The Confluence Problem for Flat TRSs

Ichiro Mitsuhashi[1], Michio Oyamaguch[1], and Florent Jacquemard[2]

[1] Mie University
{ichiro, mo}@cs.info.mie-u.ac.jp
[2] INRIA-Futurs and LSV, UMR CNRS ENS Cachan
florent.jacquemard@lsv.ens-cachan.fr

Abstract. We prove that the properties of reachability, joinability and confluence are undecidable for flat TRSs. Here, a TRS is flat if the heights of the left and right-hand sides of each rewrite rule are at most one.

Key words: Term rewriting system, Decision problem, Confluence, Flat.

1 Introduction

A term rewriting system (TRS) is a set of directed equations called rewrite rules. It defines a binary relation on terms by replacement of a subterm matching a left member of a rewrite rule by the corresponding right member. A TRS is called *confluent* (or Church-Rosser) if any two terms obtained, by the rewriting relation, from the same term are joinable. The confluence is a crucial property for the application of rewriting as a model for computation as it ensures the uniqueness of normal forms [1], and it has received much attention so far.

Confluence is undecidable in general, and even for restricted classes of TRS like monadic or semi-constructor TRSs [11]. On the other hand, decidability results have been established for several classes of TRSs, like e.g. ground (rewrite rules having no variables) TRSs [13, 4, 2], flat (left and right members of rewrite rules having height at most one) and rule-linear (a variable cannot occur more than once in a rewrite rule) TRSs [17], and more recently for flat and right-linear (a variable cannot occur more than once in a right member of rewrite rule) TRSs [6].

In this paper, we demonstrate that the above linearity restriction is necessary for decidability, showing that confluence is undecidable for flat TRSs, even with only one non-right-linear flat rewrite rule. A previous proof of this result has been published in [8]. However, we have found some technical flaws in this proof. This paper presents a correct and detailed undecidability proof, which is also significantly simpler than the one of [8].

The related properties of reachability (whether a given term can be reached from another given term by rewriting) and joinability (whether two given terms can be rewritten to the same term) are decidable for right-ground (right members of rewrite rules have no variable) TRSs [14], for right-linear monadic TRSs [15, 12], and for right-linear and finite-path-overlapping TRSs [16]. The latter two classes properly include the class of flat and right-linear TRSs. We show in this paper that reachability and joinability are undecidable if we drop the right-linearity condition, i.e. it is undecidable for general flat TRSs.

The paper is organized as follows: after giving the definitions and notations in Section 2, we show in Section 3 that reachability is undecidable for flat TRSs by reduction of the Post's correspondence problem. It follows as a corollary that joinability is also undecidable for the same class. Then, in Section 4, we show that confluence is undecidable for flat TRSs, by a reduction of reachability.

2 Preliminaries

We assume that the reader is familiar with the standard definitions of rewrite systems [5, 1] and we just recall here the main notations used in this paper.

Let ε be the empty string. Let X be a set of variables. Let F be a finite set of operation symbols graded by an arity function $\mathsf{ar}\colon F \to \mathbb{N}(= \{0, 1, 2, \cdots\})$, $F_n = \{f \in F \mid \mathsf{ar}(f) = n\}$. Let T be the set of terms built from X and F. A *substitution* is a finite mapping from X to T. As usual, we identify substitutions with their morphism extension to terms, and we use a postfix notation for the application of substitutions. We use x as a variable, f, h as function symbols, r, s, t as terms, θ as a substitution. A term s is ground if s has no variable. The *height* of a term is defined as follows: $\mathsf{height}(a) = 0$ if a is a variable or a constant and $\mathsf{height}(f(t_1, \ldots, t_n)) = 1 + \mathsf{max}\{\mathsf{height}(t_1), \ldots, \mathsf{height}(t_n)\}$ if $n > 0$.

A position in a term a sequence of positive integers, and positions are partially ordered by the prefix ordering $\geq$. Let $s|_p$ be the subterm of s at position p. Let $s \geq_{\mathsf{sub}} t$ if t is a subterm of s. For a position p and a term t, we use $s[t]_p$ to denote the term obtained from s by replacing the subterm $s|_p$ by t.

A *rewrite rule* $\alpha \to \beta$ is a directed equation over terms. A *TRS* R is a finite set of rewrite rules. A term s reduces to t at position p by the TRS R, denoted $s \xrightarrow[R]{p} t$ (p and R may be omitted), if $s|_p = \alpha\theta$ and $t = s[\beta\theta]_p$ for some rewrite rule $\alpha \to \beta$ and substitution θ. Let $\xrightarrow{=}$ be $\to \cup =$, $\leftarrow$ be the inverse of $\to$ and $\xrightarrow{*}$ be the reflexive and transitive closure of $\to$. The terms s and t are *joinable* if $s \xrightarrow{*} \cdot \xleftarrow{*} t$, which is denoted $s \downarrow t$. The term t is *reachable* from s if $s \xrightarrow{*} t$. The term r is *confluent* on the TRS R if for every peak $s \xleftarrow[R]{*} r \xrightarrow[R]{*} t$, we have $s \downarrow t$. The TRS R is *confluent* if every term is confluent on R. Let $\gamma\colon s_1 \xrightarrow{p_1} s_2 \cdots \xrightarrow{p_{n-1}} s_n$ be a *rewrite sequence*. This sequence is abbreviated by $\gamma\colon s_1 \xrightarrow{*} s_n$ and γ is called p-invariant if $p_i > p$ for every redex position p_i of γ; this is denoted by $\gamma\colon s_1 \xrightarrow{>p\ *} s_n$.

Definition 1. A rule $\alpha \to \beta$ is *flat* if $\mathsf{height}(\alpha) \leq 1$ and $\mathsf{height}(\beta) \leq 1$. A rule $\alpha \to \beta$ is *monadic* if $\mathsf{height}(\beta) \leq 1$. A term s is *shallow* if s is a variable or $s = f(s_1, \cdots, s_n)$ for some function symbol f and terms $s_1, \cdots, s_n$ such that every $s_i (1 \leq i \leq n)$ is either a variable or ground. A rule $\alpha \to \beta$ is *shallow* if both α and β are shallow. For $\mathcal{C} \in \{$flat, monadic, shallow$\}$, R is $\mathcal{C}$ if every rule in R is $\mathcal{C}$.

We are interested in the following decision problems:

Reachability Given a TRS R and two terms s, t, does there exist a rewrite sequence $s \xrightarrow[R]{*} t$?
Joinability Given a TRS R and two terms s, t, are s and t joinable, i.e., $s \downarrow_R t$?
Confluence Given a TRS R, is every term confluent on R?

Definition 2. A *finite automaton* is a 5-tuple $(Q, \Sigma, \delta, F_Q, q_0)$ where Q is a finite set of states, Σ is a finite set of input symbols, $\delta : Q \times \Sigma \to Q$ is a function, $F_Q \subseteq Q$ is a finite set of final states, and $q_0 \in Q$ is the initial state.

3 Reachability and joinability for flat TRSs

In [8], it has been reported that reachability and joinability are also undecidable for flat TRSs. But, the undecidability proof of reachability contains a flaw. We propose a repaired proof of undecidability for reachability which is simpler than the former one. The proof is a reduction of the Post's Correspondence Problem (PCP) into the reachability of a constant $\mathsf{1}$ from a constant $\mathsf{0}$ using a a flat TRS R_1. This TRS, constructed from the given instance of PCP, is such that every rewrite sequence $\mathsf{0} \xrightarrow[R_1]{*} \mathsf{1}$ contains a representation of a solution of the PCP as a term t. This property is ensured, informally, by running separately several sub-TRS of R_1 on several copies of t, where all copies have different colors and are under a function symbol of arity 6 or 7. Moreover, some equality tests are performed during the rewrite sequence using R_1, by means of a flat rewrite rules in R_1 containing some non-linear variables.

Let $P = \{\langle u_i, v_i\rangle \in \Sigma^+ \times \Sigma^+ \mid 1 \le i \le k\}$ be an instance of PCP. The goal of the problem is to find a sequence of indices $i_1, \ldots, i_n$, possibly with repetitions, such that the concatenations $u_{i_1} \ldots, u_{i_n}$ and $v_{i_1} \ldots, v_{i_n}$ are equal. Note that the alphabet Σ is fixed. Let $l_P = \mathsf{max}_{1 \le i \le k}(|u_i|, |v_i|)$. Let $\_$ be a new symbol and $\Delta = \{1, \cdots, l_P\} \times (\Sigma \cup \{\_\})^2$. We shall use a product operator $\otimes$ which associates to two non-empty strings of Σ^+ of length smaller than or equal to l_P a word of Δ^* of length l_P as follows: $a_1 \cdots a_n \otimes a'_1 \cdots a'_m = \langle 1, a_1, a'_1\rangle \cdots \langle l_P, a_{l_P}, a'_{l_P}\rangle$, where $a_1, \cdots, a_n, a'_1, \cdots, a'_m \in \Sigma$, $a_i = \_$ for all $i \in \{n+1, \cdots, l_P\}$, and $a'_j = \_$ for all $j \in \{m+1, \cdots, l_P\}$. Note that $\langle 1, \_, \_\rangle(s)$, $\langle 1, \_, a'_1\rangle(s)$, or $\langle 1, a_1, \_\rangle(s)$ can not be returned by operator $\otimes$.

Example 1. Let $l_P = 4$, then $\mathsf{a} \otimes \mathsf{bab} = \langle 1, \mathsf{a}, \mathsf{b}\rangle\langle 2, \_, \mathsf{a}\rangle\langle 3, \_, \mathsf{b}\rangle\langle 4, \_, \_\rangle$.

Let $A = (Q_A, \Delta, \delta_A, F_{Q_A}, q_A)$ and $B = (Q_B, \Sigma, \delta_B, F_{Q_B}, q_B)$ be two finite automata recognizing the respective sets $L(A) = \{u_i \otimes v_i \mid \langle u_i, v_i\rangle \in P\}^+$ and $L(B) = \Sigma^+$. We may assume that both q_A and q_B are non final. We assume that the automata A and B are clean (i.e., any state accepts some input string and is reachable from the initial state q_A(or q_B) by some input string). We associate automata A, B with the following ground TRSs T_A, T_B, respectively:

$$T_A^{(i,j)} = \{q^{(i)} \to d^{(j)}(q'^{(i)}) \mid q' \in \delta_A(q, d)\} \cup \{q^{(i)} \to \mathsf{e} \mid q \in F_{Q_A}\}$$
$$T_B^{(i,j)} = \{q^{(i)} \to a^{(j)}(q'^{(i)}) \mid q' \in \delta_B(q, a)\} \cup \{q^{(i)} \to \mathsf{e} \mid q \in F_{Q_B}\}$$

We assume given 13 disjoint copies of the above signatures, colored with color $i \in \{0, \cdots, 12\}$:

$$\Sigma^{(i)} = \{a^{(i)} \mid a \in \Sigma\} \qquad Q_A^{(i)} = \{q^{(i)} \mid q \in Q_A\}$$
$$\Delta^{(i)} = \{d^{(i)} \mid d \in \Delta\} \qquad Q_B^{(i)} = \{q^{(i)} \mid q \in Q_B\}$$

Let $\Theta^{012} = \Delta^{(0)} \cup \Sigma^{(1)} \cup \Sigma^{(2)}$, $\Theta^{345} = \Delta^{(3)} \cup \Delta^{(4)} \cup \Sigma^{(5)}$, and $Q = Q_A^{(6)} \cup Q_A^{(7)} \cup Q_A^{(8)} \cup Q_A^{(9)} \cup Q_B^{(10)} \cup Q_B^{(11)} \cup Q_A^{(12)}$. Let e be a constant. We assume that $\mathsf{ar}(f) = 1$ for every $f \in \Delta \cup \Sigma$. For a ground term t built from $\Delta \cup \Sigma \cup \{\mathsf{e}\}$, $t^{(i)}$ is defined as follows: $\mathsf{e}^{(i)} = \mathsf{e}$ and $(f(t_1))^{(i)} = f^{(i)}(t_1^{(i)})$ for $f \in \Delta \cup \Sigma$ and term t_1.

Now, we define the flat TRS R_1 on an extended signature $F_0 = Q \cup \{\mathsf{e}, \mathsf{0}, \mathsf{1}\}$, $F_1 = \Theta^{012} \cup \Theta^{345}$, $F_6 = \{\mathsf{f}\}$, and $F_7 = \{\mathsf{g}\}$. First, we color T_A and T_B:

$$T_A^{(i,j)} = \{q^{(i)} \to d^{(j)}(q'^{(i)}) \mid q' \in \delta_A(q, d)\} \cup \{q^{(i)} \to \mathsf{e} \mid q \in F_{Q_A}\}$$
$$T_B^{(i,j)} = \{q^{(i)} \to a^{(j)}(q'^{(i)}) \mid q' \in \delta_B(q, a)\} \cup \{q^{(i)} \to \mathsf{e} \mid q \in F_{Q_B}\}$$

Next, we define recoloring TRSs S, P and projection TRSs Π_1, Π_2:

$$\begin{aligned}
S^{(i,j)} &= \{a^{(i)}(x) \to a^{(j)}(x) \mid a \in \Sigma\} \\
P^{(i,j)} &= \{d^{(i)}(x) \to d^{(j)}(x) \mid d \in \Delta\} \\
\Pi_1^{(i,j)} &= \{\langle n, a, a' \rangle^{(i)}(x) \to a^{(j)}(x) \mid n \in \{1, \cdots, l_P\}, a \in \Sigma, a' \in \Sigma \cup \{_\}\} \\
&\quad \cup \{\langle n, _, a' \rangle^{(i)}(x) \to x \mid n \in \{2, \cdots, l_P\}, a' \in \Sigma \cup \{_\}\} \\
\Pi_2^{(i,j)} &= \{\langle n, a, a' \rangle^{(i)}(x) \to a'^{(j)}(x) \mid n \in \{1, \cdots, l_P\}, a \in \Sigma \cup \{_\}, a' \in \Sigma\} \\
&\quad \cup \{\langle n, a, _ \rangle^{(i)}(x) \to x \mid n \in \{2, \cdots, l_P\}, a \in \Sigma \cup \{_\}\}
\end{aligned}$$

The flat TRS R_1 is defined as follows:

$$\begin{aligned}
R_0 &= T_A^{(6,3)} \cup T_A^{(7,3)} \cup T_A^{(8,4)} \cup T_A^{(9,4)} \cup T_B^{(10,5)} \cup T_B^{(11,5)} \\
&\quad \cup P^{(3,0)} \cup \Pi_1^{(3,1)} \cup \Pi_2^{(4,2)} \cup P^{(4,0)} \cup S^{(5,1)} \cup S^{(5,2)} \cup T_A^{(12,0)} \\
R_1 &= R_0 \cup \left\{ \begin{array}{l} \mathsf{0} \to \mathsf{f}(q_A^{(6)}, q_A^{(7)}, q_A^{(8)}, q_A^{(9)}, q_B^{(10)}, q_B^{(11)}), \\ \mathsf{f}(x_3, x_3, x_4, x_4, x_5, x_5) \to \mathsf{g}(x_3, x_3, x_4, x_4, x_5, x_5, q_A^{(12)}), \\ \mathsf{g}(x_0, x_1, x_2, x_0, x_1, x_2, x_0) \to \mathsf{1} \end{array} \right\}
\end{aligned}$$

By construction of R_1, if $0 \xrightarrow[R_1]{*} 1$ then the rules of R_1 are applied as described in the following picture.

$$
\begin{array}{llllllll}
\mathsf{0} \xrightarrow{\varepsilon} \mathsf{f}(& q_A^{(6)}, & q_A^{(7)}, & q_A^{(8)}, & q_A^{(9)}, & q_B^{(10)}, & q_B^{(11)}) & \\
 & {}^*\!\downarrow_{T_A^{(6,3)}} & {}^*\!\downarrow_{T_A^{(7,3)}} & {}^*\!\downarrow_{T_A^{(8,4)}} & {}^*\!\downarrow_{T_A^{(9,4)}} & {}^*\!\downarrow_{T_B^{(10,5)}} & {}^*\!\downarrow_{T_B^{(11,5)}} & \\
\mathsf{f}(& t_3, & t_3, & t_4, & t_4, & t_5, & t_5) & \\
{}^{\varepsilon}\!\downarrow & & & & & & & \\
\mathsf{g}(& t_3, & t_3, & t_4, & t_4, & t_5, & t_5, & q_A^{(12)}) \\
 & {}^*\!\downarrow_{P^{(3,0)}} & {}^*\!\downarrow_{\Pi_1^{(3,1)}} & {}^*\!\downarrow_{\Pi_2^{(4,2)}} & {}^*\!\downarrow_{P^{(4,0)}} & {}^*\!\downarrow_{S^{(5,1)}} & {}^*\!\downarrow_{S^{(5,2)}} & {}^*\!\downarrow_{T_A^{(12,0)}} \\
\mathsf{g}(& t_0, & t_1, & t_2, & t_0, & t_1, & t_2, & t_0\ ) \quad \xrightarrow{\varepsilon} \mathsf{1}
\end{array}
$$

Indeed, each of the symbols $\mathsf{0}$ and $\mathsf{1}$ occurs in only one rewrite rule of R_1, and there is only one rule to transform the function symbol f into g.

Moreover, in the above rewrite sequence by R_1, we have a subsequence $q_A^{(12)} \xrightarrow[T_A^{(12,0)}]{*} t_0$, which means that t_0 has the form $((u_{i_1} \otimes v_{i_1}) \cdots (u_{i_m} \otimes v_{i_m})(\mathsf{e}))^{(0)}$ for some $i_1, \cdots, i_m \in \{1, \cdots, k\}$. We will show in Lemma 1 that:

$$q_A^{(6)} \xrightarrow[T_A^{(6,3)} \cup P^{(3,0)}]{*} t_0 \xleftarrow[T_A^{(9,4)} \cup P^{(4,0)}]{*} q_A^{(9)}$$

Figure 1 shows how colors are changed by the rules of R_1.

Let G_1 be the set of ground terms built from $F_0 \cup F_1 \cup F_6 \cup F_7$.

Definition 3.

(1) Let $\xrightarrow[R]{*}(s) = \{t \mid s \xrightarrow[R]{*} t\}$. For a subset $C \subseteq \{0, \cdots, 12\}$, let G^C be the intersection of G_1 and the set of ground terms built from e and colored function symbols in $\cup_{i \in C}(\Sigma^{(i)} \cup \Delta^{(i)} \cup Q_A^{(i)} \cup Q_B^{(i)})$.

(2) The index of an i-colored term built from $\Delta^{(i)} \cup \{\mathsf{e}\}$ is a string of integers defined as follows: $\mathsf{index}(\mathsf{e}) = \varepsilon$, and $\mathsf{index}(\langle n, a, a' \rangle^{(i)}(t)) = n\ \mathsf{index}(t)$.

This Lemma 1 will be useful in the proof of the following Lemma 2.

Lemma 1. Assume that $q_A^{(6)} \xrightarrow[R_0]{*} t_0$ and $q_A^{(9)} \xrightarrow[R_0]{*} t_0$ where $t_0 = ((u_{i_1} \otimes v_{i_1}) \cdots (u_{i_m} \otimes v_{i_m})(\mathsf{e}))^{(0)}$. Then, the following properties hold:

(1) $q_A^{(6)} \xrightarrow[T_A^{(6,3)} \cup P^{(3,0)}]{*} t_0$

(2) $q_A^{(9)} \xrightarrow[T_A^{(9,4)} \cup P^{(4,0)}]{*} t_0$

Proof

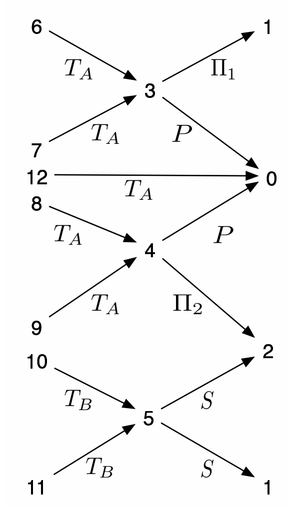

Fig. 1. Graph of the reduction of colors for R_1.

(1) By definition of R_0, $q_A^{(6)} \xrightarrow[T_A^{(6,3)} \cup P^{(3,0)} \cup \Xi_1^{(3,1)}]{*} t_0$ where $\Xi_1^{(3,1)}$ is the subset of $\Pi_1^{(3,1)}$ defined by:

$$\Xi_1^{(3,1)} = \{\langle n, \_, a' \rangle^{(3)}(x) \to x \mid n \in \{2, \cdots, l_P\}, a' \in \Sigma \cup \{\_\}\}.$$

Note that $\mathsf{index}(t_0) = (1 \cdots l_P)^m$. In this rewrite sequence, if there is at least one application of some rule of $\Xi_1^{(3,1)}$, $\mathsf{index}(t_0) = (1 \cdots l_P)^m$ does not hold, since if we applied a rule $\langle n, \_, a' \rangle^{(3)}(x) \to x$, then at most $m-1$ symbols of n would be included in $\mathsf{index}(t_0)$ whereas exactly m symbols of 1 in $\mathsf{index}(t_0)$ (since any symbol of form $\langle 1, a, a' \rangle^{(3)}$ can not be deleted). Thus, the proposition holds.

(2) Similar to (1). □

Lemma 2. $\mathsf{0} \xrightarrow[R_1]{*} \mathsf{1}$ iff the PCP P has a solution.

Proof

Only if part: by definition of R_1, we have:

$$\begin{aligned} \mathsf{0} &\xrightarrow[R_1]{} \mathsf{f}(q_A^{(6)}, q_A^{(7)}, q_A^{(8)}, q_A^{(9)}, q_B^{(10)}, q_B^{(11)}) \xrightarrow[R_0]{>\varepsilon\ *} \mathsf{f}(t_3, t_3, t_4, t_4, t_5, t_5) \\ &\xrightarrow[R_1]{} \mathsf{g}(t_3, t_3, t_4, t_4, t_5, t_5, q_A^{(12)}) \xrightarrow[R_0]{>\varepsilon\ *} \mathsf{g}(t_0, t_1, t_2, t_0, t_1, t_2, t_0) \\ &\xrightarrow[R_1]{} \mathsf{1} \end{aligned}$$

By definition of R_0:

$\xrightarrow[R_0]{*} (q_A^{(6)}) \subseteq G^{\{0,1,3,6\}}$, $\xrightarrow[R_0]{*} (q_A^{(7)}) \subseteq G^{\{0,1,3,7\}}$,
$\xrightarrow[R_0]{*} (q_A^{(8)}) \subseteq G^{\{0,2,4,8\}}$, $\xrightarrow[R_0]{*} (q_A^{(9)}) \subseteq G^{\{0,2,4,9\}}$,
$\xrightarrow[R_0]{*} (q_B^{(10)}) \subseteq G^{\{1,2,5,10\}}$, $\xrightarrow[R_0]{*} (q_B^{(11)}) \subseteq G^{\{1,2,5,11\}}$

We first show that the following condition (I) holds:

$$t_i \in G^{\{i\}} \quad \forall i \in \{0, \cdots, 5\} \tag{I}$$

Note that $t_3 \in G^{\{0,1,3\}}$ holds, since $t_3 \in \xrightarrow[R_0]{*} (q_A^{(6)}) \cap \xrightarrow[R_0]{*} (q_A^{(7)})$.
Similarly, $t_4 \in G^{\{0,2,4\}}$ and $t_5 \in G^{\{1,2,5\}}$.
Since $t_3 \xrightarrow[R_0]{*} t_0$ and $t_4 \xrightarrow[R_0]{*} t_0$, we have $t_0 \in G^{\{0,1,3\}} \cap G^{\{0,2,4\}} = G^{\{0\}}$.
Similarly, $t_1 \in G^{\{0,1,3\}} \cap G^{\{1,2,5\}} = G^{\{1\}}$ and $t_2 \in G^{\{0,2,4\}} \cap G^{\{1,2,5\}} = G^{\{2\}}$.
Hence, the condition (I) holds for $i \in \{0, 1, 2\}$.

Since $t_3 \xrightarrow[R_0]{*} t_0 \in G^{\{0\}}$ and $t_3 \xrightarrow[R_0]{*} t_1 \in G^{\{1\}}$, t_3 can not contain any symbol in $G^{\{0,1\}}$, hence $t_3 \in G^{\{3\}}$ holds.
Similarly, $t_4 \in G^{\{4\}}$ holds because $t_4 \xrightarrow[R_0]{*} t_2 \in G^{\{2\}}$ and $t_4 \xrightarrow[R_0]{*} t_0 \in G^{\{0\}}$, and $t_5 \in G^{\{5\}}$ holds because $t_5 \xrightarrow[R_0]{*} t_1 \in G^{\{1\}}$ and $t_5 \xrightarrow[R_0]{*} t_2 \in G^{\{2\}}$.
Hence, (I) holds for $i \in \{3, 4, 5\}$, as claimed.

By (I), we have: $t_3 \xrightarrow[\Pi_1^{(3,1)}]{*} t_1 \xleftarrow[S^{(5,1)}]{*} t_5$ and $t_4 \xrightarrow[\Pi_2^{(4,2)}]{*} t_2 \xleftarrow[S^{(5,2)}]{*} t_5$.
Since $q_A^{(12)} \xrightarrow[R_0]{*} t_0$, $t_0 = ((u_{i_1} \otimes v_{i_1}) \cdots (u_{i_m} \otimes v_{i_m})(\mathsf{e}))^{(0)}$ for some $i_1, \cdots, i_m \in \{1, \cdots, k\}$. We have $m > 0$ because the initial state q_A is not final.
By Lemma 1, $t_3 \xrightarrow[P^{(3,0)}]{*} t_0 \xleftarrow[P^{(4,0)}]{*} t_4$. Thus, $t_3 = ((u_{i_1} \otimes v_{i_1}) \cdots (u_{i_m} \otimes v_{i_m})(\mathsf{e}))^{(3)}$ and $t_4 = ((u_{i_1} \otimes v_{i_1}) \cdots (u_{i_m} \otimes v_{i_m})(\mathsf{e}))^{(4)}$.
Since $t_3 \xrightarrow[\Pi_1^{(3,1)}]{*} t_1$, $t_1 = (u_{i_1} \cdots u_{i_m}(\mathsf{e}))^{(1)}$,
and since $t_4 \xrightarrow[\Pi_2^{(4,2)}]{*} t_2$, $t_2 = (v_{i_1} \cdots v_{i_m}(\mathsf{e}))^{(2)}$.
Finally, $t_1 \xleftarrow[S^{(5,1)}]{*} t_5 \xrightarrow[S^{(5,2)}]{*} t_2$, hence $t_5 = (u_{i_1} \cdots u_{i_m}(\mathsf{e}))^{(5)} = (v_{i_1} \cdots v_{i_m}(\mathsf{e}))^{(5)}$.
It means that the PCP P has a solution.

If part: let $i_1 \cdots i_m$ be a solution of the PCP P, and let:

$$s = (u_{i_1} \otimes v_{i_1}) \cdots (u_{i_m} \otimes v_{i_m})(\mathsf{e}) \text{ and } t = u_{i_1} \cdots u_{i_m}(\mathsf{e})$$

Then, $t = v_{i_1} \cdots v_{i_m}(\mathsf{e})$ holds. By definition of R_1, we have:

$$\begin{aligned}
&\mathsf{0} \to \mathsf{f}(q_A^{(6)}, q_A^{(7)}, q_A^{(8)}, q_A^{(9)}, q_B^{(10)}, q_B^{(11)}) \xrightarrow{*} \mathsf{f}(s^{(3)}, s^{(3)}, s^{(4)}, s^{(4)}, t^{(5)}, t^{(5)}) \\
&\to \mathsf{g}(s^{(3)}, s^{(3)}, s^{(4)}, s^{(4)}, t^{(5)}, t^{(5)}, q_A^{(12)}) \xrightarrow{*} \mathsf{g}(s^{(0)}, t^{(1)}, t^{(2)}, s^{(0)}, t^{(1)}, t^{(2)}, s^{(0)}) \\
&\to \mathsf{1}.
\end{aligned}$$

Hence, $\mathsf{0} \xrightarrow[R_1]{*} \mathsf{1}$ □

As a consequence of Lemma 2, we have the following main theorem of this section.

Theorem 1. Reachability is undecidable for flat TRSs.

Since $\mathsf{1}$ is a normal form, $\mathsf{0} \xrightarrow[R_1]{*} \mathsf{1}$ iff $\mathsf{0} \downarrow_{R_1} \mathsf{1}$. Thus, the following corollary holds.

Corollary 1. Joinability is undecidable for flat TRSs.

Compared to the construction in [8] for Lemma 2, on the one hand, the above TRS R_1 is simpler and on the other hand, some rules have been added in order to permit the reduction $q_A^{(12)} \xrightarrow[T_A^{(12,0)}]{*} t_0$. This appeared to be necessary in order to fix a bug [8] where a reduction $\mathsf{0} \xrightarrow{*} \mathsf{1}$ was possible with the TRS associated to the PCP $\{\langle \mathsf{aa}, \mathsf{a}\rangle, \langle \mathsf{a}, \mathsf{ab}\rangle\}$ whereas it has no solution. The main reason for this counter-example is for lack of some consideration such as Lemma 1 above (derived from the existence of $q_A^{(12)} \xrightarrow{*} ((u_{i_1} \otimes v_{i_1}) \cdots (u_{i_m} \otimes v_{i_m})(\mathsf{e}))^{(0)}$ and the definition of operator $\otimes$).

4 Confluence for flat TRSs

We show that confluence for flat TRSs is undecidable by reduction of the reachability problem which has been shown undecidable in the previous section. We introduce some technical definitions in Sections 4.1.

4.1 Mapping lemma

A mapping ϕ from T to T can be extended to TRSs as follows:

$$\phi(R) = \{\phi(\alpha) \to \phi(\beta) \mid \alpha \to \beta \in R\} \setminus \{t \to t \mid t \in T\}$$

Such a mapping ϕ can also be extended to substitutions by $\phi(\theta) = \{x \mapsto \phi(x\theta) \mid x \text{ in the domain of } \theta\}$. The following lemma gives a characterization of confluence for a TRS R using $\phi(R)$.

Lemma 3. A TRS R is confluent iff there exists a mapping $\phi : T \to T$ that satisfies the following conditions (1)–(4).

(1) If $s \xrightarrow[R]{} t$ then $\phi(s) \xrightarrow[\phi(R)]{*} \phi(t)$
(2) $\xrightarrow[\phi(R)]{} \subseteq \xrightarrow[R]{*}$
(3) $t \xrightarrow[R]{*} \phi(t)$
(4) $\phi(R)$ is confluent

Proof Only if part: let ϕ be the identity mapping.
If part: assume that $s \xleftarrow[R]{*} r \xrightarrow[R]{*} t$.
By Condition (1), $\phi(s) \xleftarrow[\phi(R)]{*} \phi(r) \xrightarrow[\phi(R)]{*} \phi(t)$.
By Condition (4), $\phi(s) \downarrow_{\phi(R)} \phi(t)$.
By Condition (2), $\phi(s) \downarrow_R \phi(t)$.
By Condition (3), $s \xrightarrow[R]{*} \phi(s)$ and $t \xrightarrow[R]{*} \phi(t)$. Thus, $s \downarrow_R t$. □

This lemma is used in Section 4.2.

4.2 Proof of undecidability

Let us introduce new function symbols $\Theta_2^{012} = \{d_2 \mid d \in \Theta^{012}\}$, where each d_2 has arity 2. We add the following rules to the TRS R_1 of Section 3:

$$R_2 = R_1 \cup \{\mathsf{e} \to \mathsf{0}\} \cup \{d(x) \to d_2(\mathsf{0}, x), d_2(\mathsf{1}, x) \to x \mid d \in \Theta^{012}\}$$

Note that the TRS R_2 is flat. Let G_2 be the set of ground terms built from $F_0 \cup F_1 \cup F_6 \cup F_7 \cup \Theta_2^{012}$.

First, we show that $\mathsf{0} \xrightarrow[R_2]{*} \mathsf{1}$ iff $\mathsf{0} \xrightarrow[R_1]{*} \mathsf{1}$. For this purpose, we will introduce another reduction mapping ϕ and another TRS R_2' and show a technical lemma.

Let ψ be the mapping over G_2 defined as follows.

$$\begin{array}{rll} \psi(h(t_1, \cdots, t_n)) &= \mathsf{e} & (\text{if } h \in \{\mathsf{0}, \mathsf{1}, \mathsf{f}, \mathsf{g}\}) \\ \psi(d_2(t_1, t_2)) &= d(\psi(t_2)) & (\text{if } d \in \Theta^{012}) \\ \psi(h(t_1, \cdots, t_n)) &= h(\psi(t_1), \cdots, \psi(t_n)) & (\text{otherwise}) \end{array}$$

Let $R_2' = R_1 \cup \{\mathsf{e} \to \mathsf{0}\} \cup \{d(x) \to d_2(\mathsf{0}, x) \mid d \in \Theta^{012}\}$.

Lemma 4. For any $s \in G_2$, if $s \xrightarrow[R_2']{} t$ then $\psi(s) \xrightarrow[R_1]{=} \psi(t)$.

Proof We prove this lemma by induction on the structure of s.

Base case: if $s \in Q$ then $s = \psi(s) \xrightarrow[R_1]{} \psi(t) = t$.
If $s \in \{\mathsf{e}, \mathsf{0}\}$ then $\psi(s) = \psi(t) = \mathsf{e}$.

Induction step:
Case of $s \in \{\mathsf{f}(s_1, \cdots, s_6), \mathsf{g}(s_1, \cdots, s_7)\}$: in this case, $\psi(s) = \psi(t) = \mathsf{e}$.
Case of $s = d(s_1)$ where $d \in \Theta^{345}$: if $t = d(t_1)$ and $s_1 \xrightarrow[R_2']{} t_1$ then $\psi(s) = d(\psi(s_1)) \xrightarrow[R_1]{=} d(\psi(t_1)) = \psi(t)$ by the induction hypothesis. Otherwise either $t = d'(s_1)$ with $d' \in \Theta^{012}$ and $\psi(s) = d(\psi(s_1)) \xrightarrow[R_1]{} d'(\psi(s_1)) = \psi(t)$ or $t = s_1$ and $\psi(s) = d(\psi(s_1)) \xrightarrow[R_1]{} \psi(s_1) = \psi(t)$.
Case of $s = d(s_1)$ where $d \in \Theta^{012}$: if $t = d(t_1)$ and $s_1 \xrightarrow[R_2']{} t_1$ then $\psi(s) = d(\psi(s_1)) \xrightarrow[R_1]{=} d(\psi(t_1)) = \psi(t)$ by the induction hypothesis. Otherwise $t = d_2(\mathsf{0}, s_1)$ and $\psi(s) = \psi(t) = d(\psi(s_1))$.
Case of $s = d_2(s_1, s_2)$ where $d \in \Theta^{012}$: in this case, $t = d_2(t_1, t_2)$ holds for some t_1, t_2 and either $s_1 \xrightarrow[R_2']{} t_1$ and $s_2 = t_2$ or $s_2 \xrightarrow[R_2']{} t_2$ and $s_1 = t_1$, hence $\psi(s) = d(\psi(s_2)) \xrightarrow[R_1]{=} d(\psi(t_2)) = \psi(t)$ by the induction hypothesis. □

Lemma 5. $\mathsf{0} \xrightarrow[R_2]{*} \mathsf{1}$ iff $\mathsf{0} \xrightarrow[R_1]{*} \mathsf{1}$.

Proof The if part is obvious. For the only if part, by definition of R_2, if $\mathsf{0} \xrightarrow[R_2]{*} \mathsf{1}$ then there exists a shortest sequence γ that satisfies:

$$\begin{array}{rl} \gamma : \mathsf{0} & \xrightarrow[R_1]{} \mathsf{f}(q_A^{(6)}, q_A^{(7)}, q_A^{(8)}, q_A^{(9)}, q_B^{(10)}, q_B^{(11)}) \xrightarrow[R_2]{>\varepsilon\ *} \mathsf{f}(t_3, t_3, t_4, t_4, t_5, t_5) \\ & \xrightarrow[R_1]{} \mathsf{g}(t_3, t_3, t_4, t_4, t_5, t_5, q_A^{(12)}) \xrightarrow[R_2]{>\varepsilon\ *} \mathsf{g}(t_0, t_1, t_2, t_0, t_1, t_2, t_0) \\ & \xrightarrow[R_1]{} \mathsf{1}. \end{array}$$

Note that $d_2(\mathsf{1}, x) \to x$ can not be applied in γ. Indeed, if $d_2(\mathsf{1}, x) \to x$ is applied in γ, then γ must contain a subsequence $\mathsf{0} \xrightarrow[R_2]{*} \mathsf{1}$ since $\mathsf{1}$ appears only in the right-hand side of the rule $\mathsf{g}(x_0, x_1, x_2, x_0, x_1, x_2, x_0) \to \mathsf{1}$, g is only generated by the rule $\mathsf{f}(x_3, x_3, x_4, x_4, x_5, x_5) \to \mathsf{g}(x_3, x_3, x_4, x_4, x_5, x_5, q_A^{(12)})$, and f is only generated by the rule $\mathsf{0} \to \mathsf{f}(q_A^{(6)}, q_A^{(7)}, q_A^{(8)}, q_A^{(9)}, q_B^{(10)}, q_B^{(11)})$. This contradicts the hypothesis that γ is a shortest sequence. Thus,

$$\begin{array}{rl} & \mathsf{f}(q_A^{(6)}, q_A^{(7)}, q_A^{(8)}, q_A^{(9)}, q_B^{(10)}, q_B^{(11)}) \xrightarrow[R_2']{>\varepsilon\ *} \mathsf{f}(t_3, t_3, t_4, t_4, t_5, t_5) \\ \text{and} & \mathsf{g}(t_3, t_3, t_4, t_4, t_5, t_5, q_A^{(12)}) \xrightarrow[R_2']{>\varepsilon\ *} \mathsf{g}(t_0, t_1, t_2, t_0, t_1, t_2, t_0) \end{array}$$

By Lemma 4 (for sake of readability, we shall write below $\underline{x}$ instead of $\psi(x)$):

$$\begin{array}{rl} & \mathsf{f}\big(\underline{q_A^{(6)}}, \underline{q_A^{(7)}}, \underline{q_A^{(8)}}, \underline{q_A^{(9)}}, \underline{q_B^{(10)}}, \underline{q_B^{(11)}}\big) \xrightarrow[R_1]{*} \mathsf{f}\big(\underline{t_3}, \underline{t_3}, \underline{t_4}, \underline{t_4}, \underline{t_5}, \underline{t_5}\big) \\ \text{and} & \mathsf{g}\big(\underline{t_3}, \underline{t_3}, \underline{t_4}, \underline{t_4}, \underline{t_5}, \underline{t_5}, \underline{q_A^{(12)}}\big) \xrightarrow[R_1]{*} \mathsf{g}\big(\underline{t_0}, \underline{t_1}, \underline{t_2}, \underline{t_0}, \underline{t_1}, \underline{t_2}, \underline{t_0}\big) \end{array}$$

Since $\psi(q) = q$ for every $q \in Q$, $\mathsf{0} \xrightarrow[R_1]{} \mathsf{f}\big(\underline{q_A^{(6)}}, \underline{q_A^{(7)}}, \underline{q_A^{(8)}}, \underline{q_A^{(9)}}, \underline{q_B^{(10)}}, \underline{q_B^{(11)}}\big)$.

By definition of R_1, $\mathsf{f}\big(\underline{t_3}, \underline{t_3}, \underline{t_4}, \underline{t_4}, \underline{t_5}, \underline{t_5}\big) \xrightarrow[R_1]{} \mathsf{g}\big(\underline{t_3}, \underline{t_3}, \underline{t_4}, \underline{t_4}, \underline{t_5}, \underline{t_5}, \underline{q_A^{(12)}}\big)$ and $\mathsf{g}\big(\underline{t_0}, \underline{t_1}, \underline{t_2}, \underline{t_0}, \underline{t_1}, \underline{t_2}, \underline{t_0}\big) \xrightarrow[R_1]{} \mathsf{1}$. Altogether $\mathsf{0} \xrightarrow[R_1]{*} \mathsf{1}$. □

We shall show next that R_2 is confluent iff $\mathsf{0} \xrightarrow[R_2]{*} \mathsf{1}$ by using Lemma 3. We need the following lemma for that purpose.

Lemma 6. If $\mathsf{0} \xrightarrow[R_2]{*} \mathsf{1}$ then $t \xrightarrow[R_2]{*} \mathsf{1}$ for any $t \in G_2$.

Proof First, we note that for any $q \in Q$, there exists $s \in G_2$ which does not contain any function symbol in Q such that $q \xrightarrow[R_2]{*} s$. Since both of the automata A and B are clean, there exists $u \in {\Delta^{(3)}}^* \cup {\Delta^{(4)}}^* \cup {\Sigma^{(5)}}^* \cup {\Delta^{(0)}}^*$ such that $q \xrightarrow[R_0]{*} u(\mathsf{e})$.

Thus, it suffices to show that for any $t \in G_2$ which does not contain any function symbol in Q, $t \xrightarrow[R_2]{*} \mathsf{1}$. We show this proposition by induction on the structure of t:

Base case: by $\mathsf{e} \xrightarrow[R_2]{} \mathsf{0} \xrightarrow[R_2]{*} \mathsf{1}$.

Induction step: let $t = h(t_1, \cdots, t_n)$ where $n > 0$ and $h \in \Theta^{012} \cup \Theta^{345} \cup \{\mathsf{f}, \mathsf{g}\} \cup \Theta_2^{012}$. By the induction hypothesis, $h(t_1, \cdots, t_n) \xrightarrow[R_2]{*} h(\mathsf{1}, \cdots, \mathsf{1})$.
For every $d \in \Theta^{345}$, $d(\mathsf{1}) \xrightarrow[R_1]{} d'(\mathsf{1})$ for some $d' \in \Theta^{012}$ or $d(\mathsf{1}) \xrightarrow[R_1]{} \mathsf{1}$.
For every $d' \in \Theta^{012}$, $d'(\mathsf{1}) \xrightarrow[R_2]{} d_2'(\mathsf{0}, \mathsf{1}) \xrightarrow[R_2]{*} d_2'(\mathsf{1}, \mathsf{1})$.
For every $d_2' \in \Theta_2^{012}$, $d_2'(\mathsf{1}, \mathsf{1}) \xrightarrow[R_2]{} \mathsf{1}$.

$$\begin{array}{rl} \text{Moreover, } \mathsf{f}(\mathsf{1}, \cdots, \mathsf{1}) \xrightarrow[R_1]{} \mathsf{g}(\mathsf{1}, \cdots, \mathsf{1}, q_A^{(12)}) & \xrightarrow[R_1]{*} \mathsf{g}(\mathsf{1}, \cdots, \mathsf{1}, u(\mathsf{e})) \\ & \xrightarrow[R_2]{*} \mathsf{g}(\mathsf{1}, \cdots, \mathsf{1}, \mathsf{1}) \\ & \xrightarrow[R_1]{} \mathsf{1} \quad \text{where } u \in {\Delta^{(0)}}^* \end{array}$$

□

Let $\phi(t)$ be the term obtained from t by replacing every maximal ground subterm (w.r.t. $\geq_{\mathsf{sub}}$) by 1. Note that:

$$\begin{aligned}
\phi(R_0) &= P^{(3,0)} \cup \Pi_1^{(3,1)} \cup \Pi_2^{(4,2)} \cup P^{(4,0)} \cup S^{(5,1)} \cup S^{(5,2)} \\
\phi(R_1) &= \phi(R_0) \cup \left\{ \begin{array}{r} \mathsf{f}(x_3,x_3,x_4,x_4,x_5,x_5) \to \mathsf{g}(x_3,x_3,x_4,x_4,x_5,x_5,1), \\ \mathsf{g}(x_0,x_1,x_2,x_0,x_1,x_2,x_0) \to 1 \end{array} \right\} \\
\phi(R_2) &= \phi(R_1) \cup \{d(x) \to d_2(1,x), d_2(1,x) \to x \mid d \in \Theta^{012}\}.
\end{aligned}$$

Note also that the rules of T_A and T_B vanish in $\phi(R_0)$. The following technical lemma is used in the proof of Lemma 8.

Lemma 7. For any non-constant function symbol $h \in \Theta^{012} \cup \Theta^{345} \cup \{\mathsf{f},\mathsf{g}\} \cup \Theta_2^{012}$, $h(1,\cdots,1) \xrightarrow[\phi(R_2)]{*} 1$.

Proof
For every $d \in \Theta^{345}$, $d(1) \xrightarrow[\phi(R_1)]{} d'(1)$ for some $d' \in \Theta^{012}$ or $d(1) \xrightarrow[\phi(R_1)]{} 1$.
For every $d' \in \Theta^{012}$, $d'(1) \xrightarrow[\phi(R_2)]{} d'_2(1,1)$.
For every $d'_2 \in \Theta_2^{012}$, $d'_2(1,1) \xrightarrow[\phi(R_2)]{} 1$.
Moreover, $\mathsf{f}(1,\cdots,1) \xrightarrow[\phi(R_1)]{} \mathsf{g}(1,\cdots,1) \xrightarrow[\phi(R_1)]{} 1$. Thus, the lemma holds. □

We show now how the hypotheses of Lemma 3 hold for R_2 and ϕ.

Lemma 8. If $0 \xrightarrow[R_2]{*} 1$ then the following properties hold.

(1) If $s \xrightarrow[R_2]{} t$ then $\phi(s) \xrightarrow[\phi(R_2)]{*} \phi(t)$.
(2) $\xrightarrow[\phi(R_2)]{} \subseteq \xrightarrow[R_2]{*}$.
(3) $t \xrightarrow[R_2]{*} \phi(t)$.
(4) $\phi(R_2)$ is confluent.

Proof

(1) By induction on the structure of s. If s is a ground term then $\phi(s) = \phi(t) = 1$. Thus, we assume that s is not ground. Let $s \xrightarrow[R_2]{p} t$.
If $p = \varepsilon$ then $s = \alpha\theta \to \beta\theta = t$ where $\alpha \to \beta \in R_2$. Let $s = h(s_1,\cdots,s_n)$ for some $h \in \Theta^{012} \cup \Theta^{345} \cup \{\mathsf{f},\mathsf{g}\} \cup \Theta_2^{012}$ and $s_1,\cdots,s_n$, and $\alpha = h(a_1,\cdots,a_n)$. Since R_2 is flat, $a_1 \cdots a_n \in X \cup F_0$. If a_i is a variable then $\phi(a_i\theta) = a_i\phi(\theta)$. If a_i is a constant then $\phi(s_i) = \phi(a_i) = 1$. Thus, $\phi(s) = \phi(\alpha)\phi(\theta)$. Similarly, $\phi(t) = \phi(\beta)\phi(\theta)$, so $\phi(s) \xrightarrow[\phi(R_2)]{} \phi(t)$ holds.
If $p \neq \varepsilon$ then $s = h(s_1,\cdots,s_i,\cdots,s_n)$, $t = h(s_1,\cdots,t_i,\cdots,s_n)$, and $s_i \xrightarrow[R_2]{} t_i$ where $i \in \{1,\cdots,n\}$. Since s is not ground,

$$\phi(s) = h(\phi(s_1),\cdots,\phi(s_i),\cdots,\phi(s_n)).$$

By the induction hypothesis, $\phi(s_i) \xrightarrow[\phi(R_2)]{*} \phi(t_i)$.
If t is not ground then $h(\phi(s_1),\cdots,\phi(t_i),\cdots,\phi(s_n)) = \phi(t)$.
If t is ground then $h(\phi(s_1),\cdots,\phi(t_i),\cdots,\phi(s_n)) = h(1,\cdots,1)$.
By Lemma 7, $h(1,\cdots,1) \xrightarrow[\phi(R_2)]{*} 1 = \phi(t)$. Thus, $\phi(s) \xrightarrow[\phi(R_2)]{*} \phi(t)$ holds.

(2) Since $\phi(R_2) \setminus \Big(\{d(x) \to d_2(1,x) \mid d \in \Theta^{012}\} \cup \{\mathsf{f}(x_3,x_3,x_4,x_4,x_5,x_5) \to \mathsf{g}(x_3,x_3,x_4,x_4,x_5,x_5,1)\}\Big) \subseteq R_2$, it suffices to show that:

$$d(x\theta) \xrightarrow[R_2]{*} d_2(1,x\theta)$$
$$\text{and } \mathsf{f}(x_3\theta,x_3\theta,x_4\theta,x_4\theta,x_5\theta,x_5\theta) \xrightarrow[R_2]{*} \mathsf{g}(x_3\theta,x_3\theta,x_4\theta,x_4\theta,x_5\theta,x_5\theta,1)$$

We have that: $d(x\theta) \xrightarrow[R_2]{} d_2(0,x\theta) \xrightarrow[R_2]{*} d_2(1,x\theta)$.
By Lemma 6:

$$\begin{aligned}\mathsf{f}(x_3\theta,x_3\theta,x_4\theta,x_4\theta,x_5\theta,x_5\theta) &\xrightarrow[R_1]{} \mathsf{g}\big(x_3\theta,x_3\theta,x_4\theta,x_4\theta,x_5\theta,x_5\theta,q_A^{(12)}\big)\\ &\xrightarrow[R_2]{*} \mathsf{g}(x_3\theta,x_3\theta,x_4\theta,x_4\theta,x_5\theta,x_5\theta,1)\end{aligned}$$

(3) By Lemma 6.

(4) We can easily show that $\phi(R_2)$ is terminating by using a lexicographic path order induced by a precedence $>$ that satisfies the following conditions: for any $d \in \Theta^{345}, d' \in \Theta^{012}, d'' \in \Theta_2^{012}$, $d > d' > d'' > 1$ and $\mathsf{f} > \mathsf{g} > 1$. Thus, it suffices to show that every critical peak of $\phi(R_2)$ is joinable.
For every $a, a' \in \Sigma$, $\langle n,a,a'\rangle^{(0)}(x) \leftarrow \langle n,a,a'\rangle^{(3)}(x) \to a^{(1)}(x)$ is joinable by: $\langle n,a,a'\rangle^{(0)}(x) \to \langle n,a,a'\rangle_2^{(0)}(1,x) \to x \leftarrow a_2^{(1)}(1,x) \leftarrow a^{(1)}(x)$.
For every $a' \in \Sigma$, $\langle n,\_,a'\rangle^{(0)}(x) \leftarrow \langle n,\_,a'\rangle^{(3)}(x) \to x$ is joinable by: $\langle n,\_,a'\rangle^{(0)}(x) \to \langle n,\_,a'\rangle_2^{(0)}(1,x) \to x$.
For every $a, a' \in \Sigma$, $\langle n,a,a'\rangle^{(0)}(x) \leftarrow \langle n,a,a'\rangle^{(4)}(x) \to a'^{(2)}(x)$ is joinable by: $\langle n,a,a'\rangle^{(0)}(x) \to \langle n,a,a'\rangle_2^{(0)}(1,x) \to x \leftarrow a_2'^{(2)}(1,x) \leftarrow a'^{(2)}(x)$.
For every $a \in \Sigma$, $\langle n,a,\_\rangle^{(0)}(x) \leftarrow \langle n,a,\_\rangle^{(4)}(x) \to x$ is joinable by: $\langle n,a,\_\rangle^{(0)}(x) \to \langle n,a,\_\rangle_2^{(0)}(1,x) \to x$.
For every $a \in \Sigma$, $a^{(1)}(x) \leftarrow a^{(5)}(x) \to a^{(2)}(x)$ is joinable by: $a^{(1)}(x) \to a_2^{(1)}(1,x) \to x \leftarrow a_2^{(2)}(1,x) \leftarrow a^{(2)}(x)$.

□

Lemma 9. R_2 is confluent iff $0 \xrightarrow[R_2]{*} 1$.

Proof The if part follows from Lemmata 3 and 8.
For the only if part, by $\langle n,\_,a'\rangle^{(0)}(x) \xleftarrow[P^{(3,0)}]{} \langle n,\_,a'\rangle^{(3)}(x) \xrightarrow[\Pi_1^{(3,1)}]{} x$, the confluence ensures that $\langle n,\_,a'\rangle^{(0)}(x) \downarrow_{R_2} x$.
Since x is a normal form, $\langle n,\_,a'\rangle^{(0)}(x) \xrightarrow[R_2]{*} x$. Thus, there exists a sequence:

$$\langle n,\_,a'\rangle^{(0)}(x) \xrightarrow[R_2]{} \langle n,\_,a'\rangle_2^{(0)}(0,x) \xrightarrow[R_2]{*} \langle n,\_,a'\rangle_2^{(0)}(1,x) \xrightarrow[R_2]{} x$$

It follows that $0 \xrightarrow[R_2]{*} 1$ holds. □

By Lemmata 2, 5, 9, the following theorem holds.

Theorem 2. Confluence is undecidable for flat TRSs.

The above TRS R_2 differs from the analogous one of [8]. Indeed, in some cases, with the TRS of [8] we may have $0 \xrightarrow[R_2]{*} 1$ whereas $0 \xrightarrow[R_1]{*} 1$ does not hold, which is a problem for the correctness of the reduction. This error was corrected in [9], but Lemma 9 does not hold for the TRS of this report. Therefore, the above TRS R_2 and the above proof differ from the ones of [9].

5 Concluding remarks

We have shown that the properties of reachability, joinability and confluence are undecidable for flat TRSs. These results are negative solutions to the open problems posed in [7], and striking compared with the results that the word and unification problems for shallow TRSs are decidable [3]. The undecidability of reachability is shown by a reduction of the Post's Correspondence Problem and the case of joinability and confluence are treated both by a reduction of reachability (with a non trivial reduction in the case of confluence).

The proof techniques involved in our constructions, namely term coloring in Section 3, the criteria for confluence of Lemma 3 and the ground term mapping of Section 4 appeared to be very useful in this context and we believe that they could be of benefit to other decision problems.

Note that the only rules not linear in our TRS are $\mathsf{f}(x_3, x_3, x_4, x_4, x_5, x_5) \to \mathsf{g}(x_3, x_3, x_4, x_4, x_5, x_5, q_A^{(12)})$ and $\mathsf{g}(x_0, x_1, x_2, x_0, x_1, x_2, x_0) \to \mathtt{1}$ (both left and right members of the first rule are non-linear and they share variables, which is crucial in our reduction). Hence, we have narrowed dramatically the gap between known decidable and undecidable cases of confluence, reachability and joinability of TRS. All three properties are indeed decidable for TRSs whose left members of rules are flat and right members are flat and linear [17, 12, 16].

It will be a next step to find non-right-linear subclasses of flat (or shallow) TRSs with the decidable property for some of these decision problems. For example, what about the class of flat and semi-constructor TRSs? Here, a semi-constructor TRS is such a TRS that all defined symbols appearing in the right-hand side of each rewrite rule occur only in its ground subterms.

Another interesting question is: does there exist a subclass of TRSs such that exactly one of reachability and confluence is decidable? For the related question about whether there exists a subclass such that exactly one of reachability and joinability is decidable, the existence of such a confluent subclass has been shown in [10, 11].

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