The Ideal View on Rackoff's Coverability Technique ⋆

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Abstract. Rackoff's small witness property for the coverability problem is the standard means to prove tight upper bounds in vector addition systems (VAS) and many extensions. We show how to derive the same bounds directly on the computations of the VAS instantiation of the generic backward coverability algorithm. This relies on a dual view of the algorithm using ideal decompositions of downwards-closed sets, which exhibits a key structural invariant in the VAS case. The same reasoning readily generalises to several VAS extensions.

1 Introduction

Checking safety properties in infinite transition systems can often be reduced to *coverability* checks. The coverability problem asks, given a transition system and two configurations x and y and a quasi-ordering $\leq$ over configurations, whether x might *cover* y, i.e. reach some configuration $y' \geq y$ in finitely many steps. The problem is decidable for the large class of (effective) *well-structured transition systems* (WSTS) where $\leq$ is a *well-quasi-ordering* (wqo) compatible with the transition relation [1, 9]. The algorithm to that end is a generic *backward coverability* procedure, which computes successively the sets of configurations that can cover y in at most 0, 1, 2, ... steps. Those sets are upwards-closed and since $\leq$ is a wqo they can be represented through their finitely many minimal elements.

Nevertheless, the naive complexity upper bounds one can extract directly from the termination argument of the backward coverability algorithm—which also relies on $\leq$ being a wqo—are sometimes very far from the optimal ones. A striking illustration is provided by vector addition systems (VAS): the complexity bounds offered e.g. by [8] are in Ackermann, whereas coverability in VAS has long been known to be ExpSpace-complete thanks to a lower bound by Lipton [14] and an upper bound by Rackoff [16].

Rackoff's Technique is essentially combinatorial in nature: he shows by induction on the dimension of the VAS that, if x can reach one such $y' \geq y$, then there exists a small (doubly-exponential) run in the VAS witnessing this fact.

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A non-deterministic algorithm can then simply look for such a witness using only exponential space. The same general technique has since been extended to prove tight complexity upper bounds for coverability in numerous extensions of VASs [7, 3, 6, 13, 12]. It is however less clear how to adapt the technique for more general systems, where for instance the notion of dimension is absent or more involved.

Remarkably, Bozzelli and Ganty [5] showed that Rackoff's small witness property can be applied to the backward coverability algorithm for VAS to obtain a 2EXPTIME upper bound.[3] However, their proof uses Rackoff's analysis as a black box, and does not work directly with the structures manipulated by the backward coverability algorithm. As such, it is again unclear how this result could be translated to further classes of well-structured transition systems.

Contributions. In this paper, we revisit the backward coverability algorithm for VAS, and extract directly a 2EXPTIME upper bound for its running time. We take for this in Sec. 3 a dual view on the backward coverability algorithm, by considering successively the sets of configurations that do *not* cover y in 0, 1, 2, ... or fewer steps. Such sets are downwards-closed, and enjoy a (usually effective) canonical representation as finite unions of *ideals* [4, 10, 11]. We show in Sec. 4 that, in the case of VAS, this dual view exhibits an additional structural property of *ω-monotonicity*, which allows to derive the desired doubly-exponential bound.

Our purpose is above all pedagogical, as we hope to see this type of reasoning applied more broadly where the simple proof argument of Rackoff fails. As illustrations of the versatility of the framework, we consider in App. B and C the top-down and bottom-up coverability problems in *alternating branching* VAS. In each case, we provide an instance of the generic backward algorithm that solves the problem, and sketch why its running time matches the known optimal complexities [7, 6, 13].

We start with some preliminaries on WSTS and ideals in Sec. 2.

2 Preliminaries

We first recall the necessary background on well-quasi-orders, well-structured transition systems, and ideal decompositions, while illustrating systematically the definitions on VAS and reset VAS.

2.1 Well-Structured Transition Systems

A *well-quasi-order* (wqo) $(X, \leq)$ is a set X equipped with a transitive reflexive relation $\leq$ such that, along any infinite sequence $x_0, x_1, \ldots$ of elements from X, one can find two indices $i < j$ such that $x_i \leq x_j$. A finite or infinite sequence without such pair of indices is *bad*, and necessarily finite over a wqo. See for instance [18] for more background on wqos.

[3] In the same spirit, Majumdar and Wang [15] show that the 'expand, enlarge, and check' algorithm for bottom-up coverability in branching VASs runs in 2EXPTIME, using the combinatorial analysis of Demri et al. [7].

Example 2.1 (Dickson's Lemma). The set $\mathbb{N}^d$ of d-dimensional vectors of natural numbers forms a wqo when endowed with the product ordering $\sqsubseteq$, defined by $\boldsymbol{u} \sqsubseteq \boldsymbol{v}$ if $\boldsymbol{u}(i) \leq \boldsymbol{v}(i)$ for all $1 \leq i \leq d$.

A *well-structured transition system* (WSTS) [1, 9] is a triple $(X, \rightarrow, \leq)$ where X is a set of configurations, $\rightarrow \subseteq X \times X$ is a transition relation, and $(X, \leq)$ is a wqo with the following *compatibility* condition: if $x \leq x'$ and $x \rightarrow y$, then there exists $y' \geq y$ with $x' \rightarrow y'$. In other words, $\leq$ is a simulation relation on the transition system $(X, \rightarrow)$. We write as usual $\rightarrow^{\leq 0} \stackrel{\text{def}}{=} \{(x, x) \mid x \in X\}$ and $\rightarrow^{\leq k+1} \stackrel{\text{def}}{=} \rightarrow^{\leq k} \cup \{(x, y) \mid \exists z \in X \,.\, x \rightarrow z \rightarrow^{\leq k} y\}$ for the reachability relation in at most k steps, and $\rightarrow^* \stackrel{\text{def}}{=} \bigcup_k \rightarrow^{\leq k}$ for the reflexive transitive closure of $\rightarrow$.

Example 2.2 (VAS are WSTS). A d-dimensional *vector addition system* (VAS) is a finite set $\boldsymbol{A}$ of vectors in $\mathbb{Z}^d$. It defines a WSTS $(\mathbb{N}^d, \rightarrow, \sqsubseteq)$ with configurations space $\mathbb{N}^d$ and $\boldsymbol{u} \rightarrow \boldsymbol{u} + \boldsymbol{a}$ for all $\boldsymbol{u}$ in $\mathbb{N}^d$ and $\boldsymbol{a}$ in $\boldsymbol{A}$ such that $\boldsymbol{u} + \boldsymbol{a}$ is in $\mathbb{N}^d$.

For instance, the 2-dimensional VAS $\boldsymbol{A}_{\div 2} = \{(-2, 1)\}$ can be seen as weakly computing the halving function: from any configuration $(n, 0)$, it can reach $(n \bmod 2, \lfloor n/2 \rfloor)$ and all its reachable configurations (n', m) satisfy $m \leq n/2$.

Example 2.3 (Reset VAS are WSTS). A d-dimensional *reset VAS* is a finite subset $\boldsymbol{A}$ of $\mathbb{Z}^d \times \mathbb{P}(\{1, \ldots, d\})$. Given $R \subseteq \{1, \ldots, d\}$ and a vector $\boldsymbol{u}$, we define the vector $R(\boldsymbol{u})$ by $R(\boldsymbol{u})(i) = 0$ if $i \in R$, and $R(\boldsymbol{u})(i) = \boldsymbol{u}(i)$ otherwise. A reset VAS defines a WSTS $(\mathbb{N}^d, \rightarrow, \sqsubseteq)$ where $\boldsymbol{u} \rightarrow R(\boldsymbol{u} + \boldsymbol{a})$ if there exists $(\boldsymbol{a}, R)$ in $\boldsymbol{A}$ such that $\boldsymbol{u} + \boldsymbol{a}$ is in $\mathbb{N}^d$.

For instance, the 5-dimensional reset VAS

$$\boldsymbol{A}_{\log} = \left\{ \begin{array}{c} (0,0,-2,1,0,\emptyset), (0,0,1,-1,0,\emptyset), \\ (-1,1,-2,1,0,\{3\}), (1,-1,1,-1,1,\{4\}) \end{array} \right\}$$

is a weak computer for the logarithm function: from any configuration of the form $(1, 0, 2^n, 0, 0)$, it can reach $(1, 0, 1, 0, n)$, and all its reachable configurations of the form $(1, 0, n', m, l)$ satisfy $l \leq n$.

2.2 Ideal Decompositions

The *downward-closure* of a subset $S \subseteq X$ over a wqo $(X, \leq)$ is $\downarrow X \stackrel{\text{def}}{=} \{x \in X \mid \exists s \in S \,.\, x \leq s\}$. A subset $D \subseteq X$ is *downwards-closed* if $\downarrow D = D$. We write $\downarrow x$ for the downward-closure of the singleton set $\{x\}$. Well-quasi-orders can also be characterised by the *descending chain condition*: a quasi-order $(X, \leq)$ is a wqo if and only if every descending sequence $D_0 \supsetneq D_1 \supsetneq D_2 \supsetneq \cdots$ of downwards-closed subsets $D_i \subseteq X$ is finite.

An *ideal* of X is a non-empty downwards-closed subset $I \subseteq X$, which is *directed*: if x, x' are two elements of I, then there exists y in I with $x \leq y$ and $x' \leq y$. Over a wqo $(X, \leq)$, any downwards-closed set $D \subseteq X$ has a unique *decomposition* as a finite union of ideals $D = I_1 \cup \cdots \cup I_n$, where the I_j's are mutually incomparable for inclusion [4, 10]. Alternatively, ideals are characterised as *irreducible* downwards-closed sets: an ideal is a non-empty downwards-closed set I with the property that, if $I \subseteq D_1 \cup D_2$ for two downwards-closed sets D_1 and D_2, then $I \subseteq D_1$ or $I \subseteq D_2$.

Example 2.4 (Vector Ideals). Over $(\mathbb{N}^d, \sqsubseteq)$, observe that $\downarrow\boldsymbol{u}$ is an ideal for every $\boldsymbol{u}$ in $\mathbb{N}^d$. Those are however not the only ideals, e.g. $I \stackrel{\text{def}}{=} \{(0, n, 0) \mid n \in \mathbb{N}\}$ is also an ideal. Write $\mathbb{N}_\omega \stackrel{\text{def}}{=} \mathbb{N} \uplus \{\omega\}$ where ω is a new top element; the product ordering $\sqsubseteq$ extends naturally to $\mathbb{N}_\omega^d$. Then the ideals of $(\mathbb{N}^d, \sqsubseteq)$ are exactly the downward-closures $\downarrow\boldsymbol{u}$ inside $\mathbb{N}^d$ of vectors $\boldsymbol{u}$ from $\mathbb{N}_\omega^d$. For the previous example, $\downarrow(0, \omega, 0) = I$.

Although ideals provide finite representations for manipulating downwards-closed sets, some additional effectiveness assumptions are necessary to employ them in algorithms. In this paper, we will say that a wqo $(X, \leq)$ has *effective* ideal representations [see 10, 11, for more stringent requisites] if every ideal can be represented, and there are algorithms on those representations:

(CI) to check $I \subseteq I'$ for two ideals I and I',
(II) to compute the ideal decomposition of $I \cap I'$ for two ideals I and I',
(CU') to compute the ideal decomposition of the residual $X/x \stackrel{\text{def}}{=} \{x' \in X \mid x \not\leq x'\}$ for any x in X.

Example 2.5 (Effective Representations of Vector Ideals). We shall use vectors in $\mathbb{N}_\omega^d$ as representations. For (CI), given two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{N}_\omega^d$, $\downarrow\boldsymbol{u} \subseteq \downarrow\boldsymbol{v}$ if and only if $\boldsymbol{u} \sqsubseteq \boldsymbol{v}$. Furthermore, for (II), $\downarrow\boldsymbol{u} \cap \downarrow\boldsymbol{v} = \downarrow\boldsymbol{w}$ where $\boldsymbol{w}(i) \stackrel{\text{def}}{=} \min_{\leq}(\boldsymbol{u}(i), \boldsymbol{v}(i))$ for all $1 \leq i \leq d$. Finally, for (CU'), if $\boldsymbol{u}$ is in $\mathbb{N}^d$, then $\mathbb{N}^d/\boldsymbol{u} = \bigcup_{1 \leq j \leq d \mid \boldsymbol{u}(j) > 0} \downarrow\boldsymbol{u}_{/j}$ where $\boldsymbol{u}_{/j}(i) = \omega$ if $i \neq j$ and $\boldsymbol{u}_{/j}(j) \stackrel{\text{def}}{=} \boldsymbol{u}(j) - 1$ otherwise.

Crucially for the applicability of our approach, effective ideal representations exist for most wqos of interest [10, 11].

3 Backward Coverability

Let us recall in this section the generic backward coverability algorithm for well-structured transition systems [1, 9]. We take a dual view on this algorithm, by considering downwards-closed sets represented through their ideal decompositions, instead of the usual view using upwards-closed sets represented through their minimal elements. We present the generic algorithm, but will illustrate all the notions using the case of VAS and reset VAS in Sec. 3.2, and derive naive upper bounds for both in Sec. 3.3—which will turn out optimal for reset VAS.

3.1 Generic Algorithm

Consider a WSTS $(X, \to, \leq)$ and a target configuration y from X to be covered. Define $D_* \stackrel{\text{def}}{=} \{x \in X \mid \forall y' \geq y \,.\, x \not\to^* y'\}$ as the set of configurations that do not cover y. The purpose of the backward coverability algorithm is to compute D_*; solving a coverability instance with source configuration x_0 then amounts to checking whether x_0 belongs to D_*. The idea of the algorithm is to compute successively for every k the set D_k of configurations that do *not* cover y in k steps or fewer:

$$D_* = \bigcap_k D_k \;, \qquad\qquad D_k \stackrel{\text{def}}{=} \{x \in X \mid \forall y' \geq y \,.\, x \not\to^{\leq k} y'\} \;. \tag{1}$$

These over-approximations D_k can be computed inductively on k by

$$D_0 = X/y \;, \qquad\qquad D_{k+1} = D_k \cap \mathrm{Pre}_\forall(D_k) \;, \tag{2}$$

where for any set $S \subseteq X$,

$$\mathrm{Pre}_\forall(S) \stackrel{\text{def}}{=} \{x \in X \mid \forall z \in X \,.\, (x \to z \implies z \in S)\} \;. \tag{3}$$

The algorithm terminates as soon as $D_k \subseteq D_{k+1}$ (and thus $D_{k+j} = D_k = D_*$ for all j). This is guaranteed to arise eventually by the descending chain condition, since otherwise we would have an infinite descending chain of downwards-closed sets $D_0 \supsetneq D_1 \supsetneq D_2 \supsetneq \cdots$.

Correctness. The correctness of the algorithm hinges on the following claim:

Claim 3.1 (Correctness). Equations (1) and (2) define the same D_k.

Proof. By induction on k. For the base case, $x \not\to^{\leq 0} y'$ for all $y' \geq y$, if and only if $x \not\geq y$, i.e. if and only if x is in X/y. For the induction step and for all $y' \geq y$, $x \not\to^{\leq k+1} y'$ if and only if $x \not\to^{\leq k} y'$ and there does not exist any z with $x \to z$ and $z \to^{\leq k} y'$. The former is equivalent to x belonging to D_k by induction hypothesis. The latter occurs if and only if for all z in X, if $x \to z$ then $z \not\to^{\leq k} y'$, i.e. if and only if x belongs to $\mathrm{Pre}_\forall(D_k)$ by induction hypothesis. □

Effective Ideal Representations. The algorithm as presented above relies on the effectiveness of Eq. (2). We are going to use effective representations of the ideal decompositions of the D_k to this end. Let us first check that we are indeed dealing with downwards-closed sets:

Claim 3.2 (Downward-closure). For all k, D_k is downwards-closed.

Proof. By induction on k. For the base case, $D_0 = X/y$ is downwards-closed. For the induction step, first observe that, if D is downwards-closed, then $\mathrm{Pre}_\forall(D)$ is also downwards-closed. Indeed, let $x \leq x'$ for some x' in $\mathrm{Pre}_\forall(D)$. Consider any z such that $x \to z$. Then by WSTS compatibility, there exists $z' \geq z$ such that $x' \to z'$. Since x' belongs to $\mathrm{Pre}_\forall(D)$, z' belongs to D. Because D is downwards-closed, z also belongs to D. This shows x in $\mathrm{Pre}_\forall(D)$ as desired. We conclude by noting that downwards-closed sets are closed under intersection, hence $D_{k+1} = D_k \cap \mathrm{Pre}_\forall(D_k)$ is downwards-closed by applying the induction hypothesis to D_k. □

The only additional effectiveness assumption we make is that:

(Pre) the ideal decomposition of $\mathrm{Pre}_\forall(D)$ is computable for all downwards-closed D.

This is sufficient to compute the ideal decompositions of all the D_k. Indeed, initially D_0 is computed using (CU'). Later, $\mathrm{Pre}_\forall(D_k)$ is computable by (Pre), and its intersection with D_k is also computable by (II). Finally, recall that, by ideal irreducibility, $I_1 \cup \cdots \cup I_n \subseteq J_1 \cup \cdots \cup J_m$ for ideals $I_1, \ldots, I_n$ and downwards-closed $J_1, \ldots, J_m$ if and only if for all $1 \leq i \leq n$ there exists $1 \leq j \leq m$ such that $I_i \subseteq J_j$. Therefore, the termination check $D_k \subseteq D_{k+1}$ is effective by (CI).

3.2 Coverability for VAS and Reset VAS

In order to instantiate the backward coverability algorithm for VAS and reset VAS, we merely need to prove that they also satisfy the (Pre) effectiveness assumption: given a downwards-closed $D = {\downarrow}\boldsymbol{u}_1 \cup \cdots \cup {\downarrow}\boldsymbol{u}_m$ for some $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ in $\mathbb{N}^d_\omega$, compute a finite set of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ from $\mathbb{N}^d_\omega$ such that $\mathrm{Pre}_\forall(D) = {\downarrow}\boldsymbol{v}_1 \cup \cdots \cup {\downarrow}\boldsymbol{v}_n$. Using (CI) we can then select the maximal such $\boldsymbol{v}_j$ to obtain incomparable ideals.

Universal Predecessors in VAS. Thanks to (II) and the fact that $\boldsymbol{A}$ is finite (VAS are finitely branching), we start by reducing our computation to that of predecessors along a specific action $\boldsymbol{a}$ from $\boldsymbol{A}$: $\mathrm{Pre}_\forall(D) = \bigcap_{\boldsymbol{a} \in \boldsymbol{A}} \mathrm{Pre}_{\boldsymbol{a}}(D)$ where

$$\mathrm{Pre}_{\boldsymbol{a}}(D) \stackrel{\text{def}}{=} \{\boldsymbol{v} \in \mathbb{N}^d \mid \boldsymbol{v} + \boldsymbol{a} \in \mathbb{N}^d \implies \boldsymbol{v} + \boldsymbol{a} \in D\} \tag{4}$$

$$= \{\boldsymbol{v} \in \mathbb{N}^d \mid \boldsymbol{v} + \boldsymbol{a} \notin \mathbb{N}^d\} \cup \{\boldsymbol{v} \in \mathbb{N}^d \mid \boldsymbol{v} + \boldsymbol{a} \in D\} \tag{5}$$

$$= \mathbb{N}^d / \theta(\boldsymbol{a}) \cup \{\boldsymbol{v} \in \mathbb{N}^d \mid \boldsymbol{v} + \boldsymbol{a} \in D\}\;, \tag{6}$$

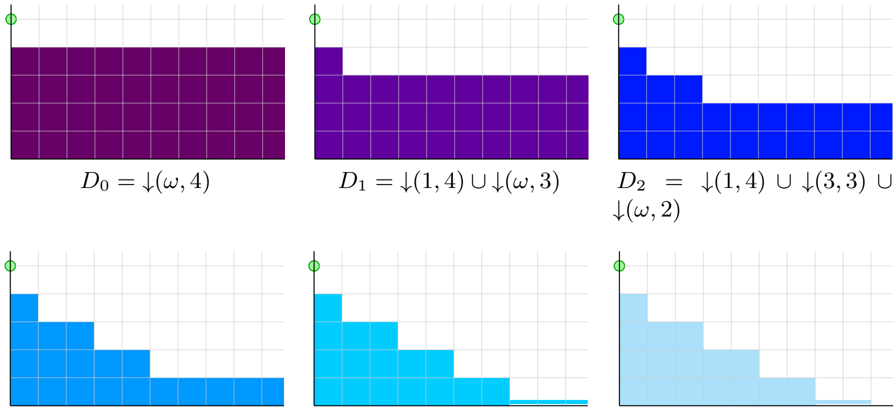
where $\theta(\boldsymbol{a}) \stackrel{\text{def}}{=} \min_{\sqsubseteq} \{\boldsymbol{v} \in \mathbb{N}^d \mid \boldsymbol{v} + \boldsymbol{a} \in \mathbb{N}^d\}$ is called the *threshold* of $\boldsymbol{a}$ and can be computed for all $1 \leq i \leq d$ by

$$\theta(\boldsymbol{a})(i) = \begin{cases} 0 & \text{if } \boldsymbol{a}(i) \geq 0 \\ -\boldsymbol{a}(i) & \text{otherwise.} \end{cases} \tag{7}$$

Thus by (CU') it only remains to compute a representation for the decomposition of $\{\boldsymbol{v} \in \mathbb{N}^d \mid \boldsymbol{v} + \boldsymbol{a} \in D\} = \bigcup_{1 \leq j \leq m} \{\boldsymbol{v} \in \mathbb{N}^d \mid \boldsymbol{v} + \boldsymbol{a} \in {\downarrow}\boldsymbol{u}_j\}$. For each ideal ${\downarrow}\boldsymbol{u}_j$ in the decomposition of D, $\{\boldsymbol{v} \in \mathbb{N}^d \mid \boldsymbol{v} + \boldsymbol{a} \in {\downarrow}\boldsymbol{u}_j\}$ is either the empty set if $\boldsymbol{u}_j \not\sqsupseteq \theta(-\boldsymbol{a})$, or ${\downarrow}(\boldsymbol{u} - \boldsymbol{a})$ otherwise, where addition is extended with $\omega + z = \omega$ for all z in $\mathbb{Z}$.

Example 3.3. Recall the VAS $\boldsymbol{A}_{\div 2} = \{(-2, 1)\}$ from Example 2.2. Setting $D_0 = {\downarrow}(\omega, 4)$, the backward coverability algorithm computes the set of all configurations from which $\boldsymbol{A}_{\div 2}$ cannot compute at least 5 in its second component; see Fig. 1.

Universal Predecessors in Reset VAS. The same reasoning holds for reset VAS as for VAS: $\mathrm{Pre}_\forall(D) = \bigcap_{(\boldsymbol{a},R) \in \boldsymbol{A}} \Big(\mathbb{N}^d/\theta(\boldsymbol{a}) \cup \bigcup_{1 \leq j \leq m} \{\boldsymbol{v} \in \mathbb{N}^d \mid R(\boldsymbol{v} + \boldsymbol{a}) \in {\downarrow}\boldsymbol{u}_j\}\Big)$.



$D_0 = \downarrow(\omega, 4)$ $D_1 = \downarrow(1,4) \cup \downarrow(\omega, 3)$ $D_2 = \downarrow(1,4) \cup \downarrow(3,3) \cup \downarrow(\omega,2)$

$D_3 = \downarrow(1,4) \cup \downarrow(3,3) \cup \downarrow(5,2) \cup \downarrow(\omega,1)$ $D_4 = \downarrow(1,4) \cup \downarrow(3,3) \cup \downarrow(5,2) \cup \downarrow(7,1) \cup \downarrow(\omega,0)$ $D_* = D_5 = \downarrow(1,4) \cup \downarrow(3,3) \cup \downarrow(5,2) \cup \downarrow(7,1) \cup \downarrow(9,0)$

Fig. 1. The successive D_k for $\boldsymbol{A}_{\div 2}$ with target $\boldsymbol{t} = (0,5)$.

In order to compute a representation for this last set, given a vector $\boldsymbol{v}$ in $\mathbb{N}_\omega^d$ and $R \subseteq \{1, \ldots, d\}$, define $\overline{\boldsymbol{v}}^R$ as the vector in $\mathbb{N}_\omega^d$ with ω's in the components of R:

$$\overline{\boldsymbol{v}}^R(i) \stackrel{\text{def}}{=} \begin{cases} \omega & \text{if } i \in R \\ \boldsymbol{v}(i) & \text{otherwise.} \end{cases} \tag{8}$$

Then $\{\boldsymbol{v} \in \mathbb{N}^d \mid R(\boldsymbol{v}+\boldsymbol{a}) \in \downarrow\boldsymbol{u}_j\}$ (where $R(\boldsymbol{v}+\boldsymbol{a})$ is defined as in Example 2.4) is either the empty set if $\overline{\boldsymbol{u}_j}^R \not\supseteq \theta(-\boldsymbol{a})$, or $\downarrow\left(\overline{\boldsymbol{u}_j}^R - \boldsymbol{a}\right)$ otherwise.

Example 3.4. Recall the reset VAS $\boldsymbol{A}_{\log}$ from Example 2.3, in which the first two vector components are used to encode two control states. Setting

$$D_0 = \downarrow(1,0,\omega,\omega,1) \cup \downarrow(0,1,\omega,\omega,0) ,$$

the backward coverability algorithm computes as follows the set of all configurations from which $\boldsymbol{A}_{\log}$ cannot compute in its last component either at least 2 in state $(1,0)$ or at least 1 in state $(0,1)$. The interesting part of the computation for the subsequent discussion occurs from $k=2$ to $k=4$:

$$\begin{aligned} D_2 = {}& \downarrow(0,0,\omega,\omega,1) \cup \downarrow(1,0,1,0,1) \cup \downarrow(1,0,0,\omega,1) \cup \downarrow(1,0,\omega,\omega,0) \cup \\ & \downarrow(0,1,\omega,0,0) \cup \downarrow(0,1,0,\omega,0) , \\ D_3 = {}& \downarrow(0,0,\omega,\omega,1) \cup \downarrow(1,0,1,0,1) \cup \downarrow(1,0,0,1,1) \cup \downarrow(1,0,\omega,\omega,0) \cup \\ & \downarrow(0,1,2,0,0) \cup \downarrow(0,1,0,1,0) , \\ D_4 = {}& \downarrow(0,0,\omega,\omega,1) \cup \downarrow(1,0,1,0,1) \cup \downarrow(1,0,0,1,1) \cup \downarrow(1,0,1,\omega,0) \cup \\ & \downarrow(1,0,\omega,0,0) \cup \downarrow(0,1,2,0,0) \cup \downarrow(0,1,0,1,0) . \end{aligned}$$

3.3 Ackermann Upper Bounds

Let us finally show how to bound the running time of the backward coverability algorithm on VAS and reset VAS. The main ingredient to that end is a combinatorial statement on the length of *controlled* descending chains of downwards-closed sets.

Controlled Descending Chains. Consider some set X with a norm $\|.\|: X \to \mathbb{N}$. Given a monotone *control* function $g: \mathbb{N} \to \mathbb{N}$ and an *initial norm* $n \in \mathbb{N}$, we say that a sequence $x_0, x_1, \ldots$ of elements from X is (g,n)-*controlled* if $\|x_i\| \leq g^i(n)$ the ith iterate of g applied to n. In particular, $\|x_0\| \leq n$ initially.

This notion can be applied to the descending chain $D_0 \supsetneq D_1 \supsetneq \cdots$ constructed by the backward coverability algorithm for a d-dimensional VAS or reset VAS $\boldsymbol{A}$ and target vector $\boldsymbol{t} \in \mathbb{N}^d$. We define for this $\|.\|$ as the infinity norm on elements and finite subsets of $\mathbb{Z}^d_\omega \stackrel{\text{def}}{=} (\mathbb{Z} \uplus \{\omega\})^d$, i.e. the maximum absolute value of any occurring integer. For instance, $\|\{(1,\omega,5),(0,\omega,\omega)\}\| = 5$, and in Example 2.2 $\|\boldsymbol{A}_{\div 2}\| = 2$. When considering a downwards-closed set D with decomposition ${\downarrow}\boldsymbol{u}_1 \cup \cdots \cup {\downarrow}\boldsymbol{u}_m$, we define $\|D\| \stackrel{\text{def}}{=} \|\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}\|$. Hence what is controlled in a descending chain $D_0 \supsetneq D_1 \supsetneq \cdots$ is its ideal representation.

Claim 3.5 (Control for VAS and Reset VAS). The descending chain $D_0 \supsetneq D_1 \supsetneq \cdots$ is (g,n)-controlled for $g(x) \stackrel{\text{def}}{=} x + \|\boldsymbol{A}\|$ and $n \stackrel{\text{def}}{=} \|\boldsymbol{t}\|$.

Proof. The fact that $\|D_0\| \leq \|\boldsymbol{t}\|$ follows from (CU'). Regarding the control function g, observe that taking unions and intersections of ideals using (II) cannot increase the norm. Hence it suffices to show that $\|\mathrm{Pre}_\forall(D)\| \leq \|D\| + \|\boldsymbol{A}\|$ for all $D = {\downarrow}\boldsymbol{u}_1 \cup \cdots \cup {\downarrow}\boldsymbol{u}_m$. Note that for reset VAS, $\|\overline{\boldsymbol{u}_j}^R - \boldsymbol{a}\| \leq \|\boldsymbol{u}_j - \boldsymbol{a}\|$. Hence for both VAS and reset VAS, $\|\mathrm{Pre}_\forall(D)\| \leq \max_{\boldsymbol{a}} \max_{1 \leq j \leq m}(\|\mathbb{N}^d/\theta(\boldsymbol{a})\|, \|\boldsymbol{u}_j - \boldsymbol{a}\|)$. We conclude by observing that $\|\mathbb{N}^d/\theta(\boldsymbol{a})\| \leq \|\boldsymbol{a}\| \leq \|\boldsymbol{A}\|$ by (CU') and $\|\boldsymbol{u}_j - \boldsymbol{a}\| \leq \|\boldsymbol{u}_j\| + \|\boldsymbol{a}\| \leq \|D\| + \|\boldsymbol{A}\|$. □

Upper Bound. Consider a computation $D_0 \supsetneq D_1 \supsetneq \cdots \supsetneq D_\ell = D_{\ell+1}$ of the backward coverability algorithm for a VAS or a reset VAS. At each step $0 \leq k \leq \ell$, the cost of computing D_{k+1} from D_k and of checking for termination is polynomial in $\|A\|$ and $\|D_k\|$. The difficulty is to evaluate how large ℓ can be.

The idea here is that, at every step $0 \leq k < \ell$, there is at least one *proper* ideal ${\downarrow}\boldsymbol{v}_k$: an ideal appearing in the representation of D_k but not in that of D_{k+1}; then ${\downarrow}\boldsymbol{v}_k \subseteq D_k$ but ${\downarrow}\boldsymbol{v}_k \not\subseteq D_{k+1}$. Note that for all $0 \leq j < k < \ell$, $\boldsymbol{v}_j \not\sqsubseteq \boldsymbol{v}_k$, since the contrary would entail ${\downarrow}\boldsymbol{v}_j \subseteq {\downarrow}\boldsymbol{v}_k \subseteq D_k \subseteq D_{j+1}$. Hence the sequence $(\boldsymbol{v}_k)_{0 \leq k < \ell}$ is a *bad* sequence, which is also controlled by (g,n) according to Claim 3.5. Using the combinatorial results from [18, Cor. 2.25 and Thm. 2.34] on such bad sequences, we obtain (see App. A for details):

Theorem 3.6. (Length Function Theorem for Descending Chains). *Let $n > 0$. Any (g,n)-controlled descending chain $D_0 \supsetneq D_1 \supsetneq \cdots$ of downwards-closed subsets of $\mathbb{N}^d$ is of length at most $h_{\omega^{d+1}}(n \cdot d!)$, where $h(x) \stackrel{\text{def}}{=} d \cdot g(x)$.*

Here h_α for an ordinal α and base function h denotes the αth Cichoń function [18]. Each of the ℓ steps of computation can furthermore be performed in time polynomial in $g^\ell(n)$.

Since g is primitive-recursive according to Claim 3.5, the overall complexity for an instance of size n is bounded by ackermann$(p(n))$ for some primitive-recursive function p, which lies within the complexity class Ackermann [17]. Such an upper bound is overly pessimistic for VAS, but is actually tight for reset VAS: coverability for reset VAS is indeed complete for Ackermann [19, 18].

4 Complexity for VAS

We know from Bozzelli and Ganty's 2ExpTime upper bound [5] for the backward coverability algorithm that the Ackermann upper bound from the previous section is far from tight in the case of VAS. We show in this section that the descending chains $D_0 \supsetneq D_1 \supsetneq \cdots$ computed by the backward coverability algorithm for VAS enjoy a structural invariant, which we dub *ω-monotonicity*, and which is absent from the chains computed for reset VAS. In turn, we show in Thm. 4.4, that controlled decreasing chains that are ω-monotone are much shorter, allowing us to derive the desired 2ExpTime bound in Cor. 4.6.

4.1 Transitions Between Proper Ideals

The proof of ω-monotonicity in the case of VAS can be shown directly, but reflects a more general *proper transition sequence* property of the generic backward coverability algorithm, which we are going to show in the general setting.

Let us first lift the transition relation $\to$ of a WSTS $(X, \to, \leq)$ to work over ideals. Define for any ideal I of X

$$\mathrm{Post}_\exists(I) \stackrel{\text{def}}{=} \{z \in X \mid \exists x \in I \,.\, x \to z\}\ . \tag{9}$$

Then $\downarrow\mathrm{Post}_\exists(I)$ is downwards-closed with a unique decomposition into maximal ideals. We follow Blondin et al. [2] and write '$I \to J$' if J is an ideal from the decomposition of $\downarrow\mathrm{Post}_\exists(I)$.

Example 4.1 (Transitions over Vector Ideals). In the case of a VAS $\boldsymbol{A}$, observe that, if $\boldsymbol{v}$ is a vector from $\mathbb{N}_\omega^d$, then $\mathrm{Post}_\exists(\downarrow\boldsymbol{v}) = \bigcup_{\boldsymbol{a}\in\boldsymbol{A}} \downarrow(\boldsymbol{v}+\boldsymbol{a})$. Each such $\downarrow(\boldsymbol{v}+\boldsymbol{a})$ is already an ideal. In the case of a reset VAS $\boldsymbol{A}$, we have similarly $\mathrm{Post}_\exists(\downarrow\boldsymbol{v}) = \bigcup_{(\boldsymbol{a},R)\in\boldsymbol{A}} \downarrow R(\boldsymbol{v}+\boldsymbol{a})$.

We can now state the result that motivates this subsection:

Claim 4.2 (Proper Transition Sequence). If I_{k+1} is a proper ideal of D_{k+1}, then there exist an ideal J and a proper ideal I_k of D_k such that $I_{k+1} \to J \subseteq I_k$.

Proof. An ideal is proper in D_k if and only if it intersects the set of elements *excluded* between steps k and $k+1$: by basic set operations, first observe that (2) is equivalent to

$$D_{k+1} = D_k \setminus \{x \in D_k \mid \exists z \notin D_k \,.\, x \to z\}\ . \tag{10}$$

Moreover, noting $D_{-1} \stackrel{\text{def}}{=} X$, z in (10) must belong to D_{k-1}, or x would have already been excluded before step k. We have therefore $D_{k+1} = D_k \setminus E_k$ where

$$E_{-1} \stackrel{\text{def}}{=} \{x \in X \mid x \geq y\} , \qquad E_k \stackrel{\text{def}}{=} \{x \in D_k \mid \exists z \in E_{k-1} \,.\, x \to z\} . \tag{11}$$

Consider now a proper ideal I_{k+1} of D_{k+1}: this means $I_{k+1} \cap E_{k+1} \neq \emptyset$. This implies in turn $\downarrow\text{Post}_\exists(I_{k+1}) \cap E_k \neq \emptyset$ by (11), thus there exists J such that $I_{k+1} \to J$ and $J \cap E_k \neq \emptyset$.

Since $I_{k+1} \subseteq D_{k+1} \subseteq \text{Pre}_\forall(D_k)$ by (2), we also know that $\text{Post}_\exists(I_{k+1}) \subseteq D_k$. By ideal irreducibility, it means that $J \subseteq I_k$ for some ideal I_k from the decomposition of D_k. Observe finally that $I_k \cap E_k \neq \emptyset$, i.e. that I_k is proper. □

4.2 ω-Monotonicity

For $\boldsymbol{u}$ in $\mathbb{N}_\omega^d$, its *ω-set* is the subset $\omega(\boldsymbol{u})$ of $\{1, \ldots, d\}$ such that $\boldsymbol{u}(i) = \omega$ if and only if $i \in \omega(\boldsymbol{u})$. We say that a descending chain $D_0 \supsetneq D_1 \supsetneq \cdots \supsetneq D_\ell$ of downwards-closed subsets of $\mathbb{N}^d$ is *ω-monotone* if for all $0 \leq k < \ell - 1$ and all proper ideals $\downarrow\boldsymbol{v}_{k+1}$ in the decomposition of D_{k+1}, there exists a proper ideal $\downarrow\boldsymbol{v}_k$ in the decomposition of D_k such that $\omega(\boldsymbol{v}_{k+1}) \subseteq \omega(\boldsymbol{v}_k)$.

Claim 4.3 (VAS Descending Chains are ω-Monotone). The descending chains computed by the backward coverability algorithm for VAS are ω-monotone.

Proof. Let $D_0 \supsetneq D_1 \supsetneq \cdots \supsetneq D_\ell$ be the descending chain computed for a VAS $\boldsymbol{A}$. Suppose $0 \leq k < \ell - 1$ and $\downarrow\boldsymbol{v}_{k+1}$ is a proper ideal in the decomposition of D_{k+1}. By Claim 4.2, there exists a proper ideal $\downarrow\boldsymbol{v}_k$ in the decomposition of D_k such that $\boldsymbol{v}_{k+1} + \boldsymbol{a} \sqsubseteq \boldsymbol{v}_k$. We conclude that $\omega(\boldsymbol{v}_{k+1}) \subseteq \omega(\boldsymbol{v}_k)$. □

As we can see with Example 3.4 however, the descending chains computed for reset VAS are in general *not* ω-monotone: $(1, 0, \omega, \omega, 0)$ is proper in D_3 and has a proper transition to $(0, 1, 0, \omega, 0)$ in D_2 using $(-1, 1, -2, 1, 0, \{3\})$ from $\boldsymbol{A}_{\log}$, but no ideal with $\{3, 4\}$ as ω-set is proper in D_2.

4.3 Upper Bound

We are now in position to state a refinement of Thm. 3.6 for ω-monotone controlled descending chains. For a control function $g\colon \mathbb{N} \to \mathbb{N}$, define the function $\widetilde{g}\colon \mathbb{N}^2 \to \mathbb{N}$ by induction on its first argument:

$$\widetilde{g}(0, n) \stackrel{\text{def}}{=} 1 , \qquad \widetilde{g}(m+1, n) \stackrel{\text{def}}{=} \widetilde{g}(m, n) + (g^{\widetilde{g}(m,n)}(n) + 1)^{m+1} . \tag{12}$$

Theorem 4.4 (Length Function Theorem for ω-Monotone Descending Chains). *Any (g, n)-controlled ω-monotone descending chain $D_0 \supsetneq D_1 \supsetneq \cdots$ of downwards-closed subsets of $\mathbb{N}^d$ is of length at most $\widetilde{g}(d, n)$.*

Proof. Note that D_ℓ the last element of the chain has the distinction of not having any proper ideal, hence it suffices to bound the index k of the last set D_k with a proper ideal $\downarrow\boldsymbol{v}_k$, and add one to get a bound on ℓ. We are going to establish by induction on $d - |I|$ that if $\downarrow\boldsymbol{v}_k$ is a proper ideal from the decomposition of D_k and its ω-set is I, then $k < \widetilde{g}(d - |I|, n)$, which by monotonicity of $\widetilde{g}$ in its first argument entails $k < \widetilde{g}(d, n)$ as desired.

For the base case, $|I| = d$ implies that $\boldsymbol{v}_k$ is the vector with ω's in every coordinate, which can only occur in D_0. The inductive step is handled by the following claim, when setting $k < \widetilde{g}(d - |I| - 1, n)$ by induction hypothesis for the last index with a proper ideal whose ω-set strictly includes I:

Claim 4.5. Let I and $k < k'$ be such that:

(i) for all $j \in \{k+1, \ldots, k'-1\}$, the decomposition of D_j does not contain a proper ideal whose ω-set strictly includes I;
(ii) the decomposition of $D_{k'}$ contains a proper ideal whose ω-set is I.

Then we have $k' - k \le (\|D_{k+1}\| + 1)^{(d-|I|)}$.

For a proof, from assumption (ii), by applying the ω-monotonicity for $j = k'-1, k'-2, \ldots, k+1$ and due to assumption (i), there exists a proper ideal $\downarrow\boldsymbol{v}_j$ in the decomposition of D_j and such that $\omega(\boldsymbol{v}_j) = I$ for all $j \in \{k+1, \ldots, k'\}$. Since they are proper, those $k' - k$ vectors are mutually distinct.

Consider any such $\boldsymbol{v}_j$. Since $D_{k+1} \supseteq D_j$, by ideal irreducibility there exists a vector $\boldsymbol{u}_j$ in the decomposition of D_{k+1} such that $\boldsymbol{v}_j \sqsubseteq \boldsymbol{u}_j$. We have that $\omega(\boldsymbol{u}_j) = I$, since otherwise $\boldsymbol{u}_j$ would be proper at $D_{j'}$ for some $j' \in \{k+1, \ldots, j-1\}$, which would contradict assumption (i). Hence $\|\boldsymbol{v}_j\| \le \|\boldsymbol{u}_j\| \le \|D_{k+1}\|$.

To conclude, note that there can be at most $(\|D_{k+1}\| + 1)^{(d-|I|)}$ mutually distinct vectors in $\mathbb{N}_\omega^d$ with I as ω-set and norm bounded by $\|D_{k+1}\|$. □

Finally, putting together Claim 3.5 (control for VAS), Claim 4.3 (ω-monotonicity), and Thm. 4.4 (lengths of controlled ω-monotone descending chains), we obtain that the backward coverability algorithm for VAS runs in 2ExpTime, and in pseudo-polynomial time if d is fixed.

Corollary 4.6. *For any d-dimensional VAS $\boldsymbol{A}$ and target vector $\boldsymbol{t}$, the backward coverability algorithm terminates after at most $((\|\boldsymbol{A}\| + 1)(\|\boldsymbol{t}\| + 2))^{(d+1)!}$ steps.*

Proof. Let $h(m, n) = \widetilde{g}(m, n)(\|\boldsymbol{A}\| + 1)(n + 2)$ where $g(x) = x + \|\boldsymbol{A}\|$. We have $h(m+1, n) \le (h(m, n))^{m+2}$, so $\widetilde{g}(m, n) \le h(m, n) \le ((\|\boldsymbol{A}\|+1)(n+2))^{(m+1)!}$. □

5 Concluding Remarks

Rackoff's technique has successfully been employed to prove tight upper bounds for the coverability problem in VAS and extensions [7, 3, 6, 13, 12]. However, the technique does not readily generalise to more complex classes of well-structured transition systems, e.g. where configurations are not vectors of natural numbers.

We have shown that the same complexity bounds can be extracted in a principled way, by considering the ideal view of the backward coverability algorithm for VAS, and by noticing a structural invariant on its computations. Essentially the same arguments suffice to re-prove several recent upper bounds [7, 6, 13].

This paves the way for future investigations on coverability problems with large complexity gaps (where different structural invariants will need to be found).

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A Ackermann Upper Bounds

One way to obtain Ackermann upper bounds for the backward coverability algorithm on VAS and reset VAS would be to consider the dual ascending chain of *upwards-closed* sets employed in the usual description of the backward coverability algorithm. The resulting bounds would be similar (and the proof somewhat simpler), see [e.g. 18, Sec. 2.2.2]. Instead, we prove the bounds directly on descending chains, thanks to Thm. 3.6:

Theorem 3.6. (Length Function Theorem for Descending Chains). *Let $n > 0$. Any (g, n)-controlled descending chain $D_0 \supsetneq D_1 \supsetneq \cdots$ of downwards-closed subsets of $\mathbb{N}^d$ is of length at most $h_{\omega^{d+1}}(n \cdot d!)$, where $h(x) \stackrel{def}{=} d \cdot g(x)$.*

The main tool to this end is the following statement, which combines Cor. 2.25 and Thm. 2.34 from [18]:

Theorem A.1 (Length Function Theorem for Bad Sequences). *Let $n > 0$. Any (g, n)-controlled bad sequence over a polynomial normed wqo $(X, \leq, |.|_X)$ with maximal order type $o(X) < \omega^{d+1}$ is of length at most $h_{o(X)}(n \cdot d)$, where $h(x) \stackrel{def}{=} d \cdot g(x)$.*

In Sec. 3.3, we have already sketched how to extract a (g, n)-controlled bad sequence $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots$ of vectors from $\mathbb{N}_\omega^d$ out of a (g, n)-controlled descending chain $D_0 \supsetneq D_1 \supsetneq \cdots$ of downwards-closed subsets of $\mathbb{N}^d$. What needs to be shown in order to apply Thm. A.1 to that bad sequence is that we can use a polynomial normed wqo $(X, \leq)$ with $o(X) < \omega^{d+1}$ instead of $\mathbb{N}_\omega^d$, and derive the $h_{\omega^{d+1}}(n \cdot d!)$ bound from it. This is a routine application of the results from [18], but we shall give a detailed account for the reader's sake.

Polynomial Normed WQOs. Let us denote by Γ_0 the empty wqo and by Γ_1 the singleton set $\{\bullet\}$ well-quasi-ordered with equality. A *polynomial* wqo is one that can be constructed from Γ_0, Γ_1, and $\mathbb{N}$ through Cartesian products and disjoint unions, using respectively the product and sum orderings [18, Sec. 2.1.2].

A *normed* quasi-order associates a *norm* function $|.|_X : X \to \mathbb{N}$ to a quasi-order $(X, \leq)$, such that $X_{\leq n} \stackrel{\text{def}}{=} \{x \in X \mid |x|_X \leq n\}$ is finite for every n. For polynomial wqos, we use the zero norm on Γ_1 and the infinity norm on Cartesian products.

Reflections. A *normed order reflection* between two normed quasi-orders $(Y, \leq_Y, |.|_Y)$ and $(Y, \leq_Y, |.|_Y)$ is a function $r : X \to Y$ such that for all x and x' from X, $r(x) \leq_Y r(x')$ implies $x \leq_X x'$, and $|r(x)|_Y \leq |x|_X$. If $(Y, \leq_Y, |.|_Y)$ is a normed wqo and there is a normed order reflection from $(X, \leq_X, |.|_X)$ to it, then $(X, \leq_X, |.|_X)$ is a normed wqo with (g, n)-controlled bad sequences of at most the same length [18, Prop. 2.16].

Observe that r defined by $r(\omega) \stackrel{\text{def}}{=} \bullet$ and $r(n) \stackrel{\text{def}}{=} n$ is a normed order reflection from $(\mathbb{N}_\omega, \leq, \|.\|)$ into $(\mathbb{N} + \Gamma_1, \leq, |.|)$. Hence $(\mathbb{N}_\omega^d, \sqsubseteq, \|.\|)$ reflects into

$$((\mathbb{N} + \Gamma_1)^d, \sqsubseteq, |.|) \tag{13}$$

by [18, Rem. 2.17]. This will be the polynomial normed wqo on which we will apply Thm. A.1.

Maximal Order Types. It remains to compute the maximal order type of $((\mathbb{N}+\Gamma_1)^d, \sqsubseteq, |.|)$. This can be done algebraically [18, Sec. 2.4.1]:

$$o(\mathbb{N}+\Gamma_1) = \omega \oplus 1 = \omega + 1 \;, \tag{14}$$

$$o((\mathbb{N}+\Gamma_1)^d) = \underbrace{(\omega+1)\otimes\cdots\otimes(\omega+1)}_{d \text{ times}} = \sum_{d\geq i\geq 0} \omega^i \binom{d}{i} \;. \tag{15}$$

Cichoń Functions. Let us recall that, given a monotone expansive $h\colon \mathbb{N}\to\mathbb{N}$, and an ordinal α, the αth *Cichoń function* h_α is defined by induction on α by

$$h_0(x) \stackrel{\text{def}}{=} 0 \;, \qquad h_\alpha(x) \stackrel{\text{def}}{=} 1 + h_{P_x(\alpha)}(h(x)) \;, \tag{16}$$

where $P_x(\alpha) < \alpha$ denotes the predecessor ordinal at x of α, defined for $0 < \alpha < \varepsilon_0$ by:

$$P_x(\alpha+1) \stackrel{\text{def}}{=} \alpha \;, \qquad P_x(\gamma+\omega^\beta) \stackrel{\text{def}}{=} \gamma + \omega^{P_x(\beta)}\cdot x + P_x(\omega^{P_x(\beta)}) \;. \tag{17}$$

For instance, $P_x(\omega^2) = \omega\cdot x + P_x(\omega) = \omega\cdot x + x + P_x(1) = \omega\cdot x + x$, and more generally $P_x(\omega^{d+1}) = \sum_{d\geq i\geq 0}\omega^i\cdot x$. Each Cichoń function h_α is monotone expansive.

Using Thm. A.1, we obtain an upper bound of

$$h_{o((\mathbb{N}+\Gamma_1)^d)}(n\cdot d) \tag{18}$$

on the length of (g,n)-controlled bad sequences over $\mathbb{N}^d_\omega$, and thus on the length of (g,n)-controlled descending chains of downwards-closed subsets of $\mathbb{N}^d$. As (15) is quite a mouthful, we are going to over-approximate this bound with a more readable one.

Recall that any ordinal α below ω^{d+1} can be written in Cantor normal form as $\alpha = \omega^d\cdot a_d + \cdots + \omega^0\cdot a_0$ where $a_d,\ldots,a_0$ are coefficients in $\mathbb{N}$. We can refine the *structural ordering* of [18, Eq. 2.70] for ordinals below ω^{d+1} by:

$$\omega^d\cdot a_d + \cdots + \omega^0\cdot a_0 \sqsubseteq \omega^d\cdot b_d + \cdots + \omega^0\cdot b_0 \text{ if } \forall 1\leq i\leq d \,.\, a_i \leq b_i \;. \tag{19}$$

By [18, Exercise 2.11], $\alpha\sqsubseteq\beta$ ensures $h_\alpha(n)\leq h_\beta(n)$ for all n.

Observe now that (15) is such that, for all $n>0$,

$$o((\mathbb{N}+\Gamma_1)^d) \sqsubseteq \sum_{d\geq i\geq 0}\omega^i\cdot nd! = P_{nd!}(\omega^{d+1}) \;. \tag{20}$$

Hence the result stated in Thm. 3.6, by (18) and

$$\begin{aligned} h_{o((\mathbb{N}+\Gamma_1)^d)}(nd) &\leq h_{o((\mathbb{N}+\Gamma_1)^d)}\left(h(nd!)\right) && \text{since } nd \leq h(nd!) \\ &\leq h_{P_{nd!}(\omega^{d+1})}\left(h(nd!)\right) && \text{by (20)} \\ &= h_{\omega^{d+1}}(nd!) - 1 \;. && \square \end{aligned}$$

B Top-Down Tree Coverability

We turn to demonstrating how easily our new proof of the doubly-exponential bound for the backward coverability algorithm on VAS can be extended to derive optimal bounds for top-down alternating branching VAS: Tower in general [13] and 2ExpTime with alternation only [6].

Recall that a top-down *alternating branching vector addition system (AB-VAS)* of dimension $d \in \mathbb{N}$ consists of: a finite set of *unary* rules $\boldsymbol{A} \subseteq \mathbb{Z}^d$, a finite set of *fork* rules $\boldsymbol{B}_\wedge \subseteq \mathbb{Z}^d$, and a finite set of *split* rules $\boldsymbol{B}_+ \subseteq \mathbb{Z}^d$. A computation is a tree whose nodes are labelled by vectors in $\mathbb{N}^d$, and such that every non-leaf node ν is obtained by applying some rule (we write $\boldsymbol{v}(\cdot)$ for node labels, and extend the min and max operations to vectors component-wise):

- either ν has one child ν' and $\boldsymbol{v}(\nu)+\boldsymbol{a} = \boldsymbol{v}(\nu')$ for some unary rule $\boldsymbol{a}$ from $\boldsymbol{A}$,
- or ν has two children ν', ν'' and $\boldsymbol{v}(\nu)+\boldsymbol{b} = \max\{\boldsymbol{v}(\nu'), \boldsymbol{v}(\nu'')\}$ for some fork rule $\boldsymbol{b}$ from $\boldsymbol{B}_\wedge$,
- or ν has two children ν', ν'' and $\boldsymbol{v}(\nu)+\boldsymbol{b} = \boldsymbol{v}(\nu')+\boldsymbol{v}(\nu'')$ for some split rule $\boldsymbol{b}$ from $\boldsymbol{B}_+$.

The coverability problem, given a d-dimensional top-down ABVAS $(\boldsymbol{A}, \boldsymbol{B}_\wedge, \boldsymbol{B}_+)$, and root and target vectors $\boldsymbol{r}, \boldsymbol{t} \in \mathbb{N}^d$, asks whether there exists a computation whose root label is $\boldsymbol{r}$ and whose every leaf label is $\sqsupseteq \boldsymbol{t}$. We remark that the problem is equivalent to the variant in which the fork rules are applied with exact alternation (i.e. $\boldsymbol{v}(\nu) + \boldsymbol{b} = \boldsymbol{v}(\nu') = \boldsymbol{v}(\nu'')$), and also to the reachability problem for *lossy* top-down ABVAS.

To solve the coverability problem, we instantiate the generic backward algorithm from Sec. 3 by the following operator on sets of vectors:

$$\mathrm{Pre}_{\forall\exists}(S) = \left\{ \boldsymbol{v} \in \mathbb{N}^d \;\middle|\; \begin{array}{c} \forall \boldsymbol{a} \in \boldsymbol{A} \,.\, \boldsymbol{v} + \boldsymbol{a} \in \mathbb{N}^d \Rightarrow \boldsymbol{v} + \boldsymbol{a} \in S \\ \wedge \begin{pmatrix} \forall \boldsymbol{b} \in \boldsymbol{B}_\wedge \,.\, \forall \boldsymbol{v}', \boldsymbol{v}'' \in \mathbb{N}^d \,. \\ \boldsymbol{v} + \boldsymbol{b} = \max\{\boldsymbol{v}', \boldsymbol{v}''\} \Rightarrow \\ \boldsymbol{v}' \in S \vee \boldsymbol{v}'' \in S \end{pmatrix} \\ \wedge \begin{pmatrix} \forall \boldsymbol{b} \in \boldsymbol{B}_+ \,.\, \forall \boldsymbol{v}', \boldsymbol{v}'' \in \mathbb{N}^d \,. \\ \boldsymbol{v} + \boldsymbol{b} = \boldsymbol{v}' + \boldsymbol{v}'' \Rightarrow \\ \boldsymbol{v}' \in S \vee \boldsymbol{v}'' \in S \end{pmatrix} \end{array} \right\} .$$

Downward-Closure and Correctness. It is straightforward to check that $\mathrm{Pre}_{\forall\exists}$ preserves the property of downward closure. Given an initial downwards-closed $D_0 \subseteq \mathbb{N}^d$, as in Sec. 3, we write D_* for the last set in the longest (necessarily finite) descending chain of downwards-closed sets $D_0 \supsetneq D_1 \supsetneq \cdots$ defined by $D_{k+1} = D_k \cap \mathrm{Pre}_{\forall\exists}(D_k)$. From the definition of $\mathrm{Pre}_{\forall\exists}$, we have the required correctness property:

Claim B.1. (a) Any D_i consists of all vectors $v \in \mathbb{N}^d$ such that all computations of $(\boldsymbol{A}, \boldsymbol{B}_\wedge, \boldsymbol{B}_+)$ whose root label is v and whose height is at most i have some leaf label in D_0.

(b) The set D_* consists of all vectors $v \in \mathbb{N}^d$ such that all computations of $(\boldsymbol{A}, \boldsymbol{B}_\wedge, \boldsymbol{B}_+)$ whose root label is v have some leaf label in D_0.

Effectiveness and Control. Similarly to the cases of VAS and reset VAS in Sections 3.2 and 3.3, we obtain:

Claim B.2. (a) The ideal decomposition of $\mathrm{Pre}_{\forall\exists}(D)$ is computable for all downwards-closed $D \subseteq \mathbb{N}^d$.

(b) The descending chain $D_0 \supsetneq D_1 \supsetneq \cdots$ is (g, n)-controlled for

$$g(x) \stackrel{\text{def}}{=} \max\{x + \|\boldsymbol{A}\|, x + \|\boldsymbol{B}_\wedge\|, 2x + \|\boldsymbol{B}_+\| + 1\}$$

and $n \stackrel{\text{def}}{=} \|\boldsymbol{t}\|$. For AVAS, i.e. when $\boldsymbol{B}_+$ is empty, the term $2x + \|\boldsymbol{B}_+\| + 1$ disappears.

ω-Monotonicity. The property that allows us to deduce that the backward coverability algorithm is optimal for top-down ABVAS, and also when restricted to AVAS, is again ω-monotonicity of the downwards-closed sets that it computes.

Claim B.3. The descending chains computed by the backward coverability algorithm for top-down ABVAS are ω-monotone.

Proof. Let $D_0 \supsetneq D_1 \supsetneq \cdots \supsetneq D_k$ be the descending chain computed for a top-down ABVAS $(\boldsymbol{A}, \boldsymbol{B}_\wedge, \boldsymbol{B}_+)$. Suppose $0 \leq j < k - 1$ and $\boldsymbol{v}_{j+1}$ is a proper vector in the decomposition of D_{j+1}. Then $\downarrow\boldsymbol{v}_{j+1} \not\subseteq \mathrm{Pre}_{\forall\exists}(D_{j+1})$, so there is a case for each of the three types of rule.

Split: Here some $\boldsymbol{b} \in \boldsymbol{B}_+$ and $\boldsymbol{v}', \boldsymbol{v}'' \in \mathbb{N}^d_\omega$ are such that $\boldsymbol{v}_{j+1} + \boldsymbol{b} = \boldsymbol{v}' + \boldsymbol{v}''$ and $\downarrow\boldsymbol{v}', \downarrow\boldsymbol{v}'' \not\subseteq D_{j+1}$. Without loss of generality, we may assume that the ω-sets of $\boldsymbol{v}$, $\boldsymbol{v}'$ and $\boldsymbol{v}''$ are the same. Since $\downarrow\boldsymbol{v}_{j+1} \subseteq \mathrm{Pre}_{\forall\exists}(D_j)$, we have that either $\downarrow\boldsymbol{v}' \subseteq D_j$ or $\downarrow\boldsymbol{v}'' \subseteq D_j$, say the former. Let $\boldsymbol{v}_j$ be any vector in the decomposition of D_j such that $\boldsymbol{v}' \sqsubseteq \boldsymbol{v}_j$. We conclude that $\boldsymbol{v}_j$ is proper and that $\omega(\boldsymbol{v}_{j+1}) \subseteq \omega(\boldsymbol{v}_j)$.
Fork: This case is similar but easier as we may assume $\boldsymbol{v}_{j+1} + \boldsymbol{b} = \boldsymbol{v}' = \boldsymbol{v}''$.
Unary: This case is as for VAS, cf. the proof of Claim 4.3. ☐

Upper Bounds. We are now equipped to establish, by applying the length function theorem for ω-monotone descending chains (Thm. 4.4) that the backward coverability algorithm for top-down ABVAS runs in TOWER in general and 2EXPTIME with alternation only. Since the ideal decomposition of each D_{k+1} is computable in time polynomial in the bound on the norm of D_k, it suffices to bound the number of iterations of the main loop.

Recall (12), and let $h(m, n) = 32(n + L)(\widetilde{g^\dagger}(m, n))^2$ where

$$L = \max\{\|\boldsymbol{A}\|, \|\boldsymbol{B}_\wedge\|, \|\boldsymbol{B}_+\| + 1\}$$
$$g^\dagger(x) = 2x + L.$$

We have

$$\widetilde{g^\dagger}(m+1, n) \leq (2^{\widetilde{g^\dagger}(m,n)}(n + L + 1))^{m+1} \leq 2^{(\widetilde{g^\dagger}(m,n)+1)(n+L+1)(m+1)} \leq 2^{h(m,n)/4}$$

and hence

$$h(m+1,n) \leq 32(n+L)2^{h(m,n)/2} \leq 2^{5+n+L+h(m,n)/2} \leq 2^{h(m,n)},$$

so $h(d,n) \leq \underbrace{2^{\cdot^{\cdot^{2^{32(n+L)}}}}}_{d}$.

Let also $h'(m,n) = \widetilde{g^{\ddagger}}(m,n)(L'+1)(n+2)$ where $L' = \max\{\|\boldsymbol{A}\|, \|\boldsymbol{B}_\wedge\|\}$ and $g^{\ddagger}(x) = x + L'$. We have $h'(m+1,n) \leq (h'(m,n))^{m+2}$, so $h'(d,n) \leq ((L'+1)(n+2))^{(d+1)!}$.

Theorem 4.4 gives us:

Corollary B.4. *For any d-dimensional top-down ABVAS $(\boldsymbol{A}, \boldsymbol{B}_\wedge, \boldsymbol{B}_+)$ and initial downwards-closed $D_0 \subseteq \mathbb{N}^d$, the backward coverability algorithm terminates after at most $\underbrace{2^{\cdot^{\cdot^{2^{32(\|D_0\|+L)}}}}}_{d}$ steps, where $L = \max\{\|\boldsymbol{A}\|, \|\boldsymbol{B}_\wedge\|, \|\boldsymbol{B}_+\| + 1\}$. For AVAS, i.e. when $\boldsymbol{B}_+$ is empty, the algorithm terminates after at most $((L'+1)(\|D_0\|+2))^{(d+1)!}$ steps, where $L' = \max\{\|\boldsymbol{A}\|, \|\boldsymbol{B}_\wedge\|\}$.*

C Bottom-Up Tree Coverability

As the last case study in this paper of the ideal view of Rackoff's technique, we consider the coverability problem for bottom-up ABVAS. We discover a backward coverability algorithm that, instead of configurations that are vectors of natural numbers as in Sections 4 and B, works with downwards-closed sets of such vectors. The ideals involved are therefore one level higher than the vector ideals, and we recall some of their properties from the literature. We follow the pattern from the preceding sections and again derive the optimal complexity bounds: Ackermann in general [13] and 2ExpTime for BVAS [7].

Bottom-up ABVAS are defined as top-down ABVAS (cf. Sec. B), except that applications of fork rules take component-wise minima. Thus, for every non-leaf node ν in a computation:

- either ν has one child ν' and $\boldsymbol{v}(\nu) + \boldsymbol{a} = \boldsymbol{v}(\nu')$ for some unary rule $\boldsymbol{a}$ from $\boldsymbol{A}$,
- or ν has two children ν', ν'' and $\boldsymbol{v}(\nu) + \boldsymbol{b} = \min\{\boldsymbol{v}(\nu'), \boldsymbol{v}(\nu'')\}$ for some fork rule $\boldsymbol{b}$ from $\boldsymbol{B}_\wedge$,
- or ν has two children ν', ν'' and $\boldsymbol{v}(\nu) + \boldsymbol{b} = \boldsymbol{v}(\nu') + \boldsymbol{v}(\nu'')$ for some split rule $\boldsymbol{b}$ from $\boldsymbol{B}_+$.

The coverability problem, given a d-dimensional bottom-up ABVAS $(\boldsymbol{A}, \boldsymbol{B}_\wedge, \boldsymbol{B}_+)$, and leaf and target vectors $\boldsymbol{l}, \boldsymbol{t} \in \mathbb{N}^d$, asks whether there exists a computation whose every leaf label is $\boldsymbol{l}$ and whose root label is $\sqsupseteq \boldsymbol{t}$. We remark that the problem is equivalent to the reachability problem for *lossy* bottom-up ABVAS[4],

[4] We mean here that the losses are in the leaves-to-root direction.

but that the variant of coverability in which the fork rules are applied with exact alternation is undecidable [20].

To solve the coverability problem, we first define an operator on sets of vectors:

$$\begin{aligned}\mathrm{Post}(S) = \{&\boldsymbol{v} - \boldsymbol{a} \in \mathbb{N}^d \mid \boldsymbol{v} \in S \wedge \boldsymbol{a} \in \boldsymbol{A}\} \cup \\ &\{\min\{\boldsymbol{v}', \boldsymbol{v}''\} - \boldsymbol{b} \in \mathbb{N}^d \mid \boldsymbol{v}', \boldsymbol{v}'' \in S \wedge \boldsymbol{b} \in \boldsymbol{B}_\wedge\} \cup \\ &\{\boldsymbol{v}' + \boldsymbol{v}'' - \boldsymbol{b} \in \mathbb{N}^d \mid \boldsymbol{v}', \boldsymbol{v}'' \in S \wedge \boldsymbol{b} \in \boldsymbol{B}_+\},\end{aligned}$$

and then instantiate the generic backward algorithm from Sec. 3 by the following operator on sets of downward-closed sets of vectors:

$$\mathcal{PRE}(\mathcal{S}) = \{D \in \mathbb{P}^{\sqsubseteq}_{\downarrow}(\mathbb{N}^d) \mid D \cup {\downarrow}\mathrm{Post}(D) \in \mathcal{S}\}.$$

Vector Set Ideals. We pause to recall some properties of the downwards-closed powerset of $\mathbb{N}^d$.

We write $\mathbb{D}(X)$ for the set of all subsets of X that are downwards-closed with respect to a quasi-ordering $\leq$. The subset relation is a quasi-ordering on $\mathbb{D}(X)$. Because $(\mathbb{N}^d, \sqsubseteq)$ is an ω^2-wqo (it is even a *better-quasi-order*), $(\mathbb{D}(\mathbb{N}^d), \subseteq)$ is a wqo.

Fact C.1 ([10, 11]). *For any $d \in \mathbb{N}$, $(\mathbb{D}(\mathbb{N}^d), \subseteq)$ is a wqo with effective ideal representations.*

Indeed, the ideals of $(\mathbb{D}(\mathbb{N}^d), \subseteq)$ are all ${\downarrow}D = \{D' \in \mathbb{D}(\mathbb{N}^d) \mid D' \subseteq D\}$ for downwards-closed $D \subseteq \mathbb{N}^d$, and can thus be represented as finite sets of vectors from $\mathbb{N}^d_\omega$.

Downward-Closure and Correctness. Resuming, it is straightforward to check that $\mathcal{PRE}$ preserves the property of downward closure. Given an initial downwards-closed $\mathcal{D}_0 \subseteq \mathbb{D}(\mathbb{N}^d)$, as in Sec. 3, we write $\mathcal{D}_*$ for the last set in the longest (necessarily finite) descending chain of downwards-closed sets $\mathcal{D}_0 \supsetneq \mathcal{D}_1 \supsetneq \cdots$ defined by $\mathcal{D}_{k+1} = \mathcal{D}_k \cap \mathcal{PRE}(\mathcal{D}_k)$. Observing that $\mathcal{PRE}(\mathcal{D}) \subseteq \mathcal{D}$ whenever $\mathcal{D}$ is downwards-closed, we actually have $\mathcal{D}_{k+1} = \mathcal{PRE}(\mathcal{D}_k)$. From the definitions of Post and $\mathcal{PRE}$, we have the required correctness property:

Claim C.2. (a) Any $\mathcal{D}_i$ consists of all downwards-closed sets of vectors D for which there exists $D' \in \mathcal{D}_0$ such that all computations of $(\boldsymbol{A}, \boldsymbol{B}_\wedge, \boldsymbol{B}_+)$ whose leaves are in D and whose height is at most i have their root in D'.

(b) The set $\mathcal{D}_*$ consists of all downwards-closed sets of vectors D for which there exists $D' \in \mathcal{D}_0$ such that all computations of $(\boldsymbol{A}, \boldsymbol{B}_\wedge, \boldsymbol{B}_+)$ whose leaves are in D have their root in D'.

Effectiveness and Control. Let the norm of a downwards-closed subset of $\mathbb{D}(\mathbb{N}^d)$ whose ideal decomposition is ${\downarrow}D_1 \cup \cdots \cup {\downarrow}D_n$ be the maximum of the norms of $D_1, \ldots, D_n$.

Similarly to the cases of VAS and reset VAS in Sections 3.2 and 3.3, we obtain:

Claim C.3. (a) The ideal decomposition of $\mathcal{PRE}(\mathcal{D})$ is computable for all downwards-closed $\mathcal{D} \subseteq \mathbb{D}(\mathbb{N}^d)$.

(b) The descending chain $\mathcal{D}_0 \supsetneq \mathcal{D}_1 \supsetneq \cdots$ is (g, n)-controlled for

$$g(x) \stackrel{\text{def}}{=} x + \max\{||\boldsymbol{A}||, ||\boldsymbol{B}_\wedge||, ||\boldsymbol{B}_+||\}$$

and $n \stackrel{\text{def}}{=} \|\boldsymbol{t}\|$.

ω-Monotonicity. The distinction between bottom-up ABVAS and BVAS appears here. Inside the descending chain of downwards-closed subsets of $\mathbb{D}(\mathbb{N}^d)$ computed by the backward coverability algorithm, we can always find a descending chain of downwards-closed subsets of $\mathbb{N}^d$, whose length is bounded by the Ackermann function. However, for bottom-up BVAS, the latter descending chain is ω-monotone, enabling us to infer that its length is then at most doubly exponential.

Claim C.4. Suppose $\mathcal{D}_0 \supsetneq \cdots \supsetneq \mathcal{D}_k$ is the descending chain computed by the backward coverability algorithm for a d-dimensional bottom-up ABVAS $(\boldsymbol{A}, \boldsymbol{B}_\wedge, \boldsymbol{B}_+)$ and an initial downwards-closed $\mathcal{D}_0 \subseteq \mathbb{D}(\mathbb{N}^d)$.

(a) There exist proper downwards-closed D_j in the ideal decomposition of $\mathcal{D}_j$ for all $j \in \{0, \ldots, k-1\}$ such that

$$D_{j+1} \cup {\downarrow}\mathrm{Post}(D_{j+1}) \subseteq D_j$$

for all $j \in \{0, \ldots, k-2\}$.

(b) For BVAS, i.e. when $\boldsymbol{B}_\wedge$ is empty, any descending chain $D_0, \ldots, D_{k-1}$ as in (a) is ω-monotone.

Proof. For (a), pick any proper D_{k-1} in the decomposition of $\mathcal{D}_{k-1}$. Assuming that D_{j+1} has been picked, we have ${\downarrow}D_{j+1} \not\subseteq \mathcal{PRE}(\mathcal{D}_{j+1})$ and thus $D_{j+1} \notin \mathcal{PRE}(\mathcal{D}_{j+1})$, so $D_{j+1} \cup {\downarrow}\mathrm{Post}(D_{j+1}) \in \mathcal{D}_j \setminus \mathcal{D}_{j+1}$. Pick any D_j in the decomposition of $\mathcal{D}_j$ such that $D_{j+1} \cup {\downarrow}\mathrm{Post}(D_{j+1}) \subseteq D_j$.

For (b), suppose $0 \le j < k-2$ and $\boldsymbol{v}_{j+1}$ is a proper vector in the decomposition of D_{j+1}, i.e. ${\downarrow}\boldsymbol{v}_{j+1} \not\subseteq D_{j+2}$. Since ${\downarrow}D_{j+2}$ is in the decomposition of $\mathcal{D}_{j+2}$, we have that $D_{j+2} \cup {\downarrow}\boldsymbol{v}_{j+1} \notin \mathcal{D}_{j+2}$. But $D_{j+2} \cup {\downarrow}\boldsymbol{v}_{j+1} \subseteq D_{j+1}$, so $\mathrm{Post}(D_{j+2} \cup {\downarrow}\boldsymbol{v}_{j+1}) \not\subseteq D_{j+1}$. Recalling that $\mathrm{Post}(D_{j+2}) \subseteq D_{j+1}$, we have $\boldsymbol{w} \in \mathbb{N}_\omega^d$ and ${\downarrow}\boldsymbol{w} \not\subseteq D_{j+1}$ where:

- either $\boldsymbol{w} = \boldsymbol{v}_{j+1} - \boldsymbol{a}$ for some $\boldsymbol{a} \in \boldsymbol{A}$,
- or $\boldsymbol{w} = \boldsymbol{v}' + \boldsymbol{v}_{j+1} - \boldsymbol{b}$ for some $\boldsymbol{v}'$ in the decomposition of D_{j+2} and $\boldsymbol{b} \in \boldsymbol{B}_+$.

In both cases, ${\downarrow}\boldsymbol{w} \subseteq {\downarrow}\mathrm{Post}(D_{j+2} \cup {\downarrow}\boldsymbol{v}_{j+1}) \subseteq D_j$. Let $\boldsymbol{v}_j$ be any vector in the decomposition of D_j such that $\boldsymbol{w} \sqsubseteq \boldsymbol{v}_j$. We conclude that $\boldsymbol{v}_j$ is proper and that $\omega(\boldsymbol{v}_{j+1}) \subseteq \omega(\boldsymbol{v}_j)$. □

Upper Bounds. Finally, by applying the length function theorem for descending chains (Thm. 3.6), or for ω-monotone descending chains (Thm. 4.4) in the case of BVAS, we establish the Ackermann and 2ExpTime bounds. Again, the ideal decomposition of each $\mathcal{D}_{k+1}$ is computable in time polynomial in $g^k(\|\mathcal{D}_0\|)$, so it suffices to bound the number of iterations of the main loop.

Corollary C.5. *For any d-dimensional bottom-up ABVAS $(\boldsymbol{A}, \boldsymbol{B}_\wedge, \boldsymbol{B}_+)$ and initial downwards-closed $\mathcal{D}_0 \subseteq \mathbb{D}(\mathbb{N}^d)$, the backward coverability algorithm terminates after at most* $\mathrm{ackermann}(p(d, L, \|\mathcal{D}_0\|))$ *steps, where p is a primitive-recursive function and $L = \max\{||\boldsymbol{A}||, ||\boldsymbol{B}_\wedge||, ||\boldsymbol{B}_+||\}$. For BVAS, i.e. when $\boldsymbol{B}_\wedge$ is empty, the algorithm terminates after at most $((L+1)(\|\mathcal{D}_0\|+2))^{(d+1)!}+1$ steps.*

Additional References

20. Lincoln, P., Mitchell, J., Scedrov, A., Shankar, N.: Decision problems for propositional linear logic. Ann. Pure App. Logic 56(1–3), 239–311 (1992)