A New Version of Focus Games for LTL Satisfiability*

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Abstract. We aim to relate game theoretic method and automata theoretic method for the satisfiability problem of LTL. We introduce a one-player focus game as an intermediate formalism between the two.

1 Introduction

Automata theoretic methods are widely used for the satisfiability problem of temporal logics [6]. In [5], Lange and Stirling employ a game theoretic approach for the same. A study on how the different methods relate is interesting on its own account. Moreover it may allow results/techniques in one paradigm to be applied in another.

In this note we try to understand and establish the connection between the game theoretic approach and the automata theoretic approach for LTL satisfiability. To this end, we define a turn-based variant of the focus games introduced in [5] and prove the completeness result for this variant. While unraveling the relations with automata, we come up with a new one player game which serves as an intermediate formalism between the two methods.

We describe focus games in the next section. A proof outline for its completeness is given in Section 3. Some connections to automata are also briefly discussed in this section. Section 4 introduces the new one player game and proves the completeness.

2 Focus Games

In this section we describe the focus game (which is a two player game) for the satisfiability of an LTL formula. We assume that the reader is familiar with LTL, see e.g. [1]. Our variant is a concrete form of the high level description of the focus games defined in [5]. We assume the formula is in negation normal form (negation appears only at the literal level) which is given by:

$$\phi_1, \phi_2 ::= q \mid \neg q \mid \phi_1 \vee \phi_2 \mid \phi_1 \wedge \phi_2 \mid X\phi_1 \mid \phi_1 \, U \, \phi_2 \mid \phi_1 \, R \, \phi_2$$

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Let $Subf(\phi)$ denote the set of all subformulas of ϕ. A set $\Gamma \subseteq Subf(\phi)$ is called a *reduced set* if for all formulas $\phi \in \Gamma$, ϕ is either a literal or a 'next' formula (that is there is a ϕ' such that $\phi \equiv X\phi'$). $\bigwedge \Gamma$ denote the conjunction of all formulas in Γ.

The players are denoted '$\exists$' and '$\forall$'. The focus game for an LTL formula ϕ is given by $G(\phi) = (V, \longmapsto, W)$ where:

- $V \subseteq Subf(\phi) \times 2^{Subf(\phi)} \times \{\exists, \forall\}$. A typical node is a tuple $([\phi_1], \Gamma, Q)$ where $\Gamma \subseteq Subf(\phi)$, $\phi_1 \in \Gamma$ and $Q \in \{\exists, \forall\}$. We may abuse the notations and write $([\phi_1], \Gamma \setminus \{\phi_1\}, Q)$ as well for the same. Intuitively Player $\exists$ tries to prove that $\bigwedge \Gamma$ is satisfiable, where as Player $\forall$ tries to prove it is not. Moreover he hopes ϕ_1 to cause an inconsistency and *focuses* on that (denoted by '$[\phi_1]$'). 'Q' says whose turn it is to make a move.
- the set of moves, $\longmapsto$, is given in the Table 1. Rule 1 allows change of focus. Rules 2–4 show the decomposition of a conjunction. Rules 5–8 show the unfolding of an until or release formula in and outside the focus. Rules 11–14 show the choices of Player $\exists$ on a disjunction. Rule 15 is applicable only at reduced positions (a game node $([\phi], \Gamma, \forall)$ with a reduced set Γ) and shows the unwinding of a next. Note that all nodes by default belong to Player $\forall$. In the presence of a disjunction, at his discretion, Player $\forall$ may give the turn to Player $\exists$ using Rules 9–11.
- W defines the winning conditions given in the Table 2. The conditions $\forall 1$ and $\exists 1$ are self-explanatory. Winning condition $\forall 2$ can be explained as follows: If a game node repeats with an until formula in the focus and focus not being changed throughout, then Player $\exists$ has failed to fulfill the until requirement. Had she been able to do it, she could have done that at the first chance given to her. Winning condition $\exists 2$ can also be explained similarly.

Note that the winning conditions make the plays finite. Moreover, they are mutually exclusive and cover all possible plays. Hence each play has a unique winner. Since exactly one player is allowed to make a move at any node, we can apply Zermelo's theorem [7] to get determinacy.

Theorem 1 (Zermelo). *The game $G(\phi)$ is determined. That is, for every LTL formula ϕ, one of the players has a winning strategy.*

3 Correctness and Completeness

We can prove the completeness of the focus games by adapting the technique in [5] with slight adjustments.

Theorem 2. *Player $\exists$ has a winning strategy in the game $G(\phi)$ if and only if ϕ is satisfiable.*

Sketch of Proof For the *only if* direction, we consider a rational strategy by Player $\forall$. The strategy uses a priority list (finite memory). Consider a linear

1.	$[\phi_1], \phi_2, \Gamma, \forall \longmapsto [\phi_2], \phi_1, \Gamma, \forall$
2.	$[\phi_1 \wedge \phi_2], \Gamma, \forall \longmapsto [\phi_1], \phi_2, \Gamma, \forall$
3.	$[\phi_1 \wedge \phi_2], \Gamma, \forall \longmapsto [\phi_2], \phi_1, \Gamma, \forall$
4.	$[\phi_1], \phi_2 \wedge \phi_3, \Gamma, \forall \longmapsto [\phi_1], \phi_2, \phi_3, \Gamma, \forall$
5.	$[\phi_1], \phi_2\ U\ \phi_3, \Gamma, \forall \longmapsto [\phi_1], \phi_3 \vee (\phi_2 \wedge X(\phi_2\ U\ \phi_3)), \Gamma, \forall$
6.	$[\phi_1\ U\ \phi_2], \Gamma, \forall \longmapsto [\phi_2 \vee (\phi_1 \wedge X(\phi_1\ U\ \phi_2))], \Gamma, \forall$
7.	$[\phi_1], \phi_2\ R\ \phi_3, \Gamma, \forall \longmapsto [\phi_1], \phi_3 \wedge (\phi_2 \vee X(\phi_2\ R\ \phi_3)), \Gamma, \forall$
8.	$[\phi_1\ R\ \phi_2], \Gamma, \forall \longmapsto [\phi_2 \wedge (\phi_1 \vee X(\phi_1\ R\ \phi_2))], \Gamma, \forall$
9.	$[\phi_1 \vee \phi_2], \Gamma, \forall \longmapsto [\phi_1 \vee \phi_2], \Gamma, \exists$
10.	$[\phi_1], \phi_2 \vee \phi_3, \Gamma, \forall \longmapsto [\phi_1], \phi_2 \vee \phi_3, \Gamma, \exists$
11.	$[\phi_1 \vee \phi_2], \Gamma, \exists \longmapsto [\phi_1], \Gamma, \forall$
12.	$[\phi_1 \vee \phi_2], \Gamma, \exists \longmapsto [\phi_2], \Gamma, \forall$
13.	$[\phi_1], \phi_2 \vee \phi_3, \Gamma, \exists \longmapsto [\phi_1], \phi_2, \Gamma, \forall$
14.	$[\phi_1], \phi_2 \vee \phi_3, \Gamma, \exists \longmapsto [\phi_1], \phi_3, \Gamma, \forall$
15.	$[X\phi_1], \ldots, X\phi_m, q_1, \ldots, q_n, \forall \longmapsto \phi_1, \ldots, \phi_m, \forall$

Table 1. Game rules.

Player $\forall$ wins the play $P_0, P_1, \ldots, P_n$ if: $\forall$**1.** P_n is $[q], \Gamma, Q$ and (q is $\bot$ or $\neg q \in \Gamma$) or $\forall$**2.** P_n is $[\phi U \psi], \Gamma, Q$ and for some $i < n$ the position $P_i = P_n$ and between $P_i, \ldots, P_n$ player $\forall$ has not applied Rule 1.
Player $\exists$ wins the play $P_0, P_1, \ldots, P_n$ if: $\exists$**1.** P_n is $[q_1], \ldots, q_n, Q$ and $\{q_1, \ldots, q_n\}$ is satisfiable or $\exists$**2.** P_n is $[\phi], \Gamma, Q$ and for some $i < n$ the position $P_i = P_n$ and between $P_i, \ldots, P_n$ player $\forall$ has applied Rule 1 or ϕ is not an until formula.

Table 2. Winning Conditions.

ordering of all the until formulas present in $Subf(\phi)$ which also preserves subformula relation. i.e, $\phi_1 \in Subf(\phi_2) \Rightarrow \phi_2 > \phi_1$. The priority list is initialized according to this ordering.

At any position $([\varphi], \Gamma, \forall)$ if Player $\forall$ can find a propositional inconsistency discarding the X formulas, he will set the focus appropriately so as to win by condition $\forall 1$ given in Table 2. If the first element of the priority list is a subformula of the current focus, he does not change focus. Else he focuses on the first available formula from the priority list, cyclically shifting the preceding ones. Note that the definition of rational strategy is not complete.

The idea is to extract a model from a play obtained when Player $\exists$ plays her winning strategy and Player $\forall$ plays the rational strategy described above. Let $P_0, \ldots, P_n$ be the resulting play. Let $P_{i_0}, P_{i_1} \ldots, P_{i_k}$ be the positions of the play where the rule 15 was applied. P_{i_j} is of the form $(A_{i_j}, B_{i_j}, \forall)$ where A_{i_j} has only X formulas and B_{i_j} has only literals.

If the play is terminated using winning condition $\exists 1$, $P_n \subseteq$ Literals. We claim that the infinite word $\sigma = B_{i_0} \cdot \ldots \cdot B_{i_k} \cdot P_n \cdot \top^\omega$ is a model for ϕ. If the play is won using winning condition $\exists 2$, for some $m < n$, $P_m = P_n$. Let $P_{i_{m'}}$ be the next reduced position after P_m. We claim that the infinite word $\sigma = B_{i_0} \cdot \ldots \cdot B_{i_{m'-1}} \cdot (B_{i_{m'}} \cdot \ldots \cdot B_{i_k})^\omega$. For each case of the winning condition reached, the proof proceeds by an induction on the structure of the formula.

The *if* direction relies on preserving satisfiability. But this could potentially lead to repetition of game nodes. In order to avoid that, we follow the adornment techniques from [5]. Player $\exists$ keeps a separate copy of the current game node in her memory. Each time a principal formula is decomposed into its subformulas, she keeps a copy of the principal formula as well in the game node (hence building a downward closed consistent set). Every time an until formula is unfolded she adorns its interpretation by the negation of conjunction of formulas in the current game node in her memory, and keep them along until the second argument of the until is present in the game node. Thus preserving satisfiability also helps her to meet the until requirements at the earliest. Note that Player $\exists$ needs polynomial space for this strategy. $\square$

Checking for the existence of a winning strategy can be done in PSPACE which matches the lower bound for LTL satisfiability.

When we look into the details of the completeness proof, the *only if* direction reveals some relation with tableaux construction in the automata theoretic method [6]. We observe that the set of all formulas present in the nodes between two consecutive application of Rule 15 forms a downward closed set. The play we considered for extracting the model corresponds to a witness path in the Büchi automaton. Player $\forall$, playing the rational strategy, changes the focus on meeting the until requirements in focus. Hence the corresponding downward closed sets will correspond to one of the acceptance conditions of the generalized Büchi automaton. The order in which the priority list is initialized in the rational strategy of Player $\forall$ may correspond to an order followed in converting the generalized Büchi acceptance to normal Büchi acceptance.

4 One player focus game

The interesting plays of a focus game are the ones in which Player $\forall$ plays sensibly (some rational strategy). We can in fact embed the rational strategy of Player $\forall$ in the two-player focus game into the definition of the game to get a one-player game. We observe that the role of focus is to force the satisfaction of the until requirement. Hence instead of focusing on just a single formula, we want to focus on the set of unfulfilled until requirements at one go.

A game node has *focused* formulas (enclosed in '[','] ') and *unfocused* ones. We have two modes: a RESET mode where we populate the 'focus', and a CHECK mode where we empty the focus. A game node looks like ($mode$: $[\Gamma_1], \Gamma_2$). The initial position of the one-player game $\mathcal{G}(\phi)$ is (RESET : $[\ \ ], \phi$) If the satisfaction of an until requirement is postponed for a later moment, then the promise to satisfy it later (*next-until* formula) gets into the focus. In the CHECK mode, if an until requirement is satisfied, it is removed from the focus. We do not add formulas into the focus in this mode. Between two consecutive applications of 'Next' rule, the game remains in the same mode. When a 'Next' rule is applied, if the focus is empty, the next mode is RESET otherwise the next mode is CHECK no matter what the current mode is. The rules are given in Table 3, where MODE stands for both CHECK and RESET and the winning and losing conditions in Table 4.

1.	MODE :	$[\Gamma_1], \phi_1 \wedge \phi_2, \Gamma_2 \longmapsto [\Gamma_1], \phi_1, \phi_2, \Gamma_2.$
2.	MODE :	$[\Gamma_1], \phi_1 \vee \phi_2, \Gamma_2 \longmapsto [\Gamma_1], \phi_1, \Gamma_2$
3.	MODE :	$[\Gamma_1], \phi_1 \vee \phi_2, \Gamma_2 \longmapsto [\Gamma_1], \phi_2, \Gamma_2.$
4.	MODE :	$[\Gamma_1], \phi_1\, U\, \phi_2, \Gamma_2 \longmapsto [\Gamma_1], \phi_2, \Gamma_2.$
5.	RESET :	$[\Gamma_1], \phi_1\, U\, \phi_2, \Gamma_2 \longmapsto [X(\phi_1\, U\, \phi_2), \Gamma_1], \phi_1, \Gamma_2.$
6.	CHECK :	$[\Gamma_1], \phi_1\, U\, \phi_2, \Gamma_2 \longmapsto [\Gamma_1], \phi_1, X(\phi_1\, U\, \phi_2), \Gamma_2.$
7.	CHECK :	$[\phi_1\, U\, \phi_2, \Gamma_1], \Gamma_2 \longmapsto [\Gamma_1], \phi_2, \Gamma_2.$
8.	CHECK :	$[\phi_1\, U\, \phi_2, \Gamma_1], \Gamma_2 \longmapsto [X(\phi_1\, U\, \phi_2), \Gamma_1], \phi_1, \Gamma_2.$
9.	MODE :	$[\Gamma_1], \phi_1\, R\, \phi_2, \Gamma_2 \longmapsto [\Gamma_1], \phi_1, \phi_2, \Gamma_2.$
10.	MODE :	$[\Gamma_1], \phi_1\, R\, \phi_2, \Gamma_2 \longmapsto [\Gamma_1], \phi_2, X(\phi_1\, R\, \phi_2), \Gamma_2.$
11.	MODE :	$[\ \ ], X\phi_1, \ldots, X\phi_m, q_1, \ldots, q_n \longmapsto$ RESET : $[\ \ ], \phi_1, \ldots, \phi_m.$
12.	MODE :	$[X\phi_1, \ldots, X\phi_l], X\phi_{l+1}, \ldots, X\phi_m, q_1, \ldots, q_n$
	.	$\longmapsto$ CHECK : $[\phi_1, \ldots, \phi_l], \phi_{l+1}, \ldots, \phi_m.$

Table 3. Game rules.

Theorem 3. *If Player $\exists$ can win the one-player game $\mathcal{G}(\phi)$, then ϕ is satisfiable.*

Proof. Let $R_1, \ldots, R_n$ be the set of reduced positions in the winning play. Let π_i be the set of literals appearing in the position R_i. If the play was terminated using winning condition $\exists 1$, let $w = \pi_1 \cdot \ldots \cdot \pi_n \cdot \top^\omega$. If the play was terminated

Player $\exists$ loses the play $P_0, P_1, \ldots, P_n$ if: $\forall$**1.** P_n is MODE $: [\Gamma_1], \Gamma_2$ and ($\bot \in \Gamma_2$ or $q, \neg q \in \Gamma_2$) for some literal q or $\forall$**2.** P_n is a reduced position with a non empty focus and for some $i < n$ the position $P_i = P_n$ disregarding mode and there is no position in between with an empty focus.
Player $\exists$ wins the play $P_0, P_1, \ldots, P_n$ if: $\exists$**1.** P_n is MODE $: [\ ], q_1, \ldots, q_n$ and $\{q_1, \ldots, q_n\}$ is satisfiable or $\exists$**2.** P_n is a reduced position and for some $i < n$ the position $P_i = P_n$ disregarding the mode and there exists a j such that $i \le j \le n$ and P_j has an empty focus.

Table 4. Winning Conditions.

using winning condition $\exists 2$, there exists a $k < n$ such that $R_n = R_k$. Let $w = \pi_1 \cdot \ldots \cdot \pi_{k-1}.(\pi_k \cdot \ldots \pi_{n-1})^\omega$ in this case.

We claim that for any formula ψ present in the play, if R_i is the nearest reduced position in its future, then $w, i \models \psi$. This is proved by an induction on the structure of ψ. The only difficult case is to show that the until requirements are eventually satisfied in the case with winning condition $\exists 2$. Recall that an until formula once put in focus is removed only when the second argument is chosen on expansion. Hence if an until formula is present in focus at some instance, it is eventually satisfied since we have a reduced set with an empty focus. If there is no reduced set with empty focus in its future, it is either satisfied before R_n or it is present in R_k as well. If an until formula never enters the focus, it is satisfied either before the empty focus is reached or in the RESET mode right after that.

Hence $w, 0 \models \phi$. $\square$

We take the following lemma from [5, 4]:

Lemma 1. *If $\gamma \wedge (\varphi U \psi)$ is satisfiable, then $\gamma \wedge (\psi \vee (\varphi \wedge X((\varphi \wedge \neg\gamma) U (\psi \wedge \neg\gamma))))$ is satisfiable.*

Theorem 4. *If ϕ is satisfiable, then Player $\exists$ has a winning strategy in the one-player focus game $\mathcal{G}(\phi)$.*

Proof. We construct a strategy which avoids both losing conditions. Player $\exists$ uses a finite memory to remember a context in order to avoid losing condition $\forall 2$. A position MODE $: [\varphi_1\, U\, \psi_1, \ldots, \varphi_k\, U\, \psi_k, X(\varphi_{k+1}\, U\, \psi_{k+1}), \ldots, X(\varphi_{k+l}\, U\, \psi_{k+l})], \Gamma$ in context c is interpreted as MODE $: [\varphi_1\, U\, \psi_1, \ldots, \varphi_k\, U\, \psi_k, X(\varphi_{k+1}\, U\, \psi_{k+1}), \ldots, X(\varphi_{k+l} U \psi_{k+l})], \Gamma, cU(c \wedge \bigvee_{i=1}^{k+l} \psi_i)$. The game starts in the context $\top$. When multiple formulas are ready to be expanded, we take them in left to right order though the order in which they are taken does not matter. To avoid losing condition $\forall 1$, Player $\exists$ preserves satisfiability respecting the context at disjunction, release and until moves. In case of ambiguity, choose the second argument.

If it is possible to choose the second argument on expanding an until formula in focus, do so and reset the context to $\top$. At a reduced position $(\textsc{Mode} : [\Gamma_0], \Gamma_j)$ in context c, update the context to $c \wedge (\neg \bigwedge \Gamma_0 \vee \neg \bigwedge \Gamma_j)$ with the next rule. If the focus is empty at a reduced position, the context is reset to $\top$. This preserves satisfiability and avoids losing condition $\forall 2$ as explained below.

At a reduced position $(\textsc{Mode} : [\Gamma_0], \Gamma_j)$ in context c, where $\Gamma_0 = \{ X(\phi_1\, U\, \psi_1), \ldots, X(\phi_k\, U\, \psi_k) \}$, the formula $c\, U\, (c \wedge \bigvee_1^k \psi_i) \wedge \bigwedge \Gamma_0 \wedge \bigwedge \Gamma_j$ is satisfiable since satisfiability respecting context is preserved, but $(c \wedge \bigvee_1^k \psi_i) \wedge \bigwedge \Gamma_0 \wedge \bigwedge \Gamma_j$ is not satisfiable. Indeed, the game rules preserve unsatisfiability, if $c \wedge \psi_i \wedge \bigwedge \Gamma_0 \wedge \bigwedge \Gamma_j$ were satisfiable for some i, then at the position $(\textsc{Mode} : [\Gamma_0'], \Gamma_j')$ with no reduced positions between the two and where $\varphi_i\, U\, \psi_i$ was going to be expanded, $(c\, U\, (c \wedge \bigvee_1^k \psi_i)), [\Gamma_0'], \Gamma_j', \psi_i$ would have been satisfiable, and Player $\exists$ playing consistent with the strategy above, would have chosen ψ_i on expansion and hence $X(\varphi_i\, U\, \psi_i)$ would not be present in Γ_0. Hence owing to Lemma 1, $\bigwedge \Gamma_0 \wedge \bigwedge \Gamma_j \wedge X((c \wedge (\neg \bigwedge \Gamma_0 \vee \neg \bigwedge \Gamma_j))\, U\, (c \wedge (\neg \bigwedge \Gamma_0 \vee \neg \bigwedge \Gamma_j) \wedge \bigvee_1^k \psi_i))$ is satisfiable. If $(\textsc{Mode} : [\Gamma_0], \Gamma_j)$ gives $(\textsc{Check} : [\Gamma_0'], \Gamma_j')$ by Rule 12, then $\bigwedge \Gamma_0' \wedge \bigwedge \Gamma_j' \wedge (c \wedge (\neg \bigwedge \Gamma_0 \vee \neg \bigwedge \Gamma_j))\, U\, (c \wedge (\neg \bigwedge \Gamma_0 \vee \neg \bigwedge \Gamma_j) \wedge \bigvee_1^k \psi_i)$ is satisfiable. Recall that the context is reset to $\top$ whenever the size of the focus reduces. Hence updating the context preserves satisfiability.

Note that when a reduced position repeats under losing condition $\forall 2$, the size of the focus is the same. If the size of the focus once reduces, it cannot increase without touching zero. Hence it is enough to show that the reduced position cannot repeat with the same (non-empty) focus throughout the loop. If it could repeat with the same non-empty focus $[\Gamma_0]$, we have a sequence of reduced positions $(\textsc{Mode} : [\Gamma_0], \Gamma_1), \ldots, (\textsc{Mode} : [\Gamma_0], \Gamma_n)$ with $\Gamma_n = \Gamma_m$ for some $m < n$. The strategy ensures that the context at the last position is $\bigwedge_{i=1}^{n-1} (\neg \bigwedge \Gamma_0 \vee \neg \bigwedge \Gamma_j)$ which is equivalent to $\neg \bigwedge \Gamma_0 \vee \bigwedge_{i=1}^{n-1} \neg \bigwedge \Gamma_j$, and that satisfiability is preserved respecting the context. Hence $[\Gamma_0], \Gamma_n, \neg \bigwedge \Gamma_0 \vee \bigwedge_{i=1}^{n-1} \neg \bigwedge \Gamma_j$ is satisfiable. Since $\bigwedge \Gamma_0 \wedge \neg \bigwedge \Gamma_0$ is not satisfiable, $\bigwedge \Gamma_n \wedge \bigwedge_{i=1}^{n-1} \neg \bigwedge \Gamma_j$ is satisfiable. This would mean that $\bigwedge \Gamma_n \wedge \neg \bigwedge \Gamma_m$ is satisfiable which is a contradiction. Thus losing condition $\forall 2$ can not be reached.

Every play is finite. Each play is either winning or losing. The strategy does not allow Player $\exists$ to lose. Hence it is winning. $\square$

In order to illustrate that preserving satisfiability alone is not enough and how context helps in the winning strategy given in the completeness proof above, we consider an example game[1] $\mathcal{G}(\mathrm{FG}p \wedge \mathrm{G}(q \vee p) \wedge (\mathrm{XF}\neg p) \wedge p)$ given in Figure 1. Note that at position P_b, both the disjuncts of $q \vee p$ are satisfiable. But the left branch is losing since R_1 and R_2' witness losing condition $\forall 2$, whereas the right branch is winning since R_3 and R_4 witness winning condition $\exists 2$. On the other hand, assume Player $\exists$ remembers the context and preserves satisfiability respecting context. The context from P_a to R_2' is $\neg p \vee \neg \mathrm{XF}\neg p \vee \neg \mathrm{XG}(q \vee p) \vee \neg \mathrm{XFG}p$. Since the position P_b has $\mathrm{XG}(q \vee p)$, the disjunct $\neg \mathrm{XG}(q \vee p)$ cannot be satisfied.

[1] $\mathrm{F}\varphi$ and $\mathrm{G}\varphi$ are shorthands for $\top\, U\, \varphi$ and $\bot\, R\, \varphi$ respectively.

The disjunct $\neg \mathrm{XFG}p$ cannot be satisfied since $\mathrm{XFG}p$ is present in the position. $p \wedge \mathrm{F}\neg p \wedge (\neg p \vee \neg \mathrm{XF}\neg p)$ is not satisfiable, hence p cannot be chosen in the disjunction $q \vee p$. Hence preserving satisfiability respecting context eliminates the losing branch.

$$
\begin{array}{lll}
(P_0) & \textsc{Reset} & : \mathrm{FG}p \wedge \mathrm{G}(q \vee p) \wedge \mathrm{XF}(\neg p) \wedge p \\
& & \downarrow * \\
(R_1) & \textsc{Reset} & : [\mathrm{XFG}p], \mathrm{XG}(q \vee p), \mathrm{XF}(\neg p), p \\
& & \downarrow \text{Rule 12} \\
(P_a) & \textsc{Check} & : [\mathrm{FG}p], \mathrm{G}(q \vee p), \mathrm{F}(\neg p) \\
& & \downarrow * \\
(P_b) & \textsc{Check} & : [\mathrm{XFG}p], q \vee p, \mathrm{XG}(q \vee p), \mathrm{F}(\neg p)
\end{array}
$$

$$
\begin{array}{lll|lll}
& \swarrow \text{Rule 3} & & & \searrow \text{Rule 2} & \\
& \textsc{Check} & : [\mathrm{XFG}p], p, \mathrm{XG}(q \vee p), \mathrm{F}(\neg p) & & \textsc{Check} & : [\mathrm{XFG}p], q, \mathrm{XG}(q \vee p), \mathrm{F}(\neg p) \\
& & \downarrow \text{Rule 6} & & & \downarrow \text{Rule 4} \\
(R_2') & \textsc{Check} & : [\mathrm{XFG}p], p, \mathrm{XG}(q \vee p), \mathrm{XF}(\neg p) & (R_2) & \textsc{Check} & : [\mathrm{XFG}p], q, \mathrm{XG}(q \vee p), \neg p \\
& & & & & \downarrow \text{Rule 12} \\
& & & & \textsc{Check} & : [\mathrm{FG}p], \mathrm{G}(q \vee p) \\
& & & & & \downarrow * \\
& & & (R_3) & \textsc{Check} & : [\quad], p, \mathrm{XG}p, \mathrm{XG}(q \vee p) \\
& & & & & \downarrow * \\
& & & (R_4) & \textsc{Check} & : [\quad], p, \mathrm{XG}p, \mathrm{XG}(q \vee p)
\end{array}
$$

Fig. 1. An example game

The game is very close to automata. But the game does not give rise to all models, and we cannot build a winning strategy from any model. It is not clear if we can build a winning strategy from some minimal ultimately periodic model or if the game yields all minimal ultimately periodic models. The length of a play is at most exponential in the size of the formula. Hence checking for the existence of a winning strategy can be done in NPSPACE and hence in PSPACE owing to Savitch's technique.

In [3] a one-player game which uses multiple foci in parallel is defined. All the until formulas in foci games are in focus, whereas our one-player game may have until formulas which never enter focus. The completeness of the foci games builds a strategy from an arbitrary model of the formula, but an arbitrary model cannot yield a strategy for the one-player focus game presented here. In [2], Brünnler and Lange introduce cut-free sequent systems for LTL which can be thought of as a one-player game. But it is technically more similar to the two-player focus games. The 'history' in their definition corresponds to the 'adornment' in the proof of Theorem 2.

5 Conclusion

We introduced a turn based and determined variant of the focus games which was originally introduced in [5]. A new one-player game for the satisfiability problem of LTL is introduced as an intermediate formalism between game theoretic approach and automata theoretic approach. This new game is very close to automata, but the completeness-proof is similar to that of two-player focus game. This manifests the relation between automata theoretic method and the focus games method. A similar study on branching time logics is an immediate future work.

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