\alpha+1}(x) \stackrel{\text{def}}{=} A_\alpha^x(1), \quad A_\lambda(x) \stackrel{\text{def}}{=} A_{\lambda(x)}(x). \quad (4.1)$$

The hierarchy differs in the treatment of successor indices, where the argument is reset to 1 instead of keeping x as in (2.3). This definition results for instance in $A_2(x) = 2^x$ and $A_3(x) = \text{tower}(x)$, and is typically used in lower bound proofs.

We can define a hierarchy of decision problems generated from the $(A_\alpha)_\alpha$ by analogy with (2.9):

$$\mathbf{A}_\alpha \stackrel{\text{def}}{=} \bigcup_{p \in \mathcal{F}_{<\alpha}} \text{DTIME}(A_\alpha(p(n))). \quad (4.2)$$

For two functions $g: \mathbb{N} \rightarrow \mathbb{N}$ and $h: \mathbb{N} \rightarrow \mathbb{N}$, let us write $g \leq h$ if $g(x) \leq h(x)$ for all x in \mathbb{N} . Because $A_\alpha \leq F_\alpha$ for all $\alpha > 0$, it follows that $\mathbf{A}_\alpha \subseteq \mathbf{F}_\alpha$. The converse inclusion also holds: in order to prove it, it suffices to exhibit for all $\alpha > 0$ a function p_α in $\mathcal{F}_{<\alpha}$ such that $F_\alpha \leq A_\alpha \circ p_\alpha$. It turns out that a uniform choice $p_\alpha(x) \stackrel{\text{def}}{=} 6x + 5$ fits those requirements—it is a linear function in \mathcal{F}_0 and $F_\alpha \leq A_\alpha \circ p_\alpha$ as shown in Lemma A.3—, thus:

Theorem 4.1. *For all $\alpha > 0$, $\mathbf{A}_\alpha = \mathbf{F}_\alpha$.*

4.1.2. *Relativized Hierarchies.* Another means for defining a variant of the fast-growing functions is to pick a different definition for F_0 : recall the relativized fast-growing functions employed in (3.5). The corresponding relativized complexity classes are then defined by

$$\mathbf{F}_{h,\alpha} \stackrel{\text{def}}{=} \bigcup_{p \in \mathcal{F}_{<\alpha}} \text{DTIME}(F_{h,\alpha}(p(n))). \quad (4.3)$$

It is easy to check that, if $g \leq h$, then $F_{g,\alpha} \leq F_{h,\alpha}$ for all α . Because we assumed h to be strictly increasing, this entails $F_\alpha \leq F_{h,\alpha}$, and we have the inclusion $\mathbf{F}_\alpha \subseteq \mathbf{F}_{h,\alpha}$ for all strictly increasing h .

The converse inclusion does not hold, since for instance $F_{h,1}$ is non-elementary for $h(x) = 2^x$. Observe however that, in this instance, $h \leq F_2$, and we can see that $F_{F_2,k} = F_{2+k}$ for all k in \mathbb{N} . This entails that $\mathbf{F}_{h,1} \subseteq \mathbf{F}_3$ for $h(x) = 2^x$. Thus, when working with relativized classes, one should somehow “offset” the ordinal index by an appropriate amount.

There is nevertheless a difficulty with relativized functions: it is rather straightforward to show that $F_{h,\alpha} \leq F_{\beta+\alpha}$ if $h \leq F_\beta$, *assuming* that the direct sum $\beta + \alpha$ does not “discard” any summand from the CNF of β ; e.g. $F_{F_1,k} = F_{k+1}$ and $F_{F_\omega,\omega} = F_\omega \cdot 2$. Observe however that $F_{F_1,\omega}(x) = F_{F_1,x+1}(x) = F_{x+2}(x) > F_{x+1}(x) = F_\omega(x)$. Thanks to the closure of \mathbf{F}_α under reductions in $\mathcal{F}_{<\alpha}$, this issue can be solved by composing with an appropriate function, e.g. $F_{F_1,\omega}(x) \leq F_\omega(x+1)$. This idea is formalized in Section A.4, and allows to show:

Theorem 4.2. *Let $h: \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function and α, β be two ordinals.*

- (i) *If $h \in \mathcal{F}_\beta$, then $\mathbf{F}_{h,\alpha} \subseteq \mathbf{F}_{\beta+1+\alpha}$.*
- (ii) *If $h \leq F_\beta$, then $\mathbf{F}_{h,\alpha} \subseteq \mathbf{F}_{\beta+\alpha}$.*

Proof. For (i), if h is in \mathcal{F}_β , then there exists x_h in \mathbb{N} such that, for all $x \geq x_h$, $h(x) \leq F_{\beta+1}(x)$ [56, Lemma 2.7]. By Lemma A.4, this entails that for all $x \geq x_h$, $F_{h,\alpha}(x) \leq F_{\beta+1+\alpha}(F_\gamma(x))$ for some $\gamma < \beta + 1 + \alpha$. Define the function f_h by $f_h(x) \stackrel{\text{def}}{=} x + x_h$; then for all x , $F_{h,\alpha}(x) \leq F_{h,\alpha}(f_h(x)) \leq F_{\beta+1+\alpha}(F_\gamma(f_h(x)))$. Observe that $F_\gamma \circ f_h$ is in $\mathcal{F}_{<\beta+1+\alpha}$, thus $\mathbf{F}_{h,\alpha} \subseteq \mathbf{F}_{\beta+1+\alpha}$.

For (ii), if $\beta + \alpha = 0$, then $\beta = \alpha = 0$, thus $h(x) = x + 1$ since it has to be strictly increasing, and $F_{h,0} = F_0$. Otherwise, Lemma A.4 shows that $F_{h,\alpha} \leq F_{\beta+\alpha} \circ F_\gamma$ for some $\gamma < \beta + \alpha$. Observe that F_γ is in $\mathcal{F}_{<\beta+\alpha}$, thus $\mathbf{F}_{h,\alpha} \subseteq \mathbf{F}_{\beta+\alpha}$. \square

The statement of Theorem 4.2 is somewhat technical, but easy to apply to concrete situations; for instance:

Corollary 4.3. *Let $h: \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing primitive recursive function and $\alpha \geq \omega$. Then $\mathbf{F}_{h,\alpha} = \mathbf{F}_\alpha$.*

Proof. The function h is in \mathcal{F}_k for some $k < \omega$, thus $\mathbf{F}_{h,\alpha} \subseteq \mathbf{F}_{k+1+\alpha} = \mathbf{F}_\alpha$ by Theorem 4.2. Conversely, since h is strictly increasing, $\mathbf{F}_\alpha \subseteq \mathbf{F}_{h,\alpha}$. \square

4.1.3. *Fundamental Sequences.* Our last example of minor variation is to change the assignment of fundamental sequences. Instead of the standard assignment of (2.1), we posit a monotone function $s: \mathbb{N} \rightarrow \mathbb{N}$ and consider the assignment

$$(\gamma + \omega^{\beta+1})(x)_s \stackrel{\text{def}}{=} \gamma + \omega^\beta \cdot s(x), \quad (\gamma + \omega^\lambda)(x)_s \stackrel{\text{def}}{=} \gamma + \omega^{\lambda(x)_s}. \quad (4.4)$$

Thus the standard assignment in (2.1) is obtained as the particular case $s(x) = x + 1$. As previously, this gives rise to new fast-growing functions

$$F_{0,s}(x) \stackrel{\text{def}}{=} x + 1, \quad F_{\alpha+1,s}(x) \stackrel{\text{def}}{=} F_{\alpha,s}^{s(x)}(x), \quad F_{\lambda,s}(x) \stackrel{\text{def}}{=} F_{\lambda(x)_s,s}(x) \quad (4.5)$$

and complexity classes

$$\mathbf{F}_{\alpha,s} \stackrel{\text{def}}{=} \bigcup_{p \in \mathcal{F}_{<\alpha}} \text{DTIME}(F_{\alpha,s}(p(n))). \quad (4.6)$$

We obtain similar results with non-standard fundamental sequences as with relativized hierarchies (thus also yielding a statement similar to that of Corollary 4.3):

Theorem 4.4. *Let $s: \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function and α, β be two ordinals.*

- (i) *If $s \in \mathcal{F}_\beta$, then $\mathbf{F}_{\alpha,s} \subseteq \mathbf{F}_{\beta+1+\alpha}$.*
- (ii) *If $s \leq F_\beta$, then $\mathbf{F}_{\alpha,s} \subseteq \mathbf{F}_{\beta+\alpha}$.*

Proof. By applying Theorem 4.2 alongside Lemma A.5. \square

4.2. Computational Models and Reductions. In order to be used together with reductions in $\mathcal{F}_{<\alpha}$, the classes \mathbf{F}_α need to be closed under such functions. The main technical lemma to this end states:

Lemma 4.5. *Let f and f' be two functions in $\mathcal{F}_{<\alpha}$. Then there exists p in $\mathcal{F}_{<\alpha}$ such that $f \circ F_\alpha \circ f' \leq F_\alpha \circ p$.*

Proof. By Corollary A.7, we know that there exists g in $\mathcal{F}_{<\alpha}$ such that $f \circ F_\alpha \leq F_\alpha \circ g$. We can thus define $p \stackrel{\text{def}}{=} g \circ f'$, which is also in $\mathcal{F}_{<\alpha}$ since the latter is closed under composition, to obtain the statement. \square

4.2.1. Computational Models. Note that because we assume $\alpha \geq 3$, $\mathcal{F}_{<\alpha}$ contains all the elementary functions, thus Lemma 4.5 also entails the robustness of the \mathbf{F}_α classes under changes in the model of computation—e.g. RAM vs. Turing machines vs. Minsky machines, deterministic or nondeterministic or alternating—or the type of resources under consideration—time or space; e.g.

$$\mathbf{F}_\alpha = \bigcup_{p \in \mathcal{F}_{<\alpha}} \text{NTIME}(F_\alpha(p(n))) = \bigcup_{p \in \mathcal{F}_{<\alpha}} \text{SPACE}(F_\alpha(p(n))). \quad (4.7)$$

4.2.2. Many-One Reductions. For a function $f: \mathbb{N} \rightarrow \mathbb{N}$ and two languages A and B , we say that A *many-one* reduces to B in time $f(n)$, written $A \leq_m^f B$, if there exists a Turing transducer T working in deterministic time $f(n)$ such that, for all x , x is in A if and only if $T(x)$ is in B . For a class of functions \mathcal{C} , we write $A \leq_m^{\mathcal{C}} B$ if there exists f in \mathcal{C} such that $A \leq_m^f B$. As could be expected given the definitions, each class \mathbf{F}_α is closed under many-one $\mathcal{F}_{<\alpha}$ reductions:

Theorem 4.6. *Let A and B be two languages. If $A \leq_m^{\mathcal{F}_{<\alpha}} B$ and $B \in \mathbf{F}_\alpha$, then $A \in \mathbf{F}_\alpha$.*

Proof. By definition, $A \leq_m^{\mathcal{F}_{<\alpha}} B$ means that there exists a Turing transducer T working in deterministic time $f(n)$ for some f in $\mathcal{F}_{<\alpha}$; note that this implies that the function implemented by T is also in $\mathcal{F}_{<\alpha}$ by (2.6). Furthermore, $B \in \mathbf{F}_\alpha$ entails the existence of a Turing machine M that accepts x if and only if x is in B and works in deterministic time $F_\alpha(p(n))$ for some p in $\mathcal{F}_{<\alpha}$. We construct $T(M)$ a Turing machine which, given an input x , first computes $T(x)$ by simulating T , and then simulates M on $T(x)$ to decide acceptance; $T(M)$ works in deterministic time $f(n) + F_\alpha(p(T(n)))$, which shows that A is in \mathbf{F}_α by Lemma 4.5. \square

4.2.3. Turing Reductions. We write similarly that $A \leq_T^f B$ if there exists a Turing machine for A working in deterministic time $f(n)$ with oracle calls to B , and $A \leq_T^{\mathcal{C}} B$ if there exists f in \mathcal{C} such that $A \leq_T^f B$. It turns out that Turing reductions in $\mathcal{F}_{<\alpha}$ can be used instead of many-one reductions:

Theorem 4.7. *Let $\alpha \geq 3$ and A and B be two languages. If $A \leq_T^{\mathcal{F}_{<\alpha}} B$ and $B \in \mathbf{F}_\alpha$, then $A \in \mathbf{F}_\alpha$.*

Proof. It is a folklore result on queries in recursion theory that, if $A \leq_T^f B$, then $A \leq_m^{2^f} B^{\text{tt}}$ where $2^f(n) \stackrel{\text{def}}{=} 2^{f(n)}$ and B^{tt} is the *truth table* version of the language B , which evaluates a Boolean combination of queries “ $x \in B$.” Indeed, we can easily simulate the oracle machine for A using a nondeterministic Turing transducer also in time $f(n)$ that guesses the answers of the B oracle and writes a conjunction of checks “ $x \in B$ ” or “ $x \notin B$ ” on the output, to be evaluated by a B^{tt} machine. This transducer can be determined by exploring both outcomes of the oracle calls, and handling them through disjunctions in the output; it now works in time $2^f(n)$.

Since $\alpha \geq 3$ and f is in $\mathcal{F}_{<\alpha}$, 2^f is also in $\mathcal{F}_{<\alpha}$. Furthermore, since B is in \mathbf{F}_α , B^{tt} is also in \mathbf{F}_α . The statement then holds by Theorem 4.6. \square

5. STRICTNESS

The purpose of this section is to establish the strictness of the $(\mathbf{F}_\alpha)_\alpha$ hierarchy (Section 5.2). As a first step, we prove that the F_α functions are “elementarily” constructible (Section 5.1), which is of independent interest for combinatorial algorithms in the line of that of §3.2.3. We end this section with a remark on the case $\alpha = 2$ (Section 5.3).

5.1. Elementary Constructivity. The functions F_α are known to be *honest*, i.e. to be computable in time \mathcal{F}_α [85, 27]. This is however not tight enough for their use in length function theorems, as in §3.2.3, where we want to compute their value in time elementary in F_α itself. We present the statement in the more general case of relativized fast-growing functions, defined in (3.5) and discussed in §4.1.2.

Theorem 5.1. *Let $h: \mathbb{N} \rightarrow \mathbb{N}$ be a time constructible strictly increasing function and α be an ordinal, then*

$$F_{h,\alpha} \in \bigcup_{e \in \mathbf{FELEM}} \text{FDTIME}(F_{h,\alpha}(e(n))).$$

Proof. Proposition A.10 shows that $F_{h,\alpha}$ can be computed in time $O(f(F_{h,\alpha}(n)))$ for the function $f(x) \stackrel{\text{def}}{=} x \cdot (G_{\omega^\alpha}(x) + x)$, where G_{ω^α} is an elementary function that takes the cost of manipulating (an encoding of) the ordinal indices into account. Lemma 4.5 then yields the result. \square

5.2. Strictness. Let us introduce yet another generalization of the $(\mathbf{F}_\alpha)_\alpha$ classes, which will allow for a characterization of the $(\mathcal{F}_\alpha^*)_ \alpha$ and $(\mathcal{F}_{<\alpha}^*)_ \alpha$ classes. For an ordinal α and a finite $c > 0$, define

$$\mathbf{F}_\alpha^c \stackrel{\text{def}}{=} \bigcup_{p \in \mathcal{F}_{<\alpha}} \text{DTIME}(F_\alpha^c(p(n))). \quad (5.1)$$

Thus \mathbf{F}_α as defined in (2.9) corresponds to the case $c = 1$.

Proposition 5.2. *For all $\alpha \geq 2$,*

$$\mathcal{F}_\alpha^* = \bigcup_c \mathbf{F}_\alpha^c .$$

Proof. The left-to-right inclusion is immediate by definition of \mathcal{F}_α^* in (2.8). The converse inclusion stems from the fact that if p is in \mathcal{F}_β for some $\beta < \alpha$, then there exists d such that $p \leq F_\alpha^d$ [56, Theorem 2.10], hence $F_\alpha^c \circ p \leq F_\alpha^{c+d}$ by monotonicity of F_α . \square

Let us prove the strictness of the $(\mathbf{F}_\alpha^c)_{c,\alpha}$ hierarchy. By Proposition 5.2 it will also prove that of $(\mathcal{F}_\alpha^*)_\alpha$ along the way (note that it is not implied by the strictness of $(\mathcal{F}_\alpha)_\alpha$, since it would be conceivable that none of the separating examples would be $\{0, 1\}$ -valued):

Theorem 5.3 (Strictness). *For all $c > 0$ and $2 \leq \beta < \alpha$,*

$$\mathbf{F}_\beta^c \subsetneq \mathbf{F}_\beta^{c+1} \subsetneq \mathbf{F}_\alpha .$$

Proof of $\mathbf{F}_\beta^{c+1} \subsetneq \mathbf{F}_\alpha$. Consider first a language L in \mathbf{F}_β^{c+1} , accepted by a Turing machine working in time $F_\beta^{c+1} \circ p$ for some p in $\mathcal{F}_{<\beta}$. Since $\beta < \alpha$ and $F_\beta^{c+1} \circ p$ is in \mathcal{F}_β , there exists n_0 such that, for all $n \geq n_0$, $F_\beta^{c+1}(p(n)) \leq F_\alpha(n)$, hence for all n , $F_\beta^{c+1}(p(n)) \leq F_\beta^{c+1}(p(n + n_0)) \leq F_\alpha(n + n_0)$ by monotonicity and expansivity of F_β . Observe that the function $n \mapsto n_0 + n$ is in $\mathcal{F}_0 \subseteq \mathcal{F}_{<\alpha}$, thus L also belongs to \mathbf{F}_α .

The strictness of the inclusion can be shown by a straightforward diagonalization argument. Define for this the language

$$L_\alpha \stackrel{\text{def}}{=} \{ \langle M \rangle \# x \mid M \text{ accepts } x \text{ in } F_\alpha(|x|) \text{ steps} \} \quad (5.2)$$

where $\langle M \rangle$ denotes a description of the Turing machine M and $\#$ is a separator. Then, by Theorem 5.1, L_α belongs to \mathbf{F}_α , thanks to a Turing machine that first computes F_α in time $F_\alpha \circ e$ for some elementary function e , and then simulates M in time elementary in $F_\alpha \circ e$. Assume now for the sake of contradiction that L_α belongs to \mathbf{F}_β^{c+1} , i.e. that there exists some c and some Turing machine K that accepts L_α in time F_β^{c+1} . Again, since $\beta < \alpha$ and $F_\beta^{c+1} \circ F_1$ is in \mathcal{F}_β , there exists n_0 such that, for all $n \geq n_0$, $F_\beta^{c+1}(2n + 1) \leq F_\alpha(n)$. We exhibit a new Turing machine N

- (1) that takes as input the description $\langle M \rangle$ of a Turing machine and simulates K on $\langle M \rangle \# \langle M \rangle$ but accepts if and only if K rejects, and
- (2) we ensure that a description $\langle N \rangle$ of N has size $n \geq n_0$.

Feeding this description $\langle N \rangle$ to N , it runs in time $F_\beta^{c+1}(2n + 1) \leq F_\alpha(n)$, and we obtain a contradiction whether it accepts or not:

- if N accepts, then K rejects $\langle N \rangle \# \langle N \rangle$ which is therefore not in L_α , thus N does not accept $\langle N \rangle$ in at most $F_\alpha(n)$ steps, which is absurd;
- if N rejects, then K accepts $\langle N \rangle \# \langle N \rangle$ which is therefore in L_α , thus N accepts $\langle N \rangle$ in at most $F_\alpha(n)$ steps, which is absurd. \square

Proof of $\mathbf{F}_\beta^c \subsetneq \mathbf{F}_\beta^{c+1}$. Similar to the previous proof; picking F_β^{c+1} as the time bound instead of F_α in (5.2) suffices to establish strictness. \square

By Proposition 5.2, a first consequence of Theorem 5.3 is that

$$\mathcal{F}_\beta^* \subsetneq \mathbf{F}_\alpha \quad (5.3)$$

for all $2 \leq \beta < \alpha$. Another consequence is that $(\mathbf{F}_\alpha)_\alpha$ “catches up” with $(\mathcal{F}_\alpha^*)_\alpha$ at every limit ordinal:

Corollary 5.4. *Let λ be a limit ordinal, then*

$$\mathcal{F}_{<\lambda}^* = \bigcup_{\beta < \lambda} \mathbf{F}_\beta \subsetneq \mathbf{F}_\lambda .$$

Proof. The equality $\mathcal{F}_{<\lambda}^* = \bigcup_{\beta < \lambda} \mathbf{F}_\beta$ and the inclusion $\mathcal{F}_{<\lambda} \subseteq \mathbf{F}_\lambda$ can be checked by considering a problem in some \mathcal{F}_β^* for $\beta < \lambda$: it is in \mathbf{F}_β^c for some $c > 0$ by Proposition 5.2, hence in $\mathbf{F}_{\beta+1}$ with $\beta + 1 < \lambda$ by Theorem 5.3, and therefore in \mathbf{F}_λ again by Theorem 5.3. Regarding the strictness of the inclusion, assume for the sake of contradiction $\mathbf{F}_\lambda \subseteq \mathcal{F}_{<\lambda}^*$: this would entail $\mathbf{F}_\lambda \subseteq \mathbf{F}_\beta$ for some $\beta < \lambda$, violating Theorem 5.3. \square

Corollary 5.4 yields yet another characterization of the primitive-recursive and multiply-recursive problems as

$$\text{PR} = \bigcup_k \mathbf{F}_k , \quad \text{MR} = \bigcup_k \mathbf{F}_{\omega^k} . \quad (5.4)$$

Note that strictness implies that there are no “ \mathcal{F}_α^* -complete” problems under $\mathcal{F}_{<\alpha}$ reductions, since by Proposition 5.2 such a problem would necessarily belong to some \mathbf{F}_α^c level, which would in turn entail the collapse of the $(\mathbf{F}_\alpha^c)_c$ hierarchy at the \mathbf{F}_α^c level and contradict Theorem 5.3.

Similarly, fix a limit ordinal λ and some reduction class \mathcal{F}_α for some $\alpha < \lambda$: there cannot be any meaningful “ $\mathcal{F}_{<\lambda}^*$ -complete” problem under \mathcal{F}_α reductions, since such a problem would be in \mathcal{F}_β^* for some $\alpha < \beta < \lambda$, hence contradicting the strictness of the $(\mathcal{F}_\beta^*)_{\beta < \alpha}$ hierarchy; in particular, there are no “PR-complete” nor “MR-complete” problems.

5.3. The Case $\alpha = 2$ is a bit particular. We did not consider it in the rest of the paper (nor the other cases for $\alpha < 2$) because it does not share the usual characteristics of the $(\mathbf{F}_\alpha)_\alpha$: for instance, the model of computation and the kind of resources become important, as

$$\mathbf{F}_2 \stackrel{\text{def}}{=} \bigcup_{p \in \mathcal{F}_1} \text{DTIME}(F_2(p(n))) \quad (5.5)$$

would a priori be different if we were to define it through NTIME or DSPACE computations; the following results are artifacts of one particular definition choice.

5.3.1. *Recursion Schemes.* In order to define \mathbf{F}_2 fully we need the original definition of the extended Grzegorzcyk hierarchy $(\mathcal{F}_\alpha)_\alpha$ by Löb and Wainer [56]—the characterization in (2.5) is only correct for $\alpha \geq 2$. This definition is based on the closure of a set of initial functions under the operations of *substitution* and *limited primitive recursion*. More precisely, the set of initial functions at level α comprises the constant *zero function* 0 , the *sum function* $+: x_1, x_2 \mapsto x_1 + x_2$, the *projections* $\pi_i^n: x_1, \dots, x_n \mapsto x_i$ for all $0 < i \leq n$, and the fast-growing function F_α . New functions are added to form the class \mathcal{F}_α through two operations:

substitution: if h_0, h_1, \dots, h_p belong to the class, then so does f if

$$f(x_1, \dots, x_n) = h_0(h_1(x_1, \dots, x_n), \dots, h_p(x_1, \dots, x_n)),$$

limited primitive recursion: if h_0, h_1 , and g belong to the class, then so does f if

$$\begin{aligned} f(0, x_1, \dots, x_n) &= h_0(x_1, \dots, x_n), \\ f(y + 1, x_1, \dots, x_n) &= h_1(y, x_1, \dots, x_n, f(y, x_1, \dots, x_n)), \\ f(y, x_1, \dots, x_n) &\leq g(\max\{y, x_1, \dots, x_n\}). \end{aligned}$$

Observe that primitive recursion is defined by ignoring the last *limitedness* condition in the previous definition. See Clote [21] for an survey of the relationships between machine-defined and recursion-defined complexity classes.

5.3.2. *Linear Exponential Time.* Let us focus for now on \mathcal{F}_1 , which is the class of reductions used in \mathbf{F}_2 . Call a function f *linear* if there exists a constant c such that $f(x_1, \dots, x_n) \leq c \cdot \max_i x_i$ for all x_1, \dots, x_n . Observe that, for all c , the function $f_c(x) \stackrel{\text{def}}{=} c \cdot x$ is in \mathcal{F}_1 since $f_c(0) = 0$, $f_c(x + 1) = s^c(0) + f_c(x)$, and $f_c(x) \leq F_1^c(x)$; thus any linear function is bounded above by a function in \mathcal{F}_1 . Conversely, if f is in \mathcal{F}_1 , then it is linear: this is true of the initial functions, and preserved by the two operations of substitution and limited primitive recursion.³

This entails that \mathbf{F}_2 matches a well-known complexity class, since furthermore $F_2(n) = 2^{n+1+\log(n+1)} - 1$ is in $2^{O(n)}$: \mathbf{F}_2 is the *weak* (aka *linear*) exponential-time complexity class:

$$\mathbf{F}_2 = \mathbf{E} \stackrel{\text{def}}{=} \text{DTIME}(2^{O(n)}). \quad (5.6)$$

6. A SHORT CATALOG

Our introduction of the fast-growing complexity classes is motivated by *already known* decidability problems, arising for instance in logic, verification, or database theory, for which no precise classification could be provided in the existing hierarchies. By listing some of these problems, we hope to initiate the exploration of this mostly uncharted area of complexity, and to foster the use of reductions from known problems, rather than proofs from

³Thus $\mathcal{F}_1 \subsetneq \mathcal{E}^2$: the latter additionally contains the function $x, y \mapsto (x + 1) \cdot (y + 1)$ as an initial function, and is equal to FLINSPACE [68, 21, Theorem 3.36].

Turing machines. The following catalog of complete problems does not attempt to be exhaustive; Friedman [36] for instance presents many problems “of enormous complexity.”

Because examples for TOWER are well-known and abound in the literature, starting with a 1975 survey by Meyer [61],⁴ we rather focus on the non primitive-recursive levels, i.e. the \mathbf{F}_α for $\alpha \geq \omega$. Interestingly, all these examples rely for their upper bound on the existence of some well-quasi-ordering (of *maximal order type* ω^α , see [22]), and on a matching length function theorem.

6.1. \mathbf{F}_ω -Complete Problems. We gather here some decision problems that can be proven decidable in \mathbf{F}_ω thanks to Dickson’s Lemma over \mathbb{N}^d for some d and to the combinatorial analyses of McAloon [59], Clote [20], Figueira et al. [31], Abriola et al. [6]. We therefore focus on the references for lower bounds.

6.1.1. *Vector Addition Systems* (VAS, and equivalently Petri nets), provided the first known Ackermannian decision problem: FCP.

A d -dimensional VAS is a pair $\langle \vec{x}_0, \vec{A} \rangle$ where \vec{x}_0 is an initial configuration in \mathbb{N}^d and \vec{A} is a finite set of transitions in \mathbb{Z}^d . A transition \vec{a} in \vec{A} can be applied to a configuration \vec{x} in \mathbb{N}^d if $\vec{x}' = \vec{x} + \vec{a}$ is in \mathbb{N}^d ; the resulting configuration is then \vec{x}' . The complexity of decision problems for VAS usually varies from EXPSPACE-complete [55, 67, 11] to \mathbf{F}_ω -complete [58, 43] to undecidable [39, 42], via a key problem, which is decidable but of unknown complexity: VAS Reachability [57, 47, 50, 54].

Finite Containment Problem (FCP)

instance: Two VAS \mathcal{V}_1 and \mathcal{V}_2 known to have finite sets $\text{Reach}(\mathcal{V}_1)$ and $\text{Reach}(\mathcal{V}_2)$ of reachable configurations.

question: Is $\text{Reach}(\mathcal{V}_1)$ included in $\text{Reach}(\mathcal{V}_2)$?

reference: Mayr and Meyer [58], from an F_ω -bounded version of Hilbert’s Tenth Problem. A simpler reduction is given by Jančar [43] from the halting problem of F_ω -bounded Minsky machines.

comment: Testing whether the set of reachable configurations of a VAS is finite is EXPSPACE-complete [55, 67]. FCP provided the initial motivation for the work of McAloon [59] and Clote [20]. FCP has been generalized by Jančar [43] to a large range of behavioral relations between two VASs. Without the finiteness condition, these questions are undecidable [39, 42, 43].

6.1.2. *Unreliable Counter Machines.* A *lossy counter machine* (LCM) is syntactically a Minsky machine, but its operational semantics are different: its counter values can decrease nondeterministically at any moment during execution.

⁴Of course Meyer does not explicitly state TOWER-completeness, but it follows immediately from the lower and upper bounds he provides.

Lossy Counter Machines Reachability (LCM)

instance: A lossy counter machine M and a configuration σ .

question: Is σ reachable in M with lossy semantics?

reference: Schnoebelen [75], by a direct reduction from F_ω -bounded Minsky machines. The first proofs were given independently by Urquhart [83] and Schnoebelen [74].

comment: Hardness also holds for terminating LCMs, for coverability in Reset or Transfer Petri nets, and for reachability in counter machines with incrementing errors.

6.1.3. *Relevance Logics* provide different semantics of implication, where a fact B is said to follow from A , written “ $A \rightarrow B$ ”, only if A is actually *relevant* in the deduction of B . This excludes for instance $A \rightarrow (B \rightarrow A)$, $(A \wedge \neg A) \rightarrow B$, etc.—see Dunn and Restall [24] for more details. Although the full logic \mathbf{R} is undecidable [82], its conjunctive-implicative fragment $\mathbf{R}_{\rightarrow, \wedge}$ is decidable, and Ackermannian:

Conjunctive Relevant Implication (CRI)

instance: A formula A of $\mathbf{R}_{\rightarrow, \wedge}$.

question: Is A a theorem of $\mathbf{R}_{\rightarrow, \wedge}$?

reference: Urquhart [83], from a variant of LCM: the emptiness problem of *alternating expansive counter systems*, for which he proved \mathbf{F}_ω -hardness directly from the halting problem in F_ω -bounded Minsky machines.

comment: Hardness also holds for $\mathbf{LR}+$ and any intermediate logic between $\mathbf{R}_{\rightarrow, \wedge}$ and $\mathbf{T}_{\rightarrow, \wedge}$ —which might include some undecidable fragments.

6.1.4. *Data Logics & Register Automata* are concerned with structures like words or trees with an additional equivalence relation over the positions. The motivation for this stems in particular from XML processing, where the equivalence stands for elements sharing the same *datum* from some infinite data domain \mathbb{D} . Enormous complexities often arise in this context, both for automata models (register automata and their variants, when extended with alternation or histories) and for logics (which include logics with *freeze* operators and XPath fragments)—the two views being tightly interconnected.

Emptiness of Alternating 1-Register Automata (ARA)

instance: An ARA \mathcal{A} .

question: Is $L(\mathcal{A})$ empty?

reference: Demri and Lazić [23], from reachability in incrementing counter machines LCM.

comment: There exist many variants of the ARA model, and hardness also holds for the corresponding data logics [e.g. 44, 23, 30, 80, 29, 81]. See ATA for the case of linearly ordered data.

6.1.5. *Interval Temporal Logics* provide a formal framework for reasoning about temporal intervals. Halpern and Shoham [41] define a logic with modalities expressing the basic relationships that can hold between two temporal intervals, $\langle B \rangle$ for “begun by”, $\langle E \rangle$ for “ended by”, and their inverses $\langle \bar{B} \rangle$ and $\langle \bar{E} \rangle$. This logic, and even small fragments of it, has an undecidable satisfiability problem, thus prompting the search for decidable restrictions and variants. Montanari et al. [63] show that the logic with relations $A\bar{A}B\bar{B}$ —where $\langle A \rangle$ expresses that the two intervals “meet”, i.e. share an endpoint—, has an \mathbf{F}_ω -complete satisfiability problem over finite linear orders:

Finite Linear Satisfiability of $A\bar{A}B\bar{B}$ Interval Temporal Logic (ITL)

instance: An $A\bar{A}B\bar{B}$ formula φ .

question: Does there exist an interval structure \mathcal{S} over some finite linear order and an interval I of \mathcal{S} s.t. $\mathcal{S}, I \models \varphi$?

reference: Montanari et al. [63], from LCM.

comment: Hardness already holds for the fragments $\bar{A}B$ and $\bar{A}\bar{B}$ [14].

6.2. **$\mathbf{F}_{\omega\omega}$ -Complete Problems.** The following problems have been proven decidable thanks to Higman’s Lemma over some finite alphabet. All the complexity upper bounds in $\mathbf{F}_{\omega\omega}$ stem from the constructive proofs of Weiermann [86], Cichoń and Tahhan Bittar [19], Schmitz and Schnoebelen [71]. Again, we point to the relevant references for lower bounds.

6.2.1. *Lossy Channel Systems* (LCS) are finite labeled transition systems $\langle Q, M, \delta, q_0 \rangle$ where transitions in $\delta \subseteq Q \times \{?, !\} \times M \times Q$ read and write on an unbounded channel. This would lead to a Turing-complete model of computation, but the operational semantics of LCS are “lossy”: the channel loses symbols in an uncontrolled manner. Formally, the configurations of an LCS are pairs (q, x) , where q in Q holds the current state and x in M^* holds the current contents of the channel. A read $(q, ?m, q')$ in δ updates this configuration into (q, x') if there exists some x'' s.t. $x' \leq_* x''$ and $mx'' \leq_* x$ —where \leq_* denotes subword embedding—, while a write transition $(q, !m, q')$ updates it into (q', x') with $x' \leq_* xm$; the initial configuration is (q_0, ε) , with empty initial channel contents.

Due to the unboundedness of the channel, there might be infinitely many configurations reachable through transitions. Nonetheless, many problems are decidable [2, 15] using Higman’s Lemma and what would later become known as the theory of *well-structured transition systems* (WSTS) [33, 4, 34]. LCS are also the primary source of problems hard for $\mathbf{F}_{\omega\omega}$:

LCS Reachability (LCS)

instance: A LCS and a configuration (q, x) in $Q \times M^*$.

question: Is (q, x) reachable from the initial configuration?

reference: Chambart and Schnoebelen [18], by a direct reduction from $\mathbf{F}_{\omega\omega}$ -bounded Minsky machines.

comment: Hardness already holds for terminating systems, and for reachability in *faulty channel systems*, where symbols are nondeterministically inserted in the channel at arbitrary positions instead of being lost. The bounds are refined and parametrized in function of the size of the alphabet M in [45].

There are many interesting applications of this question; let us mention one in particular: Atig et al. [8] show how concurrent finite programs communicating through *weak* shared memory—i.e. prone to reorderings of read or writes, modeling the actual behavior of microprocessors, their instruction pipelines and cache levels—have an \mathbf{F}_{ω} -complete control-state reachability problem, through reductions to and from LCS.

LCS Termination (LCST)

instance: A LCS.

question: Is every sequence of transitions from the initial configuration finite?

reference: Chambart and Schnoebelen [18], by an easy reduction from terminating instances of LCS.

comment: Unlike Reachability, Termination is sensible to switching from lossy semantics to faulty semantics: it becomes NL-complete in general [15], TOWER-complete when the channel system is equipped with *channel tests* [13], and ACK-complete when one asks for *fair* non-termination, where the channel contents are read infinitely often [53].

6.2.2. *Embedding Problems* have been introduced by Chambart and Schnoebelen [16], motivated by decidability problems in various classes of channel systems mixing lossy and reliable channels. These problems are centered on the subword embedding relation \leq_* and called **Post Embedding Problems**. There is a wealth of variants and applications, see e.g. [17, 46, 45].

We give here a slightly different viewpoint, taken from [9, 45], that uses regular relations (i.e. definable by synchronous finite transducers) and rational relations (i.e. definable by finite transducers):

Rational Embedding Problem (RatEP)

instance: A rational relation R included in $(\Sigma^*)^2$.

question: Is $R \cap \leq_*$ non empty?

reference: Chambart and Schnoebelen [16], from LCS.

comment: Chambart and Schnoebelen [16] call this problem the **Regular Post Embedding Problem**, but the name is misleading due to GEP. An equivalent presentation uses a rational language L included in Σ^* and two homomorphisms $u, v: \Sigma^* \rightarrow \Sigma^*$, and asks whether there exists w in L s.t. $u(w) \leq_* v(w)$. The bounds are refined and parametrized in function of the size of the alphabet Σ in [45].

Generalized Embedding Problem (GEP)

instance: A regular relation R included in $(\Sigma^*)^m$ and a subset I of $\{1, \dots, m\}^2$.

question: Does there exist (w_1, \dots, w_m) in R s.t. for all (i, j) in I , $w_i \leq_* w_j$?

reference: Barceló et al. [9], from RatEP.

comment: The Regular Embedding Problem (RegEP) corresponds to the case where $m = 2$ and $I = \{(1, 2)\}$, and is already $\mathbf{F}_{\omega^\omega}$ -hard; see [45] for refined bounds. Barceló et al. [9] use GEP to show the $\mathbf{F}_{\omega^\omega}$ -completeness of querying graph databases using particular extended conjunctive regular path queries.

6.2.3. *Metric Temporal Logic & Timed Automata* allow to reason on *timed words* over $\Sigma \times \mathbb{R}$, where Σ is a finite alphabet and the real values are non-decreasing *timestamps* on events. A *timed automaton* [NTA, 7] is a finite automaton extended with *clocks* that evolve synchronously through time, and can be reset and compared against some time interval by the transitions of the automaton. The model can be extended with alternation, and is then called an ATA.

Metric temporal logic [MTL, 48] is an extension of linear temporal logic where temporal modalities are decorated with real intervals constraining satisfaction; for instance, a timed word w satisfies the formula $F_{[3, \infty)}\varphi$ at position i , written $w, i \models F_{[3, \infty)}\varphi$, only if φ holds at some position $j > i$ of w with timestamp $\tau_j - \tau_i \geq 3$. Satisfiability problems for MTL reduce to emptiness problems for timed automata.

Ouaknine and Worrell [66] and Lasota and Walukiewicz [51] prove using WSTS techniques that, in the case of a single clock, emptiness of ATAs is decidable.

Emptiness of Alternating 1-Clock Timed Automata (ATA)

instance: An ATA \mathcal{A} .

question: Is $L(\mathcal{A})$ empty?

reference: Lasota and Walukiewicz [51], from faulty channel systems LCS.

comment: Hardness already holds for universality of nondeterministic 1-clock timed automata.

Finite Satisfiability of Metric Temporal Logic (fMTL)

instance: An MTL formula φ .

question: Does there exist a finite timed word w s.t. $w, 0 \models \varphi$?

reference: Ouaknine and Worrell [66], from faulty channel systems LCS.

comment: The related problem of satisfiability for the *safety* fragment of MTL is ACK-complete [53].

Note that recent work on data automata over linearly ordered domains has uncovered some strong ties with timed automata [32, 29].

6.3. **$\mathbf{F}_{\omega^\omega}$ -Complete Problems.** Currently, the known $\mathbf{F}_{\omega^\omega}$ -complete problems are all related to extensions of Petri nets called *enriched nets*, which include timed-arc Petri nets [3], data nets and Petri data nets [52], and constrained multiset rewriting systems [1]. Reductions between the different classes of enriched nets can be found in [5, 12]. Defining these families of nets here would take too much space; see the references for details. These

models share one characteristic: they define well-structured transition systems over finite sequences of multisets of natural numbers, which have an $\omega^{\omega^{\omega}}$ maximal order type.

Enriched Net Coverability (ENC)

instance: An enriched net \mathcal{N} and a place p of the net.

question: Is there a reachable marking with a least one token in p ?

reference: Haddad et al. [40], by a direct reduction from the halting problem in $F_{\omega^{\omega}}$ -bounded Minsky machines.

comment: Hardness already holds for bounded, terminating nets.

6.4. **F_{ε_0} -Complete Problems** have only recently been investigated in [38], with the definition of *priority channel systems* (PCS). Those are defined similarly to lossy channel systems (c.f. §6.2.1), but the message alphabet M is linearly ordered to represent message *priorities*. Rather than message losses, the unreliable behaviors are now *message supersedings*, i.e. applications of the rewrite rules $ab \rightarrow b$ for $b \geq a$ in M on the channel contents.

PCS Reachability (PCS)

instance: A PCS and a configuration (q, x) in $Q \times M^*$.

question: Is (q, x) reachable from the initial configuration?

reference: Haase et al. [38], by a direct reduction from the halting problem in F_{ε_0} -bounded Turing machines.

comment: Hardness already holds for terminating PCSs.

7. CONCLUDING REMARKS

The classical complexity hierarchies are limited to elementary problems, in spite of a growing number of natural problems that require much larger computational resources. We propose in this paper a definition for fast-growing complexity classes $(\mathbf{F}_\alpha)_\alpha$, which provide accurate enough notations for many non-elementary decision problems: they allow to express some important landmarks, like TOWER = \mathbf{F}_3 , ACK = \mathbf{F}_ω , or HACK = $\mathbf{F}_{\omega^\omega}$, and are close enough to the extended Grzegorzcyck hierarchy so that complexity statements in terms of \mathcal{F}_α can often be refined as statements in terms of \mathbf{F}_α . These definitions allow to employ the familiar vocabulary of complexity theory, reductions and completeness, instead of the more ad-hoc notions used this far. This will hopefully foster the reuse of “canonical problems” in establishing high complexity results, rather than proofs from first principles, i.e. resource-bounded Turing machines.

A pattern emerges in the list of known \mathbf{F}_α -complete problems, allowing to answer a natural concern already expressed by Clote [20]: “what do complexity classes for such rapidly growing functions really mean?” Indeed, beyond the intellectual satisfaction one might find in establishing a problem as complete for some class, being \mathbf{F}_α -complete brings additional information on the problem itself: that it relies in some essential way on the ordinal ω^α being well-ordered. All the problems in Section 6 match this pattern, as

their decision algorithms rely on well-quasi-orders with maximal order type ω^α for their termination, for which length function theorems then allow to derive \mathbf{F}_α bounds.

Finally, we remark that there are currently no known natural problem of “intermediate” complexity, for instance between ELEM and ACK, or between the latter and HACK. Parametric versions of LCM or LCS seem like good candidates for this, but so far the best lower and upper bounds do not quite match [see e.g. 45]. It would be interesting to find examples that exercise the intermediate levels of the $(\mathbf{F}_\alpha)_\alpha$ hierarchy.

APPENDIX A. SUBRECURSIVE HIERARCHIES

This section presents the technical background and proofs missing from the main text.

A.1. Hardy Functions. Let $h: \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function. The *Hardy functions* $(h^\alpha)_{\alpha \in \varepsilon_0}$ controlled by h are defined inductively by

$$h^0(x) \stackrel{\text{def}}{=} x, \quad h^{\alpha+1}(x) \stackrel{\text{def}}{=} h^\alpha(h(x)), \quad h^\lambda(x) \stackrel{\text{def}}{=} h^{\lambda(x)}(x). \quad (\text{A.1})$$

A definition related to fundamental sequences is that of the *predecessor* at x of an ordinal greater than 0, which recursively considers the x th element in the fundamental sequence of limit ordinals, until a successor ordinal is found:

$$P_x(\alpha + 1) \stackrel{\text{def}}{=} \alpha, \quad P_x(\lambda) \stackrel{\text{def}}{=} P_x(\lambda(x)). \quad (\text{A.2})$$

Using predecessors, the definition of the Hardy functions becomes even simpler: for $\alpha > 0$,

$$h^\alpha(x) \stackrel{\text{def}}{=} h^{P_x(\alpha)}(h(x)). \quad (\text{A.3})$$

Observe for instance that $h^k(x)$ for some finite k is the k th iterate of h . This intuition carries over: h^α is a transfinite iteration of the function h , using diagonalization to handle limit ordinals. The usual Hardy functions H^α from are then obtained by fixing $H(x) \stackrel{\text{def}}{=} x + 1$.

The Hardy functions enjoy a number of properties; see [26, 19]. They are *expansive*, and *monotonic* with respect to both the base function h and to the argument x : for all $g \leq h$, $x \leq y$, and α ,

$$x \leq h^\alpha(x), \quad g^\alpha(x) \leq h^\alpha(x), \quad h^\alpha(x) \leq h^\alpha(y). \quad (\text{A.4})$$

As often with subrecursive functions, what the Hardy functions lack is monotonicity in the ordinal index, see Section A.2.

By transfinite induction on ordinals, we also find several identities:

$$h^{\omega \cdot c} = F_{h,\alpha}^c, \quad (\text{A.5})$$

$$h^{\alpha+\beta} = h^\alpha \circ h^\beta. \quad (\text{A.6})$$

Note that (A.5) entails the expansiveness and monotonicity of the fast-growing functions.

Equation (A.6) is extremely valuable: it shows that—up to some extent—the *composition* of Hardy functions can be internalized in the ordinal index. Here we run however into a limitation of considering “set-theoretic” ordinal indices: informally, (A.6) is implicitly restricted to ordinals $\alpha + \beta$ “in CNF”. Formally, it requires $\alpha + \beta = \alpha \oplus \beta$, where ‘ \oplus ’ denotes the natural sum operation. For instance, it fails in $H^1(H^\omega(x)) = H^1(H^x(x+1)) = 2x+2 > 2x+1 = H^\omega(x)$, although $1 + \omega = \omega$. We will discuss this point further in Section A.6.

A.2. Monotonicity. One of the issues of most subrecursive hierarchies of functions is that they are not monotone in the ordinal index: $\beta < \alpha$ does not necessarily imply $H^\beta \leq H^\alpha$; for instance, $H^{x+2}(x) = 2x+2 > 2x+1 = H^\omega(x)$. What is however true is that they are *eventually* monotone: if $\beta < \alpha$, then there exists n_0 such that, for all $x \geq n_0$, $H^\beta(x) \leq H^\alpha(x)$. This result (and others) can be proven using a *pointwise ordering*: for all x , define the \prec_x relation as the transitive closure of

$$\alpha \prec_x \alpha + 1, \quad \lambda(x) \prec_x \lambda. \quad (\text{A.7})$$

The relation “ $\beta \prec_x \alpha$ ” is also noted “ $\beta \in \alpha[x]$ ” in [77, pp. 158–163], where the results of this section are proven.

The \prec_x relations form a strict hierarchy of refinements of the ordinal ordering $<$:

$$\prec_0 \subsetneq \prec_1 \subsetneq \cdots \subsetneq \prec_x \subsetneq \cdots \subsetneq <. \quad (\text{A.8})$$

We are going to use two main properties of the pointwise ordering:

$$x < y \quad \text{implies} \quad \lambda(x) \prec_y \lambda(y), \quad (\text{A.9})$$

$$\beta \prec_x \alpha \quad \text{implies} \quad H^\beta(x) \leq H^\alpha(x). \quad (\text{A.10})$$

For a first application, define the *norm* of an ordinal term as the maximal coefficient that appears in its normal form: if $\alpha = \omega^{\alpha_1} \cdot c_1 + \cdots + \omega^{\alpha_m} \cdot c_m$ with $\alpha_1 > \cdots > \alpha_m$ and $c_1, \dots, c_m > 0$, then $N\alpha \stackrel{\text{def}}{=} \max\{c_1, \dots, c_m, N\alpha_1, \dots, N\alpha_m\}$. Then $\beta < \alpha$ implies $\beta \prec_{N\beta} \alpha$ [77, p. 158]. Together with (A.10), this entails that, for all $x \geq N\beta$, $H^\beta(x) \leq H^\alpha(x)$.

A.3. Ackermann Functions. We prove in this section some basic properties of the Ackermann hierarchy of functions $(A_\alpha)_\alpha$ defined in §4.1.1. Its definition is less uniform than the fast-growing and Hardy functions, leading to slightly more involved proofs.

Lemma A.1. *For all $\alpha > 0$, $A_\alpha(0) \leq 1$.*

Proof. By transfinite induction over α . For $\alpha = 1$, $A_1(0) = 0 \leq 1$. For a successor ordinal $\alpha + 1$, $A_{\alpha+1}(0) = 1$. For a limit ordinal λ , $A_\lambda(0) = A_{\lambda(0)}(0) \leq 1$ by ind. hyp. \square

As usual with subrecursive hierarchies, the main issue with the Ackermann functions is to prove various monotonicity properties in the argument and in the index.

Lemma A.2. For all $\alpha, \beta > 0$ and x, y :

- (i) if $\alpha > 1$, A_α is strictly expansive: $A_\alpha(x) > x$,
- (ii) A_α is strictly monotone in its argument: if $y > x$, $A_\alpha(y) > A_\alpha(x)$,
- (iii) $(A_\alpha)_\alpha$ is pointwise monotone in its index: if $\alpha \succ_x \beta$, $A_\alpha(x) \geq A_\beta(x)$.

Proof. Let us first consider the case $\alpha = 1$: A_1 is strictly monotone, proving (ii). Regarding (i) for $\alpha = 2$, $A_2(x) = 2^x > x$ for all x .

We prove now the three statements by simultaneous transfinite induction over α . Assume they hold for all $\beta < \alpha$ (and thus for all $\beta \prec_x \alpha$ for all x).

For (i),

- if α is a successor ordinal $\beta + 1$, then $A_{\beta+1}(x) \geq A_\beta(x) > x$ by ind. hyp. (iii) and (i) on $\beta \prec_x \alpha$.
- If α is a limit ordinal λ , then $A_\lambda(x) = A_{\lambda(x)}(x) > x$ by ind. hyp. (i) on $\lambda(x) \prec_x \alpha$.

For (ii), it suffices to prove the result for $y = x + 1$.

- If α is a successor ordinal $\beta + 1$, then $A_\alpha(x+1) = A_\beta(A_\alpha(x)) > A_\alpha(x)$ by ind. hyp. (i) on $\beta \prec_x \alpha$.
- If α is a limit ordinal λ , then $A_\lambda(x+1) = A_{\lambda(x+1)}(x+1) \geq A_{\lambda(x)}(x+1)$ by ind. hyp. (iii) on $\lambda(x) \prec_{x+1} \lambda(x+1)$ (recall Equation A.9), hence the result by ind. hyp. (ii) on $\lambda_x \prec_x \alpha$.

For (iii), it suffices to prove the result for $\alpha = \beta + 1$ and $\beta = \alpha(x)$ and rely on transitivity.

- If $\alpha = \beta + 1$, then we show (iii) by induction over x : the base case $x = 0$ stems from $A_\alpha(0) = A_\beta^0(1) = 1 \geq A_\beta(0)$ by Lemma A.1; the induction step $x + 1$ stems from $A_\alpha(x+1) = A_\beta(A_\alpha(x)) \geq A_\beta(x+1)$ using the ind. hyp. on x and (ii) on $\beta \prec_{A_\alpha(x)} \alpha$.
- If $\beta = \alpha(x)$, then $A_\alpha(x) = A_\beta(x)$ by definition. \square

Our main interest in the Ackermann functions is their relation with the fast-growing ones:

Lemma A.3. For all $\alpha > 0$ and all x , $A_\alpha(x) \leq F_\alpha(x) \leq A_\alpha(6x + 5)$.

Proof. We only prove the second inequality, as the first one can be deduced from the various monotonicity properties of F_α and A_α . The case $x = 0$ is settled for all $\alpha > 0$ by checking that $F_\alpha(0) = 1 \leq 10 = A_1(5) \leq A_\alpha(5)$, since $1 \preceq_x \alpha$ for all $\alpha > 0$ and we can therefore apply Lemma A.2.(iii). Assume now $x > 0$; we prove the statement by transfinite induction over $\alpha > 0$.

- For the base case $\alpha = 1$, $F_1(x) = 2x + 1 \leq 12x + 10 = A_1(6x + 5)$.
- For the successor case $\alpha + 1$, $A_{\alpha+1}(6x + 5) = A_\alpha^{5(x+1)}(A_\alpha^x(1)) \geq A_\alpha^{5(x+1)}(x)$ by Lemma A.2.

We show by induction over j that $A_\alpha^{5j}(x) \geq F_\alpha^j(x)$. This holds for the base case $j = 0$, and for the induction step, $A_\alpha^5(A_\alpha^{5j}(x)) \geq A_\alpha^5(F_\alpha^j(x))$ by ind. hyp. on j and Lemma A.2.(ii). Furthermore, for

all $y > 0$, $A_\alpha(A_\alpha^4(y)) \geq A_\alpha(A_1^4(y)) = A_\alpha(16y) \geq A_\alpha(6y+5) \geq F_\alpha(y)$
 by ind. hyp. on α , which shows that $A_\alpha^5(F_\alpha^j(x)) \geq F_\alpha^{j+1}(x)$ when
 choosing $y = F_\alpha^j(x) > 0$. Then $A_\alpha^{5(x+1)}(x) \geq F_\alpha^{x+1}(x) = F_{\alpha+1}(x)$,
 thus completing the proof in the successor case.

- For the limit case λ , $A_\lambda(6x+5) = A_{\lambda(6x+5)}(6x+5) \geq A_{\lambda(x)}(6x+5) \geq F_{\lambda(x)}(x) = F_\lambda(x)$, using successively Lemma A.2.(iii) on $\lambda(x) \prec_{6x+5} \lambda(6x+5)$ and the ind. hyp. on $\lambda(x) < \lambda$. \square

A.4. Relativized Functions. We prove here the missing lemma from the proof of Theorem 4.2:

Lemma A.4. *Let $h: \mathbb{N} \rightarrow \mathbb{N}$ be a function α, β be two ordinals, and x_0 be a natural number. If for all $x \geq x_0$, $h(x) \leq F_\beta(x)$, then there exists an ordinal γ such that*

- (i) *for all $x \geq x_0$, $F_{h,\alpha}(x) \leq F_{\beta+\alpha}(F_\gamma(x))$, and*
- (ii) *$\gamma < \beta + \alpha$ whenever $\beta + \alpha > 0$.*

Proof. Let us first fix some notations: write $\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_m}$ with $\alpha_1 \geq \dots \geq \alpha_m$ and $\beta = \omega^{\beta_1} + \dots + \omega^{\beta_n}$ with $\beta_1 \geq \dots \geq \beta_n$, and let i be the maximal index in $\{1, \dots, n\}$ such that $\beta_i \geq \alpha_1$, or set $i = 0$ if this does not occur. Define $\beta' \stackrel{\text{def}}{=} \omega^{\beta_1} + \dots + \omega^{\beta_i}$ and $\gamma \stackrel{\text{def}}{=} \omega^{\beta_{i+1}} + \dots + \omega^{\beta_n}$ (thus $\beta' = 0$ if $i = 0$); then $\beta = \beta' + \gamma$ and $\beta + \alpha = \beta' + \alpha$. Note that this implies $\gamma < \omega^{\alpha_1} \leq \alpha \leq \beta + \alpha$, unless $\alpha = 0$ and then $\gamma = 0$, thus fulfilling (ii).

We first prove by transfinite induction over α that

$$F_{\beta'+\alpha} \circ F_\gamma \geq F_\gamma \circ F_{F_\beta, \alpha}. \quad (\text{A.11})$$

Proof of (A.11). For the base case $\alpha = 0$, then $\gamma = 0$ and $\beta' = \beta$, and indeed

$$\begin{aligned} F_\beta(F_0(x)) &= F_\beta(x+1) \\ &\geq F_\beta(x) + 1 && \text{by monotonicity of } F_\beta \\ &= F_0(F_\beta(x)) \\ &= F_0(F_{F_\beta, 0}(x)). \end{aligned}$$

For the successor case $\alpha + 1$ and assuming it holds for α , let us first show by induction over j that, for all y ,

$$F_{\beta'+\alpha}^j(F_\gamma(y)) \geq F_\gamma(F_{F_\beta, \alpha}^j(y)). \quad (\text{A.12})$$

This immediately holds for the base case $j = 0$, and for the induction step,

$$\begin{aligned} F_{\beta'+\alpha}^j(F_{\beta'+\alpha}^j(F_\gamma(y))) &\geq F_{\beta'+\alpha}^j(F_\gamma(F_{F_\beta, \alpha}^j(y))) && \text{by ind. hyp. (A.12) on } j \\ &\geq F_\gamma(F_{F_\beta, \alpha}(F_{F_\beta, \alpha}^j(y))) && \text{by ind. hyp. (A.11) on } \alpha < \alpha + 1. \end{aligned}$$

This yields the desired inequality:

$$\begin{aligned}
F_{\beta'+\alpha+1}(F_\gamma(x)) &= F_{\beta'+\alpha}^{F_\gamma(x)+1}(F_\gamma(x)) \\
&\geq F_{\beta'+\alpha}^{x+1}(F_\gamma(x)) \\
&\geq F_\gamma(F_{F_\beta,\alpha}^{x+1}(x)) \\
&= F_\gamma(F_{F_\beta,\alpha+1}(x))
\end{aligned}$$

using (A.12) with $j = x + 1$ and $y = x$.

For the limit case λ ,

$$\begin{aligned}
F_{\beta'+\lambda}(F_\gamma(x)) &= F_{\beta'+\lambda(F_\gamma(x))}(F_\gamma(x)) \\
&\leq F_{\beta'+\lambda(x)}(F_\gamma(x)) && \text{since } \lambda(x) \prec_{F_\gamma(x)} \lambda(F_\gamma(x)) \\
&\leq F_\gamma(F_{F_\beta,\lambda(x)}(x)) && \text{by ind. hyp. (A.11) on } \lambda(x) < \lambda \\
&= F_\gamma(F_{F_\beta,\lambda}(x)). && \square
\end{aligned}$$

Returning to the main proof, a simple induction over α shows that, for all $x \geq x_0$,

$$F_{h,\alpha}(x) \leq F_{F_\beta,\alpha}(x). \quad (\text{A.13})$$

We then conclude for (i) that, for all $x \geq x_0$,

$$\begin{aligned}
F_{h,\alpha}(x) &\leq F_{F_\beta,\alpha}(x) && \text{by (A.13)} \\
&\leq F_\gamma(F_{F_\beta,\alpha}(x)) && \text{by expansivity of } F_\gamma \\
&\leq F_{\beta'+\alpha}(F_\gamma(x)) && \text{by (A.11).} \quad \square
\end{aligned}$$

A.5. Non-standard Assignment of Fundamental Sequences. We show here the omitted details of the proof of Theorem 4.4:

Lemma A.5. *Let $s: \mathbb{N} \rightarrow \mathbb{N}$ be a monotone function and α be an ordinal.*

- *If s is strictly expansive, then $F_{\alpha,s} \leq F_{s,\alpha} \circ s$, and*
- *otherwise $F_{\alpha,s} \leq F_\alpha$.*

Proof. For the first point, let us show that

$$s(F_{\alpha,s}(x)) \leq F_{s,\alpha}(s(x)) \quad (\text{A.14})$$

for all monotone s with $s(x) > x$, all α and all x , which entails the lemma since s is expansive. We proceed by transfinite induction over α . For the base case, $F_{s,0}(s(x)) = s(s(x)) \geq s(x+1) = s(F_{0,s}(x))$ since s is monotone and strictly expansive. For the successor case, $F_{s,\alpha+1}(s(x)) = F_{s,\alpha}^{s(x)+1}(s(x)) \geq s(F_{\alpha,s}^{s(x)}(x)) = s(F_{\alpha+1,s}(x))$, where the middle inequality stems from the fact that $F_{s,\alpha}^j(s(x)) \geq s(F_{\alpha,s}^j(x))$, as can be seen by induction on j using the induction hypothesis on $\alpha < \alpha + 1$. For the limit case, observe that $\lambda(x)_s \prec_{s(x)} \lambda(s(x))$, thus $F_{s,\lambda}(s(x)) = F_{s,\lambda(s(x))}(s(x)) \geq F_{s,\lambda(x)_s}(s(x)) \geq s(F_{\lambda(x)_s,s}(x)) = s(F_{\lambda,s}(x))$ using the induction hypothesis on $\lambda(x)_s < \lambda$.

The second point is straightforward by induction over α . \square

A.6. Composing Hardy Functions. The purpose of this section is to provide the technical details for the proof of Lemma 4.5.

The *natural sum* $\alpha \oplus \beta$ of two ordinals written as $\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_m}$ with $\alpha_1 \geq \dots \geq \alpha_m$ and $\beta = \omega^{\beta_1} + \dots + \omega^{\beta_n}$ with $\beta_1 \geq \dots \geq \beta_n$ can be defined as the ordinal $\omega^{\gamma_1} + \dots + \omega^{\gamma_{m+n}}$ where the γ_i 's range over $\{\alpha_j \mid 1 \leq j \leq m\} \cup \{\beta_k \mid 1 \leq k \leq n\}$ in non-increasing order. For instance, $\omega^2 + \omega^\omega = \omega^\omega$ but $\omega^2 \oplus \omega^\omega = \omega^\omega + \omega^2$.

Lemma A.6. *For all ordinals α and β , and all functions h ,*

$$h^\alpha \circ h^\beta \leq h^{\alpha \oplus \beta} .$$

Proof. Write $\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_m}$ with $\alpha_1 \geq \dots \geq \alpha_m$ and $\beta = \omega^{\beta_1} + \dots + \omega^{\beta_n}$ with $\beta_1 \geq \dots \geq \beta_n$, then $\alpha \oplus \beta = \omega^{\gamma_1} + \dots + \omega^{\gamma_{m+n}}$. We prove the lemma by transfinite induction over β : it holds immediately for the base case since $\alpha \oplus 0 = \alpha$ and for the successor case since $\alpha \oplus (\beta + 1) = (\alpha \oplus \beta) + 1$. For the limit case, let i be the last index of β_n among the γ_j in the CNF of $\alpha \oplus \beta$. If $i = m + n$, then $\alpha \oplus (\beta(x)) = (\alpha \oplus \beta)(x)$ and the statement holds. Otherwise, define $\gamma \stackrel{\text{def}}{=} \omega^{\gamma_1} + \dots + \omega^{\gamma_i}$ and $\gamma' \stackrel{\text{def}}{=} \omega^{\gamma_{i+1}} + \dots + \omega^{\gamma_{m+n}}$. For all x ,

$$\begin{aligned} h^{\alpha \oplus \beta} &= h^\gamma(h^{\gamma'}(x)) && \text{by (A.6)} \\ &= h^{\gamma(h^{\gamma'}(x))}(h^{\gamma'}(x)) && \text{since } \gamma \text{ is a limit ordinal} \\ &\geq h^{\gamma(x)}(h^{\gamma'}(x)) && \text{since } \gamma(x) \prec_{[h^{\gamma'}(x)]} \gamma(h^{\gamma'}(x)) \\ &= h^{\alpha \oplus (\beta(x))}(x) && \text{by (A.6)} \\ &\geq h^\alpha(h^{\beta(x)}(x)) && \text{by ind. hyp. on } \beta(x) < \beta \\ &= h^\alpha(h^\beta(x)) . && \square \end{aligned}$$

Corollary A.7. *Let α be an ordinal and f a function in $\mathcal{F}_{<\alpha}$. Then there exists g in $\mathcal{F}_{<\alpha}$ such that $f \circ F_\alpha \leq F_\alpha \circ g$.*

Proof. As f is in some \mathcal{F}_β for $\beta < \alpha$, $f \leq F_\beta^c$ for some finite c by [56, Theorem 2.10], thus $f \leq H^{\omega^\beta \cdot c}$ by (A.5), and we let $g \stackrel{\text{def}}{=} H^{\omega^\beta \cdot c}$, which indeed belongs to $\mathcal{F}_\beta \subseteq \mathcal{F}_{<\alpha}$. Still by (A.5), $F_\alpha = H^{\omega^\alpha}$. Observe that $\omega^\beta \cdot c < \omega^\alpha$, hence $(\omega^\beta \cdot c) \oplus \omega^\alpha = \omega^\alpha + \omega^\beta \cdot c$. By (A.6), $H^{\omega^\alpha + \omega^\beta \cdot c} = H^{\omega^\alpha} \circ H^{\omega^\beta \cdot c}$. Applying (A.5) and Lemma A.6, we obtain that $f \circ F_\alpha \leq g \circ F_\alpha \leq F_\alpha \circ g$. \square

A.7. Computing Hardy Functions. We explain in this section how to compute Hardy functions, thus providing the background material for the proof of Theorem 5.1. This type of results is pretty standard—see for instance [85], [27], or [77, pp. 159–160]—, but the particular way we employ is closer in spirit to the viewpoint employed in [40, 45, 38].

A.7.1. Hardy Computations. Using (A.3), let us call a *Hardy computation* for $h^\alpha(n)$ a sequence of pairs $\langle \alpha_0, n_0 \rangle, \langle \alpha_1, n_1 \rangle, \dots, \langle \alpha_\ell, n_\ell \rangle$ where $\alpha_0 = \alpha$, $n_0 = n$, $\alpha_\ell = 0$, and at each step $0 < i \leq \ell$, $\alpha_i = P_{n_{i-1}}(\alpha_{i-1})$ and $n_i =$

$h(n_{i-1})$. An invariant of this computation is that $h^{\alpha_i}(n_i) = h^\alpha(n)$ at all steps $0 \leq i \leq \ell$, hence $n_\ell = h^\alpha(n)$. Since h is increasing, the n_i values increase throughout this computation, while the α_i values decrease, and termination is guaranteed.

Our plan is to implement the Hardy computation of $h^\alpha(n)$ using a Turing machine, which essentially needs to implement the ℓ steps $\langle \alpha_i, n_i \rangle \rightarrow \langle P_{n_{i-1}}(\alpha_{i-1}), h(n_{i-1}) \rangle$. Assuming h to be a time constructible expansive function, the complexity of a single step will depend on $h(n_{i-1}) \leq h^\ell(n)$ and on the complexity of updating α_i .

A.7.2. Cichoń Functions. In order to measure the length ℓ of a Hardy computation for $h^\alpha(n)$, we define a family $(h_\alpha)_\alpha$ of functions $\mathbb{N} \rightarrow \mathbb{N}$ by induction on the ordinal index:

$$h_0(x) \stackrel{\text{def}}{=} 0, \quad h_{\alpha+1}(x) \stackrel{\text{def}}{=} 1 + h_\alpha(h(x)), \quad h_\lambda(x) \stackrel{\text{def}}{=} h_{\lambda(x)}(x). \quad (\text{A.15})$$

This family is also known as the *length hierarchy* and was defined by Cichoń and Tahhan Bittar [19]. It satisfies several interesting identities:

$$h^\alpha(x) = h^{h_\alpha(x)}(x), \quad h^\alpha(x) \geq h_\alpha(x) + x. \quad (\text{A.16})$$

Its main interest here is that it measures the length of Hardy computations: $\ell = h_\alpha(n) \leq h^\alpha(n)$ by the above equations, which in turn implies $h^\ell(n) = h^\alpha(n)$.

A.7.3. Encoding Ordinal Terms. It remains to bound the complexity of computing $\alpha_i = P_{n_{i-1}}(\alpha_{i-1})$. Assuming some reasonable string encoding of the terms denoting the α_i [e.g. 38], we will consider that each α_i can be computed in time $|\alpha_i|$ the size of its term representation, and will rather concentrate on bounding this size. We define it by induction on the term denoting α_i :

$$|0| \stackrel{\text{def}}{=} 0, \quad |\omega^\alpha| \stackrel{\text{def}}{=} 1 + |\alpha|, \quad |\alpha + \alpha'| \stackrel{\text{def}}{=} |\alpha| + |\alpha'|. \quad (\text{A.17})$$

Let us also recall the definition of the *slow-growing hierarchy* $(G_\alpha)_\alpha$:

$$G_0(x) \stackrel{\text{def}}{=} 0, \quad G_{\alpha+1}(x) \stackrel{\text{def}}{=} 1 + G_\alpha(x), \quad G_\lambda(x) \stackrel{\text{def}}{=} G_{\lambda(x)}(x). \quad (\text{A.18})$$

The slow-growing function satisfy several natural identities

$$G_\alpha(x) = 1 + G_{P_x(\alpha)}(x), \quad (\text{A.19})$$

$$G_\alpha(x+1) > G_\alpha(x), \quad (\text{A.20})$$

$$\text{if } \beta \prec_x \alpha \text{ then } G_\beta(x) \leq G_\alpha(x). \quad (\text{A.21})$$

Furthermore,

$$G_{\alpha+\alpha'}(x) = G_\alpha(x) + G_{\alpha'}(x), \quad G_{\omega^\alpha}(x) = (x+1)^{G_\alpha(x)}. \quad (\text{A.22})$$

Hence, $G_\alpha(x)$ is the elementary function which results from substituting $x+1$ for every occurrence of ω in the Cantor normal form of α [77, p. 159].

Lemma A.8. *Let $x > 0$. Then $|\alpha| \leq G_\alpha(x)$.*

Proof. By induction over the term denoting α : $|0| = 0 = G_0(x)$, $|\omega^\alpha| = 1 + |\alpha| \leq (x+1)^{|\alpha|} \leq (x+1)^{G_\alpha(x)} = G_{\omega^\alpha}(x)$, and $|\alpha + \alpha'| = |\alpha| + |\alpha'| \leq G_\alpha(x) + G_{\alpha'}(x) = G_{\alpha+\alpha'}(x)$. \square

Lemma A.9. *If $\langle \alpha_0, n_0 \rangle, \dots, \langle \alpha_\ell, n_\ell \rangle$ is a Hardy computation for $h^\alpha(n)$ with $n > 0$, then for all $0 \leq i \leq \ell$, $|\alpha_i| \leq G_\alpha(n_\ell)$.*

Proof. We distinguish two cases. If $i = 0$, then $|\alpha_0| = |\alpha| \leq G_\alpha(n)$ by Lemma A.8 since $n > 0$, hence $|\alpha_0| \leq G_\alpha(n_\ell)$ since $n_\ell \geq n$ by (A.20). If $i > 0$, then

$$\begin{aligned} |\alpha_i| &= |P_{n_{i-1}}(\alpha_{i-1})| \\ &\leq G_{P_{n_{i-1}}(\alpha_{i-1})}(n_{i-1}) && \text{by Lemma A.8 since } n_{i-1} \geq n > 0 \\ &< G_{\alpha_{i-1}}(n_{i-1}) && \text{by (A.19)} \\ &\leq G_\alpha(n_{i-1}) && \text{since } \alpha_{i-1} \prec_{n_{i-1}} \alpha \text{ by (A.21)} \\ &\leq G_\alpha(n_\ell) && \text{since } n_{i-1} \leq n_\ell \text{ by (A.20)} \end{aligned} \quad \square$$

The restriction to $n > 0$ in Lemma A.9 is not a big issue: either $h(0) = 0$ and then $h^\alpha(0) = 0$, or $h(0) > 0$ and then $h^{\gamma+\omega^\beta}(0) = h^\gamma(h(0))$ and we can proceed from γ instead of $\gamma + \omega^\beta$ as initial ordinal of our computation.

A.7.4. *Wrapping up.* To conclude, each of the $\ell \leq h^\alpha(n)$ steps of a Hardy computation for $h^\alpha(n)$ needs to compute

- α_i , in time $|\alpha_i| \leq G_\alpha(h^\alpha(n))$, and
- n_i , in time $h(n_{i-1}) \leq h^\alpha(n)$.

This yields the following statement:

Proposition A.10. *The Hardy function h^α can be computed in time $O(h^\alpha(n) \cdot (G_\alpha(h^\alpha(n)) + h^\alpha(n)))$.*

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