Parity Games on Undirected Graphs Can Be Hard, Too

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Abstract

We discuss the computational complexity of solving parity games in the special case when the underlying game graph is undirected. For strictly alternating games, that is, when the game graph is bipartite between the nodes of the two players, we show that the games can be solved in linear time. However, when strict alternation is not imposed, we show that computing a solution is as hard in the undirected case as it is in the general, directed, case.

Keywords: Parity Games, Graph Structure Complexity

1. Introduction

Two-player parity games on finite graphs have been widely studied in the last two decades because of their close connection with the μ-calculus model-checking. More precisely, deciding the winner in a parity game and deciding validity of a μ-calculus formula for a given finite Kripke structure are two polynomially equivalent problems. So far it is only known that these two problems are in $\mathrm{NP} \cap \mathrm{co}-\mathrm{NP}$; membership in PTime is still open. See *e.g.* [5, 2, 4, 1] for textbooks on these topics.

Most of the research in the quest for a PTime algorithm to solve parity game has been done for the general case of directed graph, and several subclasses have been considered and shown to enjoy PTime algorithm for associated parity games (graphs of bounded clique-width [7], graphs of bounded dag-width [3]). In all those works the measure that is considered on graphs (clique-width, dag-width) is designed for oriented graphs, and this is mostly explained by the general idea that considering games on undirected graphs (*i.e.* graphs in which each edge comes with a back-edge) should be obvious.

In this paper, we discuss this latter question and prove that parity games played on undirected graphs are indeed simple to solve (linear time) if the two players strictly alternate their moves (*i.e.* the graph is bipartite). Nevertheless, in the case where we no longer assume that the graph is bipartite, we show the somehow surprising following result: solving parity games played on undirected graph is polynomially equivalent to solving parity games played on arbitrary graphs.

2. Definitions

In the sequel, when X is a finite set, $|X|$ denotes its cardinal of X, X^* the set of finite words over X, and X^ω the set of infinite words over X.

A *finite graph* is a pair $G = (V, E)$ where V is a finite set of *vertices* and $E \subseteq V \times V$ is a finite set of *edges*. An *undirected graph* is a graph $G = (V, E)$ such that $(v_1, v_2) \in E$ iff $(v_2, v_1) \in E$ for all $v_1, v_2 \in V$, *i.e.* whenever an edge belongs to the graph, the back-edge as well. A *dead-end* is a vertex v such that there is no vertex v' with $(v, v') \in E$. A *path* in a graph is a finite sequence $v_1, v_2, \ldots, v_\ell$ such that $(v_i, v_{i+1}) \in E$ for every $1 \leq i < \ell$; a *cycle* is a path $v_1, v_2, \ldots, v_\ell$ with $\ell > 1$ and $v_1 = v_\ell$. A graph that does not admit any cycle is said to be *acyclic*. The *size* of a graph is defined to be $|V| + |E|$.

An *arena* is a pair $\mathcal{G} = (G, V_{\mathrm{E}}, V_{\mathrm{A}})$ where $G = (V, E)$ and $V = V_{\mathrm{E}} \uplus V_{\mathrm{A}}$ is a partition of the vertices among two players, Éloïse and Abelard. The arena is *bipartite* iff $E \subseteq V_{\mathrm{E}} \times V_{\mathrm{A}} \cup V_{\mathrm{A}} \times V_{\mathrm{E}}$. An arena is depicted as a usual graph, where we represent vertices in V_{E} (*resp.* V_{A}) as circles (*resp.* squares). See Figure 1 for an example.

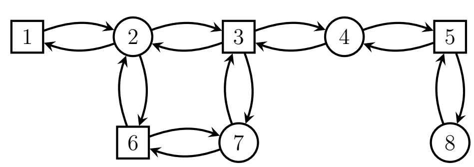

Figure 1: An undirected bipartite arena (priorities are indicated in the vertices).

A colouring function col is a mapping $\mathrm{col} : V \to \mathrm{Col} \subset \mathbb{N}$ where Col is a finite set of colours. An *infinite two-player parity game* on an arena $\mathcal{G}$ is a pair $\mathbb{G} = (\mathcal{G}, \mathrm{col})$.

Éloïse and Abelard play in $\mathbb{G}$ by moving a token between vertices. A *play* from some initial vertex v_0 proceeds as follows: the player owning v_0 (*i.e.*

Éloïse if $v_0 \in V_E$, Abelard otherwise) moves the token to a vertex v_1 such that $(v_0, v_1) \in E$. Then the player owning v_1 chooses a successor v_2 and so on. If at some point one of the players cannot move, she/he loses the play. Otherwise, the play is an infinite word $v_0 v_1 v_2 \cdots \in V^\omega$ and is won by Éloïse just in case $\liminf(\mathrm{col}(v_i))_{i \geq 0}$ is even. A *partial play* is just a prefix of a play.

A strategy for Éloïse is a function assigning, to every partial play ending in some vertex $v \in V_E$, a vertex v' such that $(v, v') \in E$. Éloïse *respects a strategy* φ during a play $\lambda = v_0 v_1 v_2 \cdots$ if $v_{i+1} = \varphi(v_0 \cdots v_i)$, for all $i \geq 0$ such that $v_i \in V_E$. A strategy φ for Éloïse is *winning* from a position $v \in V$ if she wins every play that starts from v and respects φ. Finally, a vertex $v \in V$ is *winning* for Éloïse if she has a winning strategy from v, and the winning region for Éloïse consists of all winning vertices for her. Symmetrically, one defines the corresponding notions for Abelard. For the parity game described in Figure 1, it is easily seen that the winning region for Éloïse is $\{1, 2, 6, 7\}$ and the one for Abelard is $\{3, 4, 5, 8\}$ (we identify here a vertex with its colour).

Of special interest are strategies that does not require memory. A *positional strategy* is a strategy φ such that for every partial play φ and every vertex v one has $\varphi(\lambda \cdot v) = \varphi(\lambda' \cdot v)$, *i.e.* φ only depends on the current vertex. It is a well known result that positional strategies are sufficient to win in parity games [6, 8, 5].

Theorem 1 (Positional determinacy)**.** *Let $\mathbb{G}$ be a parity game. Then for any vertex, either Éloïse or Abelard has a positional winning strategy.*

A special class of games are *reachability games* and are defined as those parity games played on arenas in which all vertices have colour 1. Hence in such a game a play is winning for Éloïse iff it eventually reaches a dead-end in V_A. It is well known that winning regions in reachability games can be computed in linear time in the size of the underlying graph [8, 5].

3. Solving Parity Games Played On Undirected Bipartite Arenas

In the whole section, we let $G = (V, E)$ be an undirected graph, $\mathcal{G} = (G, V_E, V_A)$ be a *bipartite* arena over G and $\mathbb{G} = (\mathcal{G}, \mathrm{col})$ be a parity game over $\mathcal{G}$. We prove, giving a reduction to a reachability game, that the winning regions in $\mathbb{G}$ can be computed in linear time in the size of G.

We define a new graph $\widetilde{G} = (V, \widetilde{E})$ by letting:

$$(i,j) \in \widetilde{A} \text{ iff } (i,j) \in A \text{ and } \begin{cases} \min\{\mathrm{col}(i), \mathrm{col}(j)\} \text{ is even and } i \in V_{\mathrm{E}}, \text{ or} \\ \min\{\mathrm{col}(i), \mathrm{col}(j)\} \text{ is odd and } i \in V_{\mathrm{A}} \end{cases}$$

and we let $\widetilde{\mathcal{G}} = (\widetilde{G}, V_{\mathrm{E}}, V_{\mathrm{A}})$. See Figure 2 for an example.

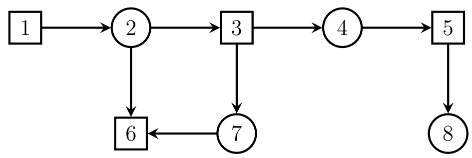

Figure 2: The arena $\widetilde{\mathcal{G}}$ associated with arena $\mathcal{G}$ depicted in Figure 1

Property 1. *The graph $\widetilde{G}$ is acyclic.*

Proof. By contradiction, assume that $\widetilde{G}$ contains a cycle (whose length is necessarily even and greater than 4). Let $v_1, \ldots, v_\ell$ be such a cycle that we choose of minimal length and let i be such that $\mathrm{col}(v_i)$ is minimal. Assume that $v_i \in V_{\mathrm{E}}$. As $(v_{i-1}, v_i) \in \widetilde{E}$ and as $\min(\mathrm{col}(v_{i-1}), \mathrm{col}(v_i)) = \mathrm{col}(v_i)$, we conclude that $\mathrm{col}(v_i)$ is odd. As $(v_i, v_{i+1}) \in \widetilde{E}$, and as $\min(\mathrm{col}(v_i), \mathrm{col}(v_{i+1}) = \mathrm{col}(v_i)$, it follows that $\mathrm{col}(v_i)$ is even, leading a contradiction. A similar argument is applied for the case where $v_i \in V_{\mathrm{A}}$. □

Define $\widetilde{\mathbb{G}} = (\widetilde{\mathcal{G}}, \widetilde{\mathrm{col}})$ as the reachability game over $\mathcal{G}$, *i.e.* $\widetilde{\mathrm{col}}(v) = 1$ for every vertex v. Then the following holds.

Property 2. *Let $v \in V$ be some vertex. Then v is winning for Éloïse in $\mathbb{G}$ iff v is winning for Éloïse in $\widetilde{\mathbb{G}}$.*

Proof. Let φ be a positional winning strategy for Éloïse in $\mathbb{G}$ over the full winning region for her (such a strategy always exists). Let us prove that when being used in $\widetilde{\mathbb{G}}$ the strategy φ is well defined provided the play start from some initial vertex that is winning in $\mathbb{G}$. Indeed, let v_0 be such a vertex and let $\lambda = v_0 \cdots v_n$ be a partial play in $\widetilde{\mathbb{G}}$ where Éloïse respects φ and such that $v_n \in V_{\mathrm{E}}$. Let $v_{n+1} = \varphi(v_n)$ and assume by contradiction that $(v_n, v_{n+1}) \notin \widetilde{E}$ (i.e. φ is not well defined in $\widetilde{\mathbb{G}}$). Therefore, $\min(\mathrm{col}(v_n), \mathrm{col}(v_{n+1}))$ is odd. Now consider the infinite play $v_0 \cdots (v_n \cdot v_{n+1})^\omega$: it is a play in $\mathbb{G}$ where Éloïse respects φ and that starts in a winning vertex for her. But the smallest colour infinitely often visited is $\min(\mathrm{col}(v_n), \mathrm{col}(v_{n+1}))$ hence is odd leading a contradiction. Therefore φ is well defined in $\widetilde{G}$, which means in particular that if Éloïse respects φ she never reaches a dead-end in V_{E} (otherwise φ would give a non-valid move from this vertex). But, as $\widetilde{G}$ is acyclic, any play ends-up in a dead-end, hence φ is winning in $\widetilde{\mathbb{G}}$ from any vertex that is winning in G (it leads to a dead-end which is not in V_{E}).

Conversely, one can make exactly the same reasoning for Abelard. □

Property 2 together with the fact that winning regions can be computed in linear time in reachability games directly imply the following result.

Theorem 2. *In a parity game played on an undirected bipartite arena, the winning region can be computed in linear time in the size of the arena.*

4. Solving Parity Games Played On General Undirected Arenas

In this section we provide a polynomial time reduction of the problem of computing the winning regions in parity games played on arbitrary arenas to the problem of computing the winning regions in parity games played on undirected (not necessarily bipartite) arenas. Hence restricting parity games to undirected arenas does not make them computationally simpler to solve.

In the whole section, we let $G = (V, E)$ be an arbitrary graph, $\mathcal{G} = (G, V_{\mathrm{E}}, V_{\mathrm{A}})$ be an arena over G and $\mathbb{G} = (\mathcal{G}, \mathrm{col})$ be a parity game over $\mathcal{G}$. We start by normalizing $\mathbb{G}$ (see Figure 4 for an example).

Lemma 1. *There exists a graph $G' = (V', E')$, an arena $\mathcal{G} = (G', V'_{\mathrm{E}}, V'_{\mathrm{A}})$ and a parity game $\mathbb{G}' = (\mathcal{G}', \mathrm{col}')$ such that the following holds:*

1. *G' is bipartite;*
2. *for every $v' \in V'$, $\mathrm{col}(v')$ is odd iff $v' \in V'_{\mathrm{E}}$;*
3. *for every $(v_1, v_2) \in E'$ either $\mathrm{col}(v_1) = \mathrm{col}(v_2)$ or $|\mathrm{col}(v_1) - \mathrm{col}(v_2)| \geq 3$;*
4. *$V \subseteq V'$.*

Moreover, the size of G' is linear in the size of G and for every vertex $v \in V$, v is winning for Éloïse in $\mathbb{G}$ iff v is winning for Éloïse in $\mathbb{G}'$.

Proof. In order to impose each of the properties we provide simple transformations on the arena. Hence, we proceed in three steps, one per property (while ensuring that $V \subseteq V'$ at all time).

First, to have G' bipartite, it suffices to insert dummy vertices having a maximal colour. Indeed, let M be the maximal colour appearing in $\mathbb{G}$. Then for every $(v_1, v_2) \in E$ with $v_1, v_2 \in V_\mathrm{E}$ (*resp.* $v_1, v_2 \in V_\mathrm{A}$) we add a new vertex $x \in V_\mathrm{A}$ (*resp.* $x \in V_\mathrm{E}$) with colour M, remove the edge (v_1, v_2) and replace it by the two edges (v_1, x) and (x, v_2); see Figure 3. It is straightforward to check that winning positions are preserved by this transformation.

Then, in order to have Éloïse (*resp.* Abelard) vertices getting only odd (*resp.* even) colour we do the following. For any vertex $v \in V_\mathrm{E}$ with even colour, one sets v to be in V'_A (then v belongs now to Abelard); adds two vertices v_{in} and v_{out} both with priority $2M+1$ and belonging to V'_E together with two edges (v_{in}, v) and (v, v_{out}); and replaces any edge (x, v) by an edge (x, v_{in}) and any edge (v, x) by an edge (v_{out}, x). We do the dual transformation for those vertices $v \in V_\mathrm{A}$ with odd colour; see Figure 3. Other vertices / edges remain unchanged. Again, it is straightforward to check that winning positions are preserved by this transformation.

Finally, to get the third property, it suffices to multiply all colours by 3.

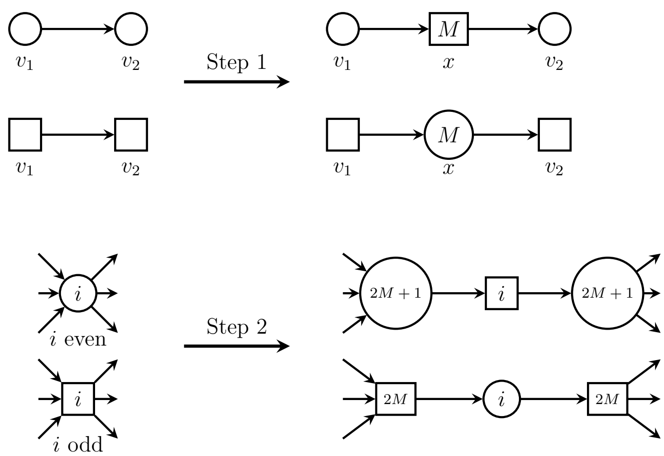

Figure 3: The successive transformations used in the proof of Lemma 1.

$\square$

From now on we assume that G, $\mathcal{G}$ and $\mathbb{G}$ are as G', $\mathcal{G}'$ and $\mathbb{G}'$ in Lemma 1. We define an undirected bipartite arena $\widetilde{\mathcal{G}} = (\widetilde{G} = (\widetilde{V}, \widetilde{E}), \widetilde{V_\mathrm{E}}, \widetilde{V_\mathrm{A}})$ together with a parity game, $\widetilde{\mathbb{G}} = (\widetilde{\mathcal{G}}, \widetilde{\mathrm{col}})$. The set of vertices $\widetilde{V}$ contains V and we will prove that winning regions (when restricted to V) coincide in both games.

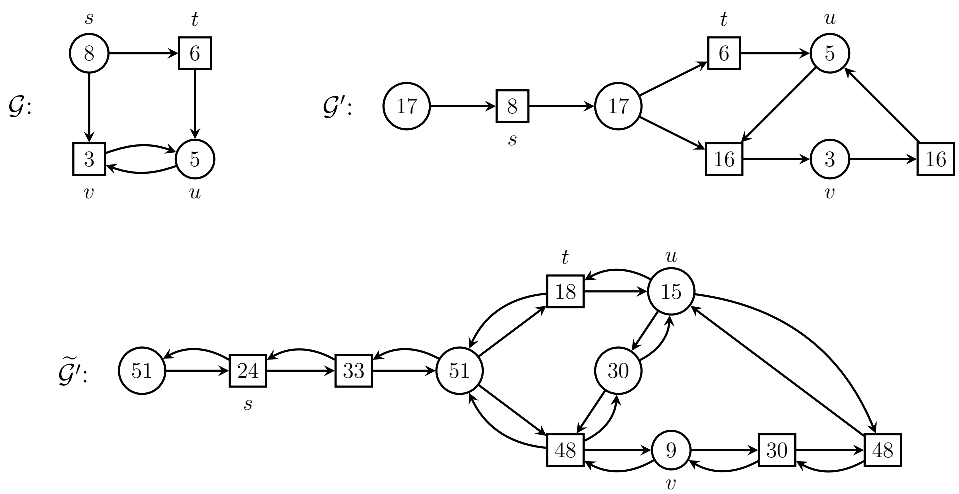

Figure 4: Example of the transformation of Lemma 1 (from $\mathcal{G}$ to $\mathcal{G}'$) and of Lemma 2 (from $\mathcal{G}'$ to $\widetilde{\mathcal{G}'}$).

The graph $\widetilde{G}$ is obtained from G as follows (see Figure 5).

- For every $(v_1, v_2) \in E \cap V_{\mathrm{E}} \times V_{\mathrm{A}}$, we let $i = \mathrm{col}(v_1)$ and $j = \mathrm{col}(v_2)$. If $i > j$ then we simply add the back-edge (v_2, v_1). If $i < j$, we pick some even number k such that $i < k < j$ (note that such a k always exists thanks to point (3) in Lemma 1). Then we add a new vertex $x \in \widetilde{V_{\mathrm{E}}}$ with colour k, we remove edge (v_1, v_2) and add the edges (v_1, x) and (x, v_2) as well as the back-edges (x, v_1) and (v_2, x).
- For every $(v_1, v_2) \in E \cap V_{\mathrm{A}} \times V_{\mathrm{E}}$, we let $i = \mathrm{col}(v_1)$ and $j = \mathrm{col}(v_2)$. If $i > j$ then we simply add the back-edge (v_2, v_1). If $i < j$, we pick some odd number k such that $i < k < j$. Then we add a new vertex $x \in \widetilde{V_{\mathrm{A}}}$ with colour k, we remove edge (v_1, v_2) and add the edges (v_1, x) and (x, v_2) as well as the back-edges (x, v_1) and (v_2, x).

Then the following holds.

Lemma 2. *Let v be some vertex in V. Then v is winning for Éloïse in $\mathbb{G}$ iff v is winning for Éloïse in $\widetilde{\mathbb{G}}$.*

Proof. Let v be some vertex in V and assume that Éloïse has a (positional) winning strategy φ in $\mathbb{G}$ from v. From φ, we define a new strategy $\widetilde{\varphi}$ for Éloïse in $\widetilde{\mathbb{G}}$, and we argue that this strategy is winning from v as well.

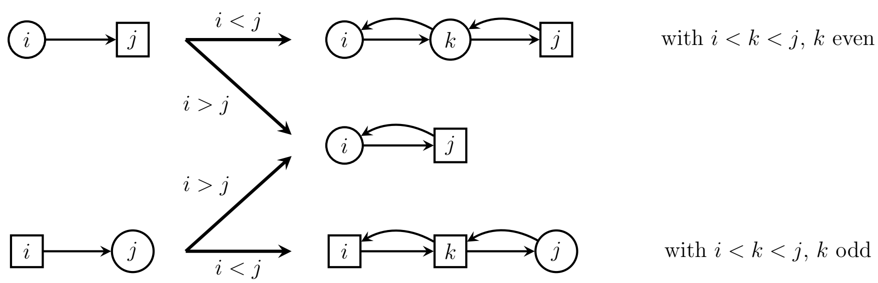

Figure 5: The transformations to define $\widetilde{\mathcal{G}}$.

The strategy $\widetilde{\varphi}$ is defined for any vertex $x \in \widetilde{V_{\mathrm{E}}}$ as follows. For any $x \in V \cap \widetilde{V_{\mathrm{E}}}$ we let $y = \varphi(x)$. If $\mathrm{col}(x) > \mathrm{col}(y)$ then $\widetilde{\varphi}(x) = y$ (note that the edge (x, y) still exists in $\widetilde{G}$). Otherwise, there is a unique vertex $z \in \widetilde{V} \cap \widetilde{V_{\mathrm{E}}}$ such that $(x, z) \in \widetilde{E}$ and $(z, y) \in \widetilde{E}$: we let $\widetilde{\varphi}(x) = z$ and $\widetilde{\varphi}(z) = y$. Then $\widetilde{\varphi}$ can take any value for those vertices for which it has not yet been defined (those will never be reached if Éloïse respects $\widetilde{\varphi}$). Note that the strategy $\widetilde{\varphi}$ never goes through a back-edge that was added when defining $\widetilde{G}$.

We argue that $\widetilde{\varphi}$ is winning from v. Indeed, assume by contradiction that Abelard has a (positional) counter-strategy $\widetilde{\psi}$, and let $\widetilde{\lambda}$ be the play obtained starting from v when Éloïse respects $\widetilde{\varphi}$ and Abelard respects $\widetilde{\psi}$. From the definition of $\widetilde{\mathcal{G}}$, and from the assumption that Abelard wins in $\widetilde{\lambda}$ it follows that $\widetilde{\lambda}$ does not eventually goes through a back-edge added in the construction of $\widetilde{\mathcal{G}}$ (otherwise this back-edge would have been chosen by Abelard, and would induce a loop of length two with an even minimal colour). In particular, it means that if we remove from $\widetilde{\lambda}$ all vertices not in V, we obtain a valid play λ in $\mathbb{G}$ starting from v and where Éloïse respects φ. As φ is winning, λ is won by Éloïse. But this leads a contradiction as, by construction of $\widetilde{\mathcal{G}}$, the smallest colour appearing infinitely often in λ is the same as the one in $\widetilde{\lambda}$. Hence v is winning for Éloïse in $\widetilde{\mathbb{G}}$.

Using a similar argument, one shows that if Abelard has a winning strategy in $\mathbb{G}$ from some vertex v then he also has one in $\widetilde{\mathbb{G}}$. □

As $\widetilde{\mathcal{G}}$ is of polynomial size in the size of $\mathcal{G}$ Lemma 2 directly induces the following.

Theorem 3. *There is a polynomial time reduction from the problem of computing the winning region in an arbitrary parity game to the problem of com-*

puting the winning region in a parity game played on an undirected arena.

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