Multiply-Recursive Upper Bounds with Higman's Lemma

Sylvain Schmitz   Philippe Schnoebelen

LSV, ENS Cachan & CNRS, Cachan, France
{schmitz,phs}@lsv.ens-cachan.fr

Abstract

We develop a new analysis for the length of controlled bad sequences in well-quasi-orderings based on Higman's Lemma. This leads to tight multiply-recursive upper bounds that readily apply to several verification algorithms for well-structured systems.

Keywords. Higman's Lemma; Well-structured systems; Verification; Complexity of algorithms.

1 Introduction

Well-quasi-orderings (wqo's) are an important tool in logic and computer science (Kruskal, 1972). They are the key ingredient to a large number of decidability (or finiteness, regularity, ...) results. In constraint solving, automated deduction, program analysis, and many more fields, wqo's usually appear under the guise of specific tools, like Dickson's Lemma (for tuples of integers), Higman's Lemma (for words and their subwords), Kruskal's Tree Theorem and its variants (for finite trees with embeddings), and recently the Robertson-Seymour Theorem (for graphs and their minors). In program verification, wqo's are the basis for *well-structured systems* (Abdulla et al., 2000; Finkel and Schnoebelen, 2001; Henzinger et al., 2005), a generic framework for infinite-state systems.

Complexity. Wqo's are seldom used in complexity analysis. In order to extract complexity upper bounds for an algorithm whose elegant termination proof rests on Dickson's or Higman's Lemma, one must be able to bound the length of so-called "controlled bad sequences" (see Definition 2.4). Here the available results are not very well known in computer science, and their current packaging does not make them easy to read and apply. In our case, investigating the complexity of verification for well-structured systems, we rely on Higman's Lemma over Γ_p^* (the words over a p-letter alphabet) and what we really need is something like:

Length Function Theorem. *Let $L_{\Gamma_p^*}(n)$ be the maximal length of bad sequences $w_0, w_1, w_2, \ldots$ over Γ_p^* with $p \geq 2$ s.t. $|w_i| < g^i(n)$ for $i = 0, 1, 2, \ldots$*

If the control function g *is primitive-recursive, then the* length function $L_{\Gamma_p^*}$ *is bounded by a function in* $\mathscr{F}_{\omega^{p-1}}$.[1]

Unfortunately, the literature contains no such clear statement (see the comparison with existing work below). The closest we can mention is (Cichoń and Tahhan Bittar, 1998) that we cited with frustrating imprecision, e.g., with "*[*$L_{\Gamma_p^*}$ *is in* $\mathscr{F}_{\omega^{\varphi(p)}}$*] for some* φ *that is left implicit but that is definitely primitive-recursive*" (Chambart and Schnoebelen, 2008, Section 6).

Our Contribution. We provide a new, *self-contained*, and reasonably clear proof of the Length Function Theorem, a fundamental result that (we think) deserves a wide audience. The exact statement we prove, Theorem 5.3, is rather general: it is parameterized by the control function g and accomodates various combinations of Γ_p^* sets without losing precision. For this we significantly extend the setting we developed for Dickson's Lemma (Figueira et al., 2011): We rely on iterated residuations with a simple but explicit algebraic framework for handling wqo's and their residuals in a compositional way. Our computations can be kept relatively simple by means of a fully explicit notion of "normed reflection" that captures the over-approximations we use, all the while enjoying good algebraic properties. We also show *how to apply* the Length Function Theorem by deriving precise multiply-recursive upper bounds, parameterized by the alphabet size, for the complexity of lossy channel systems and the Regular Post Embedding Problem (see Section 6).

Comparison with Existing Work. Here is a quick survey of results in the spirit of the Length Function Theorem, assuming a control by the successor function to avoid differences in control definitions.

For $\mathbb{N}^k$ (i.e., Dickson's Lemma), Clote gives an explicit upper bound at level $\mathscr{F}_{k+6}$ extracted from complex Ramsey-theoretical results, hence hardly self-contained (Clote, 1986). This is a simplification over an earlier analysis by McAloon, which leads to a uniform upper bound at level $\mathscr{F}_{k+1}$, but gives no explicit statement nor asymptotic analysis (McAloon, 1984). Both analyses are based on large intervals and extractions, and McAloon's is technically quite involved. With D. and S. Figueira, we improved this to an explicit and tight $\mathscr{F}_k$ (Figueira et al., 2011).

For Γ_p^* (Higman's Lemma), Cichoń and Tahhan Bittar exhibit a reduction method, deducing bounds (for tuples of) words on Γ_p from bounds on the Γ_{p-1} case (Cichoń and Tahhan Bittar, 1998). Their decomposition is clear and self-contained, with the control function made explicit. It ends up with some inequalities, collected in (Cichoń and Tahhan Bittar, 1998, Section 8), from which it is not clear what precisely are the upper bounds one can extract. Following this, Touzet claims a bound of F_{ω^p} (Touzet, 2002, Theorem 1.2) with an analysis based on iterated residuations but the proof (given in (Touzet, 1997)) is incomplete.

Finally, Weiermann gives an $\mathscr{F}_{\omega^{p-1}}$-like bound for Γ_p^* (Weiermann, 1994, Corollary 6.3) for sequences produced by term rewriting systems, but his anal-

[1]Here the functions F_α are the ordinal-indexed levels of the Fast-Growing Hierarchy (Löb and Wainer, 1970), with multiply-recursive complexity starting at level $\alpha = \omega$, i.e., Ackermannian complexity, and stopping just before level $\alpha = \omega^\omega$, i.e., hyper-Ackermannian complexity. The function classes $\mathscr{F}_\alpha$ denote their elementary-recursive closure.

ysis is considerably more involved (as can be expected since it applies to the more general Kruskal Theorem) and one cannot easily extract an explicit proof for his Corollary 6.3.

Regarding lower bounds, it is known that $F_{\omega^{p-1}}$ is essentially tight (Cichoń, 2009).

Outline of the Paper. All basic notions are recalled in Section 2, leading to the Descent Equation (3). Reflections in an algebraic setting are defined in Section 3, then transfered in an ordinal-arithmetic setting in Section 4. We prove the Main Theorem in Section 5, before illustrating its uses in Section 6. An appendix contains all the details omitted from the main text.

2 Normed Wqo's and Controlled Bad Sequences

We recall some basic notions of wqo-theory (see e.g. Kruskal, 1972). A *quasi-ordering* (a "qo") is a relation $(A; \leq)$ that is reflexive and transitive. As usual, we write $x < y$ when $x \leq y$ and $y \not\leq x$, and we denote structures $(A; P_1, \ldots, P_m)$ with just the support set A when this does not lead to ambiguities. Classically, the substructure *induced* by a subset $X \subseteq A$ is $(X; P_{1|X}, \ldots, P_{m|X})$ where, for a predicate P over A, $P_{|X}$ is its trace over X.

A qo A is a *well-quasi-ordering* (a "wqo") if every infinite sequence $x_0, x_1, x_2, \ldots$ contains an infinite increasing subsequence $x_{i_0} \leq x_{i_1} \leq x_{i_2} \ldots$ Equivalently, a qo is a wqo if it is well-founded (has no infinite strictly decreasing sequences) and contains no infinite antichains (i.e., set of pairwise incomparable elements). Every induced substructure of a wqo is a wqo.

Wqo's With Norms. A *norm function* over a set A is a mapping $|.|_A : A \to \mathbb{N}$ that provides every element of A with a positive integer, its *norm*, capturing a notion of size. For $n \in \mathbb{N}$, we let $A_{<n} \stackrel{\text{def}}{=} \{x \in A \mid |x|_A < n\}$ denote the subset of elements with norm below n. The norm function is said to be *proper* if $A_{<n}$ is finite for every n.

Definition 2.1. A *normed wqo* (a "nwqo") is a wqo $(A; \leq_A, |.|_A)$ equipped with a proper norm function.

There are no special conditions on norms, except being proper. In particular no connection is required between the ordering of elements and their norms. In applications, norms are related to natural complexity measures.

Example 2.2 (Some Basic Wqo's)**.** The set of natural numbers $\mathbb{N}$ with the usual ordering is the smallest infinite wqo. For every $p \in \mathbb{N}$, we single out two p-element wqo's: $\searrow_p$ is the p-element initial segment of $\mathbb{N}$, i.e., the set $\{0, 1, 2, \ldots, p-1\}$ ordered linearly, while Γ_p is the p-letter alphabet $\{a_1, \ldots, a_p\}$ where distinct letters are unordered. We turn them into nwqo's by fixing the following:

$$|k|_{\mathbb{N}} = |k|_{\searrow_p} \stackrel{\text{def}}{=} k\,, \qquad\qquad |a_i|_{\Gamma_p} \stackrel{\text{def}}{=} 0\,. \tag{1}$$

We write $A \equiv B$ when the two nwqo's A and B are *isomorphic* structures. For all practical purposes, isomorphic nwqo's can be identified, following a standard practice that significantly simplifies the notational apparatus we develop in Section 3. For the moment, we only want to stress that, in particular, *norm functions must be preserved* by nwqo isomorphisms.

Example 2.3 (Isomorphism Between Basic Nwqo's)**.** On the positive side, $\searrow_0 \equiv \Gamma_0$ and also $\searrow_1 \equiv \Gamma_1$ since $|a_1|_{\Gamma_1} = 0 = |0|_{\searrow_1}$. By contrast $\searrow_2 \not\equiv \Gamma_2$: not only these two have non-isomorphic order relations, they also have different norm functions.

Good, Bad, and Controlled Sequences. A sequence $\mathbf{x} = x_0, x_1, x_2, \ldots$ over a qo is *good* if $x_i \leq x_j$ for some positions $i < j$. It is *bad* otherwise. *Over a wqo*, all infinite sequences are good (equivalently, all bad sequences are finite).

We are interested in the maximal length of bad sequences for a given wqo. Here, a difficulty is that, in general, bad sequences can be arbitrarily long and there is no finite maximal length. However, in our applications we are only interested in bad sequences generated by some algorithmic method, i.e., bad sequences whose complexity is controlled in some way.

Definition 2.4 (Control Functions and Controlled Sequences)**.**
A *control function* is a mapping $g : \mathbb{N} \to \mathbb{N}$. For a *size* $n \in \mathbb{N}$, a sequence $\mathbf{x} = x_0, x_1, x_2, \ldots$ over a nwqo A is (g, n)-*controlled* $\stackrel{\text{def}}{\Leftrightarrow}$

$$\forall i = 0, 1, 2, \ldots : \ |x_i|_A < g^i(n) \ = \ \overbrace{g(g(\ldots g}^{i \text{ times}}(n)))\ .$$

Why n is called a "size" appears with Proposition 2.8 and its proof. A pair (g, n) is just called a *control* for short. We say that a sequence $\mathbf{x}$ is n-*controlled* (or just *controlled*), when g (resp. g and n) is clear from the context. Observe that the empty sequence is always a controlled sequence.

Proposition 2.5 (See App. A.1)**.** *Let A be a nwqo and (g, n) a control. There exists a finite maximal length $L \in \mathbb{N}$ for (g, n)-controlled bad sequences over A.*

We write $L_{A,g}(n)$ for this maximal length, a number that depends on all three parameters: A, g and n. However, for complexity analysis, the relevant information is how, for given A and g, the *length function* $L_{A,g} : \mathbb{N} \to \mathbb{N}$ behaves asymptotically, hence our choice of notation. Furthermore, g is a parameter that remains fixed in our analysis and applications, hence it is usually left implicit. **From now on we assume a fixed control function** g and just write $L_A(n)$ for $L_{A,g}(n)$. We further assume that g is *smooth* ($\stackrel{\text{def}}{\Leftrightarrow} g(x+1) \geq g(x)+1 \geq x+2$ for all x), which is harmless for applications but simplifies computations like (4).

Residuals Wqo's and a Descent Equation. Via residuals one expresses the length function by induction over nwqo's.

Definition 2.6 (Residuals)**.** For a nwqo A and an element $x \in A$, the *residual* A/x is the substructure (a nwqo) induced by the subset $A/x \stackrel{\text{def}}{=} \{y \in A \mid x \not\leq y\}$ of elements that are not above x.

Example 2.7 (Residuals of Basic Nwqo's)**.** For all $k < p$ and $i = 1, \ldots, p$:

$$\mathbb{N}/k = \searrow_p/k = \searrow_k\ , \qquad\qquad \Gamma_p/a_i \equiv \Gamma_{p-1}\ . \tag{2}$$

Proposition 2.8 (Descent Equation, See App. A.2)**.**

$$L_A(n) = \max_{x\in A_{<n}} \left\{1 + L_{A/x}(g(n))\right\} . \tag{3}$$

This reduces the L_A function to a finite combination of L_{A_i}'s where the A_i's are residuals of A, hence "smaller" sets. Residuation is well-founded for wqo's: a sequence of successive residuals $A \supsetneq A/x_0 \supsetneq A/x_0/x_1 \supsetneq A/x_0/x_1/x_2 \supsetneq \cdots$ is necessarily finite since $x_0, x_1, x_2, \ldots$ must be a bad sequence. Hence the recursion in the Descent Equation is well-founded and can be used to evaluate $L_A(n)$. This is our starting point for analyzing the behaviour of length functions.

For example, using induction and Eq. (2), the Descent Equation leads to:

$$L_{\Gamma_p}(n) = p\,, \qquad L_{\mathbb{N}}(n) = n\,, \qquad L_{\searrow_p}(n) = \min(n,p)\,. \tag{4}$$

3 An Algebra of Normed Wqo's

We now develop an algebraic framework for nwqos with two main goals:

1. It provides a *notation* for denoting the wqo's we encounter in algorithmic applications. These wqo's and their norm functions abstract data structures that are built inductively by combining some basic wqo's.

2. It supports a *calculus* that greatly simplifies the kind of compositional computations, based on the Descent Equation, we develop next.

Combining Normed Wqo's. Nwqo's can be combined and produce richer, or more complex, nwqo's. The constructions we use in this paper are disjoint sums, cartesian products, and Kleene stars (with Higman's order). These constructions are classic. Here we also have to define how they combine the norm functions:

Definition 3.1 (Sums, Products, Stars Nwqo's)**.** The *disjoint sum* $A_1 + A_2$ of two nwqos A_1 and A_2 is the nwqo given by

$$A_1 + A_2 = \{\langle i, x\rangle \mid i \in \{1,2\} \text{ and } x \in A_i\}\,, \tag{5}$$

$$\langle i, x\rangle \leq_{A_1+A_2} \langle j, y\rangle \overset{\text{def}}{\Leftrightarrow} i = j \text{ and } x \leq_{A_i} y\,, \tag{6}$$

$$|\langle i, x\rangle|_{A_1+A_2} \overset{\text{def}}{=} |x|_{A_i}\,. \tag{7}$$

The *cartesian product* $A_1 \times A_2$ of two nwqos A_1 and A_2 is the nwqo given by

$$A_1 \times A_2 = \{\langle x_1, x_2\rangle \mid x_1 \in A_1, x_2 \in A_2\}\,, \tag{8}$$

$$\langle x_1, x_2\rangle \leq_{A_1\times A_2} \langle y_1, y_2\rangle \overset{\text{def}}{\Leftrightarrow} x_1 \leq_{A_1} y_1 \text{ and } x_2 \leq_{A_2} y_2\,, \tag{9}$$

$$|\langle x_1, x_2\rangle|_{A_1\times A_2} \overset{\text{def}}{=} \max(|x_1|_{A_1}, |x_2|_{A_2})\,. \tag{10}$$

The *Kleene star* A^* of a nwqo A is the nwqo given by

$$A^* \overset{\text{def}}{=} \text{ all finite lists } (x_1 \ldots x_n) \text{ of elements of } A\,, \tag{11}$$

$$(x_1 \ldots x_n) \leq_{A^*} (y_1 \ldots y_m) \overset{\text{def}}{\Leftrightarrow} \begin{cases} x_1 \leq_A y_{i_1} \wedge \cdots \wedge x_n \leq_A y_{i_n} \\ \text{for some } 1 \leq i_1 < i_2 < \cdots < i_n \leq m \end{cases}\,, \tag{12}$$

$$|(x_1 \ldots x_n)|_{A^*} \overset{\text{def}}{=} \max(n, |x_1|_A, \ldots, |x_n|_A)\,. \tag{13}$$

It is well-known (and plain) that $A_1 + A_2$ and $A_1 \times A_2$ are indeed wqo's when A_1 and A_2 are. The fact that A^* is a wqo when A is, is a classical result called Higman's Lemma. We let the reader check that the norm functions defined in Equations (7), (10), and (13), are proper and turn $A_1 + A_2$, $A_1 \times A_2$ and A^* into nwqo's. Finally, we note that nwqo isomorphism is a congruence for sum, product and Kleene star.

Notation (**0** and **1**)**.** *We let* **0** *and* **1** *be short-hand notations for, respectively,* Γ_0 *(the empty nwqo) and* Γ_1 *(the singleton nwqo with the* 0 *norm).*

This is convenient for writing down the following algebraic properties:

Proposition 3.2 (See App. A.3)**.** *The following isomorphisms hold:*

$$\begin{aligned} A + B &\equiv B + A\,, & A + (B + C) &\equiv (A + B) + C\,,\\ A \times B &\equiv B \times A\,, & A \times (B \times C) &\equiv (A \times B) \times C\,,\\ \mathbf{0} + A &\equiv A\,, & \mathbf{1} \times A &\equiv A\,,\\ \mathbf{0} \times A &\equiv \mathbf{0}\,, & (A + A') \times B &\equiv (A \times B) + (A' \times B)\,,\\ \mathbf{0}^* &\equiv \mathbf{1}\,, & \mathbf{1}^* &\equiv \mathbb{N}\,. \end{aligned}$$

In view of these properties, we freely write $A \cdot k$ and A^k for the k-fold sums and products $A + \cdots + A$ and $A \times \cdots \times A$. Observe that $A \cdot k \equiv A \times \Gamma_k$.

Reflecting Normed Wqo's. Reflections are the main comparison/abstraction tool we shall use. They let us simplify instances of the Descent Equation by replacing all A/x for $x \in A_{<n}$ by a single (or a few) A' that is smaller than A but large enough to reflect all considered A/x's.

Definition 3.3. A *nwqo reflection* is a mapping $h : A \to B$ between two nwqo's that satisfies the two following properties:

$$\begin{aligned} &\forall x, y \in A : h(x) \leq_B h(y) \text{ implies } x \leq_A y\,,\\ &\forall x \in A : |h(x)|_B \leq |x|_A\,. \end{aligned}$$

In other words, a nwqo reflection is an order reflection that is also norm-decreasing (not necessarily strictly).

We write $h : A \hookrightarrow B$ when h is a nwqo reflection and say that B *reflects* A. This induces a relation between nwqos, written $A \hookrightarrow B$.

Reflection is transitive since $h : A \hookrightarrow B$ and $h' : B \hookrightarrow C$ entails $h' \circ h : A \hookrightarrow C$. It is also reflexive, hence reflection is a quasi-ordering. Any nwqo reflects its substructures since $Id : X \hookrightarrow A$ when X is a substructure of A. Thus $\mathbf{0} \hookrightarrow A$ for any A, and $\mathbf{1} \hookrightarrow A$ for any non-empty A.

Example 3.4. Among the basic nwqos from Example 2.2, we note the following relations (or absences thereof). For any $p \in \mathbb{N}$, $\searrow_p \hookrightarrow \Gamma_p$, while $\Gamma_p \not\hookrightarrow \searrow_p$ when $p \geq 2$. The reflection of substructures yields $\searrow_p \hookrightarrow \mathbb{N}$ and $\Gamma_p \hookrightarrow \Gamma_{p+1}$. Obviously, $\mathbb{N} \not\hookrightarrow \searrow_p$ and $\Gamma_{p+1} \not\hookrightarrow \Gamma_p$.

Reflections preserve controlled bad sequences. Let $h : A \hookrightarrow B$, consider a sequence $\mathbf{x} = x_0, x_1, \ldots, x_l$ over A, and write $h(\mathbf{x})$ for $h(x_0), h(x_1), \ldots, h(x_l)$, a sequence over B. Then $h(\mathbf{x})$ is bad when $\mathbf{x}$ is, and n-controlled when $\mathbf{x}$ is. Hence:

Proposition 3.5 (Monotony of Length Functions)**.**

$$A \hookrightarrow B \textit{ implies } L_A(n) \le L_B(n) \textit{ for all } n \,. \tag{14}$$

Reflections are compatible with product, sum, and Kleene star.

Proposition 3.6 (Reflection is a Precongruence, see App. A.4)**.**

$$A \hookrightarrow A' \textit{ and } B \hookrightarrow B' \textit{ imply } A + B \hookrightarrow A' + B' \textit{ and } A \times B \hookrightarrow A' \times B' \,, \tag{15}$$

$$A \hookrightarrow A' \textit{ implies } A^* \hookrightarrow A'^* \,. \tag{16}$$

Computing and Reflecting Residuals. We may now tackle our first main problem: computing residuals A/x. This is done by induction over the structure of A.

Proposition 3.7 (Inductive Rules For Residuals, see App. A.5)**.**

$$(A+B)/\langle 1, x\rangle = (A/x) + B \,, \quad (A+B)/\langle 2, x\rangle = A + (B/x) \,, \tag{17}$$

$$(A \times B)/\langle x, y\rangle \hookrightarrow \big[(A/x) \times B\big] + \big[A \times (B/y)\big] \,, \tag{18}$$

$$A^*/(x_1 \dots x_n) \hookrightarrow \Gamma_n \times A^n \times (A/x_1)^* \times \cdots \times (A/x_n)^* \,, \tag{19}$$

$$\Gamma^*_{p+1}/(x_1 \dots x_n) \hookrightarrow \Gamma_n \times (\Gamma^*_p)^n \,. \tag{20}$$

Equation (20) is a refinement of (19) in the case of finite alphabets.

Since it provides reflections instead of isomorphisms, Proposition 3.7 is not meant to support exact computations of A/x by induction over the structure of A. More to the point, it yields over-approximations that are sufficiently precise for our purposes while bringing important simplifications when we have to reflect (the max of) all A/x for all $x \in A_{<n}$.

4 Reflecting Residuals in Ordinal Arithmetic

We now introduce an *ordinal* notation for nwqo's. The purpose is twofold. Firstly, the ad-hoc techniques we use for evaluating, reflecting, and comparing residual nwqo's are more naturally stated within the language of ordinal arithmetic. Secondly, these ordinals will be essential for bounding L_A using functions in subrecursive hierarchies. For these developments, we restrict ourselves to *exponential* nwqo's, i.e., nwqo's obtained from finite Γ_p's with sums, products, and *Kleene star restricted to the* Γ_p*'s*. Modulo isomorphism, $\mathbb{N}^k \equiv \prod_{i=1}^k \Gamma_1^*$ is exponential.

Ordinal Terms. We use Greek letters like $\alpha, \beta, \dots$ to denote ordinal terms in Cantor Normal Form (CNF) built using 0, addition, and ω-exponentiation (we restrict ourselves to ordinals $< \varepsilon_0$). A term α has the general form $\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_m}$ with $\beta_1 \ge \beta_2 \ge \cdots \ge \beta_m$ (ordering defined below) and where we distinguish between three cases: α is 0 if $m = 0$, α is a *successor* if ($m > 0$ and) $\beta_m = 0$, α is a *limit* if $\beta_m \ne 0$ (in the following, λ will always denote a limit, and we write $\alpha + 1$ rather than $\alpha + \omega^0$ for a successor). We say that α is principal (additive) if $m = 1$.

Ordering among our ordinals is defined inductively by

$$\alpha < \alpha' \overset{\text{def}}{\Leftrightarrow} \begin{cases} \alpha = 0 \text{ and } \alpha' \neq 0, \text{ or} \\ \alpha = \omega^\beta + \gamma,\ \alpha' = \omega^{\beta'} + \gamma' \text{ and} \begin{cases} \beta < \beta', \text{ or} \\ \beta = \beta' \text{ and } \gamma < \gamma'. \end{cases} \end{cases} \tag{21}$$

We let $\mathrm{CNF}(\alpha)$ denote the set of ordinal terms $<\alpha$.

For $c \in \mathbb{N}$, $\omega^\beta \cdot c$ denotes the c-fold addition $\omega^\beta + \cdots + \omega^\beta$. We sometimes write terms under a "strict" form $\alpha = \omega^{\beta_1} \cdot c_1 + \omega^{\beta_2} \cdot c_2 + \cdots + \omega^{\beta^m} \cdot c_m$ with $\beta_1 > \beta_2 > \cdots > \beta_m$, where the c_i's, called *coefficients*, must be > 0.

Recall the definitions of the *natural sum* $\alpha \oplus \alpha'$ and *natural product* $\alpha \otimes \alpha'$ of two terms in $\mathrm{CNF}(\varepsilon_0)$:

$$\sum_{i=1}^{m} \omega^{\beta_i} \oplus \sum_{j=1}^{n} \omega^{\beta'_j} \overset{\text{def}}{=} \sum_{k=1}^{m+n} \omega^{\gamma_k}\,, \qquad \sum_{i=1}^{m} \omega^{\beta_i} \otimes \sum_{j=1}^{n} \omega^{\beta'_j} \overset{\text{def}}{=} \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m} \omega^{\beta_i \oplus \beta'_j}\,,$$

where $\gamma_1 \geq \cdots \geq \gamma_{m+n}$ is a rearrangement of $\beta_1, \ldots, \beta_m, \beta'_1, \ldots, \beta'_n$.

We map exponential nwqo's to ordinals in $\mathrm{CNF}(\omega^{\omega^\omega})$ using their *maximal order type* (de Jongh and Parikh, 1977). Formally $o(A)$ is defined by

$$o(\Gamma_p) \overset{\text{def}}{=} p\,, \qquad o(\Gamma^*_0) \overset{\text{def}}{=} \omega^0\,, \qquad o(\Gamma^*_{p+1}) \overset{\text{def}}{=} \omega^{\omega^p}\,, \tag{22}$$

$$o(A + B) \overset{\text{def}}{=} o(A) \oplus o(B)\,, \qquad o(A \times B) \overset{\text{def}}{=} o(A) \otimes o(B)\,. \tag{23}$$

Conversely, there is a *canonical exponential nwqo* $C(\alpha)$ for each α in $\mathrm{CNF}(\omega^{\omega^\omega})$:

$$C\Big(\omega^{\beta_1} + \cdots + \omega^{\beta_m}\Big) = C\Big(\bigoplus_{i=1}^{m} \bigotimes_{j=1}^{k_i} \omega^{\omega^{p_{i,j}}}\Big) \overset{\text{def}}{=} \sum_{i=1}^{m} \prod_{j=1}^{k_i} \Gamma^*_{(p_{i,j}+1)}\,. \tag{24}$$

Then, o and C are bijective inverses (modulo isomorphism of nwqo's), compatible with sums and products (see App. D). This correspondence equates between terms that, on one side, denote partial orderings with norms, and on the other side, ordinals in $\mathrm{CNF}(\omega^{\omega^\omega})$.

For $\alpha \in \mathrm{CNF}(\omega^{\omega^\omega})$, the decomposition $\alpha = \sum_{i=1}^{m} \omega^{\beta_i}$ uses β_i's that are in $\mathrm{CNF}(\omega^\omega)$, i.e., of the form $\beta_i = \sum_{j=1}^{k_i} \omega^{p_{i,j}}$ (with each $p_{i,j} < \omega$) so that ω^{β_i} is $\bigotimes_{j=1}^{k_i} \omega^{\omega^{p_{i,j}}}$. A term ω^{ω^p} is called a *principal multiplicative*.

Derivatives. We aim to replace the "all A/x for $x \in A_{<n}$" by a computation of "some derived $\alpha' \in \partial_n \alpha$" where $\alpha = o(A)$, see Theorem 4.1 below. For this purpose, the definition of derivatives is based on the inductive rules in Proposition 3.7.

Let $n > 0$ be some norm. We start with principal ordinals and define

$$D_n\Big(\omega^{\omega^p}\Big) \overset{\text{def}}{=} \begin{cases} n - 1 & \text{if } p = 0, \\ \omega^{(\omega^{p-1} \cdot (n-1))} \cdot (n-1) & \text{otherwise.} \end{cases} \tag{25}$$

$$D_n\left(\omega^{\omega^{p_1} + \cdots + \omega^{p_k}}\right) \overset{\text{def}}{=} \bigoplus_{j=1}^{k} \left(D_n\Big(\omega^{\omega^{p_j}}\Big) \otimes \bigotimes_{\ell \neq j} \omega^{\omega^{p_\ell}} \right)\,. \tag{26}$$

Now, with any $\alpha \in \mathrm{CNF}(\omega^{\omega^\omega})$, we associate the set of its *derivatives* $\partial_n\alpha$ with

$$\partial_n\Big(\sum_{i=1}^{m}\omega^{\beta_i}\Big) \stackrel{\text{def}}{=} \Big\{D_n\big(\omega^{\beta_i}\big) \oplus \sum_{\ell\neq i}\omega^{\beta_\ell} \;\Big|\; i=1,\ldots,m\Big\}\,. \tag{27}$$

This yields, for example, and assuming $p,k>0$:

$$D_n(1)=0,\ D_n(\omega)=n-1,\ D_n\big(\omega^{\omega^p\cdot k}\big)=\omega^{[\omega^p\cdot(k-1)+\omega^{p-1}\cdot(n-1)]}\cdot k(n-1)\,, \tag{28}$$

$$\partial_n 0=\emptyset,\ \partial_n 1=\{0\},\ \partial_n\omega=\{n-1\},\ \partial_n(\omega^\beta\cdot(k+1))=\{\omega^\beta\cdot k\oplus D_n(\omega^\beta)\}. \tag{29}$$

Thus $\partial_n\alpha$ can be a singleton even when α is not principal, e.g., $\partial_n(p+1)=\{p\}$. We sometimes write $\alpha\ \partial_n\ \alpha'$ instead of $\alpha' \in \partial_n\alpha$, seeing ∂_n as a relation. Note that $\partial_n\alpha \subseteq \mathrm{CNF}(\alpha)$ (see App. B.1), hence $\partial \stackrel{\text{def}}{=} \bigcup_{n<\omega}\partial_n$ is well-founded.

Theorem 4.1 (Reflection by Derivatives, see App. B.2)**.** *Let $x \in A_{<n}$ for some exponential A. Then there exists $\alpha' \in \partial_n o(A)$ s.t. $A/x \hookrightarrow C(\alpha')$.*

Combining with equations (3) and (14), we obtain:

$$L_{C(\alpha)}(n) \le \max_{\alpha'\in\partial_n\alpha}\big\{1+L_{C(\alpha')}(g(n))\big\}\,. \tag{30}$$

5 Classifying L using Subrecursive Hierarchies

For α in $\mathrm{CNF}(\omega^{\omega^\omega})$, define

$$M_\alpha(n) \stackrel{\text{def}}{=} \max_{\alpha'\in\partial_n\alpha}\{1+M_{\alpha'}(g(n))\}\,. \tag{31}$$

(Recall that ∂ is well-founded, thus (31) is well-defined). Comparing with (30), we see that M_α bounds the length function: $M_\alpha(n) \ge L_{C(\alpha)}(n)$.

This defines an ordinal-indexed family of functions $(M_\alpha)_{\alpha\in\mathrm{CNF}(\omega^{\omega^\omega})}$ similar to some classical subrecursive hierarchies, with the added twist of the max operation—see (Buchholz et al., 1994; Moser and Weiermann, 2003) for somewhat similar hierarchies. This is a real issue and one cannot replace a "$\max_{\alpha\in\ldots}\{M_\alpha(x)\}$" with "$M_{\sup\{\alpha\in\ldots\}}(x)$" since M_α is not always bounded by $M_{\alpha'}$ when $\alpha<\alpha'$. E.g., $M_{n+2}(n)=n+2>M_\omega(n)=n+1$.

Subrecursive Hierarchies have been introduced as generators of classes of functions. For instance, writing $\mathscr{F}_\alpha$ for the class of functions elementary-recursive in the function F_α of the *fast growing hierarchy*, we can characterize the set of primitive-recursive functions as $\bigcup_{k<\omega}\mathscr{F}_k$, or that of multiply-recursive functions as $\bigcup_{\beta<\omega^\omega}\mathscr{F}_\beta$ (Löb and Wainer, 1970).

Let us introduce (slight generalizations of) several classical hierarchies from (Löb and Wainer, 1970; Cichoń and Tahhan Bittar, 1998). Those hierarchies are defined through assignments of *fundamental sequences* $(\lambda_x)_{x<\omega}$ for limit ordinals $\lambda<\varepsilon_0$, verifying $\lambda_x<\lambda$ for all x and $\lambda=\sup_x\lambda_x$. A standard assignment is defined by:

$$(\gamma+\omega^{\beta+1})_x \stackrel{\text{def}}{=} \gamma+\omega^\beta\cdot(x+1)\,, \qquad (\gamma+\omega^\lambda)_x \stackrel{\text{def}}{=} \gamma+\omega^{\lambda_x}\,, \tag{32}$$

where γ can be 0. Note that, in particular, $\omega_x = x+1$. Given an assignment of fundamental sequences, one can define the (x-indexed) *predecessor* $P_x(\alpha) < \alpha$ of an ordinal $\alpha \neq 0$ as

$$P_x(\alpha+1) \stackrel{\text{def}}{=} \alpha\,, \qquad\qquad P_x(\lambda) \stackrel{\text{def}}{=} P_x(\lambda_x)\,. \tag{33}$$

Given a fixed smooth control function h, the *Hardy hierarchy* $(h^\alpha)_{\alpha<\varepsilon_0}$ is then defined by

$$h^0(x) \stackrel{\text{def}}{=} x, \qquad h^{\alpha+1}(x) \stackrel{\text{def}}{=} h^\alpha(h(x)), \qquad h^\lambda(x) \stackrel{\text{def}}{=} h^{\lambda_x}(x)\,. \tag{34}$$

A closely related hierarchy is the *length hierarchy* $(h_\alpha)_{\alpha<\varepsilon_0}$ defined by

$$h_0(x) \stackrel{\text{def}}{=} 0, \qquad h_{\alpha+1}(x) \stackrel{\text{def}}{=} 1 + h_\alpha(h(x)), \qquad h_\lambda(x) \stackrel{\text{def}}{=} h_{\lambda_x}(x)\,. \tag{35}$$

Last of all, the *fast growing hierarchy* $(f_\alpha)_{\alpha<\varepsilon_0}$ is defined through

$$f_0(x) \stackrel{\text{def}}{=} h(x), \qquad f_{\alpha+1}(x) \stackrel{\text{def}}{=} f_\alpha^{\omega_x}(x), \qquad f_\lambda \stackrel{\text{def}}{=} f_{\lambda_x}(x)\,. \tag{36}$$

Standard versions of these hierarchies are usually defined by setting h as the successor function, in which case they are denoted H^α, H_α, and F_α resp.

Lemma 5.1 ((Cichoń and Wainer, 1983; Cichoń and Tahhan Bittar, 1998) or App. C)**.** *For all $\alpha \in \mathrm{CNF}(\omega^{\omega^\omega})$ and $x \in \mathbb{N}$,*

1. *$h_\alpha(x) = 1 + h_{P_x(\alpha)}(h(x))$ when $\alpha > 0$,*
2. *$h_\alpha(x) \leq h^\alpha(x) - x$,*
3. *$h^{\omega^\alpha \cdot r}(x) = f_\alpha^r(x)$ for all $r < \omega$,*
4. *if h is eventually bounded by F_γ, then f_α is eventually bounded by $F_{\gamma+\alpha}$.*

Bounding the Length Function. Item 1 of Lemma 5.1 shows that M_α and h_α have rather similar expressions, based on derivatives for one and predecessors for the other; they are in fact closely related:

Proposition 5.2 (See App. B.4)**.** *For all α in $\mathrm{CNF}(\omega^{\omega^\omega})$, there is a constant k s.t. for all $n > 0$, $M_{\alpha,g}(n) \leq h_\alpha(kn)$ where $h(x) \stackrel{\text{def}}{=} x \cdot g(x)$.*

Proposition 5.2 translates for $n, p > 0$ into an

$$L_{\Gamma_p^*,g}(n) \leq h_{\omega^{\omega^{p-1}}}((p-1)n) \quad \text{for } h(x) \stackrel{\text{def}}{=} x \cdot g(x) \tag{37}$$

upper bound on bad (g,n)-controlled sequences in Γ_p^*. We believe (37) answers a wish expressed by Cichoń and Tahhan Bittar in their conclusion (Cichoń and Tahhan Bittar, 1998): "an appropriate bound should be given by the function $h_{\omega^{\omega^{p-1}}}$, for some reasonable h."

It remains to translate the bound of Proposition 5.2 into more intuitive and readily usable ones. Combined with items 2–4 of Lemma 5.1, Proposition 5.2 allows us to state a fairly general result in terms of the $(\mathscr{F}_\alpha)_\alpha$ classes in the two most relevant cases (of which both the Length Function Theorem given in the introduction and, if $\gamma \geq 2$, the $\mathscr{F}_{\gamma+k}$ bound given for $\mathbb{N}^k$ in (Figueira et al., 2011), are consequences):

Theorem 5.3 (Main Theorem)**.** *Let g be a smooth control function eventually bounded by a function in $\mathscr{F}_\gamma$, and let A be an exponential nwqo with maximal order type $< \omega^{\beta+1}$. Then $L_{A,g}$ is bounded by a function in*

- *$\mathscr{F}_\beta$ if $\gamma < \omega$ (e.g. if g is primitive-recursive) and $\beta \geq \omega$,*
- *$\mathscr{F}_{\gamma+\beta}$ if $\gamma \geq 2$ and $\beta < \omega$.*

6 Refined complexity bounds for verification problems

This section provides two *examples* where our Main Theorem leads to precise multiply-recursive complexity upper bounds for problems that were known to be decidable but not primitive-recursive. Our choice of examples is guided by our close familiarity with these problems (in fact, they have been our initial motivation for looking at subrecursive hierarchies) and by their current role as master problems for showing Ackermann complexity lower bounds in several areas of verification. (A more explicit vademecum for potential users of the Main Theorem can be found in (Figueira et al., 2011).)

Lossy Channel Systems. The wqo associated with a lossy channel system $S = (Q, M, C, \Delta)$ is the set $A_S \stackrel{\text{def}}{=} Q \times (M^*)^C$ of its configurations, ordered with embedding (see details in (Chambart and Schnoebelen, 2008)). Here Q is a set of q control locations, M is a size-m message alphabet and C is a set of c channels. Hence, we obtain $A_S \equiv q \cdot \left(\Gamma^*_m\right)^c$. For such lossy systems (Schnoebelen, 2010), reachability, safety and termination can be decided by algorithms that only need to explore bad sequences over A_S. In particular, S has a non-terminating run from configuration s_{init} iff it has a run of length $L_{A_S}(|s_{\text{init}}|)$, and the shortest run (if one exists) reaching s_{final} from s_{init} has length at most $L_{A_S}(|s_{\text{final}}|)$. Here the sequences (runs of S, forward or backward) are controlled with $g = \mathit{Succ}$. Now, since $o(A_S) = \omega^{(\omega^{m-1} \cdot c)} \cdot q$, Theorem 5.3 gives an overall complexity at level $\mathscr{F}_{\omega^{(m-1)} \cdot c}$, which is the most precise upper bound so far for lossy channel systems.

Regarding lower bounds, the construction in (Chambart and Schnoebelen, 2008) proves a $\mathscr{F}_{\omega^K}$ lower bound for systems using $m = K+2$ different symbols, $c = 2$ channels, and a quadratic $q \in O(K^2)$ number of states. If emptiness tests are allowed (an harmless extension for lossy systems, see (Schnoebelen, 2010)) one can even get rid of the `#` separator symbol in that construction (using more channels instead) and we end up with $m = K + 1$ and $c = 4$. Thus the demonstrated upper and lower bounds are very close, and tight when considering the recursivity-multiplicity level.

$\mathsf{PEP}^{\text{reg}}$**, the Regular Post Embedding Problem,** is an abstract problem that relaxes Post's Correspondence Problem by replacing the equality "$u_{i_1} \dots u_{i_n} = v_{i_1} \dots v_{i_n}$" with embedding "$u_{i_1} \dots u_{i_n} \leq_{\Gamma^*} v_{i_1} \dots v_{i_n}$" (all this under a "$\exists i_1, \dots, i_n$ in some regular R" quantification). It was introduced in (Chambart and Schnoebelen, 2007) where decidability was shown thanks to Higman's Lemma. Non-trivial reductions between $\mathsf{PEP}^{\text{reg}}$ and lossy channel systems exist. Due to its abstract nature, $\mathsf{PEP}^{\text{reg}}$ is a potentially interesting master problem for proving hardness at multiply-recursive and hyper-Ackermannian, i.e., $\mathscr{F}_{\omega^\omega}$, levels (see refs in (Chambart and Schnoebelen, 2010)).

A pumping lemma was proven at the last ICALP, that relies on the L_A function, and from which we can now derive more precise complexity upper bounds. Precisely, the proof of Lemma 7.3 in (Chambart and Schnoebelen, 2010) shows that if a $\mathsf{PEP}^{\mathrm{reg}}$ instance admits a solution $\sigma = i_1 \dots i_n$ longer than some bound H then that solution is not the shortest. Here H is defined as $2 \cdot L_{(\Gamma^* \cdot n)}(0)$ for a n that is at most exponential in the size of the instance. Since the control function is linear, Theorem 5.3 yields an $\mathscr{F}_{\omega^{p-1}}$ complexity upper bound for $\mathsf{PEP}^{\mathrm{reg}}$ on a p-letter alphabet (and a hyper-Ackermannian $\mathscr{F}_{\omega^\omega}$ when the alphabet is not fixed). This motivates a closer consideration of lower bounds (left as future work, e.g., by adapting (Chambart and Schnoebelen, 2008)).

7 Concluding Remarks

We proved a general version of the Main Theorem promised in the introduction. Our proof relies on two main components: an algebraic framework for normed wqo's and normed reflections on the one hand, leading on the other hand to descending relations between ordinals that can be captured in subrecursive hierarchies. This setting accommodates all "exponential" wqo's, i.e., finite combinations of Γ_p^*'s. This lets us derive upper bounds for the bad sequences when using Higman's Lemma on finite alphabets.

For future work, we plan to go beyond exponential wqo's and handle wqo's like $(\mathbb{N}^*)^k$ that are beyond exponential, or handle other wqo constructions like powersets, multisets, and perhaps trees. For this next move, the algebraic setting should extend smoothly. By contrast, the ordinal-arithmetical setting will require some strengthening and fine-tuning (e.g., when multiset wqo's are allowed, the maximal order types $o(A)$ can coincide for non-isomorphic wqo's).

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The following appendices provide the proofs missing from the main text (Appendices A and B), and further material not required for the main developments: Appendix D provides some technical comments on the relationships with the literature, and Appendix C proposes the full proofs of a few simple results from the literature we rely on, but for which the proofs are not easily found in print (at least we do not know where to find them).

A Proofs for Normed Wqo's and Reflections

A.1 Proof of Proposition 2.5

Since any prefix of a finite n-controlled bad sequence is n-controlled and bad, these finite sequences ordered by the prefix ordering form a tree T with the empty sequence as its root. Now T is finitely branching since $|.|_A$ is proper. Furthermore, T has no infinite branches since A is wqo. Hence, by Kőnig's Lemma, there are only finitely many branches in T, in particular finitely many maximal n-controlled bad sequences over A, and a finite maximal length for them exists.

A.2 Proof of Proposition 2.8 (Descent Equation)

If $A_{<n}$ is empty, then $L_A(n) = 0$ (the only controlled sequence over A is the empty sequence) while $\max \emptyset = 0$ by definition. If $A_{<n}$ is not empty, we prove the two directions of (3) independently.

"$(\leq)$": Write L for $L_A(n)$ and let $\mathbf{x} = x_0, x_1, x_2, \ldots, x_{L-1}$ be a maximal n-controlled bad sequence over A. The suffix sequence $\mathbf{x}' = x_1, x_2, \ldots, x_{L-1}$ is bad, is $g(n)$-controlled, and is over A/x_0: hence $L - 1 \leq L_{A/x_0}(g(n))$. Now $x_0 \in A_{<n}$ (since $\mathbf{x}$ is controlled) and we deduce one half of (3).

"$(\geq)$": Pick any $x \in A_{<n}$ and write L' for $L_{A/x}(g(n))$. This length is witnessed by a maximal $g(n)$-controlled bad sequence, of the form $\mathbf{x} = x_1, \ldots, x_{L'}$. Since $\mathbf{x}$ is over A/x, the sequence $\mathbf{y} \stackrel{\text{def}}{=} x.\mathbf{x}$ over A, obtained by prefixing $\mathbf{x}$ with x, is bad. It is also n-controlled. Hence $L_A(n) \geq 1 + L'$, which concludes the proof.

A.3 Proof of Proposition 3.2

These isomorphisms are classic for wqo's. For nwqo's, *one must check that they preserve norms.*

We consider the main cases:

$\mathbf{1} \times A \equiv A$: this relies on

$$|\langle a_1, x\rangle|_{\mathbf{1}\times A} \stackrel{(10)}{=} \max(|a_1|_{\mathbf{1}}, |x|_A) \stackrel{(1)}{=} \max(0, |x|_A) = |x|_A\,.$$

$\mathbf{0}^* \equiv \mathbf{1}$: the only element of $\mathbf{0}^*$ is $()$, the empty list. Norms are preserved:

$$|()|_{\mathbf{0}^*} \stackrel{(13)}{=} \max(0) = 0 \stackrel{(1)}{=} |a_1|_{\mathbf{1}}\,.$$

$\mathbf{1}^* \equiv \mathbb{N}$: relies on an isomorphism that links the number k in $\mathbb{N}$ with the unique length-k list in $\mathbf{1}^*$. This preserves norms:

$$|(\overbrace{a_1 \ldots a_1}^{k \text{ times}})|_{\mathbf{1}^*} \overset{(13)}{=} \max(k, |a_1|_{\mathbf{1}}, \ldots, |a_1|_{\mathbf{1}}) \overset{(1)}{=} \max(k, 0, \ldots, 0) = k \overset{(1)}{=} |k|_{\mathbb{N}} \,.$$

A.4 Proof of Proposition 3.6

Assuming $h : A \hookrightarrow A'$ and $h' : B \hookrightarrow B'$, one immediately deduces that $h + h' : A + B \hookrightarrow A' + B'$, that $h \times h' : A \times B \hookrightarrow A' \times B'$, and that $h^* : A^* \hookrightarrow A'^*$ (assuming the obvious definitions for $h + h'$, $h \times h'$ and h^*).

Let us check, for example, that h^* preserves non-comparability:

$$\begin{aligned}
& h^*(x_1 \ldots x_n) \leq_{A'^*} h^*(y_1 \ldots y_m) \\
\Leftrightarrow\ & (h(x_1) \ldots h(x_n)) \leq_{A'^*} (h(y_1) \ldots h(y_m)) && \text{(by def. of } h^*) \\
\Leftrightarrow\ & h(x_1) \leq_{A'} h(y_{i_1}) \ \wedge \ \cdots \ \wedge \ h(x_n) \leq_{A'} h(y_{i_n}) && \text{(by def. of } \leq_{A'^*})
\end{aligned}$$

for some $1 \leq i_1 < i_2 < \cdots < i_n \leq m$,

$$\begin{aligned}
\Rightarrow\ & x_1 \leq_A y_{i_1} \ \wedge \ \cdots \ \wedge \ x_n \leq_A y_{i_n} && \text{(since } h \text{ is a reflection)} \\
\Leftrightarrow\ & (x_1 \ldots x_n) \leq_{A^*} (y_1 \ldots y_m) \,.
\end{aligned}$$

And that h^* is norm-decreasing:

$$\begin{aligned}
|h^*(x_1 \ldots x_n)|_{A'^*} &= |(h(x_1) \ldots h(x_n))|_{A'^*} && \text{(by def. of } h^*) \\
&= \max(n, |h(x_1)|_{A'}, \ldots, |h(x_n)|_{A'}) && \text{(by (13))} \\
&\leq \max(n, |x_1|_A, \ldots, |x_n|_A) && \text{(since } h \text{ is norm-decreasing)} \\
&= |(x_1 \ldots x_n)|_{A^*} \,. && \text{(by (13))}
\end{aligned}$$

Using $\mathbf{0} \hookrightarrow A$ and $(A \equiv \mathbf{0}) \vee (\mathbf{1} \hookrightarrow A)$, we deduce for all $k \in \mathbb{N}$:

$$A \cdot k \hookrightarrow A \cdot (k+1) \,, \qquad\qquad A^{k+1} \hookrightarrow A^{k+2} \,. \tag{38}$$

A.5 Proof of Proposition 3.7

The reflections in (17) are in fact equalities and are obvious.

For (18), the reflection relies on

$$\langle x, y\rangle \not\leq_{A \times B} \langle x', y'\rangle \text{ iff } \big(x \not\leq_A x' \text{ or } y \not\leq_B y'\big) \,,$$

which is just a rewording of (9). It only provides a reflection, not an isomorphism, because the "or" is not exclusive.

For (19) and (20), we first observe the obvious equality

$$(A^*)/() = \mathbf{0} \,, \tag{†}$$

that applies to empty lists. For non-empty lists, the following lemma will be useful:

$$A^*/(x_1\, x_2 \ldots x_n) \hookrightarrow (A/x_1)^* \times \Big[\mathbf{1} + \uparrow_A x_1 \times (A^*/(x_2 \ldots x_n))\Big] \,, \tag{$\star$}$$

where $\uparrow_A x \overset{\text{def}}{=} \{y \in A \mid x \leq_A y\}$ denotes the *upward closure* of an element x of A, seen as a substructure of A.

Proof of $(\star)$. We let $X \stackrel{\text{def}}{=} (A/x_1)^*$, $Y \stackrel{\text{def}}{=} A^*/(x_2 \dots x_n)$, and exhibit a nwqo reflection to $X + X\times \uparrow_A x_1 \times Y$, which is isomorphic to the target in $(\star)$. Write u for $(x_1 \dots x_n)$ and consider some $v = (y_1 \dots y_m) \in A^*$. Then $(x_1 \dots x_n) \leq_{A^*} (y_1 \dots y_m)$ is equivalent to

$$\exists p \in \{1, \dots, m\} \text{ s.t. } \begin{cases} x_1 \leq_A y_p \\ (x_2 \dots x_n) \leq_{A^*} (y_{p+1} \dots y_m) \\ \forall 1 \leq i < p : \ x_1 \not\leq_A y_i \end{cases} \tag{$\ddagger$}$$

The third condition, "$x_1 \not\leq_A y_i$ for all $i < p$", states that p is the leftmost position in v where x_i can be embedded. By negating $(\ddagger)$, we see that $u \not\leq_{A^*} v$ iff there is no p with $x_1 \leq_A y_p$, or there is a leftmost one but $(x_2 \dots x_n) \not\leq_{A^*} (y_{p+1} \dots y_m)$. Therefore, any $v \in A^*/u$ (i.e., any v s.t. $u \not\leq_{A^*} v$) is either a list in $(A/x_1)^*$, or can be decomposed as a triple $\langle (y_1 \dots y_{p-1}), y_p, (y_{p+1} \dots y_m)\rangle$ belonging to $(A/x_1)^* \times \uparrow_A x_1 \times (A^*/(x_2 \dots x_n))$. This provides the required $h : A^*/u \to X + X\times \uparrow_A x_1 \times Y$.

We now check that h is an order-reflection. For this assume that $h(v) \leq_{X+X\times A\times Y} h(v')$. This requires that v and v' are mapped to the same summand, and leads to two cases. If they map to X, the left-hand summand, then $h(v) = v$, $h(v') = v'$, and $h(v) \leq_{(A/x_1)^*} h(v')$, i.e., $h(v) \leq_{(A/x_1)^*} h(v')$, implies $v \leq_{A^*} v'$. If they map to $X\times \uparrow_A x_1 \times Y$, then $h(v)$ is some $\langle v_1, y, v_2\rangle$, $h(v')$ is some $\langle v'_1, y', v'_2\rangle$, and $h(v) \leq h(v')$ implies $v_1 \leq_X v'_1$, $y \leq_{\uparrow_A x_1} y'$, and $v_2 \leq_Y v'_2$. Now, since X, Y, and $\uparrow_A x_1$, are substructures of A^* and A, we deduce $v_1 \leq_{A^*} v'_1$, $v_2 \leq_{A^*} v'_2$, and $y \leq_A y'$. Since v and v' are exactly $v_1.y.v_2$ and, respectively, $v'_1.y'.v'_2$, we deduce $v \leq_{A^*} v'$ from the compatibility of embedding with concatenation.

Finally, we let the reader check that h is norm-decreasing, and observe that the reason why Eq. $(\star)$ is not an isomorphism is because the norms are not preserved in the decomposition $v \mapsto \langle v_1, y, v_2\rangle$. □

Proof of (19). We now prove (19) by induction on n. The base case, $n = 0$, is provided by (†) since $\Gamma_0 \times A^0 \equiv \mathbf{0}$. For the inductive case we assume $n > 0$, which implies $A \neq \mathbf{0}$. By ind. hyp., $A^*/(x_2 \dots x_n) \hookrightarrow \Gamma_{n-1} \times A^{n-1} \times \prod_{i=2}^n (A/x_i)^*$. Replacing in $(\star)$, we deduce

$$\begin{aligned} A^*/(x_1 \dots x_n) &\hookrightarrow (A/x_1)^* \times \Big[\mathbf{1} + \uparrow_A x_1 \times \Gamma_{n-1} \times A^{n-1} \times \prod_{i=2}^n (A/x_i)^*\Big] \\ &\hookrightarrow (A/x_1)^* \times \Big[\mathbf{1} + \Gamma_{n-1} \times A^n \times \prod_{i=2}^n (A/x_i)^*\Big] \quad (\text{by } \uparrow_A x \hookrightarrow A) \end{aligned}$$

and since $\mathbf{1} \hookrightarrow A^n \times \prod_{i=2}^n (A/x_i)^*$ (recall that $A \neq \mathbf{0}$):

$$\begin{aligned} &\hookrightarrow (A/x_1)^* \times \Big[(\mathbf{1} + \Gamma_{n-1}) \times A^n \times \prod_{i=2}^n (A/x_i)^*\Big] \\ &\equiv \Gamma_n \times A^n \times \prod_{i=1}^n (A/x_i)^* . \end{aligned}$$
□

Proof of (20). By induction on n. The base case, $n = 0$, is provided by Eq. (†) since $\Gamma_0 \times (\Gamma_p^*)^0 \equiv \mathbf{0}$.

For the inductive case, $n > 0$, we first simplify $(\star)$ using $\Gamma_{p+1}/x \equiv \Gamma_p$ from Eq. (2), and noting that $\uparrow_{\Gamma_{p+1}} x \equiv \mathbf{1}$ since it amounts to the singleton $\{x\}$. This yields

$$\Gamma^*_{p+1}/(x_1\,x_2\dots x_n) \hookrightarrow \Gamma^*_p \times \big(\mathbf{1} + \Gamma^*_{p+1}/(x_2\dots x_n)\big)\,.$$

Using the ind. hyp., $\Gamma^*_{p+1}/(x_2\dots x_n) \hookrightarrow \Gamma_{n-1}\times(\Gamma^*_p)^{n-1}$, one obtains

$$\begin{aligned}\Gamma^*_{p+1}/(x_1\dots x_n) &\hookrightarrow \Gamma^*_p \times \big(\mathbf{1} + \Gamma_{n-1}\times(\Gamma^*_p)^{n-1}\big)\\ &\equiv \Gamma^*_p + \Gamma_{n-1}\times(\Gamma^*_p)^{n}\\ &\hookrightarrow (\Gamma^*_p)^n + \Gamma_{n-1}\times(\Gamma^*_p)^{n} && \text{(by Eq. (38))}\\ &\equiv \Gamma_n\times(\Gamma^*_p)^n\,. && \square\end{aligned}$$

B Proofs for Ordinals and Subrecursive Hierarchies

B.1 Well-Foundedness of Derivatives

This requires basic monotonicity properties of natural sums and products:

$$\alpha < \alpha' \text{ implies } \alpha\oplus\beta < \alpha'\oplus\beta\,, \tag{39}$$

$$\alpha < \alpha' \;\wedge\; 0<\beta \text{ implies } \alpha\otimes\beta < \alpha'\otimes\beta\,, \tag{40}$$

and a direct consequence of Eq. (21), the defining property of principal ordinals:

$$\Big(\bigoplus_{i=1}^{n}\alpha_i\Big) < \omega^\beta \text{ iff } \alpha_1<\omega^\beta \;\wedge\;\cdots\;\wedge\;\alpha_n<\omega^\beta\,. \tag{41}$$

We can now prove that $\alpha'\in\partial_n\alpha$ implies $\alpha'<\alpha$.

One first checks that $D_n(\alpha)<\alpha$ for all principal ordinals. This is immediate in the case of multiplicative principals: see Eq. (25). For the more general case $\alpha = \bigotimes_i \omega^{\omega^{p_i}}$, Eq. (26) gives $D_n(\alpha)$ as a sum of terms that are individually smaller than α thanks to Eq. (40). One concludes with Eq. (41).

Finally, knowing that $D_n(\omega^\beta)<\omega^\beta$, Eq. (27) combined with Eq. (39) proves $\alpha'<\alpha$ when $\alpha'\in\partial_n\alpha$.

B.2 Proof of Theorem 4.1

We want to prove that, for $x\in A_{<n}$, A/x can be reflected in $C(\alpha')$ for some $\alpha'\in\partial_n o(A)$.

We write A in canonical form $A\equiv\sum_{i=1}^m\prod_{j=1}^{k_i}\Gamma^*_{p_{i,j}+1}$, as *per* Eq. (24), and consider special cases first:

Case 1, A is finite (i.e., $k_i=0$ for all i): Then $A\equiv\sum_{i=1}^m\mathbf{1}\equiv\Gamma_m$. If $m=0$, i.e., $A\equiv\mathbf{0}$, the claim holds vacuously since there is no $x\in A$. If $m>0$ then $o(A)\overset{(22)}{=}m$, $\partial_n m\overset{(29)}{=}\{m-1\}$, $C(m-1)\overset{(24)}{=}\Gamma_{m-1}$ and we conclude with Eq. (2): $\Gamma_m/x\equiv\Gamma_{m-1}$.

Case 2, A is some Γ^*_{p+1} (i.e., $m=1=k_1$): Then $o(A)\overset{(22)}{=}\omega^{\omega^p}$. By Eq. (25), $\partial_n o(A)$ gives a single α' that is $n-1$ if $p=0$, and $\omega^{(\omega^{p-1}\cdot(n-1))}\cdot(n-1)$ if $p>0$. In the first case, $C(\alpha')=\Gamma_{n-1}$. In the second case

$$C\Big(\omega^{(\omega^{p-1}\cdot(n-1))}\cdot(n-1)\Big)\overset{(24)}{=}(\Gamma^*_p)^{n-1}\cdot(n-1)\equiv(\Gamma^*_p)^{n-1}\times\Gamma_{n-1}\,.$$

Thus, in view of $\Gamma_0^* \equiv \mathbf{1}$, we can write $C(\alpha') \equiv (\Gamma_p^*)^{n-1} \times \Gamma_{n-1}$ even when $p = 0$. On the other hand, Eq. (20) yields $A/x \hookrightarrow A' \stackrel{\text{def}}{=} (\Gamma_p^*)^{|x|} \times \Gamma_{|x|}$. But since $|x|_A < n$, one deduces $A' \hookrightarrow C(\alpha')$ from the algebraic properties of reflection.

Case 3, A is a product $\prod_{i=1}^k \Gamma_{p_i+1}^*$, i.e., $m = 1 \leq k_1$: Then

$$o(A) \stackrel{(23)}{=} \bigotimes_{i=1}^{k} o(\Gamma_{p_i+1}^*) \stackrel{(22)}{=} \bigotimes_{i=1}^{k} \omega^{\omega^{p_i}} = \omega^{(\omega^{p_1} \oplus \cdots \oplus \omega^{p_k})} . \tag{42}$$

Hence $\partial_n o(A)$ gives a single $\alpha' = D_n(o(A))$ and

$$C(\alpha') \stackrel{(26)}{=} C\Big[\bigoplus_{i=1}^{k} \Big(\overbrace{D_n\big(\omega^{\omega^{p_i}}\big)}^{\beta_i=} \otimes \overbrace{\bigotimes_{\ell \neq i} \omega^{\omega^{p_\ell}}}^{\alpha'_i=} \Big)\Big] \equiv \sum_{i=1}^{k} C(\beta_i) \times C(\alpha'_i) . \tag{43}$$

Now $C(\alpha'_i) \equiv A_i \stackrel{\text{def}}{=} \prod_{\ell \neq i} \Gamma_{p_\ell+1}^*$ since C is the inverse of o. On the other hand, $x \in A$ must have the form $\langle x_1, \ldots, x_k \rangle$. With Eq. (18) we see that

$$A/x \hookrightarrow \sum_{i=1}^{k} \Big((\Gamma_{p_i+1}^*)/x_i \times \prod_{l \neq i} \Gamma_{p_l+1}^* \Big) = \sum_{i=1}^{k} \Big((\Gamma_{p_i+1}^*)/x_i \times C(\alpha'_i) \Big) . \tag{44}$$

We saw (Case 2) that $\Gamma_{p_i+1}^*/x_i \hookrightarrow C(\beta_i)$. Combining with Eq. (43) and (44), we conclude that $A/x \hookrightarrow C(\alpha')$.

Case 4, A is $\sum_{i=1}^m \prod_{j=1}^{k_i} \Gamma_{p_{i,j}+1}^*$ with $m > 1$: We write $A = \sum_{i=1}^m A_i$, so that $o(A) = \bigoplus_{i=1}^m o(A_i)$ and

$$\partial_n o(A) = \Big\{ D_n(o(A_i)) \oplus \bigoplus_{\ell \neq i} o(A_\ell) \;\Big|\; i = 1, \ldots, m \Big\} .$$

On the other hand, we know that x is $\langle i, x' \rangle$ for some $i \in \{1, \ldots, m\}$. With Eq. (17), we deduce that $A/x \hookrightarrow A_i/x' + \sum_{\ell \neq i} A_\ell$. By picking $\alpha' = D_n(o(A_i)) \oplus \bigoplus_{\ell \neq i} o(A_\ell)$, we deduce $A_i/x' + \sum_{\ell \neq i} A_l \hookrightarrow C(\alpha')$ since $A_\ell \equiv C(o(A_\ell))$ and $A_i/x' \hookrightarrow D_n(o(A_i))$ as we saw with Case 3.

B.3 Lean Ordinals and Pointwise Ordering

We present some intermediate results before we can prove Proposition 5.2.

A key issue with hierarchies like $(h_\alpha)_{\alpha<\varepsilon_0}$ and $(f_\alpha)_{\alpha<\varepsilon_0}$ is that, in general, $\alpha < \alpha'$, does not imply $h_\alpha(x) \leq h_{\alpha'}(x)$ or $f_\alpha(x) \leq f_{\alpha'}(x)$. Such monotonicity w.r.t. α only holds "eventually", or for "sufficiently large x". This issue appears very quickly since just proving monotonicity in the x argument *requires* some monotonicity in the α index in the case where α is a limit.

Indeed, we will use some of that monotonicity w.r.t. α in order to handle the "$\max_{\alpha' \in \partial_n \alpha} M_{\alpha'}(\ldots)$" in (31) and majorize it by some "$M_{\max(\partial_n \alpha)}(\ldots)$".

A Refined Ordering. In order to deal with these issues, a standard solution goes through a ternary relation between x, α and α', as we now explain.

For each x in $\mathbb{N}$, define a relation $\prec_x$ between ordinals, called "*pointwise-at-x ordering*" in (Cichoń and Tahhan Bittar, 1998), as the smallest transitive relation s.t. for all α, λ:

$$\alpha \prec_x \alpha + 1 \;, \qquad\qquad \lambda_x \prec_x \lambda \;. \tag{45}$$

The inductive definition of $\prec_x$ implies

$$\alpha' \prec_x \alpha \text{ iff } \begin{cases} \alpha = \beta + 1 \text{ is a successor and } \alpha' \preccurlyeq_x \beta, \text{ or} \\ \alpha = \lambda \text{ is a limit and } \alpha' \preccurlyeq_x \lambda_x. \end{cases} \tag{46}$$

Obviously $\prec_x$ is a restriction of $<$, the linear ordering of ordinals. For example, $x+1 = \omega_x \prec_x \omega$ but $x+2 \not\prec_x \omega$. The $\prec_x$ relations are linearly ordered themselves, and $<$, can be recovered in view of:

$$\prec_0 \subset \cdots \subset \prec_x \subset \prec_{x+1} \subset \cdots \subset \Big(\bigcup_{x \in \mathbb{N}} \prec_x \Big) = < \;. \tag{47}$$

More precisely, we prove in Section C.2 the following results when $\omega_x = x + 1$, from which the inclusions in (47) follow:

$$0 \preccurlyeq_x \alpha \;, \tag{48}$$
$$\alpha' \prec_x \alpha \text{ implies } \gamma + \alpha' \prec_x \gamma + \alpha \;, \tag{49}$$
$$\alpha' \prec_x \alpha \text{ implies } \omega^{\alpha'} \prec_x \omega^{\alpha} \;, \tag{50}$$
$$x < y \text{ implies } \lambda_x \prec_y \lambda_y \;. \tag{51}$$

With this, one can show (see Cichoń and Tahhan Bittar, 1998, Theorem 2, or Appendix C.4) that, for smooth h:

$$x < y \text{ implies } h_\alpha(x) \le h_\alpha(y) \;, \tag{52}$$
$$\alpha' \prec_x \alpha \text{ implies } h_{\alpha'}(x) \le h_\alpha(x) \;. \tag{53}$$

Lean Ordinals. Now, in order to use Eq. (53) in the analysis of M, we need to show that $\alpha' \prec_x \alpha$ when $\alpha' \in \partial_n \alpha$. This is handled through a notion of *lean* ordinals, as we now explain.

Let k be in $\mathbb{N}$. We say that an ordinal α in $\mathrm{CNF}(\varepsilon_0)$ is *k-lean* if it only uses coefficients $\le k$, or, more formally, when it is written under the strict form $\alpha = \omega^{\beta_1} \cdot c_1 + \cdots + \omega^{\beta_m} \cdot c_m$ with $c_i \le k$ and, inductively, with k-lean β_i, this for all $i = 1, ..., m$. Observe that only 0 is 0-lean, and that if α is k-lean and α' is k'-lean, then $\alpha \oplus \alpha'$ is $(k + k')$-lean.

Leanness is a fundamental tool when it comes to understanding the $\prec_x$ relation:

Lemma B.1 (see Section C.2)**.** *Let α be x-lean. Then*

$$\alpha < \gamma \;\; \textit{iff} \;\; \alpha \preccurlyeq_x P_x(\gamma) \; \Big[\textit{also: iff } \alpha \prec_x \gamma \;\; \textit{iff} \;\; \alpha \le P_x(\gamma) \Big] \;. \tag{54}$$

B.4 Bounding M: Proof of Proposition 5.2

We first bound the leanness of derivatives $\alpha' \in \partial_n \alpha$ in function of n and α.

Proposition B.2. *Assume $k, n > 0$ and $\alpha \in \mathrm{CNF}(\omega^{\omega^\omega})$ is k-lean. If $\alpha \ \partial_n \ \alpha'$ then α' is kn-lean.*

Proof. We first show that $D_n(\omega^\beta)$ is $k(n-1)$-lean when $\beta \in \mathrm{CNF}(\omega^\omega)$ is k-lean. For this, write β under the strict form $\beta = \sum_{i=1}^m \omega^{p_i} \cdot c_i$. Now Eq. (26) gives:

$$D_n(\omega^\beta) = \bigoplus_{i=1}^m \Big[D_n(\omega^{\omega^{p_i}}) \cdot c_i \otimes \omega^{\omega^{p_i} \cdot (c_i - 1)} \otimes \bigotimes_{\ell \neq i} \omega^{\omega^{p_\ell} \cdot c_\ell} \Big]$$

$$= \bigoplus_{i=1}^m \big[\omega^{\beta_i} \cdot c_i(n-1) \big] , \tag{55}$$

with

$$\beta_i \stackrel{\text{def}}{=} \omega^{p_i} \cdot (c_i - 1) \oplus \underbrace{\omega^{p_i - 1} \cdot (n-1)}_{\text{or } 0 \text{ if } p_i = 0} \oplus \bigoplus_{\ell \neq i} \omega^{p_\ell} \cdot c_\ell \, . \tag{56}$$

We can assume $n > 1$ since otherwise $D_n(\omega^\beta) = 0$ and we are done. Inspecting Eq. (56), we see that the coefficients in β_i are $c_i - 1$, $n - 1$, c_ℓ for $\ell \neq i$, and can even be $(n-1) + c_{i+1}$ in the case where $p_i - p_{i+1} = 1$. Since β is k-lean, these coefficients are $\leq k + n - 1 \leq k(n-1)$. Hence β_i is $k(n-1)$-lean. The same holds of $D_n(\omega^\beta)$ since all the $c_i(n-1)$'s in Eq. (55) are $\leq k(n-1)$, and cannot get combined in view of $\beta_m > \beta_{m-1} > \cdots > \beta_1$.

Now assume $\alpha' \in \partial_n \alpha$ and write α under the form $\alpha = \gamma \oplus \omega^\beta$ such that $\alpha' = \gamma \oplus D_n(\omega^\beta)$. We just proved that $D_n(\omega^\beta)$ is $k(n-1)$-lean, and since γ is k-lean, α' is $(k(n-1) + k)$-lean, i.e., kn-lean. □

Corollary B.3. *Let $k, n > 0$, α, α' be in $\mathrm{CNF}(\omega^{\omega^\omega})$, and h be a smooth function. If α is k-lean and $\alpha' \in \partial_n \alpha$, then for all $x \geq kn$,*

$$h_{\alpha'}(x) \leq h_{P_{kn}(\alpha)}(x) \, .$$

Proof. Since α is k-lean, then α' is kn-lean by Proposition B.2, hence $\alpha' \preccurlyeq_{kn} P_{kn}(\alpha)$ by Lemma B.1 and thus $\alpha' \preccurlyeq_x P_{kn}(\alpha)$ by (47). One concludes by Eq. (53). □

We can now bound $M_\alpha(n)$. Let $h(x) \stackrel{\text{def}}{=} x \cdot g(x)$ and note that h is smooth since g is. The following claim proves Proposition 5.2, e.g., by choosing k as the leanness of α.

Claim B.3.1. Let $n > 0$ and α in $\mathrm{CNF}(\omega^{\omega^\omega})$. If α is k-lean, then

$$M_\alpha(n) \leq h_\alpha(kn) \, .$$

Proof. By induction on α. If $\alpha = 0$, then $M_0(n) = 0 = h_0(kn)$. Otherwise, $k > 0$ and there exists α' in $\partial_n \alpha$ s.t. $M_\alpha(n) = 1 + M_{\alpha'}(g(n))$. Observe that

$\alpha' < \alpha$, and that by Proposition B.2, α' is kn-lean, which means that we can use the induction hypothesis:

$$\begin{aligned} M_\alpha(n) &= 1 + M_{\alpha'}(g(n)) \\ &\leq 1 + h_{\alpha'}(kn \cdot g(n)) && \text{(by ind. hyp.)} \\ &\leq 1 + h_{\alpha'}(kn \cdot g(kn)) && \text{(by monotonicity of } g \text{ and } h_{\alpha'}\text{, since } k > 0) \\ &= 1 + h_{\alpha'}(h(kn)) && \text{(by def. of } h) \\ &\leq 1 + h_{P_{kn}(\alpha)}(h(kn)) && \text{(by Corollary B.3 since } h(x) \geq x) \\ &= h_\alpha(kn)\,. && \text{(by Lemma 5.1)} \end{aligned}$$

□

Note that a close inspection of the proof of Proposition B.2 would allow to refine this bound to

$$M_\alpha(n) \leq h_\alpha((k-1)n - k + 1)$$

at the expense of readability. See Section D.2 for detailed comparisons with other bounds in the literature.

B.5 Proof of Theorem 5.3

Proof sketch for Theorem 5.3. Observe that $h(x) \stackrel{\text{def}}{=} x \cdot g(x)$ is in $\mathscr{F}_{\max(\gamma,2)}$. We apply Proposition 5.2, items 2–3 of Lemma 5.1, and Lemma C.15 to show that $L_{A,g}$ is bounded above by a function in

- $\mathscr{F}_\beta$ if $\gamma < \omega$ and $\beta \geq \omega$, and in
- $\mathscr{F}_{\max(\gamma,2)+\beta}$ if $\beta < \omega$.

See Section C.7 for details on the $(\mathscr{F}_\alpha)_\alpha$ hierarchy. □

C Subrecursive Hierarchies

In a few occasions (namely Lemma 5.1, (52), and (53)), we refer to results stated *without proof* by Cichoń and Wainer (1983) and Cichoń and Tahhan Bittar (1998). As we do not know where to find these proofs (they are certainly too trivial to warrant being published in full), we provide these missing proofs in this appendix, which might still be helpful for readers unaccustomed to subrecursive hierarchies. The appendix is also the occasion of checking that minor variations we have made in the definitions of the hierarchies are harmless, and of proving useful results on lean ordinal terms.

C.1 Ordinal Terms

We work as is most customary with the set Ω of *ordinal terms* following the abstract syntax

$$\alpha ::= 0 \mid \omega^\alpha \mid \alpha + \alpha$$

for ordinals below ε_0. We write 1 for ω^0 and $\alpha \cdot n$ for $\overbrace{\alpha + \cdots + \alpha}^{n \text{ times}}$. We work modulo associativity ($(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$) and idempotence ($\alpha + 0 =$

$\alpha = 0 + \alpha$) of $+$. An ordinal term α of form $\gamma + 1$ is called a *successor ordinal*. Otherwise, if not 0, it is a *limit ordinal*, usually denoted λ.

Each ordinal term α denotes a unique ordinal $ord(\alpha)$ (by interpretation into ordinal arithmetic, with $+$ denoting direct sum), from which we can define a well-ordering on terms by $\alpha' \leq \alpha$ if $ord(\alpha') \leq ord(\alpha)$. Note that the mapping of terms to ordinals is not injective, so the ordering on terms is not antisymmetric.

Ordinal terms can be in Cantor Normal Form (CNF), i.e. sums

$$\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$$

with $\alpha > \beta_1 \geq \cdots \geq \beta_m \geq 0$ with each β_i in CNF itself. We also use at times the *strict* form

$$\alpha = \omega^{\beta_1} \cdot c_1 + \cdots + \omega^{\beta_m} \cdot c_m$$

where $\alpha > \beta_1 > \cdots > \beta_m \geq 0$ and $\omega > c_1, \ldots, c_m > 0$ and each β_i in strict form—we call the c_i's *coefficients*. Terms α in CNF are in bijection with their denoted ordinals $ord(\alpha)$. We write $\mathrm{CNF}(\alpha)$ for the set of ordinal terms $\alpha' < \alpha$ in CNF; thus $\mathrm{CNF}(\varepsilon_0)$ is a subset of Ω in this view.

When working with terms in $\mathrm{CNF}(\varepsilon_0)$, the ordering has a syntactic characterization as

$$\alpha < \alpha' \Leftrightarrow \begin{cases} \alpha = 0 \text{ and } \alpha' \neq 0, \text{ or} \\ \alpha = \omega^\beta + \gamma,\ \alpha' = \omega^{\beta'} + \gamma' \text{ and} \begin{cases} \beta < \beta', \text{ or} \\ \beta = \beta' \text{ and } \gamma < \gamma'. \end{cases} \end{cases}$$

(see (21))

C.2 Predecessors and Pointwise Ordering

Fundamental Sequences. Subrecursive hierarchies are defined through assignments of *fundamental sequences* $(\lambda_x)_{x<\omega}$ for limit ordinal terms λ in Ω, verifying $\lambda_x < \lambda$ for all x and $\lambda = \sup_x \lambda_x$. One way to obtain families of fundamental sequences is to fix a particular sequence ω_x for ω and to define

$$(\gamma + \omega^{\beta+1})_x \stackrel{\text{def}}{=} \gamma + \omega^\beta \cdot \omega_x\,, \qquad (\gamma + \omega^\lambda)_x \stackrel{\text{def}}{=} \gamma + \omega^{\lambda_x}\,. \qquad \text{(see 32)}$$

We assume ω_x to be the value in x of some monotone function s s.t. $s(x) \geq x$ for all x, typically $s(x) = x$ or $s(x) = x + 1$ (as in the main text). We will see in Section C.5 how different assignments of fundamental sequences for ω influence the hierarchies of functions built from them.

Predecessors. Given an assignment of fundamental sequences, one defines the (x-indexed) *predecessor* $P_x(\alpha) < \alpha$ of an ordinal $\alpha \neq 0$ in Ω as

$$P_x(\alpha + 1) \stackrel{\text{def}}{=} \alpha\,, \qquad P_x(\lambda) \stackrel{\text{def}}{=} P_x(\lambda_x)\,. \qquad \text{(see 33)}$$

Lemma C.1. *Assume $\alpha > 0$ in Ω. Then for all x*

$$P_x(\gamma + \alpha) = \gamma + P_x(\alpha)\,, \tag{57}$$

$$\text{if } \omega_x > 0 \text{ then } P_x(\omega^\alpha) = \omega^{P_x(\alpha)} \cdot (\omega_x - 1) + P_x(\omega^{P_x(\alpha)})\,. \tag{58}$$

Proof of (57). By induction over α. For the successor case $\alpha = \beta + 1$, this goes

$$P_x(\gamma + \beta + 1) \overset{(33)}{=} \gamma + \beta \overset{(33)}{=} \gamma + P_x(\beta + 1) .$$

For the limit case $\alpha = \lambda$, this goes

$$P_x(\gamma + \lambda) \overset{(33)}{=} P_x((\gamma + \lambda)_x) \overset{(32)}{=} P_x(\gamma + \lambda_x) \overset{ih}{=} \gamma + P_x(\lambda_x) \overset{(33)}{=} \gamma + P_x(\lambda) . \square$$

Proof of (58). By induction over α. For the successor case $\alpha = \beta + 1$, this goes

$$\begin{aligned} P_x(\omega^{\beta+1}) &\overset{(33)}{=} P_x((\omega^{\beta+1})_x) \overset{(32)}{=} P_x(\omega^\beta \cdot \omega_x) \overset{(57)}{=} \omega^\beta \cdot (\omega_x - 1) + P_x(\omega^\beta) \\ &\overset{(33)}{=} \omega^{P_x(\beta+1)} \cdot (\omega_x - 1) + P_x(\omega^{P_x(\beta+1)}) . \end{aligned}$$

For the limit case $\alpha = \lambda$, this goes

$$\begin{aligned} P_x(\omega^\lambda) &\overset{(33)}{=} P_x((\omega^\lambda)_x) \overset{(32)}{=} P_x(\omega^{\lambda_x}) \overset{ih}{=} \omega^{P_x(\lambda_x)} \cdot (\omega_x - 1) + P_x(\omega^{P_x(\lambda_x)}) \\ &\overset{(33)}{=} \omega^{P_x(\lambda)} \cdot (\omega_x - 1) + P_x(\omega^{P_x(\lambda)}) . \end{aligned} \qquad \square$$

Pointwise ordering. Recall that, for $x \in \mathbb{N}$, $\prec_x$ is the smallest transitive relation satisfying:

$$\alpha \prec_x \alpha + 1 , \qquad \lambda_x \prec_x \lambda . \qquad \text{(see 45)}$$

In particular, using induction on α, one immediately sees that

$$0 \preccurlyeq_x \alpha , \qquad \text{(see 48)}$$

$$P_x(\alpha) \prec_x \alpha . \qquad (59)$$

Lemma C.2. *For all α in Ω and all x*

$$\alpha' \prec_x \alpha \textit{ implies } \gamma + \alpha' \prec_x \gamma + \alpha , \qquad \text{(see 49)}$$

$$\omega_x > 0 \textit{ and } \alpha' \prec_x \alpha \textit{ imply } \omega^{\alpha'} \prec_x \omega^\alpha . \qquad \text{(see 50)}$$

Proof. All proofs are by induction over α (NB: the case $\alpha = 0$ is impossible).
(49)**:** For the successor case $\alpha = \beta + 1$, this goes through

$$\begin{aligned} \alpha' \prec_x \beta + 1 &\text{ implies } \alpha' \preccurlyeq_x \beta && \text{(by Eq. (46))} \\ &\text{ implies } \gamma + \alpha' \preccurlyeq_x \gamma + \beta \overset{(46)}{\prec_x} \gamma + \beta + 1 . && \text{(by ind. hyp.)} \end{aligned}$$

For the limit case $\alpha = \lambda$, this goes through

$$\begin{aligned} \alpha' \prec_x \lambda &\text{ implies } \alpha' \preccurlyeq_x \lambda_x && \text{(by Eq. (46))} \\ &\text{ implies } \gamma + \alpha' \preccurlyeq_x \gamma + \lambda_x \overset{(32)}{=} (\gamma + \lambda)_x \overset{(45)}{\prec_x} \gamma + \lambda . && \text{(by ind. hyp.)} \end{aligned}$$

(50)**:** For the successor case $\alpha = \beta + 1$, we go through

$$\begin{aligned} \alpha' \prec_x \beta + 1 &\text{ implies } \alpha' \preccurlyeq_x \beta && \text{(by Eq. (46))} \\ &\text{ implies } \omega^{\alpha'} \preccurlyeq_x \omega^\beta = \omega^\beta + 0 && \text{(by ind. hyp.)} \\ &\text{ implies } \omega^{\alpha'} \preccurlyeq_x \omega^\beta + \omega^\beta \cdot (\omega_x - 1) && \text{(by Eq. (49) and (48))} \\ &\text{ implies } \omega^{\alpha'} \preccurlyeq_x \omega^\beta \cdot \omega_x = (\omega^{\beta+1})_x \overset{(45)}{\prec_x} \omega^{\beta+1} . \end{aligned}$$

For the limit case $\alpha = \lambda$, this goes through

$$\alpha' \prec_x \lambda \text{ implies } \alpha' \preccurlyeq_x \lambda_x \qquad \text{(by Eq. (46))}$$

$$\text{implies } \omega^{\alpha'} \preccurlyeq_x \omega^{\lambda_x} \stackrel{(32)}{=} (\omega^\lambda)_x \stackrel{(45)}{\prec_x} \omega^\lambda \,. \qquad \text{(by ind. hyp.)}$$

□

Lemma C.3. *Let λ be a limit ordinal in Ω and $x < y$ in $\mathbb{N}$. Then $\lambda_x \prec_y \lambda_y$, and if furthermore $\omega_x > 0$, then $\lambda_x \prec_x \lambda_y$.*

Proof. By induction over λ. Write $\omega_y = \omega_x + z$ for some $z \geq 0$ by monotonicity and $\lambda = \gamma + \omega^\alpha$ with $0 < \alpha$.

If $\alpha = \beta + 1$ is a successor, then $\lambda_x = \gamma + \omega^\beta \cdot \omega_x \preccurlyeq_y \gamma + \omega^\beta \cdot \omega_x + \omega^\beta \cdot z$ by Eq. (49) since $0 \preccurlyeq_y \omega^\beta \cdot z$. We conclude by noting that $\lambda_y = \gamma + \omega^\beta \cdot (\omega_x + z)$; the same arguments also show $\lambda_x \prec_x \lambda_y$.

If α is a limit ordinal, then $\alpha_x \prec_y \alpha_y$ by ind. hyp., hence $\lambda_x = \gamma + \omega^{\alpha_x} \prec_y \gamma + \omega^{\alpha_y} = \lambda_y$ by Eqs. (50) (applicable since $\omega_y \geq y > x \geq 0$) and (49). If $\omega_x > 0$, then the same arguments show $\lambda_x \prec_x \lambda_y$. □

Now, using Eq. (46) together with Lemma C.3, we see

Corollary C.4. *Let α, β in Ω and x, y in $\mathbb{N}$. If $x \leq y$ then $\alpha \prec_x \beta$ implies $\alpha \prec_y \beta$.*

In other words, $\prec_x \subseteq \prec_{x+1} \subseteq \prec_{x+2} \subseteq \cdots$.

One sees $\left(\bigcup_{x\in\mathbb{N}} \prec_x\right) = <$ over terms in $\mathrm{CNF}(\varepsilon_0)$ as a result of Lemma B.1 that we now set to prove. Since $\alpha \preccurlyeq_x P_x(\gamma)$ directly entails all the other statements of Lemma B.1, it is enough to prove:

Claim C.4.1. Let α, γ in $\mathrm{CNF}(\varepsilon_0)$ and x in $\mathbb{N}$. If α is x-lean, then

$$\alpha < \gamma \text{ implies } \alpha \preccurlyeq_x P_x(\gamma) \,.$$

Proof. If $\alpha = 0$, we are done so we assume $\alpha > 0$ and hence $x > 0$, thus $\alpha = \sum_{i=1}^m \omega^{\beta_i} \cdot c_i$ with $m > 0$ and $\omega_x \geq x > 0$. Working with terms in CNF allows us to employ the syntactic characterization of $<$ given in (21).

We prove the claim by induction on γ, considering two cases:

1. if $\gamma = \gamma' + 1$ is a successor then $\alpha < \gamma$ implies $\alpha \leq \gamma'$, hence $\alpha \stackrel{ih}{\preccurlyeq_x} \gamma' \stackrel{(33)}{=} P_x(\gamma)$.
2. if γ is a limit, we claim that $\alpha < \gamma_x$, from which we deduce $\alpha \stackrel{ih}{\preccurlyeq_x} P_x(\gamma_x) \stackrel{(33)}{=} P_x(\gamma)$. We consider three subcases for the claim:
   (a) if $\gamma = \omega^\lambda$ with λ a limit, then $\alpha < \gamma$ implies $\beta_1 < \lambda$, hence $\beta_1 \stackrel{ih}{\preccurlyeq_x} P_x(\lambda_x) < \lambda_x$ (since β_1 is x-lean). Thus $\alpha < \omega^{\lambda_x} = (\omega^\lambda)_x = \gamma_x$.
   (b) if $\gamma = \omega^{\beta+1}$ then $\alpha < \gamma$ implies $\beta_1 < \beta + 1$, hence $\beta_1 \leq \beta$. Now $c_1 \leq x$ since α is x-lean, hence $\alpha < \omega^{\beta_1} \cdot (c_1 + 1) \leq \omega^{\beta_1} \cdot (x+1) \leq \omega^\beta \cdot (x+1) = (\omega^{\beta+1})_x = \gamma_x$.
   (c) if $\gamma = \gamma' + \omega^\beta$ with $0 < \gamma', \beta$, then either $\alpha \leq \gamma'$, hence $\alpha < \gamma' + (\omega^\beta)_x = \gamma_x$, or $\alpha > \gamma'$, and then α can be written as $\alpha = \gamma' + \alpha'$ with $\alpha' < \omega^\beta$. In that case $\alpha' \stackrel{ih}{\preccurlyeq_x} P_x(\omega^\beta) \stackrel{(33)}{=} P_x((\omega^\beta)_x) < (\omega^\beta)_x$, hence $\alpha = \gamma' + \alpha' \stackrel{(21)}{<} \gamma' + (\omega^\beta)_x \stackrel{(32)}{=} (\gamma' + \omega^\beta)_x = \gamma_x$. □

C.3 Ordinal Indexed Functions

Let us recall several classical hierarchies from (Cichoń and Wainer, 1983; Cichoń and Tahhan Bittar, 1998). Let us fix a unary *control function* $h : \mathbb{N} \to \mathbb{N}$; we will see later in Section C.6 how hierarchies with different control functions can be related.

Inner Iteration Hierarchies. We define the hierarchy $(h^\alpha)_{\alpha\in\Omega}$ by

$$h^0(x) \stackrel{\text{def}}{=} x, \qquad h^{\alpha+1}(x) \stackrel{\text{def}}{=} h^\alpha(h(x)), \qquad h^\lambda(x) \stackrel{\text{def}}{=} h^{\lambda_x}(x). \qquad \text{(see 34)}$$

An example of inner iteration hierarchy is the *Hardy hierarchy* $(H^\alpha)_{\alpha\in\Omega}$ defined for $h(x) = x + 1$:

$$H^0(x) \stackrel{\text{def}}{=} x, \qquad H^{\alpha+1}(x) \stackrel{\text{def}}{=} H^\alpha(x+1), \qquad H^\lambda(x) \stackrel{\text{def}}{=} H^{\lambda_x}(x). \tag{60}$$

Inner and Outer Iteration Hierarchies. Again for a unary h, we can define a variant $(h_\alpha)_{\alpha\in\Omega}$ of the inner iteration hierarchies called the *length hierarchy* by Cichoń and Tahhan Bittar (1998) and defined by

$$h_0(x) \stackrel{\text{def}}{=} 0, \qquad h_{\alpha+1}(x) \stackrel{\text{def}}{=} 1 + h_\alpha(h(x)), \qquad h_\lambda(x) \stackrel{\text{def}}{=} h_{\lambda_x}(x). \qquad \text{(see 35)}$$

As before, for the successor function $h(x) = x + 1$, this yields

$$H_0(x) \stackrel{\text{def}}{=} 0, \qquad H_{\alpha+1}(x) \stackrel{\text{def}}{=} 1 + H_\alpha(x+1), \qquad H_\lambda(x) \stackrel{\text{def}}{=} H_{\lambda_x}(x). \tag{61}$$

Those hierarchies are the most closely related to the hierarchies of functions we define for the length of bad sequences.

Fast Growing Hierarchies. Last of all, the *fast growing hierarchy* $(f_\alpha)_{\alpha\in\Omega}$ is defined through

$$f_0(x) \stackrel{\text{def}}{=} h(x), \qquad f_{\alpha+1}(x) \stackrel{\text{def}}{=} f_\alpha^{\omega_x}(x), \qquad f_\lambda \stackrel{\text{def}}{=} f_{\lambda_x}(x), \qquad \text{(see 36)}$$

while its standard version (for $h(x) = x + 1$) is defined by

$$F_0(x) \stackrel{\text{def}}{=} x + 1, \qquad F_{\alpha+1}(x) \stackrel{\text{def}}{=} F_\alpha^{\omega_x}(x), \qquad F_\lambda(x) \stackrel{\text{def}}{=} F_{\lambda_x}(x). \tag{62}$$

Lemma 5.1 and a few other properties of these hierarchies can be proved by rather simple induction arguments.

Lemma C.5 (Lemma 5.1.1)**.** *For all $\alpha > 0$ in Ω and x,*

$$h_\alpha(x) = 1 + h_{P_x(\alpha)}(h(x)) \ .$$

Proof. By transfinite induction over α. For a successor ordinal $\alpha+1$, $h_{\alpha+1}(x) = 1 + h_\alpha(h(x)) = 1 + h_{P_x(\alpha+1)}(h(x))$. For a limit ordinal λ, $h_\lambda(x) = h_{\lambda_x}(x)$ is equal to $1 + h_{P_x(\lambda_x)}(h(x))$ by ind. hyp. since $\lambda_x < \lambda$, which is the same as $1 + h_{P_x(\lambda)}(h(x))$ by definition. □

The same argument shows that for all $\alpha > 0$ in Ω and x,

$$h^\alpha(x) = h^{P_x(\alpha)}(h(x)) = h^{P_x(\alpha)+1}(x) \; , \tag{63}$$

$$f_\alpha(x) = f^{\omega_x}_{P_x(\alpha)}(x) = f_{P_x(\alpha)+1}(x) \; . \tag{64}$$

Lemma C.6 (Lemma 5.1.2)**.** *Let $h(x) > x$. Then for all α in Ω and x,*

$$h_\alpha(x) \le h^\alpha(x) - x \; .$$

Proof. By induction over α. For $\alpha = 0$, $h_0(x) = 0 = x - x = h^0(x) - x$. For $\alpha > 0$,

$$\begin{aligned}
h_\alpha(x) &= 1 + h_{P_x(\alpha)}(h(x)) && \text{(by Lemma 5.1.1)}\\
&\le 1 + h^{P_x(\alpha)}(h(x)) - h(x) && \text{(by ind. hyp. since } P_x(\alpha) < \alpha)\\
&\le h^{P_x(\alpha)}(h(x)) - x && \text{(since } h(x) > x)\\
&= h^\alpha(x) - x \; . && \text{(by (63))}
\end{aligned}$$

□

Using the same argument, one can check that in particular for $h(x) = x + 1$,

$$H_\alpha(x) = H^\alpha(x) - x \; . \tag{65}$$

Lemma C.7 ((Cichoń and Wainer, 1983))**.** *For all α, γ in Ω, and x,*

$$h^{\gamma+\alpha}(x) = h^\gamma(h^\alpha(x)) \, .$$

Proof. By transfinite induction on α. For $\alpha = 0$, $h^{\gamma+0}(x) = h^\gamma(x) = h^\gamma(h^0(x))$. For a successor ordinal $\alpha + 1$, $h^{\gamma+\alpha+1}(x) = h^{\gamma+\alpha}(h(x)) \stackrel{ih}{=} h^\gamma(h^\alpha(h(x))) = h^\gamma\big(h^{\alpha+1}(x)\big)$. For a limit ordinal λ, $h^{\gamma+\lambda}(x) = h^{(\gamma+\lambda)_x}(x) = h^{\gamma+\lambda_x}(x) \stackrel{ih}{=} h^\gamma\big(h^{\lambda_x}(x)\big) = h^\gamma\big(h^\lambda(x)\big)$. □

Lemma C.8 (Lemma 5.1.3)**.** *For all β in Ω, $r < \omega$, and x,*

$$h^{\omega^\beta \cdot r}(x) = f^r_\beta(x) \; .$$

Proof. In view of Lemma C.7 and $h^0 = f^0 = Id_{\mathbb{N}}$, it is enough to prove $h^{\omega^\beta} = f_\beta$, i.e., the $r = 1$ case. We proceed by induction over β.

For the base case. $h^{\omega^0}(x) = h^1(x) \stackrel{(36)}{=} f_0(x)$.

For a successor $\boldsymbol{\beta + 1}$. $h^{\omega^{\beta+1}}(x) \stackrel{(34)}{=} h^{(\omega^{\beta+1})_x}(x) = h^{\omega^\beta \cdot \omega_x}(x) \stackrel{ih}{=} f^{\omega_x}_\beta(x) \stackrel{(36)}{=} f_{\beta+1}(x)$.

For a limit $\boldsymbol{\lambda}$. $h^{\omega^\lambda}(x) \stackrel{(34)}{=} h^{\omega^{\lambda_x}}(x) \stackrel{ih}{=} f_{\lambda_x}(x) \stackrel{(36)}{=} f_\lambda(x)$. □

C.4 Pointwise Ordering and Monotonicity

We set to prove in this section the two equations (52) and (53) stated in (Cichoń and Tahhan Bittar, 1998, Theorem 2).

Lemma C.9 (Equations (52) and (53))**.** *Let h be a monotone function with $h(x) \geq x$. Then, for all α, α' in Ω and x, y in $\mathbb{N}$,*

$$x < y \textit{ implies } h_\alpha(x) \leq h_\alpha(y) \,, \tag{see 52}$$

$$\alpha' \prec_x \alpha \textit{ implies } h_{\alpha'}(x) \leq h_\alpha(x) \,. \tag{see 53}$$

Proof. Let us first deal with $\alpha' = 0$ for (53). Then $h_0(x) = 0 \leq h_\alpha(x)$ for all α and x.

Assuming $\alpha' > 0$, the proof now proceeds by simultaneous transfinite induction over α.

For 0. Then $h_0(x) = 0 = h_0(y)$ and $\alpha' \prec_x \alpha$ is impossible.

For a successor $\boldsymbol{\alpha + 1}$. For (52), $h_{\alpha+1}(x) = 1 + h_\alpha(h(x)) \overset{ih(52)}{\leq} 1 + h_\alpha(h(y)) = h_{\alpha+1}(y)$ where the ind. hyp. on (52) can be applied since h is monotone.

For (53), we have $\alpha' \preccurlyeq_x \alpha \prec_x \alpha + 1$, hence $h_{\alpha'}(x) \overset{ih(53)}{\leq} h_\alpha(x) \overset{ih(52)}{\leq} h_\alpha(h(x)) \overset{(35)}{<} h_{\alpha+1}(x)$ where the ind. hyp. on (52) can be applied since $h(x) \geq x$.

For a limit $\boldsymbol{\lambda}$. For (52), $h_\lambda(x) = h_{\lambda_x}(x) \overset{ih(52)}{\leq} h_{\lambda_x}(y) \overset{ih(53)}{\leq} h_{\lambda_y}(y) = h_\lambda(y)$ where the ind. hyp. on (53) can be applied since $\lambda_x \prec_y \lambda_y$ by Lemma C.3.

For (53), we have $\alpha' \preccurlyeq_x \lambda_x \prec_x \lambda$ with $h_{\alpha'}(x) \overset{ih(53)}{\leq} h_{\lambda_x}(x) = h_\lambda(x)$. □

Essentially the same proof can be carried out to prove the same monotonicity properties for h^α and f_α. As the monotonicity properties of f_α will be handy in the remainder of the section, we prove them now:

Lemma C.10 ((Löb and Wainer, 1970))**.** *Let h be a function with $h(x) \geq x$. Then, for all α, α' in Ω, x, y in $\mathbb{N}$ with $\omega_x > 0$,*

$$f_\alpha(x) \geq h(x) \geq x \,. \tag{66}$$

$$\alpha' \prec_x \alpha \textit{ implies } f_{\alpha'}(x) \leq f_\alpha(x) \,, \tag{67}$$

$$x < y \textit{ and } h \textit{ monotone imply } f_\alpha(x) \leq f_\alpha(y) \,. \tag{68}$$

Proof of (66). By transfinite induction on α. For the base case, $f_0(x) = h(x) \geq x$ by hypothesis. For the successor case, assuming $f_\alpha(x) \geq h(x)$, then by induction on $n > 0$, $f_\alpha^n(x) \geq h(x)$: for $n = 1$ it holds since $f_\alpha(x) \geq h(x)$, and for $n+1$ since $f_\alpha^{n+1}(x) = f_\alpha(f_\alpha^n(x)) \geq f_\alpha(x)$ by ind. hyp. on n. Therefore $f_{\alpha+1}(x) = f_\alpha^{\omega_x}(x) \geq x$ since $\omega_x > 0$. Finally, for the limit case, $f_\lambda(x) = f_{\lambda_x}(x) \geq x$ by ind. hyp. □

Proof of (67). Let us first deal with $\alpha' = 0$. Then $f_0(x) = h(x) \leq f_\alpha(x)$ for all $x > 0$ and all α by Eq. (66).

Assuming $\alpha' > 0$, the proof proceeds by transfinite induction over α. The case $\alpha = 0$ is impossible. For the successor case, $\alpha' \preccurlyeq_x \alpha \prec_x \alpha + 1$ with

$f_{\alpha+1}(x) = f_\alpha^{\omega_x - 1}(f_\alpha(x)) \overset{(66)}{\geq} f_\alpha(x) \overset{ih}{\geq} f_{\alpha'}(x)$. For the limit case, we have $\alpha' \preccurlyeq_x \lambda_x \prec_x \lambda$ with $f_{\alpha'}(x) \overset{ih}{\leq} f_{\lambda_x}(x) = f_\lambda(x)$. □

Proof of (68). By transfinite induction over α. For the base case, $f_0(x) = h(x) \leq h(y) = f_0(y)$ since h is monotone. For the successor case, $f_{\alpha+1}(x) = f_\alpha^{\omega_x}(x) \overset{(66)}{\leq} f_\alpha^{\omega_y}(x) \overset{ih}{\leq} f_\alpha^{\omega_y}(y) = f_{\alpha+1}(y)$ using $\omega_x \leq \omega_y$. For the limit case, $f_\lambda(x) = f_{\lambda_x}(x) \overset{ih}{\leq} f_{\lambda_x}(y) \overset{(67)}{\leq} f_{\lambda_y}(y) = f_\lambda(y)$, where (67) can be applied thanks to Lemma C.3. □

C.5 Relating Different Assignments of Fundamental Sequences

The way we employ ordinal-indexed hierarchies is as *standard* ways of classifying the growth of functions, allowing to derive meaningful complexity bounds for algorithms relying on wqo's for termination. It is therefore quite important to use a standard assignment of fundamental sequences in order to be able to compare results from different sources. The definition provided in (32) is standard, and the choices $\omega_x = x$ and $\omega_x = x + 1$ can be deemed as "equally standard" in the literature. We employed $\omega_x = x + 1$ in the main text, but the reader might desire to compare this to bounds using $\omega_x = x$.

A bit of extra notation is needed: we want to compare the length hierarchies $(h_{s,\alpha})_{\alpha\in\Omega}$ for different choices of s. Recall that s is assumed to be monotone with $s(x) \geq x$, which is fulfilled by the identity function *id*.

Lemma C.11. *Let α in Ω. If $s(h(x)) \leq h(s(x))$ for all x, then $h_{s,\alpha}(x) \leq h_{id,\alpha}(s(x))$ for all x.*

Proof. By induction on α. For 0, $h_{s,0}(x) = 0 = h_{id,0}(s(x))$. For a successor ordinal $\alpha + 1$, $h_{s,\alpha+1}(x) = 1 + h_{s,\alpha}(h(x)) \overset{ih}{\leq} 1 + h_{id,\alpha}(s(h(x))) \overset{(52)}{\leq} 1 + h_{id,\alpha}(h(s(x))) = h_{id,\alpha+1}(s(x))$ since $s(h(x)) \leq h(s(x))$. For a limit ordinal λ, $h_{s,\lambda}(x) = h_{s,\lambda_x}(x) \overset{ih}{\leq} h_{id,\lambda_x}(s(x)) \overset{(53)}{\leq} h_{id,\lambda_{s(x)}}(s(x)) = h_{id,\lambda}(s(x))$ where $s(x) \geq x$ implies $\lambda_x \prec_{s(x)} \lambda_{s(x)}$ by Lemma C.3 and allows to apply (53). □

In particular, for a smooth h and $s(x) = x+1$, $h(x)+1 \leq h(x+1)$ and we can apply Lemma C.11 together with Proposition 5.2 to get a uniform bound using the standard assignment with $\omega_x = x$ instead of $\omega_x = x+1$: for all α in $\mathrm{CNF}(\omega^{\omega^\omega})$ and $n > 0$,

$$M_{\alpha,g}(n) \leq h_\alpha(kn + 1) \tag{69}$$

where k is the leanness of α and $h(x) = x \cdot g(x)$.

C.6 Relating Different Control Functions

As in Section C.5, if we are to obtain bounds in terms of a *standard* hierarchy of functions, we ought to provide bounds for $h(x) = x + 1$ as control. We are now in position to prove a statement of Cichoń and Wainer (1983):

Lemma C.12 (Lemma 5.1.4)**.** *For all γ and α in Ω, if h is monotone eventually bounded by F_γ, then f_α is eventually bounded by $F_{\gamma+\alpha}$.*

Proof. By hypothesis, there exists x_0 (which we can assume wlog. verifies $x_0 > 0$) s.t. for all $x \geq x_0$, $h(x) \leq F_\gamma(x)$. We keep this x_0 constant and show by transfinite induction on α that for all $x \geq x_0$, $f_\alpha(x) \leq F_{\gamma+\alpha}(x)$, which proves the lemma. Note that $\omega_x \geq x \geq x_0 > 0$ and thus that we can apply Lemma C.10.

For the base case 0 for all $x \geq x_0$, $f_0(x) = h(x) \leq F_\gamma(x)$ by hypothesis.

For a successor ordinal $\alpha + 1$ we first prove that for all n and all $x \geq x_0$,

$$f^n_\alpha(x) \leq F^n_{\gamma+\alpha}(x) \; . \tag{70}$$

Indeed, by induction on n, for all $x \geq x_0$,

$$\begin{aligned} f^0_\alpha(x) &= x = F^0_{\gamma+\alpha}(x) \\ f^{n+1}_\alpha(x) &= f_\alpha(f^n_\alpha(x)) \\ &\leq f_\alpha\big(F^n_{\gamma+\alpha}(x)\big) && \text{(by (68) on } f_\alpha \text{ and the ind. hyp. on } n) \\ &\leq F_{\gamma+\alpha}\big(F^n_{\gamma+\alpha}(x)\big) \\ & \qquad \text{(since by (66) } F_{\gamma+\alpha}(x) \geq x \geq x_0 \text{ and by ind. hyp. on } \alpha) \\ &= F^{n+1}_{\gamma+\alpha}(x) \; . \end{aligned}$$

Therefore

$$\begin{aligned} f_{\alpha+1}(x) &= f^x_\alpha(x) \\ &\leq F^x_{\gamma+\alpha}(x) && \text{(by (70) for } n = x) \\ &= F_{\gamma+\alpha+1}(x) \; . \end{aligned}$$

For a limit ordinal $\boldsymbol{\lambda}$ for all $x \geq x_0$, $f_\lambda(x) = f_{\lambda_x}(x) \overset{ih}{\leq} F_{\gamma+\lambda_x}(x) = F_{(\gamma+\lambda)_x}(x) = F_{\gamma+\lambda}(x)$. □

Remark C.13. Observe that the statement of Lemma C.12 is one of the few instances in this appendix where ordinal term notations matter. Indeed, nothing forces $\gamma + \alpha$ to be an ordinal term in CNF. Note that, with the exception of Lemma B.1, all the definitions and proofs given in this appendix are compatible with arbitrary ordinal terms in Ω, and not just terms in CNF, so this is not a formal issue.

The issue lies in the intuitive understanding the reader might have of a term "$\gamma+\alpha$", by interpreting $+$ as the direct sum in ordinal arithmetic. **This would be a mistake:** in a situation where two different terms α and α' denote the same ordinal $ord(\alpha) = ord(\alpha')$, we do not necessarily have $F_\alpha(x) = F_{\alpha'}(x)$: for instance, $\alpha = \omega^{\omega^0}$ and $\alpha' = \omega^0 + \omega^{\omega^0}$ denote the same ordinal ω, but $F_\alpha(2) = F_2(2) = 2^2 \cdot 2 = 2^3$ and $F_{\alpha'}(2) = F_3(2) = 2^{2^2 \cdot 2} \cdot 2^2 \cdot 2 = 2^{11}$. The reader is therefore kindly warned that the results on ordinal-indexed hierarchies in this appendix should be understood *syntactically* on ordinal terms, and not semantically on their ordinal denotations.

The natural question at this point is: how do these new fast growing functions compare to the functions indexed by terms in CNF? Indeed, we should check that e.g. $F_{\gamma+\omega^p}$ with $\gamma < \omega^\omega$ is multiply-recursive if our results are to be of any use. The most interesting case is the one where γ is finite but α infinite (which is used in the proof of Theorem 5.3):

Lemma C.14. *Let $\alpha \geq \omega$ and $0 < \gamma < \omega$ be in* CNF(ε_0)*, and $\omega_x \overset{\text{def}}{=} x$. Then, for all x, $F_{\gamma+\alpha}(x) \leq F_\alpha(x + \gamma)$.*

Proof. We first show by induction on $\alpha \geq \omega$ that

Claim C.14.1. Let $s(x) \stackrel{\text{def}}{=} x + \gamma$. Then for all x, $F_{id,\gamma+\alpha}(x) \leq F_{s,\alpha}(x)$.

base case for ω $F_{id,\gamma+\omega}(x) = F_{id,\gamma+x}(x) = F_{s,\omega}(x)$,

successor case $\alpha + 1$ with $\alpha \geq \omega$, an induction on n shows that $F^n_{id,\gamma+\alpha}(x) \leq F^n_{s,\alpha}(x)$ for all n and x using the ind. hyp. on α, thus $F_{id,\gamma+\alpha+1}(x) = F^x_{id,\gamma+\alpha}(x) \stackrel{(66)}{\leq} F^{x+\gamma}_{id,\gamma+\alpha}(x) \leq F^{x+\gamma}_{s,\alpha}(x) = F_{s,\alpha+1}(x)$,

limit case $\lambda > \omega$ $F_{id,\gamma+\lambda}(x) = F_{id,\gamma+\lambda_x}(x) \stackrel{ih}{\leq} F_{s,\lambda_x}(x) \stackrel{(67)}{\leq} F_{s,\lambda_{x+\gamma}}(x) = F_{s,\lambda}(x)$ where (67) can be applied since $\lambda_x \prec_x \lambda_{x+\gamma}$ by Lemma C.3 (applicable since $s(x) = x + \gamma > 0$).

Returning to the main proof, note that $s(x+1) = x + 1 + \gamma = s(x) + 1$, allowing to apply Lemma C.11, thus for all x,

$$\begin{aligned} F_{id,\gamma+\alpha}(x) &\leq F_{s,\alpha}(x) && \text{(by the previous claim)} \\ &= H_s^{\omega^\alpha}(x) && \text{(by Lemma 5.1.3)} \\ &\leq H_{id}^{\omega^\alpha}(s(x)) && \text{(by Lemma C.11 and (65))} \\ &= F_{id,\alpha}(s(x)) \,. && \text{(by Lemma 5.1.3)} \end{aligned}$$

□

C.7 Classes of Subrecursive Functions

We finally consider how some natural classes of recursive functions can be characterized by closure operations on subrecursive hierarchies. The best-known of these classes is the *extended Grzegorczyk hierarchy* $(\mathscr{F}_\alpha)_{\alpha\in\mathrm{CNF}(\varepsilon_0)}$ defined by Löb and Wainer (1970) on top of the fast-growing hierarchy $(F_\alpha)_{\alpha\in\mathrm{CNF}(\varepsilon_0)}$ for $\omega_x \stackrel{\text{def}}{=} x$.

Let us first provide some background on the definition and properties of $\mathscr{F}_\alpha$. The class of functions $\mathscr{F}_\alpha$ is the closure of the constant, addition, projection—including identity—, and F_α functions, under the operations of

substitution if $h_0, h_1, \ldots, h_n$ belong to the class, then so does f if

$$f(x_1, \ldots, x_n) = h_0(h_1(x_1, \ldots, x_n), \ldots, h_n(x_1, \ldots, x_n)) \,,$$

limited recursion if h_1, h_2, and h_3 belong to the class, then so does f if

$$\begin{aligned} f(0, x_1, \ldots, x_n) &= h_1(x_1, \ldots, x_n) \,, \\ f(y+1, x_1, \ldots, x_n) &= h_2(y, x_1, \ldots, x_n, f(y, x_1, \ldots, x_n)) \,, \\ f(y, x_1, \ldots, x_n) &\leq h_3(y, x_1, \ldots, x_n) \,. \end{aligned}$$

The hierarchy is strict for $\alpha > 0$, i.e. $\mathscr{F}_{\alpha'} \subsetneq \mathscr{F}_\alpha$ if $\alpha' < \alpha$, because in particular $F_{\alpha'} \notin \mathscr{F}_\alpha$. For small finite values of α, the hierarchy characterizes some well-known classes of functions:

- $\mathscr{F}_0 = \mathscr{F}_1$ contains all the linear functions, like $\lambda x.x + 3$ or $\lambda x.2x$,

- $\mathscr{F}_2$ contains all the elementary functions, like $\lambda x.2^{2^x}$,
- $\mathscr{F}_3$ contains all the tetration functions, like $\lambda x.\ \underbrace{2^{2^{\cdot^{\cdot^{\cdot^2}}}}}_{x\text{ times}}$, etc.

The union $\bigcup_{\alpha<\omega}\mathscr{F}_\alpha$ is the set of primitive-recursive functions, while F_ω is an Ackermann-like non primitive-recursive function; we call *Ackermannian* such functions that lie in $\mathscr{F}_\omega\backslash\bigcup_{\alpha<\omega}\mathscr{F}_\alpha$. Similarly, $\bigcup_{\alpha<\omega^\omega}\mathscr{F}_\alpha$ is the set of multiply-recursive functions with F_{ω^ω} a non multiply-recursive function.

The following properties (resp. Theorem 2.10 and Theorem 2.11 in (Löb and Wainer, 1970)) are useful: for all α, unary f in $\mathscr{F}_\alpha$, and x,

$$\alpha > 0 \text{ implies } \exists p,\ f(x) \leq F_\alpha^p(x)\ , \tag{71}$$

$$\exists p,\ \forall x \geq p,\ f(x) \leq F_{\alpha+1}(x)\ . \tag{72}$$

Also note that by (71), if a unary function g is bounded by some function g' in $\mathscr{F}_\alpha$ with $\alpha > 0$, then there exists p s.t. for all x, $g(x) \leq g'(x) \leq F_\alpha^p(x)$. Similarly, (72) shows that for all $x \geq p$, $g(x) \leq g'(x) \leq F_{\alpha+1}(x)$.

Let us conclude this appendix with the following slight extension of Lemma 5.1.4:

Lemma C.15. *For all $\gamma > 0$ and α, if h is monotone and eventually bounded by a function in $\mathscr{F}_\gamma$, then*

(i) if $\alpha < \omega$, f_α is bounded by a function in $\mathscr{F}_{\gamma+\alpha}$, and

(ii) if $\gamma < \omega$ and $\alpha \geq \omega$, f_α is bounded by a function in $\mathscr{F}_\alpha$.

Proof of (i). We proceed by induction on $\alpha < \omega$.

For the base case $\alpha = 0$ we have $f_0 = h$ bounded by a function in $\mathscr{F}_\gamma$ by hypothesis.

For the successor case $\alpha = k + 1$ by ind. hyp. f_k is bounded by a function in $\mathscr{F}_{\gamma+k}$, thus by (71) there exists p s.t. $f_k(x) \leq F_k^p(x)$. By induction on n, we deduce

$$f_k^n(x) \leq F_{\gamma+k}^{pn}(x)\ ; \tag{73}$$

indeed

$$f_k^0(x) = x = F_{\gamma+k}^0(x)\ ,$$

$$f_k^{n+1}(x) = f_k(f_k^n(x)) \overset{ih}{\leq} f_k(F_{\gamma+k}^{pn}(x)) \overset{(71)}{\leq} F_{\gamma+k}^p(F_{\gamma+k}^{pn}(x)) = F_{\gamma+k}^{p(n+1)}(x)\ .$$

Therefore,

$$f_{k+1}(x) = f_k^x(x) \overset{(73)}{\leq} F_{\gamma+k}^{px}(x) \overset{(68)}{\leq} F_{\gamma+k}^{px}(px) = F_{\gamma+k+1}(px)\ ,$$

where the latter function $x \mapsto F_{\gamma+k+1}(px)$ is defined by substitution from $F_{\gamma+k+1}$ and p-fold addition, and therefore belongs to $\mathscr{F}_{\gamma+k+1}$. □

Proof of (ii). By (72), there exists x_0 s.t. for all $x \geq x_0$, $h(x) \leq F_{\gamma+1}(x)$. By Lemma 5.1.4 and Lemma C.14, $f_\alpha(x) \overset{(68)}{\leq} f_\alpha(x + x_0) \leq F_\alpha(x + x_0 + \gamma + 1)$ for all x, where the latter function $x \mapsto F_\alpha(x + x_0 + \gamma + 1)$ is in $\mathscr{F}_\alpha$. □

D Additional Comments

We gather in this appendix several additional remarks comparing some of the more technical aspects of the main text with the literature.

D.1 Maximal Order Types

Definitions of Maximal Order Types. Our definition of $o(A)$ in Section 4 is the same as that of the *maximal order type* of the wpo A, which is defined as the sup of all the order types of the linearizations of $\langle A; \leq \rangle$ (de Jongh and Parikh, 1977), or equivalently as the height of the tree of bad sequences of A (Hasegawa, 1994)—this is not a mere coincidence, as we will see at the end of the section when introducing reifications.

Consider a well partial order $\langle A; \leq \rangle$. The first definition of the maximal order type of A is through *linearizations*, i.e. linear orderings $\leq'$ extending $\leq$:

$$o(A) = \sup\{|A; \leq'| \mid \leq' \text{ is a linearization of } \leq\} . \qquad \text{((de Jongh and Parikh, 1977, Definition 1.4))}$$

This definition uses the fact that well-linear orders and ordinal terms in $\mathrm{CNF}(\varepsilon_0)$ can be identified. For the second definition, organize the set of bad sequences over $\langle A; \leq \rangle$ as a prefix tree Bad, and associate an ordinal $|\sigma|$ to each node σ respecting

$$|\sigma| = \sup\{|\sigma'| + 1 \mid \sigma' \text{ is an immediate successor of } \sigma\} .$$

Write |Bad| for the root ordinal:

$$o(A) = |\mathrm{Bad}| . \qquad \text{((Hasegawa, 1994, Definition 2.7))}$$

Bijection With Algebra. The bijection between exponential nwqo's and ordinal terms in $\mathrm{CNF}(\omega^{\omega^\omega})$ is not extremely surprising. A bijection for an algebra on wpo's with fixed points instead of Kleene star is shown to hold by Hasegawa (1994), and applied to Kruskal's Theorem. The novelty in Section 4 is that everything also works for *normed* wqo's.

Finally note that using different algebraic operators can easily break this bijection. For instance, $\searrow_p$, the p-element initial segment of $\mathbb{N}$, has order type $p = o(\Gamma_p)$, but for $p \geq 2$ the two nwqo's $\searrow_p$ and Γ_p are not isomorphic.

Reifications. Equation 30 can be viewed as a controlled variant of the reification techniques usually employed to prove upper bounds on maximal order types (Simpson, 1988; Hasegawa, 1994).

A *reification* of a partial order $\langle A; \leq \rangle$ by an ordinal α is a map Bad $\to \alpha + 1$ s.t. if σ' is a suffix of σ, then $r(\sigma') < r(\sigma)$ (Simpson, 1988, Def. 4.1). If there exists such a reification, then $o(A) < \alpha + 1$ and $\langle A; \leq \rangle$ is a wpo.

Given a normed partial order A and any bad sequence $\mathbf{x} = x_0, x_1, \ldots$, we can define a control $g(x) = \max\{|x_{i+1}| + 1 \mid x = |x_i| + 1\}$ (remember that $|.|_A$ is proper) such that $\mathbf{x}$ is $(g, |x_0| + 1)$-controlled, and use (30) to associate with each (bad) suffix $x_{i+1}, x_{i+2}, \ldots$ an ordinal α_{i+1} that maximizes $L_{C(\alpha_i)}$ in (30). Since $\alpha \; \partial_n \; \alpha'$ implies $\alpha > \alpha'$ for all n, this mapping yields a well-founded sequence $\alpha_0 > \alpha_1 > \cdots$ of ordinals, of length at most $\alpha_0 = o(A)$. While not a reification stricto sensu, this association of decreasing ordinals to each suffix of any bad sequence $\mathbf{x}$ implies every bad sequence $\mathbf{x}$ to be finite and A to be a wqo, i.e., (30) implies Higman's Lemma in the finite alphabet case. A second consequence is that no choice of $o(A)$ smaller than the maximal order type of A can be compatible with an inequality like (30), since the particular linearization that gave rise to $o(A)$ yields one particular bad sequence.

D.2 Comparisons with the Literature

We provide some elements of comparison between our bounds and similar bounds found in the literature.

Lower Bounds. Let us compare our bound with the lower bound of Cichoń (2009), who constructs a (g, n)-controlled bad sequence $\mathbf{x} = x_0, x_1, \ldots$ of length

$$g_{\omega^{\omega^{p-1}}}(n) \leq L_{\Gamma_p^*,g}(n^p)$$

for $\omega_x = x$.

The n^p bound on the length of x_0 in this sequence results from an alternative definition of the norm over Γ_p^*. Let $\Gamma_{p+1} = \{a_1, \ldots, a_p, a_{p+1}\}$, and $\pi_p : \Gamma_{p+1}^* \to \Gamma_p^*$ be the projection defined by $\pi_p(a_{p+1}) = \varepsilon$ (the empty string) and $\pi_p(a_i) = a_i$ for all $1 \leq i \leq p$. The norm $\|.\|_{\Gamma_{p+1}}$ is defined by Cichoń (2009) for all x in Γ_{p+1} by

$$\|x\|_{\Gamma_{p+1}} = \max(\{\|\pi_p(x)\|_{\Gamma_p}\} \cup \{|y| \mid \exists z, z' \in \Gamma_{p+1}, x = zyz' \wedge y \in \{a_{p+1}\}^*\})$$

for $p \geq 0$ and $\|x\|_{\Gamma_0} = 0$. For instance, $\|a_2^i a_1 a_1 a_2^j a_1^k\|_{\Gamma_2} = \max(i, j, k+2)$. Observe that, if a sequence in $(\Gamma_p^*, |.|_{\Gamma_p}; \leq_{\Gamma_p})$ is (g, n)-controlled, then seeing it as a sequence in $(\Gamma_p^*, \|.\|_{\Gamma_p}; \leq_{\Gamma_p})$, it remains (g, n)-controlled; thus despite being more involved this norm could be used seamlessly in applications. But, most importantly, using this new norm does not break (20), and the entire analysis we conducted still holds. Thus, by (69), in $(\Gamma_p^*, \|.\|_{\Gamma_p}; \leq_{\Gamma_p})$,

$$g_{\omega^{\omega^{p-1}}}(n) \leq L_{\Gamma_p^*,g}(n) \leq h_{\omega^{\omega^{p-1}}}((p-1)n + 1) \ .$$

Upper Bounds. Because previous authors employed various modified subrecursive hierarchies (as we do with h_α) but did not provide any translation into the standard ones (like our Theorem 5.3), comparing our bound with theirs is very difficult. Cichoń and Tahhan Bittar (1998) show a $g_{\alpha_p}(n)$ upper bound where α_p is an ordinal with a rather complex definition (see Cichoń and Tahhan Bittar, 1998, Section 8). Weiermann (1994, Corollary 6.3) assumes $g(x+1) = g(x) + d$ for some constant d and shows a $\bar{H}^{\omega^{\omega^{p-1}}}\big((4 + p + 12 \cdot (n + 2 + d))^3\big)$ upper bound using a modified Hardy hierarchy $(\bar{H}^\alpha)_\alpha$; it is actually not clear whether this would be eventually bounded by $F_{\omega^{p-1}}$. See the next section for a discussion of how the techniques of Buchholz et al. (1994); Weiermann (1994) could be applied to our case.

D.3 Normed Systems of Fundamental Sequences

We discuss in this subsection an alternative proof of Proposition 5.2 (with an additional hygienic condition on g), which relies on the work of Buchholz et al. (1994) on alternative definitions of hierarchies, and in particular on their Theorem 4. The proof reuses some results given in Appendix B.4 (namely (52) and Proposition B.2), and the interplay between leanness and the predecessor function (Lemma B.1). Throughout the section, fix $\omega_x \overset{\text{def}}{=} x + 1$.

Let us first define a *norm* N over $\mathrm{CNF}(\varepsilon_0)$ by

$$N\alpha = \min\{k \in \mathbb{N} \mid \alpha \text{ is } k\text{-lean}\} . \tag{74}$$

One can verify

$$\forall\alpha,\ N0 \leq N\alpha \text{ and } \forall\alpha,\ N(\alpha + 1) \leq N(\alpha) + 1 . \tag{75}$$

Note that for each k and each α in $\mathrm{CNF}(\varepsilon_0)$ (thus $\alpha < \varepsilon_0$), there are only finitely many ordinal terms $\alpha' < \alpha$ with $N\alpha' \leq k$. This is useful in definitions like Eq. (76) below, where it ensures that the max operation is applied to a finite set.

Given a monotone control function g, define an alternative length hierarchy by

$$\tilde{H}_0(x) = 0, \quad \tilde{H}_\alpha(x) = \max\{1 + \tilde{H}_{\alpha'}(g(x)) \mid \alpha' < \alpha \wedge N\alpha' \leq (N\alpha) \cdot x\} . \tag{76}$$

One easily proves, by induction over α, that each $\tilde{H}_\alpha$ is monotone. By Proposition B.2, if $\alpha' \,\partial_n\, \alpha$, then $N\alpha' \leq (N\alpha)n$, and as seen in Appendix B.1 $\alpha' < \alpha$, thus for all α and all n,

$$M_\alpha(n) \leq \tilde{H}_\alpha(n) . \tag{77}$$

We could stop here: after all, which hierarchy definition constitutes the appropriate one is debatable. Nevertheless, we shall continue toward a more "standard" understanding (see (85)) of the alternative length hierarchy $(\tilde{H}_\alpha)_\alpha$ defined in (76), from which Proposition 5.2 will be quite easy to derive (see (86)).

An alternative assignment of fundamental sequences. Consider the following alternative assignment of fundamental sequences (also defined on zero and successor ordinals):

$$[0]_x = 0, \qquad [\alpha]_x = \max\{\alpha' < \alpha \mid N\alpha' \leq (N\alpha) \cdot x\} . \tag{78}$$

This almost fits the statement of (Buchholz et al., 1994, Theorem 4), which defines fundamental sequences using a "$N\alpha' \leq p(\alpha + x)$" condition for a suitable function p, instead of "$N\alpha' \leq (N\alpha) \cdot x$" as in (78). Nevertheless, we can follow the proof of (Buchholz et al., 1994, Theorem 4) and adapt it quite easily to our case.

Lemma D.1. *Let $g(x) \geq 2x$ for all x be a control function. Then, for all $\alpha > 0$ in $\mathrm{CNF}(\varepsilon_0)$ and all x,*

$$\tilde{H}_\alpha = 1 + \tilde{H}_{[\alpha]_x}(g(x)) .$$

Proof. One inequality is immediate since $[\alpha]_x$ verifies the conditions of (76) by definition, thus

$$1 + \tilde{H}_{[\alpha]_x}(g(x)) \leq \max\{1 + \tilde{H}_{\alpha'}(g(x)) \mid \alpha' < \alpha \wedge N\alpha' \leq (N\alpha) \cdot x\} \ . \qquad (79)$$

The proof of the converse inequality is more involved. Let us first show the following:

Claim D.1.1. Let $g(x) \geq 2x$ for all x be a control function. If $\alpha' < [\alpha]_x$ and $N\alpha' \leq (N\alpha) \cdot x$, then

$$N\alpha' \leq N[\alpha]_x \cdot g(x) \ . \qquad (80)$$

Proof of (80). First note that, for a successor ordinal,

$$[\alpha + 1]_x = \alpha \qquad (81)$$

since α satisfies the conditions of (78) and is the maximal ordinal to do so. Thus

$$N[\alpha + 1]_x = N\alpha \ . \qquad (82)$$

Also note that, if λ is a limit, then

$$N[\lambda]_x = (N\lambda) \cdot x \ . \qquad (83)$$

Indeed, assume $N[\lambda]_x \neq (N\lambda) \cdot x$. By definition (78), we have $[\lambda]_x < \lambda$ and $N[\lambda]_x \leq (N\lambda) \cdot x$, so this would mean $N[\lambda]_x < (N\lambda) \cdot x$. But in that case $[\lambda]_x+1 < \lambda$ since $[\lambda]_x < \lambda$ and λ is a limit, and $N([\lambda]_x+1) \leq N[\lambda]_x+1 \leq (N\lambda) \cdot x$ by (75), hence $[\lambda]_x + 1$ also satisfies the conditions of (78) with $[\lambda]_x < [\lambda]_x + 1$, a contradiction.

Let us now prove the claim itself. Note that $\alpha' < [\alpha]_x$ implies $[\alpha]_x > 0$. If α is a limit ordinal $\alpha'' + 1$, then

$$\begin{aligned}
N\alpha' &< (N(\alpha'' + 1)) \cdot x && \text{(by hyp.)}\\
&\leq (N\alpha'' + 1) \cdot x && \text{(by (75))}\\
&= (N[\alpha]_x + 1) \cdot x && \text{(by (82))}\\
&\leq (N[\alpha]_x) \cdot 2x && \text{(since } [\alpha]_x > 0)\\
&\leq (N[\alpha]_x) \cdot g(x) \ . && \text{(since } g(x) \geq 2x)
\end{aligned}$$

If α is a limit ordinal, then

$$\begin{aligned}
N\alpha' &< (N\alpha) \cdot x && \text{(by hyp.)}\\
&= (N[\alpha]_x) \cdot x && \text{(by (83))}\\
&\leq (N[\alpha]_x) \cdot g(x) \ . && \text{(since } g(x) \geq x)
\end{aligned}$$

□

Returning to the proof of Lemma D.1, we show by induction on α that

Claim D.1.2.

$$\alpha' < \alpha \text{ and } N\alpha' \leq (N\alpha) \cdot x \text{ imply } 1 + \tilde{H}_{\alpha'}(g(x)) \leq \tilde{H}_\alpha(x) \ . \qquad (84)$$

Proof of (84). We have $\tilde{H}_\alpha(x) = 1 + \tilde{H}_{[\alpha]_x}(g(x))$ by definition, and by the hypotheses of (84) we get $\alpha' \le [\alpha]_x$. If $\alpha' = [\alpha]_x$, then (84) holds. Otherwise, i.e. if $\alpha' < [\alpha]_x$, (80) shows $N\alpha' \le (N[\alpha]_x) \cdot g(x)$, and we can apply the ind. hyp. on $[\alpha]_x$:

$$\begin{aligned} 1 + \tilde{H}_{\alpha'}(g(x)) &< 2 + \tilde{H}_{\alpha'}(g^2(x)) && \text{(by monotonicity of } g \text{ and } \tilde{H}_{\alpha'}) \\ &\le 1 + \tilde{H}_{[\alpha]_x}(g(x)) && \text{(by ind. hyp.)} \\ &= \tilde{H}_\alpha(x)\ . && \square \end{aligned}$$

The previous claim implies the desired inequality and concludes the proof of Lemma D.1. □

Relating with Predecessors. We first revisit the relationship between leanness and predecessor computations (this also provides an alternative proof of Lemma B.1).

Lemma D.2. *Let α be in* CNF(ε_0) *and k in $\mathbb{N}$. If α is k-lean, then $P_k(\alpha)$ is also k-lean, and furthermore $P_k(\alpha) = \max\{\alpha'$ k-lean $\mid \alpha' \prec_k \alpha\}$.*

Proof. Let us introduce a slight variant of k-lean ordinals: let $\alpha = \omega^{\beta_1} \cdot c_1 + \cdots + \omega^{\beta_m} \cdot c_m$ be an ordinal in CNF(ε_0) with $\alpha > \beta_1 > \cdots > \beta_m$ and $\omega > c_1, \ldots, c_m > 0$. We say that α is *almost k-lean* if either (i) $c_m = k+1$ and both $\sum_{i<m} \omega^{\beta_i}$ and β_m are k-lean, or (ii) $c_m \le k$, $\sum_{i<m} \omega^{\beta_i}$ is k-lean, and β_m is almost k-lean. Note that an almost k-lean ordinal term is *not* k-lean. Here are several properties of note on almost k-lean ordinals:

Claim D.2.1. If λ is k-lean, then λ_k is almost k-lean.

By induction on λ, letting $\lambda = \omega^{\beta_1} \cdot c_1 + \cdots + \omega^{\beta_m} \cdot c_m$ as above. If β_m is a successor ordinal $\beta + 1$ (thus β is k-lean), $\lambda_k = \omega^{\beta_1} \cdot c_1 + \cdots + \omega^{\beta_m} \cdot (c_m - 1) + \omega^\beta \cdot (k+1)$ is almost k-lean. If β_m is a limit ordinal, $\lambda_k = \omega^{\beta_1} \cdot c_1 + \cdots + \omega^{\beta_m} \cdot (c_m - 1) + \omega^{(\beta_m)_k}$ is k-lean by ind. hyp. on β_m.

Claim D.2.2. If $\alpha + 1$ is almost k-lean, then α is k-lean.

If $\alpha + 1 = \omega^{\beta_1} \cdot c_1 + \cdots + \omega^{\beta_m} \cdot c_m$ as above, it means $\beta_m = 0$, thus we are in case (i) of almost k-lean ordinals with $c_m = k+1$, and $\alpha = \omega^{\beta_1} \cdot c_1 + \cdots + \omega^{\beta_m} \cdot (c_m - 1)$ is k-lean.

Claim D.2.3. If λ is almost k-lean, then λ_k is almost k-lean.

By induction on λ, letting $\lambda = \omega^{\beta_1} \cdot c_1 + \cdots + \omega^{\beta_m} \cdot c_m$ as above.

If β_m is a successor ordinal $\beta + 1$, $\lambda_k = \omega^{\beta_1} \cdot c_1 + \cdots + \omega^{\beta_m} \cdot (c_m - 1) + \omega^\beta \cdot (k+1)$, and either (i) $c_m = k+1$ and β_m is k-lean, and then λ_k also verifies (i), or (ii) $c_m \le k$ and $\beta + 1$ is almost k-lean and thus β is k-lean by the previous claim, and λ_k is again almost k-lean verifying condition (i).

If β_m is a limit ordinal, then $\lambda_k = \omega^{\beta_1} \cdot c_1 + \cdots + \omega^{\beta_m} \cdot (c_m - 1) + \omega^{(\beta_m)_k}$. Either (i) $c_m = k+1$ and β_m is k-lean, and by the previous claims $(\beta_m)_k$ is almost k-lean and λ_k is almost k-lean by condition (ii), or (ii) $c_m \le k$ and β_m is almost k-lean, and by ind. hyp. $(\beta_m)_k$ is almost k-lean, and λ_k almost k-lean by condition (ii).

The proof of the lemma is then straightforward by applications of the previous claims and the definition of the predecessor function in (33). □

Lemma D.3. *For all $\alpha > 0$ in* $\mathrm{CNF}(\varepsilon_0)$ *and x,*

$$[\alpha]_x = P_{N\alpha\cdot x}(\alpha)\;.$$

Proof. First observe that $P_{N\alpha\cdot x}(\alpha) < \alpha$, and furthermore $N(P_{N\alpha\cdot x}(\alpha)) = N\alpha$ by Lemma D.2, hence $P_{N\alpha\cdot x}(\alpha)$ satisfies the conditions of (78): $[\alpha]_x \geq P_{N\alpha\cdot x}(\alpha)$.

Conversely, let α' be such that $\alpha' < \alpha$ and $N\alpha' \leq N\alpha\cdot x$, i.e. α' is $(N\alpha\cdot x)$-lean. By Lemma B.1, $\alpha' \prec_{N\alpha\cdot x} \alpha$. Still by Lemma B.1, $\alpha' \leq P_{N\alpha\cdot x}(\alpha)$, hence $[\alpha]_x \leq P_{N\alpha\cdot x}(\alpha)$. □

Wrapping up. Combining Lemma D.1 and Lemma D.3, we obtain that for g monotone with $g(x) \geq 2x$ and for all $\alpha > 0$ and all x,

$$\tilde{H}_\alpha(x) = 1 + \tilde{H}_{P_{N\alpha\cdot x}(\alpha)}(g(x))\;. \tag{85}$$

Let us show by induction on α that

$$\tilde{H}_\alpha(x) \leq h_\alpha(N\alpha\cdot x) \tag{86}$$

where $h(x) = x\cdot g(x)$. Proposition 5.2 will then follow from (77) and (86). For $\alpha = 0$, $\tilde{H}_0(x) = 0 = h_0(0\cdot x)$. For the induction step with $\alpha > 0$,

$$\begin{aligned}
\tilde{H}_\alpha(x) &= 1 + \tilde{H}_{P_{N\alpha\cdot x}(\alpha)}(g(x)) && \text{(by (85))}\\
&\leq 1 + h_{P_{N\alpha\cdot x}(\alpha)}(N(P_{N\alpha\cdot x}(\alpha))\cdot g(x)) && \\
& && \text{(by ind. hyp. since } P_{N\alpha\cdot x}(\alpha) < \alpha)\\
&= 1 + h_{P_{N\alpha\cdot x}(\alpha)}(N\alpha\cdot x\cdot g(x)) && \text{(by Lemma D.2)}\\
&\leq 1 + h_{P_{N\alpha\cdot x}(\alpha)}(N\alpha\cdot x\cdot g(N\alpha\cdot x)) && \\
& && \text{(since } N\alpha > 0 \text{ by monotonicity of } g \text{ and } h_{P_{N\alpha\cdot x}(\alpha)})\\
&= h_\alpha(N\alpha\cdot x)\;.
\end{aligned}$$

Comparisons. Weiermann (1994) expresses his upper bound in terms of an alternative Hardy hierarchy $(\bar{H}^\alpha)_\alpha$ defined by

$$\bar{H}^0(x) = 0, \qquad \bar{H}^\alpha(x) = \max\{\bar{H}^{\alpha'}(x+1) \mid \alpha' < \alpha \wedge N'\alpha' \leq 2^{N'\alpha+x}\}\;, \tag{87}$$

where the norm function N' measures the "depth and width" of ordinal terms and is defined by

$$N'0 = 0, \qquad N'(\omega^{\beta_1} + \cdots + \omega^{\beta_m}) = 1 + \max\{m, N'\beta_1, \ldots, N'\beta_m\}\;. \tag{88}$$

Defining an assignment of fundamental sequences by

$$\{0\}_x = 0 \qquad\qquad \{\alpha\}_x = \max\{\alpha' < \alpha \mid N'\alpha' \leq 2^{N'\alpha+x}\} \tag{89}$$

we obtain by (Buchholz et al., 1994, Theorem 4) that for $\alpha > 0$

$$\bar{H}^\alpha(x) = \bar{H}^{\{\alpha\}_x}(x+1)\;. \tag{90}$$

However the similarity with the previous developments stops here: with this Hardy hierarchy and this norm, there is no direct relationship with predecessors as in Lemma D.3: Consider for instance $\alpha = \omega\cdot n$ for some $n > 1$, and thus with $N'\alpha = n$, then $P_{2^{N'\alpha+x}+1}(\alpha) = \omega\cdot(n-1)+2^{n+x}$ with norm $N'(\omega\cdot(n-1)+2^{n+x}) = 2^{n+x} + (n-1) > 2^{n+x}$, thus $P_{2^{N'\alpha+x}+1}(\alpha) \neq \{\alpha\}_x$. This makes the translation of bounds in terms of $\bar{H}^\alpha$ into more "standard" hierarchies significantly harder (in fact Weiermann does not provide any)—our particular variations of the ideas present in the work of Buchholz et al. (1994) might actually be of independent interest.

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