Symmetry Reduction in Infinite Games with Finite Branching

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Abstract. Symmetry reductions have been applied extensively for the verification of finite-state concurrent systems and hardware designs using model-checking of temporal logics such as $\mathsf{LTL}, \mathsf{CTL}$ and CTL^*, as well as real-time and probabilistic-system model-checking. In this paper we extend the technique to handle infinite-state games on graphs with finite branching where the objectives of the players can be very general. As particular applications, it is shown that the technique can be applied to reduce the state space in parity games as well as when doing model-checking of the temporal logic ATL^*.

1 Introduction

Symmetry reduction techniques have been introduced in model-checking around twenty years ago for combatting the state-space explosion in systems that posses some amount of symmetry [6, 9, 11, 5]. The idea is to merge states of a system that behave in the same way with respect to a given property φ. This provides a smaller model of the system which exhibits the same behaviors as the original model with respect to φ; therefore model-checking can be performed on the smaller model, yielding a more efficient verification procedure since the original model need not be constructed. While the technique does not guarantee a great efficiency improvement in general, it has been applied to a large number of practical cases with great success [11, 5, 6, 10, 13, 15]. These applications include extensions from traditional model-checking of finite-state transition systems to real-time systems [10] and probabilistic systems [13]. It seems that many naturally occuring instances of model-checking of concurrent and hardware systems contain symmetry and therefore the technique is very applicable.

In this paper, we extend symmetry reduction for transition systems to symmetry reduction of games. Games can be used to naturally model concurrent and reactive systems and have applications in the synthesis of programs. We expect that on practical instances, symmetry reduction in games should be as applicable as it has been in model-checking of temporal logics. Our contribution is to extend the symmetry reduction technique introduced in [9, 6] to games. A central

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result in these papers is a correspondence lemma that describes a correspondence between paths in an original model M and in a reduced model M^G. This correspondence is used to conclude that CTL^* model-checking can be performed in the reduced model instead of the original model. In our setting, the correspondence lemma describes a correspondence between *strategies* in an original game $\mathcal{M}$ and in a reduced game $\mathcal{M}^G$. This lemma can then be used to establish a correspondence between winning strategies in the original game and in the reduced game for many different types of objectives. In particular, it follows from this that ATL^* model-checking can be performed in the reduced game, and that parity games can be reduced while preserving existence of winning strategies. However, the technique is applicable for a much more general set of objectives. The proof that the reduction works for games is technically more involved than for finite-state transition systems, due to the possible irregular behaviours of an opponent player. This phenomenon leads us to apply König's Lemma [12] in order to prove the correspondence between the original game and the reduced game. In addition, our approach does not restrict to finite-state games but also works for games played on infinite graphs, provided that they have finite branching. This includes weighted games (e.g. with energy or mean-payoff objectives), pushdown games, games played on VASS, etc.

2 Preliminaries

In this paper, we consider turn-based games played by two players I and II on a graph with finite branching.

Definition 1. *A 2-player turn-based game structure is a tuple* $\mathcal{M} = (S, R, S_{\mathrm{I}}, S_{\mathrm{II}})$ *where*

- S *is a set of states;*
- $R \subseteq S \times S$ *is a total transition relation such that for each state* $s \in S$ *there is only a finite number of states* $t \in S$ *such that* $(s,t) \in R$*;*
- S_{I} *and* S_{II} *is a partition of* S*, i.e.* $S_{\mathrm{I}} \cup S_{\mathrm{II}} = S$ *and* $S_{\mathrm{I}} \cap S_{\mathrm{II}} = \emptyset$*.*

Whenever we write *game structure* (or simply *game*) in the following, we mean 2-player turn-based game structure with finite branching, unless otherwise stated. We say that a state s is owned by player $P \in \{\mathrm{I}, \mathrm{II}\}$ if $s \in S_P$. A game is played by placing a token on an initial state s_0. Then it proceeds for an infinite number of rounds where in each round, the player owning the current state (the state on which the token is currently placed) must choose to move the token to a state t such that $(s,t) \in R$.

We denote by S^*, S^+ and S^ω the set of finite sequences of states, the set of non-empty finite sequences of states and the set of infinite sequences of states respectively. For a sequence $\rho = s_0 s_1 \dots$ of states we define $\rho_i = s_i$, $\rho_{\leq i} = s_0 \dots s_i$ and $\rho_{\geq i} = s_i s_{i+1} \dots$ When ρ is finite, i.e. $\rho = s_0 \dots s_\ell$ we write $\mathrm{last}(\rho) = s_\ell$ and $|\rho| = \ell$. A play is a sequence $s_0 s_1 \dots \in S^\omega$ such that $(s_i, s_{i+1}) \in R$ for all $i \geq 0$. The set of all plays is denoted $\mathrm{Play}_{\mathcal{M}}$. For $s_0 \in S$, the set of plays with initial state s_0 is denoted $\mathrm{Play}_{\mathcal{M}}(s_0)$. A history is a prefix of a play. The set of all histories (resp. histories with initial state s_0) is denoted $\mathrm{Hist}_{\mathcal{M}}$ (resp. $\mathrm{Hist}_{\mathcal{M}}(s_0)$). A strategy for player $P \in \{\mathrm{I}, \mathrm{II}\}$ is a partial mapping $\sigma_P \colon \mathrm{Hist}_{\mathcal{M}} \to S$ defined for all histories $h \in \mathrm{Hist}_{\mathcal{M}}$ such that $\mathrm{last}(h) \in S_P$, with the requirement that $(\mathrm{last}(h), \sigma_P(h)) \in R$. We say that a play (resp. history) $\rho = s_0 s_1 \dots$ (resp. $\rho = s_0 \dots s_\ell$) is compatible with a strategy σ_P for player $P \in \{\mathrm{I}, \mathrm{II}\}$ if $\sigma_P(\rho_{\le i}) = \rho_{i+1}$ for all $i \ge 0$ (resp. $0 \le i < \ell$) such that $\rho_i \in S_P$. We write $\mathrm{Play}(s_0, \sigma_P)$ (resp. $\mathrm{Hist}(s_0, \sigma_P)$) for the set of plays (resp. histories) starting in s_0 that are compatible with σ_P. An objective is a set $\Omega \subseteq \mathrm{Play}_{\mathcal{M}}$ of plays. A play ρ satisfies an objective Ω iff $\rho \in \Omega$. We say that σ_P is a winning strategy for player $P \in \{\mathrm{I}, \mathrm{II}\}$ from state s_0 with objective Ω if $\mathrm{Play}(s_0, \sigma_P) \subseteq \Omega$. If such a strategy exists, we say that s_0 is a winning state for player P with objective Ω. The set of winning states for player P with objective Ω in game $\mathcal{M}$ is denoted $W^P_{\mathcal{M}}(\Omega)$.

3 Symmetry Reduction

In the following we fix a game $\mathcal{M} = (S, R, S_{\mathrm{I}}, S_{\mathrm{II}})$.

Definition 2. *A permutation π of S is a* symmetry *for $\mathcal{M}$ if for all $s, s' \in S$*

1. $(s, s') \in R \Leftrightarrow (\pi(s), \pi(s')) \in R$
2. $s \in S_{\mathrm{I}} \Leftrightarrow \pi(s) \in S_{\mathrm{I}}$

Let $\mathrm{Sym}_{\mathcal{M}}$ be the set of all symmetries in $\mathcal{M}$. We call a set G of symmetries a *symmetry group* if $(G, \circ)$ is a group, where $\circ$ is the composition operator defined by $(f \circ g)(x) = f(g(x))$. We consider G to be a fixed symmetry group in the rest of this section.

Definition 3. *The* orbit *$\theta(s)$ of a state s induced by G is given by*

$$\theta(s) = \{s' \in S \mid \exists \pi \in G.\ \pi(s) = s'\}.$$

Notice that when $s' \in \theta(s)$, then also $s \in \theta(s')$. The orbits induce an equivalence relation $\sim_G$ defined by $s \sim_G s'$ if, and only if, $s \in \theta(s')$. The reason for $\sim_G$ being an equivalence relation is that G is a group. The orbit $\theta(s)$ can be thought of as a set of states that have the same behavior as s with respect to the symmetry defined by G. For a sequence $\rho = s_0 s_1 \dots$ of states we define $\theta(\rho) = \theta(s_0)\theta(s_1)\dots$ From each orbit $\theta(s)$, we choose a unique state $\mathrm{rep}(\theta(s)) \in \theta(s)$ as a representative of the orbit. For a strategy σ of player $P \in \{\mathrm{I}, \mathrm{II}\}$, an initial state s_0 and a sequence $t_0 \dots t_\ell$ of orbits, we choose a unique representative history $\mathrm{rep}_{s_0,\sigma}(t_0 \dots t_\ell) = s_0 \dots s_\ell$ that is compatible with σ and such that $s_i \in t_i$ for all $0 \le i \le \ell$, provided that such a history exists; notice that the sequence $t_0 \dots t_\ell$ is arbitrary, so that it could be the case that no such representative exists. In the later case, we let $\mathrm{rep}_{s_0,\sigma}(t_0 \dots t_\ell) = \bot$.

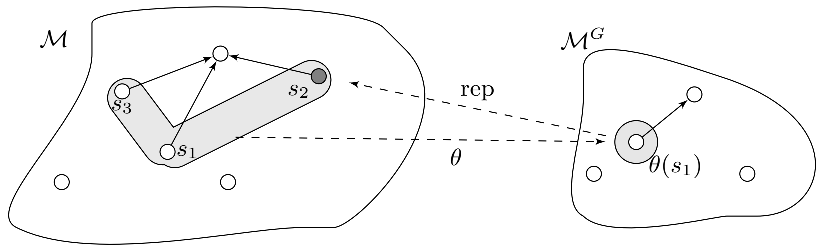
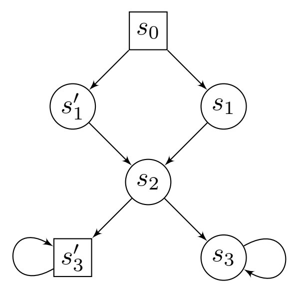


Fig. 1. Schematic representation of symmetry reduction, with three states s_1, s_2 and s_3 being in the same orbit in $\mathcal{M}$, and identified as the same state $\theta(s_1)$ in $\mathcal{M}^G$, with s_2 as its representative.

Notice that representative histories can not always "respect" prefixes (in the sense that the k-th prefix of the representative of a history is the representative of k-th prefix of that history): consider the game opposite, and the strategy σ that in s_2 goes to s_3 if s_1 was visited and to s'_3 otherwise. There is a symmetry exchanging states s_1 and s'_1 (and leaving the other states unchanged). Now, consider the sequence $h = \theta(s_0)\theta(s_1)\theta(s_2)$, and fix some representative for it (either $s_0 s_1 s_2$ or $s_0 s'_1 s_2$). Then the extensions of h with $\theta(s_3)$ and $\theta(s'_3)$ both have σ-compatible representatives, but one of them will not respect prefixes.



We are now ready to define the notion of a quotient game.

Definition 4. *Given a game $\mathcal{M} = (S, R, S_{\mathrm{I}}, S_{\mathrm{II}})$ and a symmetry group G, we define the quotient game $\mathcal{M}^G = (S^G, R^G, S^G_{\mathrm{I}}, S^G_{\mathrm{II}})$ by*

- $S^G = \{\theta(s) \mid s \in S\}$
- $R^G = \{(\theta(s), \theta(s')) \mid (s, s') \in R\}$
- $S^G_P = \{\theta(s) \mid s \in S_P\}$ *for* $P \in \{\mathrm{I}, \mathrm{II}\}$

Notice that $\mathcal{M}^G$ is indeed a game structure: symmetries respect the partition of S into S_{I} and S_{II}, and therefore S^G_{I} and S^G_{II} also constitute a partition of S^G. Also, R^G is total and has finite branching.

Example 1. Consider the game $\mathcal{M} = (S, R, S_{\mathrm{I}}, S_{\mathrm{II}})$ to the left in Fig. 2 and define

$$G = \left\{ \pi \in \mathrm{Sym}_{\mathcal{M}} \;\middle|\; \begin{array}{l} \pi(s_0, s_1, s_2, s_3, s_4, s_5) = (s_0, s_1, s_2, s_3, s_4, s_5) \\ \pi(s_0, s_1, s_2, s_3, s_4, s_5) = (s_0, s_4, s_2, s_3, s_1, s_5) \\ \pi(s_0, s_1, s_2, s_3, s_4, s_5) = (s_0, s_1, s_3, s_2, s_4, s_5) \\ \pi(s_0, s_1, s_2, s_3, s_4, s_5) = (s_0, s_4, s_3, s_2, s_1, s_5) \end{array} \right\}$$

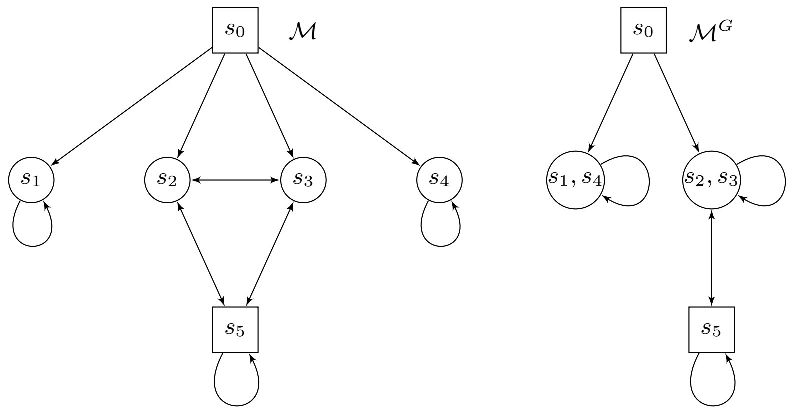


Fig. 2. A game $\mathcal{M}$ to the left that has symmetric properties and the quotient game $\mathcal{M}^G$ induced by G on the right

It is easy to see that G is a symmetry group. G now induces the orbits $\{s_0\}, \{s_5\}, \{s_2, s_3\}, \{s_1, s_4\}$. This gives rise to the quotient game $\mathcal{M}^G$ to the right in Fig. 2. Note how the construction gives us a smaller game that still has many of the structural properties of the original game.

We begin with two simple lemmas, which are not particular to our game setting and actually correspond to Lemma 3.1 of [9]. We reprove them here for the sake of completeness.

The first lemma shows a correspondence between transitions in the reduced game and transitions in the original game:

Lemma 1. *Let $(t, t') \in R^G$ be a transition in $\mathcal{M}^G$, and $s \in t$. Then there is a state s' of $\mathcal{M}$ such that $s' \in t'$ and $(s, s') \in R$.*

Proof. By definition of R^G, from the transition (t, t') in R^G, we get the existence of a transition (u, u') in R, with $u \in t$ and $u' \in t'$. Now, since s and u are in t, there is a symmetry π such that $s = \pi(u)$. By definition of a symmetry, we then have $(\pi(u), \pi(u')) \in R$ and $\pi(u') \in t'$ (because $u' \in t'$), so that letting $s' = \pi(u')$ proves the lemma.

We can extend the above correspondence to plays:

Lemma 2. *Let $\mathcal{M} = (S, R, S_{\mathrm{I}}, S_{\mathrm{II}})$ be a game and G be a symmetry group. Then*

1. *For each play $\rho \in \mathrm{Play}_{\mathcal{M}}$, there exists a play $\rho' \in \mathrm{Play}_{\mathcal{M}^G}$ such that $\rho_i \in \rho'_i$ for all $i \geq 0$;*
2. *For each play $\rho' \in \mathrm{Play}_{\mathcal{M}^G}$, and for each $s \in \rho'_0$, there exists a play $\rho \in \mathrm{Play}_{\mathcal{M}}(s)$ such that $\rho_i \in \rho'_i$ for all $i \geq 0$.*

Proof. (1) Suppose $\rho \in \mathrm{Play}_{\mathcal{M}}$. Then for every $i \geq 0$ we have $(\rho_i, \rho_{i+1}) \in R$. This implies that $(\theta(\rho_i), \theta(\rho_{i+1})) \in R^G$. Thus, $\theta(\rho) \in \mathrm{Play}_{\mathcal{M}^G}$. Since $\rho_i \in \theta(\rho_i)$ the result follows.

(2) Pick $\rho' \in \mathrm{Play}_{\mathcal{M}^G}$, and $s \in \rho'_0$. We construct a play ρ as follows. First, we let $\rho_0 = s$. Next, suppose that the history $\rho_{\leq i}$ has been constructed for some $i \geq 0$ such that $\rho_j \in \rho'_j$ for all $0 \leq j \leq i$. We have that $(\rho'_i, \rho'_{i+1}) \in R^G$, and $\rho_i \in \rho'_i$; applying Lemma 1, there must exist a state s' such that $s' \in \rho'_{i+1}$ and $(\rho_i, s') \in R$. Letting $\rho_{i+1} = s'$, we have extended our prefix $\rho_{\leq i}$ by one transition. This entails our result. □

We now show a correspondence lemma between strategies in the original game $\mathcal{M}$ and the quotient game $\mathcal{M}^G$.

Lemma 3. *Let $\mathcal{M} = (S, R, S_{\mathrm{I}}, S_{\mathrm{II}})$ be a game, G be a symmetry group, $s_0 \in S$ be an initial state, $t_0 = \theta(s_0)$ and $P \in \{\mathrm{I}, \mathrm{II}\}$. Then*

1. *For any strategy σ of player P in $\mathcal{M}$, there exists a strategy σ' of player P in $\mathcal{M}^G$ such that, for all $t_0t_1 \ldots \in \mathrm{Play}_{\mathcal{M}^G}(t_0, \sigma')$, there exists $s_0s_1 \ldots \in \mathrm{Play}_{\mathcal{M}}(s_0, \sigma)$ where $s_i \in t_i$ for all $i \geq 0$;*
2. *For any strategy σ' of player P in $\mathcal{M}^G$, there exists a strategy σ of player P in $\mathcal{M}$ such that, for all $s_0s_1 \ldots \in \mathrm{Play}_{\mathcal{M}}(s_0, \sigma)$, there exists a play $t_0t_1 \ldots \in \mathrm{Play}_{\mathcal{M}^G}(t_0, \sigma')$ where $s_i \in t_i$ for all $i \geq 0$.*

Proof. (1) Let σ be a strategy for player $P \in \{\mathrm{I}, \mathrm{II}\}$ in the original game $\mathcal{M}$. From this we construct a strategy σ' for player P in the quotient game $\mathcal{M}^G$ by

$$\sigma'(h) = \theta(\sigma(\mathrm{rep}_{s_0,\sigma}(h)))$$

for all $h \in \mathrm{Hist}_{\mathcal{M}^G}$ such that $\mathrm{rep}_{s_0,\sigma}(h) \neq \perp$ and arbitrarily when $\mathrm{rep}_{s_0,\sigma}(h) = \perp$. This strategy is well-defined, i.e., it is coherent with the transition relation. Indeed, when $\mathrm{rep}_{s_0,\sigma}(h) \neq \perp$, we have

$$\begin{aligned}
&(\mathrm{last}(\mathrm{rep}_{s_0,\sigma}(h)), \sigma(\mathrm{rep}_{s_0,\sigma}(h))) \in R \\
&\qquad \Rightarrow (\theta(\mathrm{last}(\mathrm{rep}_{s_0,\sigma}(h))), \theta(\sigma(\mathrm{rep}_{s_0,\sigma}(h)))) \in R^G \\
&\qquad \Rightarrow (\mathrm{last}(h), \sigma'(h)) \in R^G.
\end{aligned}$$

This means that there is a legal transition to the successor state prescribed by the strategy σ'.

Now, let $\rho = t_0t_1 \ldots \in \mathrm{Play}_{\mathcal{M}^G}(t_0, \sigma')$ be an arbitrary play compatible with σ' in $\mathcal{M}^G$ from t_0. We construct a directed tree T where the root is labelled by $u_0 = s_0$ and where the labelling of the infinite paths in T are exactly the plays compatible with σ in $\mathcal{M}$ from s_0. From this tree we obtain a new tree T_ρ by cutting away from T part of the branches labelled $u_0u_1 \ldots$ on which there exists $i \geq 0$ such that $u_i \notin t_i$. If j is the smallest number such that $u_j \notin t_j$ then the nodes labelled $u_ju_{j+1} \ldots$ are removed. The situation is illustrated in Fig. 3.

We assume for a contradiction that T_ρ has finite height ℓ. This means that there must be a branch in the tree labelled by the history $\mathrm{rep}_{s_0,\sigma}(t_0, \ldots, t_\ell) =$

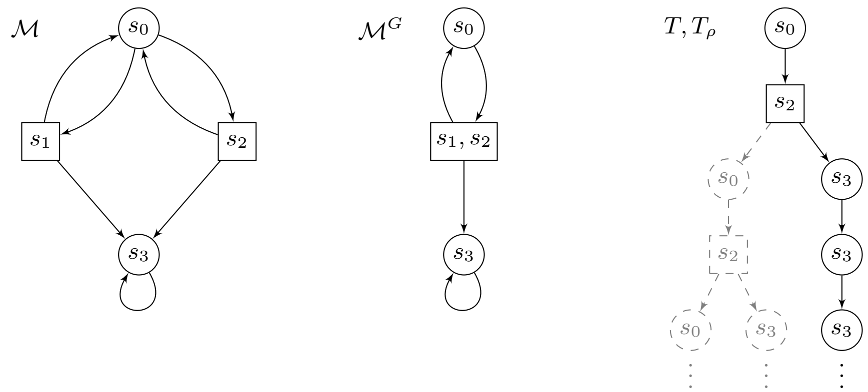


Fig. 3. From left to right is drawn the original game $\mathcal{M}$, the quotient arena $\mathcal{M}^G$ and the trees T, T_ρ where $G = \{(s_0, s_1, s_2, s_3), (s_0, s_2, s_1, s_3)\}, \sigma(h) = s_2$ for all histories h ending in s_0, and $\rho = \theta(s_0)\theta(s_1)\theta(s_3)^\omega$. T and T_ρ are drawn together: T is the whole tree, while T_ρ only consists of the solid black nodes.

$u_0 \dots u_\ell$, because if we had $\mathrm{rep}_{s_0,\sigma}(t_0, \dots, t_\ell) = \bot$ then T_ρ would have had height smaller than ℓ. There are now two cases to consider:

- Suppose $u_\ell \in S_P$. Then due to the definition of σ' we get

$$\sigma(u_0 \dots u_\ell) = \sigma(\mathrm{rep}_{s_0,\sigma}(t_0 \dots t_\ell)) \in \sigma'(t_0 \dots t_\ell) = t_{\ell+1}.$$

  Since $u_0 \dots u_\ell \sigma(s_0 \dots s_\ell)$ is compatible with σ and $u_i \in t_i$ for $0 \leq i \leq \ell$ then $u_0 \dots u_\ell \sigma(s_0 \dots s_\ell)$ is the labelling of a path in T_ρ, which gives a contradiction since it has length $\ell + 1$.
- Suppose $u_\ell \notin S_P$. Applying Lemma 1 for $(t_\ell, t_{\ell+1}) \in R_G$ and u_l, we get a state $v \in t_{\ell+1}$ such that $(u_\ell, v) \in R$. Since u_ℓ is not in S_P, we get that $u_0 \dots u_\ell v$ is compatible with σ, so that it is the labelling of a path in T_ρ of length $\ell + 1$. This gives a contradiction as well.

This means that the height of T_ρ is unbounded. Still, it could be the case that all branches are finite, in case the tree has infinite branching. Assuming T_ρ is finitely branching, it must have an infinite path according to König's Lemma. Let the labelling of such a path be $s_0 s_1 \dots$ Since $s_0 s_1 \dots$ is the labelling of an infinite path in T_ρ, it is a play compatible with σ, since all infinite paths in T_ρ are infinite paths in T. Moreover, since it is an infinite path in T_ρ, it satisfies $s_i \in t_i$ for all $i \geq 0$, because otherwise it would not be present in T_ρ. This proves the first part since $t_0 t_1 \dots$ was an arbitrary play compatible with σ'.

(2) Let σ' be a strategy for player P in $\mathcal{M}^G$. Define σ from this in such a way that

$$\sigma(s_0 \dots s_\ell) \in \sigma'(\theta(s_0) \dots \theta(s_\ell))$$

for all histories $s_0 \dots s_\ell$ in $\mathcal{M}$ with $s_\ell \in S_{\mathrm{I}}$. Note that when $s_0 \dots s_\ell$ is a history in $\mathcal{M}$ then $\theta(s_0) \dots \theta(s_\ell)$ is a history in $\mathcal{M}^G$. Further, we need to check that there exists a state $s \in \sigma'(\theta(s_0) \dots \theta(s_\ell))$ such that $(s_\ell, s) \in R$ in order for the definition to make sense. This can be seen as follows. Since $(\theta(s_\ell), \sigma'(\theta(s_0) \dots \theta(s_\ell))) \in R^G$ there exists $(u, v) \in R$ such that $u \in \theta(s_\ell)$ and $v \in \sigma'(\theta(s_0) \dots \theta(s_\ell))$. This means that there exists $\pi \in G$ with $\pi(u) = s_\ell$. Now, $(u, v) \in R \Rightarrow (\pi(u), \pi(v)) \in R \Rightarrow (s_\ell, \pi(v)) \in R$. Since $\pi(v) \in \theta(v) = \sigma'(\theta(s_0) \dots \theta(s_\ell))$ the state $s = \pi(v)$ satisfies the property.

Now, suppose that $s_0 s_1 \dots \in \mathrm{Play}_{\mathcal{M}}(\sigma)$. We prove that $\theta(s_0)\theta(s_1) \dots \in \mathrm{Play}_{\mathcal{M}^G}(\sigma')$, which entails (2) since $s_i \in \theta(s_i)$ for all $i \geq 0$. For any prefix $\theta(s_0) \dots \theta(s_\ell)$ we have that

- If $\theta(s_\ell) \notin S_P^G$ then $(s_\ell, s_{\ell+1}) \in R$ implies that $(\theta(s_\ell), \theta(s_{\ell+1})) \in R^G$.
- If $\theta(s_\ell) \in S_P^G$ then $s_{\ell+1} = \sigma(s_0 \dots s_\ell) \in \sigma'(\theta(s_0) \dots \theta(s_\ell)) \Rightarrow \theta(s_{\ell+1}) = \sigma'(\theta(s_0) \dots \theta(s_\ell))$

This means that $\theta(s_0)\theta(s_1) \dots$ is indeed compatible with σ'. □

This lemma leads to desirable properties of the quotient game when certain types of objectives are considered.

Definition 5. *A symmetry group G preserves the objective Ω if for any two plays $s_0 s_1 \dots$ and $s'_0 s'_1 \dots$ in $\mathrm{Play}_{\mathcal{M}}$, if $s_0 s_1 \dots \in \Omega$ and $s_i \sim_G s'_i$ for all $i \geq 0$, then also $s'_0 s'_1 \dots \in \Omega$.*

If Ω is an objective and G is a symmetry group that preserves it, then we denote by Ω^G the objective in the quotient game $\mathcal{M}^G$ defined as $\Omega^G = \{\theta(s_0)\theta(s_1) \dots \mid s_0 s_1 \dots \in \Omega\}$. Lemma 3 gives us the following.

Theorem 1. *Let $\mathcal{M}$ be a game, G be a symmetry group that preserves the objective Ω, $P \in \{\mathrm{I}, \mathrm{II}\}$ and $s_0 \in S$. Then*

$$s_0 \in W^P_{\mathcal{M}}(\Omega) \text{ if, and only if, } \theta(s_0) \in W^P_{\mathcal{M}^G}(\Omega^G).$$

Proof. ($\Rightarrow$) Suppose player P has a winning strategy σ in $\mathcal{M}$ with objective Ω from state s_0. Then $\mathrm{Play}_{\mathcal{M}}(s_0, \sigma) \subseteq \Omega$. According to Lemma 3 there is a strategy σ' for player P in $\mathcal{M}^G$ such that for a given play $t_0 t_1 \dots \in \mathrm{Play}_{\mathcal{M}^G}(\theta(s_0), \sigma')$ there exists a play $s_0 s_1 \dots \in \mathrm{Play}_{\mathcal{M}}(s_0, \sigma)$ with $s_i \in t_i$ for all $i \geq 0$. Since G preserves Ω and $\mathrm{Play}_{\mathcal{M}}(s, \sigma) \subseteq \Omega$ this means that $t_0 t_1 \dots \in \Omega^G$. Since $t_0 t_1 \dots$ is an arbitrary play compatible with σ' from $\theta(s_0)$ we have $\mathrm{Play}_{\mathcal{M}^G}(\theta(s_0), \sigma') \subseteq \Omega^G$ and thus $\theta(s_0) \in W^P_{\mathcal{M}^G}(\theta(s_0), \sigma')$.

($\Leftarrow$) Suppose player P has a winning strategy σ' in $\mathcal{M}^G$ with objective Ω^G from state $\theta(s_0)$. Then $\mathrm{Play}_{\mathcal{M}^G}(\theta(s_0), \sigma') \subseteq \Omega^G$. According to Lemma 3 there is a strategy σ for player P in $\mathcal{M}$ such that for a given play $s_0 s_1 \dots \in \mathrm{Play}_{\mathcal{M}}(s_0, \sigma)$ there exists a play $t_0 t_1 \dots \in \mathrm{Play}_{\mathcal{M}^G}(\theta(s_0), \sigma')$ with $s_i \in t_i$ for all $i \geq 0$. Since G preserves Ω and $\mathrm{Play}_{\mathcal{M}^G}(\theta(s_0), \sigma') \subseteq \Omega^G$ this means that $s_0 s_1 \dots \in \Omega$. Since $s_0 s_1 \dots$ is an arbitrary play compatible with σ from s_0 we have $\mathrm{Play}_{\mathcal{M}}(s_0, \sigma) \subseteq \Omega$ and thus $s_0 \in W^P_{\mathcal{M}}(s_0, \sigma)$. □

Corollary 1. *Let $\mathcal{M}$ be a game, G be a symmetry group that preserves the objective Ω, $P \in \{\mathrm{I}, \mathrm{II}\}$ and $s, s' \in S$ be such that $s \sim_G s'$. Then*

$$s \in W^P_{\mathcal{M}}(\Omega) \text{ if, and only if, } s' \in W^P_{\mathcal{M}}(\Omega).$$

We have now shown the main result of this paper, namely that a winning strategy exists in the original game if, and only if, it exists in the quotient game. This also implies that there is a winning strategy from a state s in the original game if, and only if, there is a winning strategy from another state s' that belongs to the same orbit. For transition systems the correspondence between existence of paths in the original system and the quotient system as shown in Lemma 2 was enough to show that model-checking of a CTL^* formula in the original system can be reduced to model-checking the same formula in the quotient system if the symmetry group preserves the labelling [9, 6]. However, due to the possible behaviors of an opponent player we have had to generalize this result in Lemma 3 which directly leads to Theorem 1. It will be used in Section 4 to show that we can extend the symmetry reduction approach to ATL^*, even for infinite-state games. Since we apply König's Lemma in the proof, we have assumed that the games are finitely branching. We leave it as an open problem whether the technique can be generalized to infinitely branching games as well.

4 Applications

In this section we illustrate some examples of applications of Theorem 1. We look at symmetry reductions for parity games and games with properties defined in temporal logics. We also consider an example of an infinite game with a corresponding quotient game that is finite. This makes it possible for us to decide existence of winning strategies in the original game by using standard techniques on the quotient game. Notice that this could be applied to infinite-state games such as games on counter- or pushdown systems, etc. (provided that we have a suitable symmetry group at hand).

4.1 Parity games

Let $\mathcal{M} = (S, R, S_{\mathrm{I}}, S_{\mathrm{II}})$ be a game and let $c \colon S \to \{0, \ldots, k\}$ be a coloring function that assigns a color to each state of the game. From this, the corresponding parity objective is given by $\Omega_c = \{s_0 s_1 \ldots \in \mathrm{Play}_{\mathcal{M}} \mid \min \mathrm{Inf}\{c(s_i) \mid i \in \mathbb{N}\} \text{ is odd}\}$, where Inf takes as input an infinite sequence and returns the set of items that appear infinitely many times in this sequence. A parity game is a game with a parity objective [8]. We say that a symmetry group G preserves c if for all $s, s' \in S$ we have $s \sim_G s' \Rightarrow c(s) = c(s')$. When G preserves c, we define a coloring function c^G on the set of orbits by $c^G(t) = c(\mathrm{rep}(t))$ for all orbits t. Using Theorem 1 we now get the following result for parity games when we have a symmetry group preserving the coloring function.

Proposition 1. *Let $\mathcal{M} = (S, R, S_{\mathrm{I}}, S_{\mathrm{II}})$ be a game, $c\colon S \to \{0,\dots,k\}$ be a coloring function, G be a symmetry group that preserves c, $s \in S$, and $P \in \{\mathrm{I}, \mathrm{II}\}$. Then*

1. *G preserves the objective Ω_c,*
2. *$\Omega_c^G = \{\theta(s_0)\theta(s_1)\ldots \in \mathrm{Play}_{\mathcal{M}^G} \mid \min \mathrm{Inf}\{c^G(\theta(s_i) \mid i \in \mathbb{N}\}$ is odd$\}$,*
3. *$s \in W^P_{\mathcal{M}}(\Omega_c)$ if, and only if, $\theta(s) \in W^P_{\mathcal{M}^G}(\Omega_c^G)$.*

Proof. (1) Suppose $s_0 s_1 \ldots \in \Omega$ and $s'_0 s'_1 \ldots \in \mathrm{Play}_{\mathcal{M}}$ satisfy $s_i \sim_G s'_i$ for all $i \geq 0$. Then $\min \mathrm{Inf}\{c(s'_i) \mid i \in \mathbb{N}\} = \min \mathrm{Inf}\{c(s_i) \mid i \in \mathbb{N}\}$ is odd since G preserves c. Thus, $s'_0 s'_1 \ldots \in \Omega_c$ and G preserves Ω_c.
(2) This can be seen as follows

$$\begin{aligned}
\Omega_c^G &= \{\theta(s_0)\theta(s_1)\ldots \in \mathrm{Play}_{\mathcal{M}^G} \mid s_0 s_1 \ldots \in \Omega_c\} \\
&= \{\theta(s_0)\theta(s_1)\ldots \in \mathrm{Play}_{\mathcal{M}^G} \mid \min \mathrm{Inf}\{c(s_i) \mid i \in \mathbb{N}\} \text{ is odd}\} \\
&= \{\theta(s_0)\theta(s_1)\ldots \in \mathrm{Play}_{\mathcal{M}^G} \mid \min \mathrm{Inf}\{c(\mathrm{rep}(\theta(s_i))) \mid i \in \mathbb{N}\} \text{ is odd}\} \\
&= \{\theta(s_0)\theta(s_1)\ldots \in \mathrm{Play}_{\mathcal{M}^G} \mid \min \mathrm{Inf}\{c^G(\theta(s_i)) \mid i \in \mathbb{N}\} \text{ is odd}\}
\end{aligned}$$

(3) From (1), we have that G preserves Ω_c and thus, we get the result by applying Theorem 1. ☐

This means that if we have a symmetry group that preserves the coloring function we can decide existence of winning strategies in a parity game by deciding existence of winning strategies in the quotient game. Furthermore, the quotient game is also a parity game and it has the same number of colors as the original game.

Example 2. Consider again the game $\mathcal{M}$ from Example 1. Let a coloring function c be defined by $c(s_0) = c(s_1) = c(s_5) = 0$ and $c(s_2) = c(s_3) = c(s_4) = 1$. Then the symmetry group G defined in the example does not preserve c since $s_1 \sim_G s_4$ but $c(s_1) \neq c(s_4)$. However, we can define a (smaller) symmetry group G' that preserves c by

$$G' = \left\{ \pi \in \mathrm{Sym}_{\mathcal{M}} \,\middle|\, \begin{matrix} \pi(s_0,s_1,s_2,s_3,s_4,s_5) = (s_0,s_1,s_2,s_3,s_4,s_5) \\ \pi(s_0,s_1,s_2,s_3,s_4,s_5) = (s_0,s_1,s_3,s_2,s_4,s_5) \end{matrix} \right\}$$

This does not give as great a reduction as G, but on the other hand it preserves the existence of winning strategies for parity conditions defined by c.

4.2 Alternating-time temporal logic

We will show that the symmetry reduction technique can be applied for model-checking of the alternating-time temporal logic ATL^* [1, 2] as well. In this section let $\mathrm{Agt} = \{\mathrm{I}, \mathrm{II}\}$ be a fixed set of players and let AP be a finite set of proposition symbols. Then ATL^* state formulas are defined by the grammar

$$\varphi ::= p \mid \neg\varphi_1 \mid \varphi_1 \vee \varphi_2 \mid \langle\!\langle A \rangle\!\rangle \psi_1$$

where $p \in \mathrm{AP}$ is a proposition symbol, $A \subseteq \mathrm{Agt}$ is a set of players, φ_1, φ_2 are ATL^* state formulas and ψ_1 is an ATL^* path formula. ATL^* path formulae are defined by the grammar

$$\psi ::= \varphi_1 \mid \neg\psi_1 \mid \psi_1 \vee \psi_2 \mid \mathbf{X}\psi_1 \mid \psi_1 \mathbf{U} \psi_2$$

where φ_1 is an ATL^* state formula and ψ_1 and ψ_2 are ATL^* path formulas. State formulas are interpreted over states of a game whereas path formulas are interpreted over plays of a game. For all games $\mathcal{M} = (S, R, S_{\mathrm{I}}, S_{\mathrm{II}})$, labelling functions $L\colon S \to 2^{\mathrm{AP}}$, all states $s \in S$, all plays $\rho \in \mathrm{Play}_{\mathcal{M}}$, all propositions $p \in \mathrm{AP}$, all state formulas φ_1, φ_2 and all path formulas ψ_1, ψ_2 and all coalitions $A \in \mathrm{Agt}$ define the satisfaction relation $\models$ by

$$\begin{array}{ll}
\mathcal{M}, s \models p & \text{if } p \in L(s) \\
\mathcal{M}, s \models \neg\varphi_1 & \text{if } \mathcal{M}, s \not\models \varphi_1 \\
\mathcal{M}, s \models \varphi_1 \vee \varphi_2 & \text{if } \mathcal{M}, s \models \varphi_1 \text{ or } \mathcal{M}, s \models \varphi_2 \\
\mathcal{M}, s \models \langle\!\langle A \rangle\!\rangle \psi_1 & \text{if there exist strategies } (\sigma_i)_{i\in A} \text{ so that} \\
& \qquad \text{for all } \rho \in \mathrm{Play}_{\mathcal{M}}(s, (\sigma_i)_{i\in A}), \text{ we have } \mathcal{M}, \rho \models \psi_1 \\
\mathcal{M}, \rho \models \varphi_1 & \text{if } \mathcal{M}, \rho \models \varphi_1 \\
\mathcal{M}, \rho \models \neg\psi_1 & \text{if } \mathcal{M}, \rho \not\models \psi_1 \\
\mathcal{M}, \rho \models \psi_1 \vee \psi_2 & \text{if } \mathcal{M}, \rho \models \psi_1 \text{ or } \mathcal{M}, \rho \models \psi_2 \\
\mathcal{M}, \rho \models \mathbf{X}\psi_1 & \text{if } \mathcal{M}, \rho_{\geq 1} \models \psi_1 \\
\mathcal{M}, \rho \models \psi_1 \mathbf{U} \psi_2 & \text{if } \exists i \geq 0. \mathcal{M}, \rho_{\geq i} \models \psi_2 \text{ and } \forall 0 \leq j < i. \rho_{\geq j} \models \psi_1
\end{array}$$

As usual, we define the abbrevations $\psi_1 \wedge \psi_2 = \neg(\neg\psi_1 \vee \neg\psi_2)$, $\mathbf{F}\psi_1 = \top \mathbf{U} \psi_1$ and $\mathbf{G}\psi_1 = \neg\mathbf{F}\neg\psi_1$ where $\top$ is a special proposition that is true in all states. We say that a symmetry group G preserves the labelling function L if, for all $s, s' \in S$, we have $s \sim_G s' \Rightarrow L(s) = L(s')$. When G preserves L we define a labelling function L^G on the set of orbits by $L^G(t) = L(\mathrm{rep}(t))$ for all orbits t. By applying Theorem 1 we can now show that the symmetry reduction works for ATL^*.

In order to prove this result, we rely on a characterization of ATL^* equivalence in terms of alternating bisimulation [3].

Definition 6. *Let* AP *be a finite set of atomic propositions. Let $\mathcal{M} = (S, R, S_{\mathrm{I}}, S_{\mathrm{II}})$ be a game, with a labelling function $L\colon S \to 2^{\mathrm{AP}}$. Two states s and s' of S are* alternating bisimilar *if there exists a binary relation $\mathcal{B}$ over S such that*

- *$(s, s') \in \mathcal{B}$;*
- *for every $(t, t') \in \mathcal{B}$, it holds that $L(t) = L(t')$;*
- *for every $(t, t') \in \mathcal{B}$, if it holds that $t \in S_{\mathrm{I}}$ if and only if $t' \in S_{\mathrm{I}}$ then*
  - *for every u s.t. $(t, u) \in R$, there exists u' such that $(t', u') \in R$ and $(u, u') \in \mathcal{B}$;*
  - *for every u' s.t. $(t', u') \in R$, there exists u such that $(t, u) \in R$ and $(u, u') \in \mathcal{B}$;*

– *for every* $(t, t') \in \mathcal{B}$*, if it holds that* $t \in S_{\mathrm{I}}$ *if and only if* $t' \in S_{\mathrm{II}}$ *then*
  - *for every* u, u' *s.t.* $(t, u) \in R$ *and* $(t', u') \in R$ *it holds that* $(u, u') \in \mathcal{B}$*;*

Proposition 2. *Let* AP *be a finite set of atomic propositions. Let* $\mathcal{M} = (S, R, S_{\mathrm{I}}, S_{\mathrm{II}})$ *be a game, with labelling function* $L\colon S \to 2^{\mathrm{AP}}$*. Let* G *be a symmetry group that preserves* L*, and* L^G *be the quotient labelling function for* S^G*. Then for any* $s \in S$*,* s *and* $\theta(s)$ *are alternating bisimilar.*

Proof. Consider the disjoint union of $\mathcal{M}$ and $\mathcal{M}^G$, and the relation $\mathcal{B}$ defined by

$$(s, s') \in \mathcal{B} \quad \text{if, and only if,} \quad s' = \theta(s).$$

Then the first two conditions in the definition of alternating bisimilarity are fulfilled.

Now, pick $(t, t') \in \mathcal{B}$, assuming that t (hence also $t' = \theta(t)$) belongs to Player I. First, pick a successor u of t, i.e. $(t, u) \in R$. Then $(\theta(t), \theta(u)) \in R^G$ and since $(u, \theta(u)) \in \mathcal{B}$ the first condition is satisfied. Second, pick a successor u' of t', i.e. $(t', u') \in R^G$. Then there exists $v, w \in S$ such that $(v, w) \in R$, $v \in t'$ and $w \in u'$. Then there exists $\pi \in G$ such that $\pi(v) = t$. This means that $(\pi(v), \pi(w)) = (t, \pi(w)) \in R$. Since $\pi(w) \in u'$ we also have $(\pi(w), u') \in \mathcal{B}$ which means the second condition is satisfied. The proof is the same if t belongs to Player II. □

Proposition 3. *Let* $\mathcal{M} = (S, R, S_{\mathrm{I}}, S_{\mathrm{II}})$ *be a game,* $L\colon S \to 2^{\mathrm{AP}}$ *be a labelling function and* G *be a symmetry group that preserves* L*. Then for every* $s \in S$*, every* $\rho \in \mathrm{Play}_{\mathcal{M}}$*, every* ATL^* *state formula* φ *and every* ATL^* *path formula* ψ *over* AP *we have*

– $\mathcal{M}, s \models \varphi$ *if, and only if,* $\mathcal{M}^G, \theta(s) \models \varphi$
– $\mathcal{M}, \rho \models \psi$ *if, and only if,* $\mathcal{M}^G, \theta(\rho) \models \psi$

where the satisfaction relation $\models$ *in* $\mathcal{M}^G$ *is defined with respect to the labelling function* L^G*.*

Proof. This is a consequence of the results of [3] for the case of finite state games since s and $\theta(s)$ are alternating bisimilar acccording to Prop. 2. This can also be proven directly by induction on the structure of the formula, using Lemma 3 for infinite games with finite branching.

The most interesting case is $\psi = \langle\!\langle \{P\} \rangle\!\rangle \psi_1$ with $P \in \{\mathrm{I}, \mathrm{II}\}$; define the objective $\Omega_{\psi_1} = \{\rho \in \mathrm{Play}_{\mathcal{M}} \mid \mathcal{M}, \rho \models \psi_1\}$ as the set of plays in $\mathcal{M}$ satisfying ψ_1. We will first show that G preserves Ω_{ψ_1}. Suppose $\rho \in \Omega_{\psi_1}$ and $\rho' \in \mathrm{Play}_{\mathcal{M}}$ is a play such that $\rho \sim_G \rho'$. According to the induction hypothesis, $\mathcal{M}, \rho \models \psi_1$ if and only if $\mathcal{M}^G, \theta(\rho) \models \psi_1$ but also that $\mathcal{M}, \rho' \models \psi_1$ if and only if $\mathcal{M}^G, \theta(\rho') \models \psi_1$. Since $\theta(\rho) = \theta(\rho')$ we have that ρ' satisfies ψ_1 since ρ does. Thus, $\rho' \in \Omega_{\psi_1}$ which means that G preserves Ω_{ψ_1}. Then by the induction hypothesis we have

$$\begin{aligned}
\Omega^G_{\psi_1} &= \{\theta(\rho) \in \mathrm{Play}_{\mathcal{M}^G} \mid \rho \in \Omega_{\psi_1}\} \\
&= \{\theta(\rho) \in \mathrm{Play}_{\mathcal{M}^G} \mid \mathcal{M}, \rho \models \psi_1\} \\
&= \{\theta(\rho) \in \mathrm{Play}_{\mathcal{M}^G} \mid \mathcal{M}^G, \theta(\rho) \models \psi_1\}
\end{aligned}$$

Using this and Theorem 1 we have for all $s \in S$

$$\begin{aligned}\mathcal{M}, s \models \langle\!\langle \{P\} \rangle\!\rangle \psi_1 &\text{ iff } s \in W^P_{\mathcal{M}}(\Omega_{\psi_1}) \\ &\text{ iff } \theta(s) \in W^P_{\mathcal{M}^G}(\Omega^G_{\psi_1}) \\ &\text{ iff } \mathcal{M}^G, \theta(s) \models \langle\!\langle \{P\} \rangle\!\rangle \psi_1\end{aligned}$$ □

Remark 1. Even though the result for ATL^* was only proved in two-player games above, this can easily be extended to handle n-player games for $n \geq 3$ as well. This is the case since formulas of the form $\langle\!\langle A \rangle\!\rangle \psi$ can be evaluated at a state by letting one player control the players in coalition A and let another player control the players in coalition $\mathrm{Agt} \setminus A$.

Remark 2. Notice that the result of Prop. 3 does not extend to Strategy Logic [4, 14] or ATL with strategy contexts [7]. Considering the game depicted on Fig. 2, assume that s_2 and s_3 are labelled with p and s_5 is labelled with q. One can notice that there is a strategy of the circle player (namely, playing from s_2 to s_3 and from s_3 to s_5) under which the following two propositions hold in s_0:

- there is a strategy for the square player to end up in a p-state after two steps (namely, playing to s_2),
- there is a strategy for the square player to end up in a q-state after two steps (namely, playing to s_3).

This obviously fails in the reduced game.

Example 3. Consider the infinite game illustrated in Fig. 4 which is played on an infinite grid. Player I controls the circle states and player II controls the square states. The games starts in $(0,0)$ and in each state the player controlling the state can move up, down, left or right. The proposition p is true exactly when the first coordinate is odd. Formally, the game is defined by $\mathcal{M} = (S, R, S_{\mathrm{I}}, S_{\mathrm{II}})$ where

- $S = \mathbb{Z}^2$
- $R = \{((x_1, y_1), (x_2, y_2)) \in S \times S \mid |x_1 - x_2| + |y_1 - y_2| = 1\}$
- $S_{\mathrm{I}} = \{(x, y) \in S \mid \text{ y is even}\}$
- $S_{\mathrm{II}} = \{(x, y) \in S \mid \text{ y is odd}\}$

The labelling is defined by $L((x, y)) = \{p\}$ if x is odd and $L((x, y)) = \emptyset$ if x is even. Suppose we want to check if some ATL^* formula φ over the set $\mathrm{AP} = \{p\}$ is true in $(0,0)$. This is not necessarily easy to do in an automatic way since $\mathcal{M}$ is infinite. However, we can use symmetry reduction to obtain a finite quotient game as follows. Let us define

$$G = \{\pi \in \mathrm{Sym}_{\mathcal{M}} \mid \exists a, b \in \mathbb{Z}.\ \forall (x, y) \in S.\ \pi(x, y) = (x' + 2 \cdot a, y' + 2 \cdot b)\}.$$

It is simple to show that G is a group and also that it preserves the labelling L. Further, G induces four orbits $\theta((0,0)), \theta((0,1)), \theta((1,0))$ and $\theta((1,1))$. The corresponding quotient game can be seen in Fig. 5.

According to Prop. 3 we can just do model-checking in the quotient game since $\mathcal{M}, (0,0) \models \varphi$ if and only if $\mathcal{M}^G, \theta((0,0)) \models \varphi$. This shows how the original game can be infinite but still have a finite quotient game.

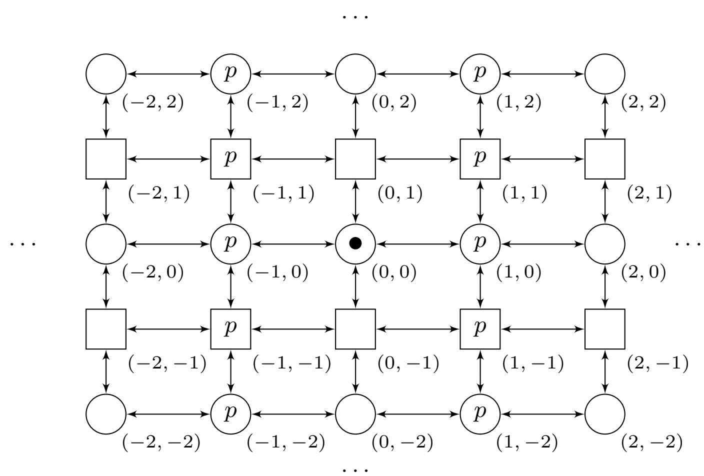


Fig. 4. Game on an infinite grid

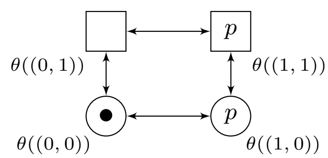


Fig. 5. Finite quotient game

5 Where do the symmetry groups come from?

Until now we have just assumed that a symmetry group G was known, but we have not mentioned how to obtain it. The short answer is that it is not tractable to find the symmetry group that gives the largest reduction in general. Indeed, even for the special case of finite-state transition systems, this problem is computationally hard. For a detailed discussion of this, see Section 6 in [6]. There it is shown that the orbit problem is as hard as the Graph Isomorphism problem when the transition system is finite: the orbit problem is to decide, for a given group G generated by a set $\{\pi_1, \ldots, \pi_n\}$ of permutations, whether two states s and s' belong to the same orbit. According to the knowledge of the authors, there is still no known polynomial time algorithm for the graph isomorphism problem. Unless the aim is to apply algorithms having high complexity in the size of the model, computing symmetries this way might not be so interesting.

While this may look quite negative, the approach has given very large speed-ups on practical verification instances. Here, it is typically the responsibility of the engineer designing the system to provide the symmetry groups as well as the orbits to the program. The main reason why this is possible is that many natural

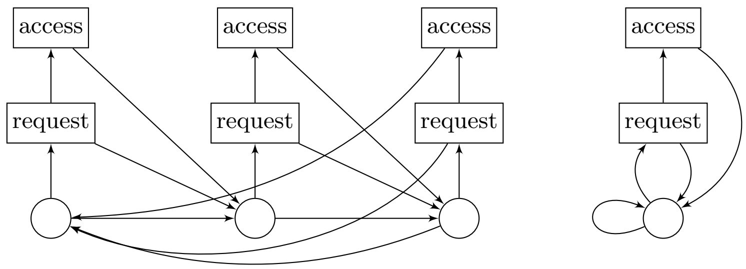


Fig. 6. A simple game $\mathcal{M}$ modeling a situation with a server and three clients is shown to the left. The smallest quotient game $\mathcal{M}^G$ such that G preserves the labelling of the propositions $\{\text{request}, \text{ access}\}$ is shown to the right.

instances of embedded, concurrent and distributed systems have a number of identical components or processes. A simple example of this can be seen in Fig. 6.

This gives rise to symmetry in the model which is quite easy to detect for a human with some amount of experience. Another approach is to design modeling languages and data structures where certain forms of symmetry can be detected automatically. For discussions of this in different contexts, see [11, 10, 13]. We have no reason to believe that the symmetry reduction technique will be less applicable for model-checking properties of games.

6 Concluding Remarks

We have proved that the symmetry reduction technique can be generalized to infinite-state turn-based games with finite branching and provided particular applications of this result in the areas of parity games and model-checking of ATL^*. The technique has not yet been implemented and tested on practical examples, but we expect that it should be as applicable as it has been in the context of model-checking of temporal logics, model-checking of real-time systems and probabilistic systems. It is still open whether the technique can be generalized to games with infinite branching since our application of König's Lemma requires that the games have finite branching.

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