REACHABILITY IN VECTOR ADDITION SYSTEMS DEMYSTIFIED

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Abstract. More than 30 years after their inception, the decidability proofs for reachability in vector addition systems (VAS) still retain much of their mystery. These proofs rely crucially on a decomposition of runs successively refined by Mayr, Kosaraju, and Lambert, which appears rather magical, and for which no complexity upper bound is known.

We first offer a justification for this decomposition technique, by showing that it emerges naturally in the study of the ideals of a well quasi ordering of VAS runs. In a second part, we apply recent results on the complexity of termination thanks to well quasi orders and well orders to obtain fast-growing complexity upper bounds for the decomposition algorithms, thus providing the first known upper bounds for general VAS reachability.

Keywords. Vector addition system, reachability, well quasi order, ideal, fast-growing complexity

1. Introduction

Vector addition systems (VAS), or equivalently Petri nets, find a wide range of applications in the modelling of concurrent, chemical, biological, or business processes. Their algorithmics, and in particular the decidability of their *reachability problem*, is a central component to many decidability results spanning from the verification of asynchronous programs [13] to the decidability of data logics [4, 9, 7]. Considered as one of the great achievements of theoretical computer science, the original 1981 decidability proof of Mayr [31] is the culmination of more than a decade of research into the topic, and builds notably on an incomplete proof by Sacerdote and Tenney [35]. The proof was simplified a year later by Kosaraju [21]; see also the account by Müller [32] and the self-contained and detailed monograph of Reutenauer [34] on this second proof. In spite of this success, as put by Lambert [23] "the complexity of the two proofs (especially in [31]) wrapped the result in mystery and no use of their original ideas" was made before he provided a further simplification ten years later in 1992, and employed it to prove results on VAS languages.

At the heart of the various proofs lies a *decomposition technique*, which we dub the Kosaraju-Lambert-Mayr-Sacerdote-Tenney (KLMST) decomposition in this article after its inventors. In a nutshell, the KLMST decomposition defines both a structure and a condition for this structure to represent in some way the set of all runs witnessing reachability. The algorithms advanced by Mayr, Kosaraju, and Lambert compute this decomposition by

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successive refinements of the structure until the condition is fulfilled. The KLMST decomposition is a powerful tool when reasoning about VAS runs, and it has notably been employed

- by Habermehl, Meyer, and Wimmel [15] to show that the downward-closure of a labelled VAS language is effectively computable, and
- by Leroux [26] to derive a new algorithm for reachability based on Presburger inductive invariants—he would later re-prove the correction of this new algorithm *without* referring to the KLMST decomposition, yielding a compact self-contained decidability proof for VAS reachability [27].

Our feeling however is that the decidability of VAS reachability, and especially the KLMST decomposition, is still shrouded in mystery. The result is highly complex on two accounts:

On a conceptual level the various instances of the KLMST decomposition seem rather magical. How did Mayr come up with *regular constraint graphs* with a *consistent marking*? How did Kosaraju come up with *generalised VASS* and his θ *condition*? How did Lambert come up with his *perfect* condition on *marked graph-transition sequences*? Most importantly, which guidelines to follow in order to develop similar concepts for VAS extensions where the decidability of reachability is still open, e.g. for unordered data Petri nets [25], pushdown VASS [24], or branching VAS [36]?

Arguably, the issue here is not to understand how these structures and conditions are used in the algorithms themselves, nor to check that they indeed yield the decidability of VAS reachability. Rather, the issue is to explain how these structures and conditions can be derived in a principled manner.

On a computational complexity level no complexity upper bound is known for the general VAS reachability problem, while the best known lower bound is ExpSpace-hardness [29]. The only known tight bounds pertain to the very specific case of 2-dimensional VAS with states, which were recently shown to have a PSpace-complete reachability problem [3].

As observed e.g. by Müller [32] the algorithms computing the KLMST decomposition are not primitive-recursive, but no one has been able to derive a complexity upper bound for these algorithms, while the new algorithm of Leroux [26, 27] using Presburger inductive invariants seems even harder to analyse from a complexity viewpoint.

Our contribution in this paper is first to propose an explanation for the KLMST decomposition. Using a well quasi ordering of VAS runs defined by Jančar [17] and Leroux [27] and recalled in Section 5, we show that the KLMST algorithm can be understood as computing an *ideal* decomposition of the set of runs, i.e. a decomposition into irreducible downward-closed sets (see Section 8). The effective presentation of those ideals through finite structures turns out to match exactly the structures and conditions expressed by Lambert [23], see sections 6 and 7. This provides a full formal framework in which the reachability problem in various VAS extensions might be cast, offering some hope to see progress on those open issues.

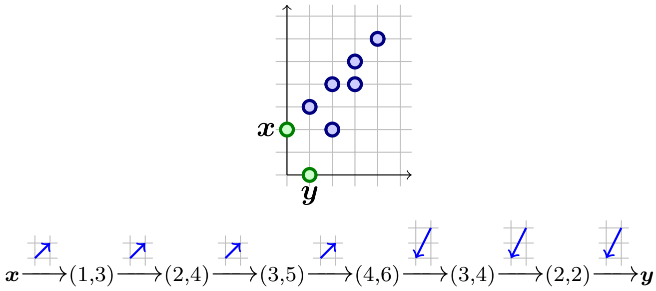

Figure 1. A run from $\boldsymbol{x} = (0, 2)$ to $\boldsymbol{y} = (1, 0)$ labelled by $(1, 1)^4(-1, -2)^3$.

The second contribution in Section 9 is the proof of a multiply-recursive $\mathbf{F}_{\omega^3}$ upper bound on the complexity of the KLMST decomposition algorithm, using the *fast-growing* complexity classes $(\mathbf{F}_\alpha)_\alpha$ defined in [37]. This upper bound applies the recent results on bounding the length of controlled bad sequences over well quasi orders in [39, 38]. It yields the first known upper bound on VAS reachability. As a byproduct, it also yields the first complexity upper bound for numerous problems known decidable thanks to a reduction to VAS reachability, e.g. [4, 13, 9, 7] among many others.

We start in sections 2, 3, and 4 by presenting the necessary background on VAS, well quasi orders, and ideals.

2. Vector Addition Systems

A *vector addition system* of dimension d in $\mathbb{N}$ is a finite set $\boldsymbol{A}$ of *actions* $\boldsymbol{a}$ in $\mathbb{Z}^d$. The operational semantics of VASs operates on *configurations*, which are vectors $\boldsymbol{c}$ in $\mathbb{N}^d$. A *transition* is then a triple $(\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{v}) \in \mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$ such that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}$, where addition operates componentwise; the set of transitions of $\boldsymbol{A}$ is denoted by $\mathrm{Trans}_{\boldsymbol{A}}$.

A *prerun* over $\boldsymbol{A}$ is a triple $\rho = (\boldsymbol{u}, w, \boldsymbol{v})$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are two configurations in $\mathbb{N}^d$ and w is a sequence of triples $(\boldsymbol{u}_1, \boldsymbol{a}_1, \boldsymbol{v}_1) \cdots (\boldsymbol{u}_k, \boldsymbol{a}_k, \boldsymbol{v}_k)$ in $(\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d)^*$. The configurations $\boldsymbol{u}$ and $\boldsymbol{v}$ are called respectively the *source* and *target* of ρ, and are denoted respectively by $\mathrm{src}(\rho)$ and $\mathrm{tgt}(\rho)$. The action sequence $\sigma = \boldsymbol{a}_1 \cdots \boldsymbol{a}_k$ is called the *label* of ρ. We write $\mathrm{PreRuns}_{\boldsymbol{A}}$ for the set of preruns over $\boldsymbol{A}$.

A prerun $(\boldsymbol{u}, w, \boldsymbol{v})$ is *connected* if $w = (\boldsymbol{u}_1, \boldsymbol{a}_1, \boldsymbol{v}_1) \cdots (\boldsymbol{u}_k, \boldsymbol{a}_k, \boldsymbol{v}_k)$ is a transition sequence in $\mathrm{Trans}^*_{\boldsymbol{A}}$ such that

- either $w = \varepsilon$ is the empty sequence and then $\boldsymbol{u} = \boldsymbol{v}$,
- or $k > 0$ and $\boldsymbol{u} = \boldsymbol{u}_1$, $\boldsymbol{v} = \boldsymbol{v}_k$, and $\boldsymbol{u}_{j+1} = \boldsymbol{v}_j$ for all $0 \leq j < k$.

We call a connected prerun ρ a *run*. If there exists a run ρ from source $\boldsymbol{u}$ to target $\boldsymbol{v}$ labelled by σ, we denote by $\boldsymbol{u} \xrightarrow{\sigma} \boldsymbol{v}$ this unique run ρ. Notice that it implies $\boldsymbol{v} = \boldsymbol{u} + \sum_{j=1}^{k} \boldsymbol{a}_j$; note however that given $\boldsymbol{u}$, $\boldsymbol{v}$, and σ, $\boldsymbol{v} = \boldsymbol{u} + \sum_{j=1}^{k} \boldsymbol{a}_j$ does not necessarily imply that $\boldsymbol{u} \xrightarrow{\sigma} \boldsymbol{v}$.

We are interested in this paper in the following decision problem:

Problem: VAS Reachability.

input: A d-dimensional VAS $\boldsymbol{A}$, a source configuration $\boldsymbol{x}$, and a target configuration $\boldsymbol{y}$.
question: $\exists\sigma \in \boldsymbol{A}^*.\boldsymbol{x} \xrightarrow{\sigma} \boldsymbol{y}$?

Given two configurations $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{N}^d$, we define the *set of runs* of $\boldsymbol{A}$ from $\boldsymbol{x}$ to $\boldsymbol{y}$ as

$$\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}) \stackrel{\text{def}}{=} \{\boldsymbol{x} \xrightarrow{\sigma} \boldsymbol{y} \mid \sigma \in \boldsymbol{A}^*\} \ . \tag{1}$$

The VAS reachability problem can then be recast as asking whether the set $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$ is non empty.

3. Well Quasi Orders

A *quasi-order* (qo) is a pair $(X, \leq)$ where X is a set and $\leq$ is a reflexive and transitive binary relation over X. We write $x < y$ if $x \leq y$ but $y \not\leq x$. Given a set $S \subseteq X$, we define its *upward-closure* $\uparrow S \stackrel{\text{def}}{=} \{x \in X \mid \exists s \in S. s \leq x\}$ and *downward-closure* $\downarrow S \stackrel{\text{def}}{=} \{x \in X \mid \exists s \in S. x \leq s\}$. When $S = \{s\}$ is a singleton, we write more succinctly $\uparrow s$ and $\downarrow s$. An *upward-closed* set $U \subseteq X$ is such that $U = \uparrow U$ and a *downward-closed* set $D \subseteq X$ such that $D = \downarrow D$. Observe that upward- and downward-closed sets are closed under arbitrary union and intersection, and that the complement over X of an upward-closed set is downward-closed and vice versa.

3.1. **Characterisations.** A finite or infinite sequence $x_0, x_1, x_2, \ldots$ of elements of a qo $(X, \leq)$ is *good* if there exist two indices $i < j$ such that $x_i \leq x_j$, and *bad* otherwise. A *well quasi order* (wqo) is a qo with the additional property that all its bad sequences are finite.

Example 3.1 (Finite sets). As an example, a set X ordered by equality is a wqo if and only if it is finite: if finite, by the pigeonhole principle its bad sequences have length at most $|X|$; if infinite, any enumeration of infinitely many distinct elements yields an infinite bad sequence. □

There are many equivalent characterisations of wqos [22, 39]. For instance, $(X, \leq)$ is a wqo if and only if it is *well-founded*, i.e. there are no infinite descending sequences $x_0 > x_1 > \cdots$ of elements from X, and it has the *finite antichain* (FAC) property, i.e. any set of mutually incomparable elements from X is finite.

Example 3.2 (Well orders). Any well-founded *linear* order, i.e. where $\leq$ is furthermore antisymmetric and total, is a wqo: in that case, antichains have cardinal at most one. Examples include $(\mathbb{N}, \leq)$ the set of natural numbers, i.e. the ordinal ω. □

We will also be interested in the following characterisation:

Fact 3.3 (Descending Chain Property). *A qo $(X, \leq)$ is a wqo if and only if any non-ascending chain $D_0 \supseteq D_1 \supseteq D_2 \supseteq \cdots$ of downward-closed subsets of X eventually stabilises, i.e. there exists a finite rank k such that $\bigcap_{i\in\mathbb{N}} D_i = D_k$.*

Proof. For the direct implication, assume that there exists a non-ascending chain that does not stabilise, i.e. there exists an infinite descending sub-chain $D_{i_0} \supsetneq D_{i_1} \supsetneq D_{i_2} \subsetneq \cdots$. This means that there exists an infinite sequence of elements $x_{i_j} \in D_{i_j} \setminus D_{i_{j+1}}$. Note that, if $j < k$, then x_{i_j} is in $D_{i_j} \setminus D_{i_k}$, hence $x_{i_j} \not\leq x_{i_k}$, and therefore $(X, \leq)$ is not a wqo.

Conversely, consider any infinite sequence $x_0, x_1, \ldots$ of elements of X. Let then $U_i \stackrel{\text{def}}{=} \bigcup_{j \leq i} {\uparrow} x_j$ and $D_i \stackrel{\text{def}}{=} X \setminus U_i$. Observe that if the non-ascending chain of D_i's stabilises at some rank k, then $U_k = U_{k+1} = U_k \cup {\uparrow} x_{k+1}$, hence there exists $i \leq k$ such that $x_i \leq x_{k+1}$, showing that $(X, \leq)$ is a wqo. $\square$

Another consequence of the definition of wqos is:

Fact 3.4 (Finite Basis Property)**.** *Let $(X, \leq)$ be a wqo. If $U \subseteq X$ is upward-closed, then there exists a* finite basis *$B \subseteq U$ such that ${\uparrow} B = U$.*

3.2. **Elementary Operations.** Many constructions are known to yield new wqos from existing ones. In this paper we will employ the following elementary operations:

3.2.1. *Cartesian Products.* If $(X, \leq_X)$ and $(Y, \leq_Y)$ are wqos, then their Cartesian product $X \times Y$ is well quasi ordered by the *product (quasi-) ordering* defined by $(x, y) \leq (x', y')$ if and only if $x \leq_X x'$ and $y \leq_Y y'$. For instance, vectors in $\mathbb{N}^d$ along with the product ordering form a wqo. This result is also known as *Dickson's Lemma.*

3.2.2. *Finite Sequences.* If $(X, \leq_X)$ is a wqo, then the set X^* of finite sequences over X is well quasi ordered by the *sequence embedding* defined by $\sigma \leq_* \sigma'$ if and only if $\sigma = x_1 \cdots x_k$ and $\sigma' = \sigma'_0 x'_1 \sigma'_1 \cdots \sigma'_{k-1} x'_k \sigma'_k$ for some $x_j \leq_X x'_j$ in X for $1 \leq j \leq k$ and some σ'_j in X^* for $0 \leq j \leq k$. For instance, finite sequences in Σ^* for a finite alphabet $(\Sigma, =)$ form a wqo. This result is also known as *Higman's Lemma.*

In the following, we call *elementary* those wqos obtained from finite sets $(X, =)$ through finitely many applications of Dickson's and Higman's lemmas. Note that $(\mathbb{N}, \leq)$ is elementary since it is isomorphic with finite sequences over some unary alphabet with equality.

4. WQO Ideals

Downward-closed sets D can be denoted by a finite set of elements in X: since $X \setminus D$ is upward closed, it is the upward closure of a finite set $B \subseteq X \setminus D$ thanks to Fact 3.4. We deduce the following decomposition:

$$D = \bigcap_{x \in B} (X \setminus {\uparrow} x) . \tag{2}$$

In this section, we recall an alternative way of decomposing downward-closed sets, namely as finite unions of *ideals.* This is a classical notion—Fraïssé [12, Section 4.5] attributes finite ideal decompositions to Bonnet [5]—which has been rediscovered in the study of well structured transition systems [11]. Let us review the basic theory of ideals, as can be found in [5, 12, 19, 11]; see in particular [14] for a gentle introduction.

4.1. **Ideals.** A subset S of a qo $(X, \leq)$ is *directed* if for every $x_1, x_2 \in S$ there exists $x \in S$ such that both $x_1 \leq x$ and $x_2 \leq x$. An *ideal* I is a directed non-empty downward-closed set. The class of ideals is denoted by $\mathrm{Idl}(X)$.[1]

Example 4.1 (Well orders). In an ordinal α seen in set-theoretic terms as $\{\beta \mid \beta < \alpha\}$, any $\beta \leq \alpha$ is a downward-closed directed subset of α, and conversely any downward-closed directed subset of α is some $\beta \leq \alpha$. Hence the ideals of α are exactly the elements of $\alpha + 1$ except 0. □

4.1.1. *Ideals as Irreducible Downward-Closed Sets.* An alternative characterisation of ideals shows that they are the *irreducible* downward-closed sets of a qo $(X, \leq)$:

Fact 4.2. *Let I be a non-empty downward-closed set. The following are equivalent:*

(1) *I is an ideal,*
(2) *for every pair of downward-closed sets (D_1, D_2), if $I = D_1 \cup D_2$, then $I = D_1$ or $I = D_2$, and*
(3) *for every pair of downward-closed sets (D_1, D_2), if $I \subseteq D_1 \cup D_2$, then $I \subseteq D_1$ or $I \subseteq D_2$.*

Because we find the proof of this fact in [19, 11, 14] enlightening, we recall the main ideas here:

Proof of $1 \implies 2$. Assume that I is an ideal and let (D_1, D_2) be two downward-closed sets such that $I = D_1 \cup D_2$. If $I = D_1$ we are done, so we can assume that there exists $x \in I \setminus D_1$. Because $D_2 \subseteq I$, it remains to prove that $I \subseteq D_2$.

Consider any $y \in I$. Because I is directed, there exists $m \in I$ such that $x, y \leq m$. Observe that $m \in I \subseteq D_1 \cup D_2$ but $m \notin D_1$ since D_1 is downward-closed, $x \leq m$ and $x \notin D_1$. Thus $m \in D_2$, and since D_2 is downward-closed, $y \in D_2$. We have shown that $I \subseteq D_2$. □

Proof of $2 \implies 3$. Let I be a non-empty downward-closed set satisfying item 2 and let (D_1, D_2) be a pair of downward-closed sets with $I \subseteq D_1 \cup D_2$. Define $D_1' \stackrel{\text{def}}{=} D_1 \cap I$ and $D_2' \stackrel{\text{def}}{=} D_2 \cap I$: then $I = D_1'$ or $I = D_2'$ by item 2, and therefore $I \subseteq D_1$ or $I \subseteq D_2$. □

Proof of $3 \implies 1$. Let I be a non-empty downward-closed set satisfying item 3. Consider $x_1, x_2 \in I$ along with the downward-closed sets $D_1 \stackrel{\text{def}}{=} X \setminus {\uparrow} x_1$ and $D_2 \stackrel{\text{def}}{=} X \setminus {\uparrow} x_2$. Observe that, if $I \subseteq D_1 \cup D_2$, by item 3 $I \subseteq D_1$ or $I \subseteq D_2$, and in both cases we get a contradiction with $x_1, x_2 \in I$. Hence, there exists $m \in I \setminus (D_1 \cup D_2)$, thus $x_1, x_2 \leq m$ and we have shown that I is directed. □

Example 4.3 (Finite sets). In a finite wqo $(X, =)$, any subset of X is downward-closed. The ideals are thus exactly the singletons over X: any other non-empty subset of X can be split into simpler sets. □

[1] The set of ideals equipped with the inclusion relation is called the *completion* of the wqo $(X, \leq)$, see [11].

Corollary 4.4. *An ideal I is included in a finite union $D_1 \cup \ldots \cup D_k$ of downward-closed sets $D_1, \ldots, D_k$ if and only if $I \subseteq D_j$ for some $1 \leq j \leq k$.*

Proof. By induction on k using Fact 4.2. □

4.1.2. *Finite Decompositions.* Observe that any downward-closed set of the form $\downarrow x$ is an ideal, hence any downward-closed set is a union of ideals. However, the main interest we find with ideals is that they provide *finite* decompositions for downward-closed subsets of wqos:

Fact 4.5 (Canonical Ideal Decompositions)**.** *Every downward-closed set over a wqo is the union of a unique finite family of incomparable (for the inclusion) ideals.*

Let us again recall the proof as found in [19, 11, 14]:

Proof. Assume for the sake of contradiction that there exists a downward-closed set D of a wqo $(X, \leq)$, for which only infinite ideal decompositions exist. Because $(X, \leq)$ is a wqo, by Fact 3.3 $(\mathrm{Idl}(X), \subseteq)$ is well-founded and we can choose D minimal for inclusion. Observe that D is nonempty (or it would be an empty union of ideals). Whenever $D = D_1 \cup D_2$ for some downward-closed sets D_1 and D_2, there is i in $\{1, 2\}$ such that D_i requires an infinite ideal decomposition, and thus by minimality of D, $D = D_i$. By Fact 4.2, D is an ideal, contradiction. Finally, the unicity of the decomposition follows from Corollary 4.4. □

The statement of Fact 4.5 can be strengthened: it already holds for FAC partial orders [see 5, 12, 19].

4.1.3. *Induced Ideals.* Consider some non-empty directed set Δ in $(X, \leq)$. Then its downward-closure $\downarrow\Delta$ is downward-closed, non-empty, and directed, i.e., is an ideal of X. This characterises ideals, since an ideal I is also equal to $\downarrow I$.

As a related notion, consider some subset S of X: by Fact 4.5, its downward-closure has a canonical ideal decomposition. The following lemma shows that the ideals in this decomposition are downward-closures of some directed subsets of S itself:[2]

Lemma 4.6. *Let $(X, \leq)$ be a wqo and $S \subseteq X$. Then every maximal ideal of $\downarrow S$ is the downward-closure $\downarrow\Delta$ of a directed set $\Delta \subseteq S$.*

Proof. Assume that S is non-empty—or the lemma holds trivially. Let us write $\downarrow S = J \cup J_1 \cup \cdots \cup J_k$ for the canonical decomposition of $\downarrow S$. By minimality of this decomposition, there exists x_J in J such that $x_J \notin J_j$ for all $1 \leq j \leq k$. Thus any element s in $\uparrow x_J \cap S$ must belong to J.

Let us show that $J \cap S$ is directed: for $s, s' \in J \cap S$, because J is directed we first find y in J larger or equal to s, s', and x_J. Since $J \subseteq \downarrow S$, we then find $s'' \geq y$ in S. By the remark made in the previous paragraph, since $s'' \geq x_J$, s'' also belongs to J.

It remains to show that $J = \downarrow(J \cap S)$. It suffices to show the inclusion $J \subseteq \downarrow(J \cap S)$ since the converse inclusion is immediate. Consider any y from

[2]In more topological terms, the ideals of the decomposition of $\downarrow S$ are the *adherent points* of $\{\downarrow s \mid s \in S\}$ inside $\mathrm{Idl}(X)$ equipped with the upper topology for inclusion.

J. Then there exists y' in J larger or equal to both y and x_J, and again since $J \subseteq {\downarrow}S$ and by definition of x_J there exists $s \geq y'$ in $J \cap S$. □

4.2. **Effective Ideal Representations.** Thanks to Fact 4.5, any downward-closed set has a presentation using finitely many ideals. Should we manage to find *effective* representations of wqo ideals, this will provide us with algorithmic means to manipulate downward-closed sets. This endeavour is the subject of [11, 14], and we merely provide pointers to their results here.

4.2.1. *Natural Numbers.* As seen in Example 4.1, the ideals of $(\mathbb{N}, \leq)$ are either ${\downarrow}n$ for some finite $n \in \mathbb{N}$, or the whole of $\mathbb{N}$ itself. As done classically in the VAS literature, we represent the latter using a new element noted "ω" with $n < \omega$ for all $n \in \mathbb{N}$, and denote the new set $\mathbb{N}_\omega \stackrel{\text{def}}{=} \mathbb{N} \uplus \{\omega\}$. For notational convenience, we write ${\downarrow}\omega$ for $\mathbb{N}$, so that an ideal of $(\mathbb{N}, \leq)$ can be written as ${\downarrow}x$ for x in $\mathbb{N}_\omega$.

4.2.2. *Cartesian Products.* Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two wqos, and assume that we know how to represent the ideals in $\mathrm{Idl}(X)$ and $\mathrm{Idl}(Y)$. Then the ideals of $X \times Y$ equipped with the product ordering have a simple enough representation:

$$\mathrm{Idl}(X \times Y) = \{I \times J \mid I \in \mathrm{Idl}(X) \wedge J \in \mathrm{Idl}(Y)\} . \tag{3}$$

Configurations. For example, configuration ideals can be represented as ${\downarrow}\boldsymbol{v}$ for a vector $\boldsymbol{v}$ in $\mathbb{N}_\omega^d$.

In this paper we often find it convenient to identify *partial* vectors $\boldsymbol{u}$ in $\mathbb{N}^F$ for some subset $F \subseteq \{1, \ldots, d\}$ with vectors $\boldsymbol{v}$ in $\mathbb{N}_\omega^d$ with finite values over F, such that $\boldsymbol{v}(i) = \omega$ if $i \notin F$ and $\boldsymbol{v}(i) = \boldsymbol{u}(i)$ otherwise. Then *projections* $\pi_F \colon \mathbb{N}_\omega^d \to \mathbb{N}_\omega^d$ on a set $F \subseteq \{1, \ldots, d\}$ can be defined for all $1 \leq i \leq d$ by

$$\pi_F(\boldsymbol{u})(i) \stackrel{\text{def}}{=} \begin{cases} \boldsymbol{u}(i) & \text{if } i \in F \\ \omega & \text{otherwise.} \end{cases} \tag{4}$$

Transitions. By Dickson's Lemma, the product ordering over $\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$ is a wqo.

A *transition ideal* is an ideal of $\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$ that is the downward closure of a set of transitions of $\mathrm{Trans}_{\boldsymbol{A}}$. As seen in Example 4.3, the ideals of $\boldsymbol{A}$ are the singletons $\{\boldsymbol{a}\}$ for $\boldsymbol{a} \in \boldsymbol{A}$. By (3), the ideals of $\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$ can thus be presented as downward-closures of triples $(\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{v})$ in $\mathbb{N}_\omega^d \times \boldsymbol{A} \times \mathbb{N}_\omega^d$.

The transition ideals are going to form a particular class of such triples. Let us define addition over $\mathbb{Z} \uplus \{\omega\}$ by $k + \omega = \omega + k = \omega + \omega = \omega$. A *partial transition* is a triple $(\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{v})$ in $\mathbb{N}_\omega^d \times \boldsymbol{A} \times \mathbb{N}_\omega^d$ such that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}$. The following is immediate by continuity, but can also be given a non-topological proof:

Lemma 4.7. *The transitions ideals of* $\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$ *are exactly the sets* ${\downarrow}t$ *with* t *a partial transition.*

Proof. First notice that ${\downarrow}(\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{u} + \boldsymbol{a})$ for some $\boldsymbol{u}$ in $\mathbb{N}_\omega^d$ and $\boldsymbol{a}$ in $\boldsymbol{A}$ is a transition ideal of $\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$: it is non-empty, directed, and the downward closure of a set of transitions in $\mathrm{Trans}_{\boldsymbol{A}}$.

Conversely, let $I \subseteq \mathrm{Trans}_{\boldsymbol{A}}$ be a transition ideal. There exists a set $T \subseteq \mathrm{Trans}_{\boldsymbol{A}}$ such that $I = {\downarrow} T$. Then $I = {\downarrow}(\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{v})$ for $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbb{N}_\omega^d$ and $\boldsymbol{a}$

in $\boldsymbol{A}$. Let us show that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}$. Assume for the sake of contradiction that there exists $1 \le i \le d$ such that $\boldsymbol{v}(i) < \boldsymbol{u}(i) + \boldsymbol{a}(i)$. There exists $\boldsymbol{u}'$ in $\downarrow\boldsymbol{u}$ such that $\boldsymbol{v}(i) < \boldsymbol{u}'(i) + \boldsymbol{a}(i)$. Moreover, since $\boldsymbol{u}'$ is in $\downarrow\boldsymbol{u}$, there exists $(\boldsymbol{u}'', \boldsymbol{a}, \boldsymbol{u}'' + \boldsymbol{a}) \in T$ such that $\boldsymbol{u}' \le \boldsymbol{u}''$. But then $\boldsymbol{u}'' + \boldsymbol{a}$ does not belong to $\downarrow\boldsymbol{v}$ since $\boldsymbol{u}''(i) + \boldsymbol{a}(i) \ge \boldsymbol{u}'(i) + \boldsymbol{a}(i) > \boldsymbol{v}(i)$. This is a contradiction. The case where there exists $1 \le i \le d$ such that $\boldsymbol{v}(i) > \boldsymbol{u}(i) + \boldsymbol{a}(i)$ is similar. □

The concept of partial configurations can be lifted to partial transitions as follows:

$$\pi_F((\boldsymbol{u}, \boldsymbol{a}, \boldsymbol{v})) \stackrel{\text{def}}{=} (\pi_F(\boldsymbol{u}), \boldsymbol{a}, \pi_F(\boldsymbol{v})) . \tag{5}$$

4.2.3. *Finite Sequences.* In the case of sequences over a finite alphabet $(\Sigma, =)$, Jullien [18] first characterised the ideals using a simple form of regular expressions, which was later rediscovered by Abdulla et al. [1] for the verification of lossy channel systems. A representation of ideals for sequences over an arbitrary wqo $(X, \le)$ was given by Kabil and Pouzet [19] and also rediscovered in the context of well-structured systems by Finkel and Goubault-Larrecq [11].

Assume as before that we know how to represent the ideals in $\mathrm{Idl}(X)$. Define an *atom* A over X as a language $A \subseteq X^*$ of the form $A = D^*$ where D is a downward-closed set of X—i.e. a finite union of ideals from $\mathrm{Idl}(X)$—, or form $A = I \cup \{\varepsilon\}$ where I is an ideal from $\mathrm{Idl}(X)$ and ε denotes the empty sequence. A *product* $P \subseteq X^*$ over X is a finite concatenation $P = A_1 \cdots A_k$ of atoms $A_1, \dots, A_k$ over X. We denote by $\mathrm{Prod}(X)$ the set of products over X.

Fact 4.8. *The ideals of X^* are the products over X.*

It is convenient for algorithmic tasks to have a canonical representation of ideals. In the case of products over X, there is no uniqueness of presentation, e.g. $(a + b)^* \cdot b^*$ denotes the same ideal as $(a + b)^*$ over $X = \{a, b\}$. We can avoid such redundancies by considering *reduced* products $P = A_1 \cdots A_k$, where for every j, the following conditions hold:

(1) $A_j \neq \emptyset^*$,
(2) if $j + 1 \le k$ and A_{j+1} is some D^*, then $A_j \not\subseteq A_{j+1}$,
(3) if $j - 1 \ge 1$ and A_{j-1} is some D^*, then $A_j \not\subseteq A_{j-1}$.

Because inclusion tests between effective representations of ideals are usually decidable, these conditions can effectively be enforced.

Fact 4.9. *Every ideal of X^* admits a canonical presentation as a reduced product over X.*

4.2.4. *Effectiveness.* In order to be usable in algorithms, wqo ideals need to be effectively presented. Following Goubault-Larrecq et al. [14], one can check that all the elementary wqos $(X, \le)$ enjoy a number of effectiveness properties. Besides some basic desiderata, among which being able to decide whether (the presentation of) two elements of X coincide or are related through $\le$, and similarly for $\mathrm{Idl}(X)$ and the inclusion ordering, our elementary wqos are in particular equipped with (see [14] for details):

II: an algorithm taking any pair of (presentations of) ideals I and J in $\mathrm{Idl}(X)$ and returning (a presentation of) an ideal decomposition of $I \cap J$, and

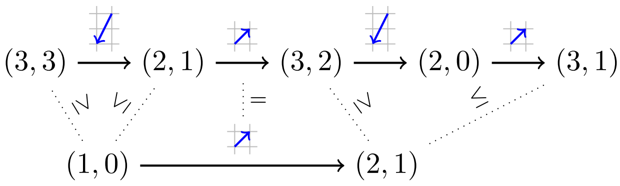

Figure 2. A run embedding.

CU': an algorithm taking any (presentation of an) element x in X and returning (a presentation of) an ideal decomposition of $X \setminus \uparrow x$.

By combining those two algorithms, we get:

Corollary 4.10. *Let $(X, \leq)$ be an elementary wqo. Then there is an algorithm taking any (presentation of an) ideal I in $\mathrm{Idl}(X)$ and any (presentation of an) element x in X and returning (a presentation of) an ideal decomposition of $I \setminus \uparrow x$.*

5. A WQO on Runs

The key idea in our explanation of the KLMST decomposition is to see it as building the ideals of the downward-closure of $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ for an appropriate well quasi ordering defined by Jančar [17] and Leroux [27]. The reachability problem can then be restated as asking whether $\downarrow \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ is non empty, i.e. whether the ideal decomposition of $\downarrow \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ is empty or not.

5.1. **Ordering Preruns and Runs.** There is a natural ordering $\trianglelefteq$ of preruns. The product ordering over $\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$ can be lifted to an embedding between sequences of tuples in $(\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d)^*$. Finally, we denote by $\trianglelefteq$ the natural ordering over $\mathrm{PreRuns}_{\boldsymbol{A}}$, which induces a wqo $(\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}), \trianglelefteq)$ for any $\boldsymbol{x}$ and $\boldsymbol{y}$—see Figure 2 for an illustration. For a set of runs Ω, we write $\downarrow\Omega$ for its downward-closure *inside* $\mathrm{PreRuns}_{\boldsymbol{A}}$, i.e.

$$\downarrow\Omega \stackrel{\text{def}}{=} \{\rho' \in \mathrm{PreRuns}_{\boldsymbol{A}} \mid \exists \rho \in \Omega . \rho' \trianglelefteq \rho\} \ . \tag{6}$$

5.1.1. *Transformer Relations.* Embeddings between runs can also be understood in terms of *transformer relations* (aka production relations) à la Hauschildt [16] and Leroux [27, 28]: the relation $\overset{\boldsymbol{c}}{\curvearrowright}$ with *capacity* $\boldsymbol{c}$ in $\mathbb{N}^d$ is the relation included in $\mathbb{N}^d \times \mathbb{N}^d$ defined by $\boldsymbol{u} \overset{\boldsymbol{c}}{\curvearrowright} \boldsymbol{v}$ if there exists a run from $\boldsymbol{u} + \boldsymbol{c}$ to $\boldsymbol{v} + \boldsymbol{c}$.

Then by definition, for two runs ρ and ρ' we have $\rho \trianglelefteq \rho'$ if, and only if, $\rho = \boldsymbol{c}_0 \xrightarrow{\boldsymbol{a}_1} \boldsymbol{c}_1 \cdots \boldsymbol{c}_{k-1} \xrightarrow{\boldsymbol{a}_k} \boldsymbol{c}_k$ and $\rho' = \boldsymbol{v}_0 + \boldsymbol{c}_0 \xrightarrow{\sigma_0} \boldsymbol{v}_1 + \boldsymbol{c}_0 \xrightarrow{\boldsymbol{a}_1} \boldsymbol{v}_1 + \boldsymbol{c}_1 \xrightarrow{\sigma_1} \boldsymbol{v}_2 + \boldsymbol{c}_1 \cdots \boldsymbol{v}_k + \boldsymbol{c}_{k-1} \xrightarrow{\boldsymbol{a}_k} \boldsymbol{v}_k + \boldsymbol{c}_k \xrightarrow{\sigma_k} \boldsymbol{v}_{k+1} + \boldsymbol{c}_k$ for some vectors $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{k+1}$ in $\mathbb{N}^d$ and action sequences $\sigma_0, \ldots, \sigma_k$. Observe that each run $\boldsymbol{v}_j + \boldsymbol{c}_j \xrightarrow{\sigma_j} \boldsymbol{v}_{j+1} + \boldsymbol{c}_j$ witnesses an element of the transformer relation $\boldsymbol{v}_j \overset{\boldsymbol{c}_j}{\curvearrowright} \boldsymbol{v}_{j+1}$.

5.1.2. *Run Amalgamation.* Leroux [27] observed that, thanks to monotonicity, each $\overset{\boldsymbol{c}}{\curvearrowright}$ is a *periodic* relation, meaning that: $\boldsymbol{0} \overset{\boldsymbol{c}}{\curvearrowright} \boldsymbol{0}$, as witnessed by the empty run, and if $\boldsymbol{u} \overset{\boldsymbol{c}}{\curvearrowright} \boldsymbol{v}$ and $\boldsymbol{u}' \overset{\boldsymbol{c}}{\curvearrowright} \boldsymbol{v}'$ (witnessed by $\boldsymbol{u} + \boldsymbol{c} \xrightarrow{\sigma} \boldsymbol{v} + \boldsymbol{c}$ and $\boldsymbol{u}' + \boldsymbol{c} \xrightarrow{\sigma'} \boldsymbol{v}' + \boldsymbol{c}$ respectively), then $\boldsymbol{u} + \boldsymbol{u}' \overset{\boldsymbol{c}}{\curvearrowright} \boldsymbol{v} + \boldsymbol{v}'$ as witnessed by $\boldsymbol{u} + \boldsymbol{u}' + \boldsymbol{c} \xrightarrow{\sigma} \boldsymbol{v} + \boldsymbol{u}' + \boldsymbol{c} \xrightarrow{\sigma'} \boldsymbol{v} + \boldsymbol{v}' + \boldsymbol{c}$. Translated in terms of embeddings, the same reasoning shows:

Proposition 5.1. *Let ρ_0, ρ_1, and ρ_2 be runs with $\rho_0 \trianglelefteq \rho_1, \rho_2$. Then there exists a run ρ_3 such that $\rho_1, \rho_2 \trianglelefteq \rho_3$.*

Proof. Let $\rho_0 = \boldsymbol{c}_0 \xrightarrow{\boldsymbol{a}_1} \boldsymbol{c}_1 \cdots \boldsymbol{c}_{k-1} \xrightarrow{\boldsymbol{a}_k} \boldsymbol{c}_k$, then we can write ρ_1 and ρ_2 as $\rho_1 = \boldsymbol{v}_0 + \boldsymbol{c}_0 \xrightarrow{\sigma_0} \boldsymbol{v}_1 + \boldsymbol{c}_0 \xrightarrow{\boldsymbol{a}_1} \boldsymbol{v}_1 + \boldsymbol{c}_1 \cdots \boldsymbol{v}_k + \boldsymbol{c}_{k-1} \xrightarrow{\boldsymbol{a}_k} \boldsymbol{v}_k + \boldsymbol{c}_k \xrightarrow{\sigma_k} \boldsymbol{v}_{k+1} + \boldsymbol{c}_k$ and $\rho_2 = \boldsymbol{v}'_0 + \boldsymbol{c}_0 \xrightarrow{\sigma'_0} \boldsymbol{v}'_1 + \boldsymbol{c}_0 \xrightarrow{\boldsymbol{a}_1} \boldsymbol{v}'_1 + \boldsymbol{c}_1 \cdots \boldsymbol{v}'_k + \boldsymbol{c}_{k-1} \xrightarrow{\boldsymbol{a}_k} \boldsymbol{v}'_k + \boldsymbol{c}_k \xrightarrow{\sigma'_k} \boldsymbol{v}'_{k+1} + \boldsymbol{c}_k$. Define $\rho_3 = \boldsymbol{v}_0 + \boldsymbol{v}'_0 + \boldsymbol{c}_0 \xrightarrow{\sigma_0} \boldsymbol{v}_1 + \boldsymbol{v}'_0 + \boldsymbol{c}_0 \xrightarrow{\sigma'_0} \boldsymbol{v}_1 + \boldsymbol{v}'_1 + \boldsymbol{c}_0 \xrightarrow{\boldsymbol{a}_1} \boldsymbol{v}_1 + \boldsymbol{v}'_1 + \boldsymbol{c}_1 \cdots \boldsymbol{v}_k + \boldsymbol{v}'_k + \boldsymbol{c}_{k-1} \xrightarrow{\boldsymbol{a}_k} \boldsymbol{v}_k + \boldsymbol{v}'_k + \boldsymbol{c}_k \xrightarrow{\sigma_k} \boldsymbol{v}_{k+1} + \boldsymbol{v}'_k + \boldsymbol{c}_k \xrightarrow{\sigma'_k} \boldsymbol{v}_{k+1} + \boldsymbol{v}'_{k+1} + \boldsymbol{c}_k$. ☐

Note that the proof of Proposition 5.1 further shows that when $\rho_0, \rho_1, \rho_2 \in \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$, then $\rho_3 \in \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ as well.

5.1.3. *Prerun Ideals.* By Fact 4.8 and Equation 3, the ideals of $\mathrm{PreRuns}_{\boldsymbol{A}}$ are of the form ${\downarrow}\boldsymbol{u} \times P \times {\downarrow}\boldsymbol{v}$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are in $\mathbb{N}^d_\omega$ and P is a product over $\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d$, i.e. can be presented as a regular expression over the alphabet $\mathbb{N}^d_\omega \times \boldsymbol{A} \times \mathbb{N}^d_\omega$.

5.2. **Abstraction Refinement Procedure.** Because runs are particular preruns, we can look at the downward-closure of $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ *inside* $\mathrm{PreRuns}_{\boldsymbol{A}}$. By Fact 4.5, this set has a finite decomposition using prerun ideals from $\mathrm{Idl}(\mathrm{PreRuns}_{\boldsymbol{A}})$. This suggests an abstraction refinement procedure to compute the ideal decomposition of ${\downarrow}\,\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$.

5.2.1. *A Procedure for Reachability.* An idea that looks promising is to build a descending sequence of downward-closed sets $D_0 \supsetneq D_1 \supsetneq \cdots$ inside $\mathrm{PreRuns}_{\boldsymbol{A}}$ while maintaining ${\downarrow}\,\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) \subseteq D_n$ at all steps, until we find the ideal decomposition of ${\downarrow}\,\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$. By Fact 4.5 we can work with finite sets of incomparable ideals to represent the D_n's.

We start therefore with

$$D_0 \stackrel{\text{def}}{=} \mathrm{PreRuns}_{\boldsymbol{A}} \ . \tag{7}$$

Assume we are provided with an oracle to decide whether an ideal from D_n is included in ${\downarrow}\,\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ and extract a counter-example if not. If $I \subseteq {\downarrow}\,\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ for all the (finitely many) maximal ideals I in D_n we stop; otherwise we find a maximal ideal I from the decomposition of D_n s.t.

$$\exists w \in I \setminus {\downarrow}\,\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) \tag{8}$$

and thanks to Corollary 4.10 we construct an ideal decomposition of

$$D' \stackrel{\text{def}}{=} I \setminus {\uparrow}w \tag{9}$$

and we can refine D_n and construct the downward-closed set for the next iteration—which involves removing redundant ideals—by

$$D_{n+1} \stackrel{\text{def}}{=} D' \cup (D_n \setminus I) . \tag{10}$$

The procedure terminates by Fact 3.3 but depends on an oracle to perform (8).

5.2.2. *Connected Prerun Ideals.* Turning the previous abstraction refinement procedure into an algorithm hinges on the effective checking of $I \subseteq {\downarrow}\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ for a maximal prerun ideal I of D_n. We show that, in the particular context of the abstraction refinement procedure, this inclusion check is equivalent to a *connectedness* condition: we call a prerun ideal I *connected* if it equals ${\downarrow}\Delta$ for some directed subset Δ of $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$.

Let us show that connectedness is really equivalent to the inclusion check. If I is connected then it is necessarily included into ${\downarrow}\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$. For the converse, assume that $I \subseteq {\downarrow}\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$. Because I is non-empty, this means that ${\downarrow}\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ has a non-empty ideal decomposition into maximal ideals. By Corollary 4.4, I is included into one of those maximal ideals J of ${\downarrow}\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$. Because $J \subseteq D_n$ in the abstraction refinement procedure, by Corollary 4.4 again there exists I' maximal with $J \subseteq I'$. Hence $I = J = I'$ or I would not be maximal. Then Lemma 4.6 allows to conclude that $J = {\downarrow}\Delta$ for a directed subset Δ of $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$. One can show that a connected ideal is necessarily of the form ${\downarrow}\boldsymbol{x} \times P \times {\downarrow}\boldsymbol{y}$ with P a product of transition ideals.

5.2.3. *Deciding Connectedness.* The following decision problem captures the inclusion check in the abstraction refinement procedure:

Problem: Connectedness of Prerun Ideals.

input: A d-dimensional VAS $\boldsymbol{A}$, two configurations $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{N}^d$, and an ideal I in $\mathrm{Idl}(\mathrm{PreRuns}_{\boldsymbol{A}})$.

question: Is I connected?

As we show in App. A this problem in its full generality is undecidable:

Theorem 5.2. *The connectedness of prerun ideals is already undecidable for ideals of the form ${\downarrow}\boldsymbol{x} \times D^* \times {\downarrow}\boldsymbol{x}$ for D a downward-closed subset of* $\mathrm{Trans}_{\boldsymbol{A}}$ *and $\boldsymbol{x}$ in $\mathbb{N}^d$.*

All is not lost however: we ask with the connectedness problem for more than really needed. In the decomposition algorithm, I presents some further structure that can be exploited towards an algorithm. This motivates a deeper investigation of the properties of run ideals, which will be the object of the next sections.

6. Local Run Ideals

We start our investigation of the ideals of ${\downarrow}\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ by looking at rather restricted classes of runs. The treatment of this restricted case will turn out to contain most of the technical challenges of the next section on general run ideals, where we will assemble those local ideals into global ones.

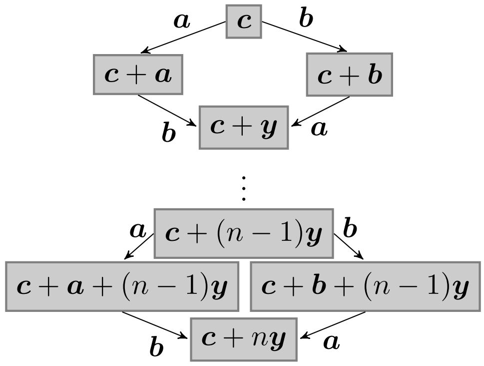

FIGURE 3. The set of runs Ω_γ in Example 6.1.

More precisely, we focus on sets Ω_γ of runs of the form

$$\boldsymbol{c}+\boldsymbol{u} \xrightarrow{\sigma} \boldsymbol{c}+\boldsymbol{v} \tag{11}$$

where $\boldsymbol{c}$ is a configuration in $\mathbb{N}^d$, σ is a sequence in $\boldsymbol{A}^*$, and $(\boldsymbol{u},\boldsymbol{v})$ is a pair of configurations in a periodic set $\boldsymbol{P}$ included in the transformer relation $\overset{\boldsymbol{c}}{\curvearrowright}$. We write γ for the pair $(\boldsymbol{c},\boldsymbol{P})$. As we are going to see in Lemma 6.3, $\downarrow\Omega_\gamma$ is an ideal of a particular form, for which an effective representation can be found, see Section 6.2.

6.1. **Periodic Transformer Subrelations.** Formally, let γ denote a pair $(\boldsymbol{c},\boldsymbol{P})$ where $\boldsymbol{c}$ is in $\mathbb{N}^d$ and $\boldsymbol{P}\subseteq \overset{\boldsymbol{c}}{\curvearrowright}$ is periodic. This is a familiar object, and we will reuse several statements from the literature. Following the notations from [28], let

- Ω_γ denote the set of runs of the form (11),
- $\boldsymbol{Q}_\gamma \subseteq \mathbb{N}^d$ denote the set of configurations $\boldsymbol{q}$ that appear along some run in Ω_γ—thus in particular $\boldsymbol{c}+\boldsymbol{u}$ and $\boldsymbol{c}+\boldsymbol{v}$ belong to $\boldsymbol{Q}_\gamma$ whenever $(\boldsymbol{u},\boldsymbol{v})$ are in $\boldsymbol{P}$.

Example 6.1. Let us consider the 3-dimensional VAS $\boldsymbol{A}=\{\boldsymbol{a},\boldsymbol{b}\}$ where $\boldsymbol{a}=(1,1,-1)$ and $\boldsymbol{b}=(-1,0,1)$, and let us consider the pair $\gamma=(\boldsymbol{c},\boldsymbol{P})$ where $\boldsymbol{c}=(1,0,1)$ and $\boldsymbol{P}=\mathbb{N}(\boldsymbol{0},\boldsymbol{y})$ with $\boldsymbol{y}=(0,1,0)$. Note that $\boldsymbol{P}$ is included in $\overset{\boldsymbol{c}}{\curvearrowright}$ since there exists a run $\boldsymbol{c}\xrightarrow{(\boldsymbol{ab})^n}\boldsymbol{c}+n\boldsymbol{y}$ for every n. We have

$$\Omega_\gamma = \{\boldsymbol{c}\xrightarrow{w_1\ldots w_n}\boldsymbol{c}+n\boldsymbol{y} \mid n\in\mathbb{N}, w_j\in\{\boldsymbol{ab},\boldsymbol{ba}\}\}\;,$$
$$\boldsymbol{Q}_\gamma = (\boldsymbol{c}+\boldsymbol{a}+\mathbb{N}\boldsymbol{y})\cup(\boldsymbol{c}+\mathbb{N}\boldsymbol{y})\cup(\boldsymbol{c}+\boldsymbol{b}+\mathbb{N}\boldsymbol{y})\;.$$

The set Ω_γ is depicted in Figure 3. □

6.1.1. *Saturated Pairs.* We denote by F_γ^{in} (resp. F_γ^{out}) the sets of indices i such that $\boldsymbol{u}(i)=0$ (resp. $\boldsymbol{v}(i)=0$) for every pair $(\boldsymbol{u},\boldsymbol{v})\in\boldsymbol{P}$. We say that a pair $(\boldsymbol{u},\boldsymbol{v})$ in $\boldsymbol{P}$ saturates $(F_\gamma^{\text{in}},F_\gamma^{\text{out}})$ if $\boldsymbol{u}(i)=0$ implies $i\in F_\gamma^{\text{in}}$ and $\boldsymbol{v}(i)=0$ implies $i\in F_\gamma^{\text{out}}$. Since $\boldsymbol{P}$ is periodic, by summing at most $2d$ pairs in $\boldsymbol{P}$, we see that there exist pairs in $\boldsymbol{P}$ that saturate $(F_\gamma^{\text{in}},F_\gamma^{\text{out}})$.

By projecting $\boldsymbol{c}$, we obtain two partial configurations $\boldsymbol{s}_\gamma^{\text{in}}$ and $\boldsymbol{s}_\gamma^{\text{out}}$:

$$\boldsymbol{s}_\gamma^{\text{in}} \overset{\text{def}}{=} \pi_{F_\gamma^{\text{in}}}(\boldsymbol{c})\;, \qquad\qquad \boldsymbol{s}_\gamma^{\text{out}} \overset{\text{def}}{=} \pi_{F_\gamma^{\text{out}}}(\boldsymbol{c})\;. \tag{12}$$

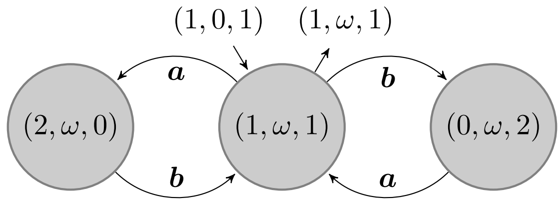

FIGURE 4. The graph G_γ with its input $\boldsymbol{s}_\gamma^{\text{in}}$ and output $\boldsymbol{s}_\gamma^{\text{out}}$ for Example 6.1.

Example 6.1 (continued). We have for our example:

$$F_\gamma^{\text{in}} = \{1,2,3\} \qquad F_\gamma^{\text{out}} = \{1,3\}\ ,$$
$$\boldsymbol{s}_\gamma^{\text{in}} = (1,0,1)\ , \qquad \boldsymbol{s}_\gamma^{\text{out}} = (1,\omega,1)\ . \qquad \square$$

6.2. **Representation through Marked Witness Graphs.** We investigate in this section how to effectively represent $\downarrow\Omega_\gamma$. In the sequel, we show that this ideal can be represented using the set of edges of a strongly connected graph called a *witness graph* (see Lemma 6.2) enjoying some *pumping* properties with respect to $\boldsymbol{s}_\gamma^{\text{in}}$ and $\boldsymbol{s}_\gamma^{\text{out}}$ (see Lemma 6.4). Such graphs will turn out to be exactly the ones employed by Lambert [23] in his variant of the KLMST decomposition (see also [26]).

6.2.1. *Marked Witness Graphs.* A *witness graph* is a strongly connected directed graph $G = (\boldsymbol{S}, E, \boldsymbol{s})$ where $\boldsymbol{S}$ is a non-empty finite set of partial configurations in $\mathbb{N}^F$ for some $F \subseteq \{1,\ldots,d\}$, $E \subseteq \boldsymbol{S} \times \boldsymbol{A} \times \boldsymbol{S}$ is a finite set of partially defined transitions, and $\boldsymbol{s}$ is a distinguished state in $\boldsymbol{S}$.

A *marked witness graph* is a triple $M = (\boldsymbol{s}^{\text{in}}, G, \boldsymbol{s}^{\text{out}})$ where G is a witness graph, and $\boldsymbol{s}^{\text{in}}$ and $\boldsymbol{s}^{\text{out}}$ are partial configurations in $\mathbb{N}^{F^{\text{in}}}$ and $\mathbb{N}^{F^{\text{out}}}$ for some $F^{\text{in}}, F^{\text{out}} \supseteq F$ such that $\pi_F(\boldsymbol{s}^{\text{in}}) = \pi_F(\boldsymbol{s}^{\text{out}}) = \boldsymbol{s}$. We associate with M the set Ω_M of runs ρ of the form $\boldsymbol{x} \xrightarrow{\sigma} \boldsymbol{y}$ where σ is the label of a cycle on $\boldsymbol{s}$ in G, and such that $\boldsymbol{s}^{\text{in}} = \pi_{F^{\text{in}}}(\boldsymbol{x})$ and $\boldsymbol{s}^{\text{out}} = \pi_{F^{\text{out}}}(\boldsymbol{y})$.

6.2.2. *Projected Graphs.* Let $F_\gamma \subseteq \{1,\ldots,d\}$ denote the set of indices i such that $\{\boldsymbol{q}(i) \mid \boldsymbol{q} \in \boldsymbol{Q}_\gamma\}$ is finite, i.e. the indices where $\boldsymbol{Q}_\gamma$ remains bounded. Note that this entails $F_\gamma \subseteq F_\gamma^{\text{in}}$ and $F_\gamma \subseteq F_\gamma^{\text{out}}$. We denote by π_γ the projection function π_{F_γ}.

Observe that the projection $S_\gamma \stackrel{\text{def}}{=} \pi_\gamma(\boldsymbol{Q}_\gamma)$ of $\boldsymbol{Q}_\gamma$ is finite, and so is E_γ the set of partial transitions $(\pi_\gamma(\boldsymbol{q}), \boldsymbol{a}, \pi_\gamma(\boldsymbol{q}'))$ where $(\boldsymbol{q}, \boldsymbol{a}, \boldsymbol{q}')$ appears in some run in Ω_γ. We distinguish $\boldsymbol{s}_\gamma \stackrel{\text{def}}{=} \pi_\gamma(\boldsymbol{c})$ as a particular state in S_γ. We denote by $G_\gamma \stackrel{\text{def}}{=} (\boldsymbol{S}_\gamma, E_\gamma, \boldsymbol{s}_\gamma)$ the finite labelled directed graph defined by projecting the runs in Ω_γ, and $M_\gamma \stackrel{\text{def}}{=} (\boldsymbol{s}_\gamma^{\text{in}}, G_\gamma, \boldsymbol{s}_\gamma^{\text{out}})$ the corresponding marked graph with input $\boldsymbol{s}_\gamma^{\text{in}}$ and output $\boldsymbol{s}_\gamma^{\text{out}}$.

Example 6.1 (continued). Projecting $\boldsymbol{Q}_\gamma$ on $F_\gamma = \{1,3\}$ yields $\pi_\gamma(\boldsymbol{c} + \boldsymbol{a} + n\boldsymbol{y}) = (2,\omega,0)$, $\pi_\gamma(\boldsymbol{c} + n\boldsymbol{y}) = (1,\omega,1)$, and $\pi_\gamma(\boldsymbol{c} + \boldsymbol{b} + n\boldsymbol{y}) = (0,\omega,2)$:

$$\boldsymbol{s}_\gamma = (1,\omega,1)\ , \qquad \boldsymbol{S}_\gamma = \{(2,\omega,0),(1,\omega,1),(0,\omega,2)\}\ .$$

The graph G_γ is depicted on Figure 4. $\square$

We associate to a prerun $\rho = (\boldsymbol{x}, t_1 \cdots t_k, \boldsymbol{y})$ and a set $F \subseteq \{1, \ldots, d\}$, the partial prerun:

$$\pi_F(\rho) \stackrel{\text{def}}{=} (\pi_F(\boldsymbol{x}), \pi_F(t_1) \cdots \pi_F(t_k), \pi_F(\boldsymbol{y}))$$

If ρ is a run in Ω_γ, then $\pi_\gamma(\rho)$ is a path inside G_γ, and by [28, Corollary VIII.5], $\pi_\gamma(\boldsymbol{q}_0) = \pi_\gamma(\boldsymbol{q}_m) = s_\gamma$, which means that this path is actually a cycle in G_γ. This in turn shows that G_γ is strongly connected. This proves:

Lemma 6.2. *The marked graph M_γ is a marked witness graph such that $\Omega_\gamma \subseteq \Omega_{M_\gamma}$.*

6.2.3. *Intraproductions.* An *intraproduction* for γ is a vector $\boldsymbol{h}$ in $\mathbb{N}^d$ such that $\boldsymbol{c} + \boldsymbol{h}$ belongs to $\boldsymbol{Q}_\gamma$. We denote by $\boldsymbol{H}_\gamma$ the set of intraproductions for γ; note that it contains in particular $\boldsymbol{u}$ and $\boldsymbol{v}$ if $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{P}$.

Leroux [28, Lemma VIII.3] shows that $\boldsymbol{H}_\gamma$ is periodic and $\boldsymbol{Q}_\gamma + \boldsymbol{H}_\gamma \subseteq \boldsymbol{Q}_\gamma$. Following the proof of that lemma we deduce that if $t = (\boldsymbol{p}, \boldsymbol{a}, \boldsymbol{q})$ is a transition occurring in a run of Ω_γ and $\boldsymbol{h}$ in $\boldsymbol{H}_\gamma$ is an intraproduction, then the transition $t + \boldsymbol{h} \stackrel{\text{def}}{=} (\boldsymbol{p} + \boldsymbol{h}, \boldsymbol{a}, \boldsymbol{q} + \boldsymbol{h})$ also occurs in a run of Ω_γ. It follows that, if $\boldsymbol{h}$ in $\boldsymbol{H}_\gamma$ is such that $\boldsymbol{h}(i) > 0$ for some index i, then i cannot belong to F_γ, since $\boldsymbol{c} + n\boldsymbol{h}$ is in $\boldsymbol{Q}_\gamma$ for all n. This entails that $\boldsymbol{h} = \boldsymbol{0}$ if $F_\gamma = \{1, \ldots, d\}$. A kind of converse property sometimes holds: we say that an intraproduction $\boldsymbol{h}$ in $\boldsymbol{H}_\gamma$ *saturates* F_γ if whenever $\boldsymbol{h}(i) = 0$, then i belongs to F_γ, and therefore $F_\gamma = \{i \mid \boldsymbol{h}(i) = 0\}$. Leroux [28, Lemma VIII.3] shows there exist intraproductions $\boldsymbol{h}$ in $\boldsymbol{H}_\gamma$ that saturate F_γ.

Example 6.1 (continued). To continue with our example, the set of intraproductions is $\boldsymbol{H}_\gamma = \mathbb{N}\boldsymbol{y}$. The only non-saturated intraproduction is $\boldsymbol{0}$, as any $n\boldsymbol{y}$ with $n > 0$ saturates F_γ. □

We deduce the form of $\downarrow\Omega_\gamma$:

Lemma 6.3. *The following equality holds:*

$$\downarrow\Omega_\gamma = \downarrow\boldsymbol{s}_\gamma^{\text{in}} \times (\downarrow E_\gamma)^* \times \downarrow\boldsymbol{s}_\gamma^{\text{out}}$$

Proof. The inclusion $\subseteq$ is immediate. For the converse inclusion, let us introduce the set T_γ of transitions occurring in runs of Ω_γ. Now, let us consider a word $w = t_1 \cdots t_k$ of transitions in T_γ^*, an intraproduction $\boldsymbol{h}$, and a pair $(\boldsymbol{u}_0, \boldsymbol{v}_0)$ in $\boldsymbol{P}$. We denote by $w + \boldsymbol{h}$ the word $(t_1 + \boldsymbol{h}) \cdots (t_k + \boldsymbol{h})$. Since $t_j + \boldsymbol{h}$ is a transition in T_γ, it occurs along a run $\boldsymbol{c} + \boldsymbol{u}_j \xrightarrow{\sigma_j} \boldsymbol{c} + \boldsymbol{v}_j$ of Ω_γ. Moreover, as $(\boldsymbol{u}_0, \boldsymbol{v}_0)$ is in $\boldsymbol{P}$, there exists a run $\boldsymbol{c} + \boldsymbol{u}_0 \xrightarrow{\sigma_0} \boldsymbol{c} + \boldsymbol{v}_0$. Let $\boldsymbol{u} \stackrel{\text{def}}{=} \sum_{j=0}^{k} \boldsymbol{u}_j$, $\boldsymbol{v} \stackrel{\text{def}}{=} \sum_{j=0}^{k} \boldsymbol{v}_j$, and $\sigma \stackrel{\text{def}}{=} \sigma_0 \cdots \sigma_k$. Because $\boldsymbol{P}$ is periodic, it follows that $(\boldsymbol{u}, \boldsymbol{v})$ is a pair in $\boldsymbol{P}$. Notice that $\rho \stackrel{\text{def}}{=} (\boldsymbol{c} + \boldsymbol{u} \xrightarrow{\sigma} \boldsymbol{c} + \boldsymbol{v})$ is a run in Ω_γ and $(\boldsymbol{c} + \boldsymbol{u}_0, w + \boldsymbol{h}, \boldsymbol{c} + \boldsymbol{v}_0) \in \downarrow\rho$. As there exists intraproduction that saturate F_γ, and pairs $(\boldsymbol{u}, \boldsymbol{v})$ in $\boldsymbol{P}$ that saturate $(F_\gamma^{\text{in}}, F_\gamma^{\text{out}})$, we deduce that $\downarrow(\boldsymbol{s}_\gamma^{\text{in}}, \pi_\gamma(w), \boldsymbol{s}_\gamma^{\text{out}}) \subseteq \downarrow\Omega_\gamma$. We have proved the converse inclusion. □

Leroux [28, Lemma VIII.11] shows that $\boldsymbol{S}_\gamma$ is a set of incomparable partial configurations. Therefore the partial transitions in E_γ are incomparable. The previous lemma then shows that E_γ is the unique finite set of incomparable elements in $\mathbb{N}_\omega^d \times \boldsymbol{A} \times \mathbb{N}_\omega^d$ satisfying Lemma 6.3.

6.2.4. *Pumpable Configurations.* A partial configuration $\boldsymbol{x}$ in $\mathbb{N}_\omega^d$ is said to be *forward pumpable* by a witness graph $G = (\boldsymbol{S}, E, \boldsymbol{s})$ if there exists a cycle on $\boldsymbol{s}$ labelled by a word σ_+, and a run using this label $\boldsymbol{x} \xrightarrow{\sigma_+} \boldsymbol{x}'$ with $\boldsymbol{x} \leq \boldsymbol{x}'$ such that $\downarrow\boldsymbol{s} = \lim_n \boldsymbol{x}_n$, where $\boldsymbol{x}_n$ is the configuration defined by $\boldsymbol{x} \xrightarrow{\sigma_+^n} \boldsymbol{x}_n$ (such a configuration exists by monotonicity). Symmetrically, a partial configuration $\boldsymbol{y}$ in $\mathbb{N}_\omega^d$ is said to be *backward pumpable* by a witness graph $G = (\boldsymbol{S}, E, \boldsymbol{s})$ if there exists a cycle on $\boldsymbol{s}$ labelled by a word σ_-, and a run $\boldsymbol{y}' \xrightarrow{\sigma_-} \boldsymbol{y}$ with $\boldsymbol{y} \leq \boldsymbol{y}'$ such that $\downarrow\boldsymbol{s} = \lim_n \boldsymbol{y}_n$ where $\boldsymbol{y}_n$ is the configuration defined by $\boldsymbol{y}_n \xrightarrow{\sigma_-^n} \boldsymbol{y}$.

Saturated intraproductions also provide a way to prove that the graph input $\boldsymbol{s}_\gamma^{\text{in}}$ and output $\boldsymbol{s}_\gamma^{\text{out}}$ are pumpable.

Lemma 6.4. *The input $\boldsymbol{s}_\gamma^{\text{in}}$ is forward pumpable by G_γ, and the output $\boldsymbol{s}_\gamma^{\text{out}}$ is backward pumpable by G_γ.*

Proof. Let $\boldsymbol{h}$ be an intraproduction that saturates F_γ. There exists a run $\rho \stackrel{\text{def}}{=} \boldsymbol{c} + \boldsymbol{u_h} \xrightarrow{\sigma_+} \boldsymbol{c} + \boldsymbol{h} \xrightarrow{\sigma_-} \boldsymbol{c} + \boldsymbol{v_h}$ in Ω_γ. The projection $\pi_\gamma(\rho)$ shows that σ_+, σ_- are cycles on $\boldsymbol{s}_\gamma$. Moreover, by projecting over F_γ^{in} the run $\boldsymbol{c} + \boldsymbol{u_h} \xrightarrow{\sigma_+} \boldsymbol{c} + \boldsymbol{h}$ we see that $\boldsymbol{s}_\gamma^{\text{in}} \xrightarrow{\sigma_+} \boldsymbol{s}_\gamma^{\text{in}} + \boldsymbol{h}$. Hence $\boldsymbol{s}_\gamma^{\text{in}}$ is forward pumpable by G_γ. Symmetrically $\boldsymbol{s}_\gamma^{\text{out}}$ is backward pumpable by G_γ. □

7. Global Run Ideals

Our understanding of the KLMST decomposition is that it builds an ideal decomposition of $\downarrow \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ inside $\text{PreRuns}_{\boldsymbol{A}}$. We have seen in §5.1.3 how to represent prerun ideals. However we should expect the maximal ideals of $\downarrow \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ to have additional properties besides connectedness, and indeed we shall see they can be represented using the structures employed in the KLMST decomposition.

The starting point for our characterisation of run ideals is to consider some finite basis B of $(\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}), \trianglelefteq)$: if we consider the upward closure $\uparrow\rho \cap \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ of each run ρ in B *inside* $\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$, we obtain again

$$\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) = \bigcup_{\rho \in B} \uparrow\rho \cap \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) \ . \tag{13}$$

Taking the downward-closure inside $\text{PreRuns}_{\boldsymbol{A}}$ then yields

$$\downarrow \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) = \bigcup_{\rho \in B} \downarrow(\uparrow\rho \cap \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})) \ , \tag{14}$$

prompting the study of $\downarrow(\uparrow\rho \cap \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}))$.

7.1. Maximal Ideals.

Observe that each set $\downarrow(\uparrow\rho \cap \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}))$ for a run ρ is downward-closed and non-empty, and that by Proposition 5.1 it is also directed, and is therefore an ideal.

We can further see that those ideals are exactly the *maximal ideals* in the canonical decomposition of $\downarrow \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$.

Proposition 7.1. *The maximal ideals from the canonical decomposition of $\downarrow \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ are exactly the sets $\downarrow(\uparrow\rho \cap \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}))$ for some runs ρ in $\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$.*

Proof. For any run ρ, because $\downarrow(\uparrow\rho \cap \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}))$ is an ideal, it is included into some maximal ideal I. By Lemma 4.6, $I = \downarrow\Delta$ for some directed subset Δ of $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$. Let us show that $I \subseteq \downarrow(\uparrow\rho \cap \Delta)$, which will show that $I \subseteq \downarrow(\uparrow\rho \cap \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}))$ and thereby the maximality of $\downarrow(\uparrow\rho \cap \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}))$. Since ρ is in I, there is a run ρ_Δ in Δ such that $\rho \trianglelefteq \rho_\Delta$. Then, for any prerun ρ_0 in I, since I is directed there exists ρ_1 in I with $\rho_\Delta, \rho_0 \trianglelefteq \rho_1$. Finally, since $I = \downarrow\Delta$, there exists ρ_2 in Δ such that $\rho_1 \trianglelefteq \rho_2$, i.e. $\rho_2 \in \uparrow\rho \cap \Delta$ as desired.

Conversely, if I is a maximal ideal of $\downarrow \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$, then by Lemma 4.6 it is connected and equal to $\downarrow\Delta$ for some directed subset Δ of runs in $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$. Pick some ρ_0 in Δ; then $I \subseteq \downarrow(\uparrow\rho_0 \cap \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}))$, and equality follows from the maximality of I. □

Note that the sets $\downarrow(\uparrow\rho \cap \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}))$ and $\downarrow(\uparrow\rho' \cap \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}))$ for $\rho \neq \rho'$ might coincide, even for minimal ρ and ρ', so there is no canonicity in terms of those basic runs.

What we seek now is a more syntactic representation for such ideals, which would not require to explicitly exhibit a run ρ.

7.2. **Perfect Runs.** Let us accordingly fix a run $\rho = \boldsymbol{c}_0 \xrightarrow{\boldsymbol{a}_1} \boldsymbol{c}_1 \cdots \boldsymbol{c}_{k-1} \xrightarrow{\boldsymbol{a}_k} \boldsymbol{c}_k$ throughout this subsection.

7.2.1. *Transformer Relations Along a Run.* Consider the relation $\boldsymbol{R}$ of tuples $((\boldsymbol{u}_0,\boldsymbol{v}_0),\ldots,(\boldsymbol{u}_k,\boldsymbol{v}_k))$ of pairs in $\mathbb{N}^d \times \mathbb{N}^d$ such that:

$$\mathbf{0} = \boldsymbol{u}_0 \overset{\boldsymbol{c}_0}{\curvearrowright} \boldsymbol{v}_0 = \boldsymbol{u}_1 \overset{\boldsymbol{c}_1}{\curvearrowright} \boldsymbol{v}_1 \cdots = \boldsymbol{u}_k \overset{\boldsymbol{c}_k}{\curvearrowright} \boldsymbol{v}_k = \mathbf{0} \tag{15}$$

and let us introduce the relation $\boldsymbol{P}_j$ defined for $0 \leq j \leq k$ by:

$$\boldsymbol{P}_j \stackrel{\text{def}}{=} \{(\boldsymbol{u}_j,\boldsymbol{v}_j) \mid ((\boldsymbol{u}_0,\boldsymbol{v}_0),\ldots,(\boldsymbol{u}_k,\boldsymbol{v}_k)) \in \boldsymbol{R}\}\ . \tag{16}$$

Informally, each $\boldsymbol{P}_j$ is the subset of $\overset{\boldsymbol{c}_j}{\curvearrowright}$ that can be completed into some run in $\uparrow\rho \cap \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$. We can check that $\boldsymbol{R}$ and each $\boldsymbol{P}_j$ is a periodic relation since each transformer relation is periodic.

7.2.2. *Global Ideal Representation.* Denoting by γ_j the pair $(\boldsymbol{c}_j,\boldsymbol{P}_j)$, we derive from Lemma 6.3 the following equality:

$$\downarrow\Omega_{\gamma_j} \;=\; \downarrow \boldsymbol{s}^{\text{in}}_{\gamma_j} \times (\downarrow E_{\gamma_j})^* \times \downarrow \boldsymbol{s}^{\text{out}}_{\gamma_j}\ . \tag{17}$$

Notice that $\boldsymbol{s}^{\text{in}}_{\gamma_0} = \boldsymbol{x}$ and $\boldsymbol{s}^{\text{out}}_{\gamma_k} = \boldsymbol{y}$. Moreover, the triple $e_j \stackrel{\text{def}}{=} (\boldsymbol{s}^{\text{out}}_{\gamma_{j-1}}, \boldsymbol{a}_j, \boldsymbol{s}^{\text{in}}_{\gamma_j})$ is a partial transition for every $1 \leq j \leq k$.

Observe that $\downarrow(\uparrow\rho \cap \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}))$ is included in

$$\downarrow\boldsymbol{x} \times (\downarrow E_{\gamma_0})^* \cdot A_0 \cdot (\downarrow E_{\gamma_1})^* \cdots A_k \cdot (\downarrow E_{\gamma_k})^* \times \downarrow\boldsymbol{y} \tag{18}$$

where A_j is the atom $\downarrow e_j \cup \{\varepsilon\}$. The converse inclusion will be a consequence of Lemma 7.3 and Lemma 7.5.

In the upcoming subsection, we derive a condition satisfied by the following sequence ξ_ρ of interspersed marked witness graphs and actions, which allows to represent the ideal (18):

$$\xi_\rho \stackrel{\text{def}}{=} M_{\gamma_0}, \boldsymbol{a}_1, M_{\gamma_1}, \ldots, \boldsymbol{a}_k, M_{\gamma_k}\ . \tag{19}$$

7.3. **Perfect Marked Witness Graph Sequences.** A *marked witness graph sequence* ξ is a sequence

$$\xi \;=\; M_0, \boldsymbol{a}_1, M_1, \ldots \boldsymbol{a}_k, M_k \;, \tag{20}$$

where $M_0, \ldots, M_k$ are marked witness graphs and $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$ are actions in $\boldsymbol{A}$. In the sequel, M_j denotes the marked witness graph $(\boldsymbol{s}_j^{\mathrm{in}}, G_j, \boldsymbol{s}_j^{\mathrm{out}})$ where G_j is the witness graph $(\boldsymbol{S}_j, E_j, \boldsymbol{s}_j)$. The sets $F_j^{\mathrm{in}}, F_j, F_j^{\mathrm{out}}$ denote the finite coordinates of $\boldsymbol{s}_j^{\mathrm{in}}, \boldsymbol{s}_j, \boldsymbol{s}_j^{\mathrm{out}}$. The two partial configurations $\boldsymbol{s}_0^{\mathrm{in}}$ and $\boldsymbol{s}_k^{\mathrm{out}}$ are assumed to be respectively $\boldsymbol{x}$ and $\boldsymbol{y}$. Such sequences ξ are also called *marked graph-transition sequences* in [23], and are the structures maintained throughout the KLMST decomposition algorithm.

7.3.1. *Ideals and Runs.* A marked witness graph sequence ξ defines a prerun ideal

$$I_\xi \stackrel{\mathrm{def}}{=} {\downarrow}\boldsymbol{x} \times ({\downarrow}E_0)^* \cdot A_1 \cdot ({\downarrow}E_1)^* \cdots A_k \cdot ({\downarrow}E_k)^* \times {\downarrow}\boldsymbol{y} \tag{21}$$

where $A_j \stackrel{\mathrm{def}}{=} {\downarrow}(\boldsymbol{s}_{j-1}^{\mathrm{out}}, \boldsymbol{a}_j, \boldsymbol{s}_j^{\mathrm{in}}) \cup \{\varepsilon\}$ for all $1 \leq j \leq k$. It is also associated with a set of runs Ω_ξ of the form

$$\boldsymbol{x}_0 \xrightarrow{\sigma_0} \boldsymbol{y}_0 \xrightarrow{\boldsymbol{a}_1} \boldsymbol{x}_1 \xrightarrow{\sigma_1} \boldsymbol{y}_1 \cdots \xrightarrow{\boldsymbol{a}_k} \boldsymbol{x}_k \xrightarrow{\sigma_k} \boldsymbol{y}_k \tag{22}$$

where each $\boldsymbol{x}_j \xrightarrow{\sigma_j} \boldsymbol{y}_j$ is a run in Ω_{M_j}. Note that ${\downarrow}\Omega_\xi \subseteq I_\xi$.

We show next in Lemma 7.3 that for marked witness graph sequences ξ which satisfy the *perfectness* condition of Lambert [23]—which is mostly equivalent to Kosaraju's θ *condition*—, the prerun ideal I_ξ associated with ξ is connected. This condition is not arbitrary, but stems from the properties of the sequences ξ_ρ we derived in sections 6 and 7.

7.3.2. *Perfectness Condition.* Perfectness is defined by introducing a linear system over the natural numbers that denotes a set L_ξ of solutions. This linear system relies on a binary relation $\overset{\psi}{\dashrightarrow}$ over configurations in $\mathbb{N}^d$, where $\psi\colon E \to \mathbb{N}$ denotes some function defined on a finite set E of partial transitions. The relation is defined by $\boldsymbol{x}\overset{\psi}{\dashrightarrow}\boldsymbol{y}$ if $\boldsymbol{y} = \boldsymbol{x} + \sum_{e\in E} \psi(e)\Delta(e)$, where $\Delta(e) \stackrel{\mathrm{def}}{=} \boldsymbol{a}$ for a partial transition e labelled by $\boldsymbol{a}$.

Let L_ξ be the set of tuples $(\boldsymbol{x}_0, \psi_0, \boldsymbol{y}_0, \ldots, \boldsymbol{x}_k, \psi_k, \boldsymbol{y}_k)$ where $\psi_j\colon E_j \to \mathbb{N}$ is a function satisfying for every $\boldsymbol{s} \in \boldsymbol{S}_j$:

$$\sum_{e\in E_j \mid \mathrm{tgt}(e)=\boldsymbol{s}} \psi_j(e) = \sum_{e\in E_j \mid \mathrm{src}(e)=\boldsymbol{s}} \psi_j(e)$$

and $\boldsymbol{x}_0, \boldsymbol{y}_0, \ldots, \boldsymbol{x}_k, \boldsymbol{y}_k$ are configurations in $\mathbb{N}^d$ such that

$$\boldsymbol{x}_0 \overset{\psi_0}{\dashrightarrow} \boldsymbol{y}_0 \xrightarrow{\boldsymbol{a}_1} \boldsymbol{x}_1 \overset{\psi_1}{\dashrightarrow} \boldsymbol{y}_1 \cdots \boldsymbol{x}_k \overset{\psi_k}{\dashrightarrow} \boldsymbol{y}_k$$

and such that for every $0 \leq j \leq k$

$$\pi_{F_j^{\mathrm{in}}}(\boldsymbol{x}_j) = \boldsymbol{s}_j^{\mathrm{in}} \;\wedge\; \pi_{F_j^{\mathrm{out}}}(\boldsymbol{y}_j) = \boldsymbol{s}_j^{\mathrm{out}} \;.$$

Notice that L_ξ is defined as solutions of a linear system. Moreover, for every run in Ω_ξ of the form (22), by introducing the Parikh image $\psi_j\colon E_j \to \mathbb{N}$ of the cycle on $\boldsymbol{s}_j$ labelled by σ_j, we get a sequence $((\boldsymbol{x}_0, \psi_1, \boldsymbol{x}_1), \ldots, (\boldsymbol{x}_k, \psi_k, \boldsymbol{y}_k))$ in L_ξ.

Definition 7.2. *A marked witness graph sequence is said to be* perfect *if it satisfies the following conditions for all j:*

- $\boldsymbol{s}_j^{\text{in}}$ *and* $\boldsymbol{s}_j^{\text{out}}$ *are respectively forward and backward pumpable by* G_j,
- $\sup \boldsymbol{X}_j = \boldsymbol{s}_j^{\text{in}}$ *and* $\sup \boldsymbol{Y}_j = \boldsymbol{s}_j^{\text{out}}$,
- $\sup \Psi_j(e) = \omega$ *for every* $e \in E_j$, *and*

where $\boldsymbol{X}_j$, Ψ_j, *and* $\boldsymbol{Y}_j$ *are respectively the elements* $\boldsymbol{x}_j$, ψ_j, *and* $\boldsymbol{y}_j$ *satisfying:*

$$((\boldsymbol{x}_0, \psi_0, \boldsymbol{y}_0), \ldots, (\boldsymbol{x}_k, \psi_k, \boldsymbol{y}_k)) \in L_\xi \ .$$

Perfect witness graph sequences denote connected ideals:

Lemma 7.3. *If ξ is a perfect marked witness graph sequence, then I_ξ is connected and $I_\xi = \downarrow\Omega_\xi$.*

Proof. The proof comes from [23, Lemma 4.1] and shows that a family of runs of the following form can always be extracted from a perfect marked witness graph sequence:

$$\boldsymbol{x}_{0,n} \xrightarrow{\sigma_{+,0}^n \sigma_0^n w_0 \sigma_{-,0}^n} \boldsymbol{y}_{0,n} \xrightarrow{\boldsymbol{a}_1} \boldsymbol{x}_{1,n} \cdots \boldsymbol{x}_{k,n} \xrightarrow{\sigma_{+,k}^n \sigma_k^n w_k \sigma_{-,k}^n} \boldsymbol{y}_{k,n} \tag{23}$$

such that each run family $\boldsymbol{x}_{j,n} \xrightarrow{\sigma_{+,j}^n \sigma_j^n w_j \sigma_{-,j}^n} \boldsymbol{y}_{j,n}$ is non-decreasing with $\downarrow\Omega_{M_j}$ as limit. Intuitively, $\sigma_{+,j}$ pumps up the components in $F_j^{\text{in}} \setminus F_j$, $\sigma_{-,j}$ pumps down those in $F_j^{\text{out}} \setminus F_j$, and σ_j is the label of a cycle on $\boldsymbol{s}_j$ such that every transition in E_j occurs at least once along the cycle. The sequence w_j comes from a solution of the linear system L_ξ. □

7.3.3. *Deciding Perfectness.* We can decide if a marked witness graph sequence is perfect as follows. First of all, observe that checking if a partial configuration $\boldsymbol{x} \in \mathbb{N}_\omega^d$ is pumpable (either backward or forward) by a witness graph $G = (\boldsymbol{S}, E, \boldsymbol{s})$ can be performed in exponential space since this problem reduces to the place boundedness problem for vector addition systems [2, 8]. Moreover, since we can compute the unbounded components of the set of solutions of a linear system on $\mathbb{N}$ in non-deterministic polynomial time, we can effectively do this computation on sets L_ξ of solutions for marked witness graph sequences ξ. Hence:

Lemma 7.4. *The perfectness of a marked witness graph sequence is decidable in exponential space.*

7.4. **Run Ideals.** We have seen that the downward closed set $\downarrow \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ can be decomposed as a finite union of ideals I_{ξ_ρ} where ξ_ρ is the marked witness graph sequence associated to ρ. By the following lemma, this implies that $\downarrow \text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ can be represented using a finite set of perfect marked witness graph sequences.

Lemma 7.5. *The marked witness graph sequence ξ_ρ is perfect for every every run ρ.*

Proof. By Lemma 6.4, for all j, $\boldsymbol{s}_{\gamma_j}^{\text{in}}$ and $\boldsymbol{s}_{\gamma_j}^{\text{in}}$ are resp. forward and backward pumpable by G_{γ_j}.

Regarding the conditions on L_{ξ_ρ}, for every tuple $((\boldsymbol{u}_0, \boldsymbol{v}_0), \ldots, (\boldsymbol{u}_k, \boldsymbol{v}_k))$ in $\boldsymbol{R}$, every sequence family $(\sigma_j)_{1 \leq j \leq k}$ in $\boldsymbol{A}^*$ such that $\rho_j \stackrel{\text{def}}{=} (\boldsymbol{c}_j + \boldsymbol{u}_j \xrightarrow{\sigma_j} \boldsymbol{c}_j + \boldsymbol{v}_j)$, and every $n \in \mathbb{N}$, we observe that

$$((\boldsymbol{c}_0 + n\boldsymbol{u}_0, n\psi_0, \boldsymbol{c}_0 + n\boldsymbol{v}_0), \ldots, (\boldsymbol{c}_k + n\boldsymbol{u}_k, n\psi_k, \boldsymbol{c}_k + n\boldsymbol{v}_k))$$

is in L_{ξ_ρ} where $\psi_j : E_j \to \mathbb{N}$ is the Parikh image of the cycle $\pi_{\gamma_j}(\rho_j)$ on $\boldsymbol{s}_j$ in G_j. In particular, if $\boldsymbol{s}_j^{\text{in}}(i) = \omega$ for some $i \in F_{\gamma_j}^{\text{in}}$ and some $0 \leq j \leq k$. There exists $(\boldsymbol{u}_j, \boldsymbol{v}_j) \in \boldsymbol{P}_j$ such that $\boldsymbol{u}_j(i) > 0$. By completing this pair as a tuple $((\boldsymbol{u}_0, \boldsymbol{v}_0), \ldots, (\boldsymbol{u}_k, \boldsymbol{v}_k))$ in $\boldsymbol{R}$, we deduce that $\sup \boldsymbol{X}_j(i) = \omega$ and thus $\sup \boldsymbol{X}_j = \boldsymbol{s}_{\gamma_j}^{\text{in}}$. We get similarly $\sup \boldsymbol{Y}_j = \boldsymbol{s}_{\gamma_j}^{\text{out}}$ and $\sup \Psi_j(e) = \omega$ for every $e \in E_j$. Thus ξ_ρ is perfect. □

Theorem 7.6. *For any perfect marked witness graph sequence ξ, $I_\xi \subseteq {\downarrow}\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$. Moreover, there exists a* finite *set Ξ of perfect marked witness graph sequences such that*

$${\downarrow}\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) = \bigcup_{\xi \in \Xi} I_\xi \ .$$

8. The Decomposition Algorithm

We explain succinctly in this section how the classical KLMST algorithm of Mayr, Kosaraju, and Lambert computes the decomposition of ${\downarrow}\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ into ideals. By Theorem 7.6 these ideals can be presented as finite families of perfect marked witness graph sequences.

The KLMST algorithm operates along the same general lines as the abstraction refinement procedure of Section 5.2. It refines successively a finite family Ξ_n of marked witness graph sequences from $\boldsymbol{x}$ to $\boldsymbol{y}$ while maintaining as an invariant

$$\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) = \bigcup_{\xi \in \Xi_n} \Omega_\xi \tag{24}$$

for all i. Because ${\downarrow}\Omega_\xi \subseteq I_\xi$ for all ξ, this implies

$${\downarrow}\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) \subseteq D_n \stackrel{\text{def}}{=} \bigcup_{\xi \in \Xi_n} I_\xi \tag{25}$$

as in the abstraction refinement procedure.

If every marked witness graph sequence in Ξ_n is perfect (which is decidable by Lemma 7.4), the algorithm stops since by Lemma 7.3

$${\downarrow}\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y}) = \bigcup_{\xi \in \Xi_n} I_\xi \ . \tag{26}$$

Otherwise, the family Ξ_n is decomposed into a new family Ξ_{n+1} as follows: we pick a marked witness graph sequence $\xi \in \Xi_n$ that is not perfect. The imperfectness of ξ provides a way of computing a new finite family $\text{dec}(\xi)$ of marked witness graph sequences from $\boldsymbol{x}$ to $\boldsymbol{y}$ (see Section 8.2) with

$$\Omega_\xi = \bigcup_{\xi' \in \text{dec}(\xi)} \Omega_{\xi'} \ . \tag{27}$$

The family Ξ_{n+1} is then defined as

$$\Xi_{n+1} \stackrel{\text{def}}{=} (\Xi_n \setminus \{\xi\}) \cup \text{dec}(\xi) \ . \tag{28}$$

Termination is ensured through a ranking function relating ξ with each sequence in $\mathrm{dec}(\xi)$, see Section 8.3. The KLMST algorithm shows:

Proposition 8.1. *The ideal decomposition of* $\downarrow\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$ *inside* $\mathrm{PreRuns}_{\boldsymbol{A}}$ *is effectively computable.*

Because $\downarrow\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}) = \emptyset$ if and only if $\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}) = \emptyset$, this yields:

Theorem 8.2 (Mayr [31], Kosaraju [21], Lambert [23])**.** *VAS reachability is decidable.*

8.1. **Initial Family.** The KLMST algorithm starts with an initial family Ξ_0 containing a single marked witness graph sequence ξ_0, itself reduced to a single marked witness graph $M \stackrel{\text{def}}{=} (\boldsymbol{x}, G, \boldsymbol{y})$ where $G \stackrel{\text{def}}{=} (\boldsymbol{S}, E, \boldsymbol{s})$ is defined by $\boldsymbol{s} = (\omega, \ldots, \omega)$, $\boldsymbol{S} = \{\boldsymbol{s}\}$, and $E = \boldsymbol{S} \times \boldsymbol{A} \times \boldsymbol{S}$. Note that $\Omega_{\xi_0} = \mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y})$ and

$$\downarrow\mathrm{Runs}_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{y}) \subseteq D_0 = {\downarrow}\boldsymbol{x} \times (\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d)^* \times {\downarrow}\boldsymbol{y} \ . \tag{29}$$

8.2. **Decomposition.** Let us fix a marked witness graph sequence ξ that is not perfect, and let us recall how the finite family $\mathrm{dec}(\xi)$ is obtained in the KLMST algorithm. We assume that

$$\xi \ = \ M_0, \boldsymbol{a}_1, M_1, \ldots \boldsymbol{a}_k, M_k \ ,$$

where $M_0, \ldots, M_k$ are marked witness graphs, and $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$ are actions in $\boldsymbol{A}$. In the sequel, M_j denotes the marked witness graph $(\boldsymbol{s}_j^{\text{in}}, G_j, \boldsymbol{s}_j^{\text{out}})$ and G_j is the witness graph $(\boldsymbol{S}_j, E_j, \boldsymbol{s}_j)$. We let F_j^{in}, F_j, F_j^{out} be respectively the finite components of $\boldsymbol{s}_j^{\text{in}}$, $\boldsymbol{s}_j$ and $\boldsymbol{s}_j^{\text{out}}$.

Remark 8.3. The main difference between the KLMST algorithm and the abstraction refinement procedure from Section 5.2 lies in the decomposition step. Because some of the ideals I_ξ denoted by the various sequences ξ in Ξ_n might be comparable, a decomposition step (28) might leave $D_n = D_{n+1}$ unchanged. This explains why we cannot use Fact 3.3 to prove termination but rely instead on a ranking function in Section 8.3. It would be interesting to provide a variant of the KLMST decomposition algorithm that follows more closely the abstraction refinement procedure. □

8.2.1. *Unpumpable Case.* If $\boldsymbol{s}_j^{\text{in}}$ is not forward pumpable by G_j, the algorithm of Karp and Miller [20] provides an effective way of computing an index $i \notin F_j$ and a constant c such that configurations occurring in any run ρ in Ω_{M_j} are bounded by c oncomponent i. The same property holds if symmetrically $\boldsymbol{s}_j^{\text{out}}$ is not backward pumpable by G_j.

In such cases the graph G_j can be synchronised with a finite state automaton $\mathcal{A}$ with states in $S = \{0, \ldots, c\}$ and transitions of form $(n, \boldsymbol{a}, m) \in S \times \boldsymbol{A} \times S$ satisfying $m = \boldsymbol{a}(i) + n$. This synchronisation might produce a graph that is no longer strongly connected, but it can be decomposed into strongly connected components. This way we obtain a finite family $\mathrm{dec}(\xi)$ of marked witness graph sequences where the graph G_j in ξ is replaced by sequences of subgraphs of $G_j \times \mathcal{A}$ where the finite components F_j of G_j are replaced by a larger set $F_j \cup \{i\}$.

8.2.2. *Input/Output Bounded Solutions.* Now, let us assume that ξ is not perfect due to the conditions on the set of solutions L_ξ. Following the notations introduced in Definition 7.2, recall that we can check in non deterministic polynomial time whether $\sup \boldsymbol{X}_j(i) < \omega$ for a component i such that $\boldsymbol{s}_j^{\text{in}}(i) = \omega$. If it is not the case, we obtain a component $i \notin F^{\text{in}}$ such that $\sup \boldsymbol{X}_j(i) = c$ is finite. Such a bound is computable in polynomial time. Now, just observe that component i of $\boldsymbol{s}_j^{\text{in}}$ can be replaced by all the possible values up to c. We obtain in this way a finite family $\text{dec}(\xi)$ where the set F_j^{in} is replaced by $F_j^{\text{in}} \cup \{i\}$. The same construction can be applied symmetrically when $\sup \boldsymbol{Y}_j$ does not match $\boldsymbol{s}_j^{\text{out}}$.

8.2.3. *Edge Bounded Solutions.* Finally, assume that $\{\psi_j(e) \mid \psi_j \in \Psi_j\}$ is bounded. Once again, we can effectively compute in polynomial time an upper bound c of this set. Notice that in this case, every run $\rho_j \in \Omega_{M_j}$ labelled by a word σ provides a cycle on $\boldsymbol{s}_j$ in G_j in such a way that e occurs at most c times. By removing from G_j the edge e we obtain a graph that may not be strongly connected any more. However, by computing strongly connected components, we obtain in this way a finite family $\text{dec}(\xi)$ such that the graph G_j has been replaced by a sequences of up to c graphs, each with a set of edges included in $E_j \setminus \{e\}$.

8.3. **Ranking Function.** We present the usual termination argument for the KLMST algorithm by explicitly giving a ranking function r from marked witness graph sequences into an ordinal, such that $r(\xi) > r(\xi')$ for all ξ' in $\text{dec}(\xi)$.

8.3.1. *Ordinals.* Rather than the usual multiset ordering over triples in $\mathbb{N}^3$ ordered lexicographically used in the KLMST algorithm, we use an equivalent formulation using ordinals. Recall that an ordinal $\alpha < \varepsilon_0$ can be written in Cantor normal form (CNF) as $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ where $\alpha > \alpha_1 \geq \cdots \geq \alpha_n$. One can compare two ordinals $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ using their CNFs: $\alpha < \beta$ if and only if there exists $k \leq m$ such that $\alpha_j = \beta_j$ for all $1 \leq j < k$ with $j \leq n$, and $n < k$ or $\alpha_k < \beta_k$. The natural sum of two ordinals $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ is defined as $\alpha \oplus \beta \stackrel{\text{def}}{=} \omega^{\gamma_1} + \cdots + \omega^{\gamma_{n+m}}$ such that $\gamma_1 \geq \cdots \geq \gamma_{n+m}$ is a reordering of the α_i's and β_j's.

8.3.2. *Rank of a Marked Witness Graph.* We associate with a marked witness graph $M = (\boldsymbol{s}^{\text{in}}, G, \boldsymbol{s}^{\text{out}})$ an ordinal β_M in ω^3 defined as

$$\beta_M \stackrel{\text{def}}{=} \omega^2 \cdot (d - |F|) + \omega \cdot |E| + (2d - |F^{\text{in}}| - |F^{\text{out}}|) \tag{30}$$

where $G = (\boldsymbol{S}, E, \boldsymbol{s})$, and F^{in}, F, F^{out} are respectively the defined components of $\boldsymbol{s}^{\text{in}}$, $\boldsymbol{s}$, $\boldsymbol{s}^{\text{out}}$. Note that this is equivalent to a lexicographic ordering over triples in $\mathbb{N}^3$.

8.3.3. *Rank of a Sequence.* We associate with a marked witness graph sequence $\xi = M_0\boldsymbol{a}_1, M_1, \ldots, \boldsymbol{a}_k, M_k$ the ordinal $r(\xi)$ in ω^{ω^3} defined by

$$r(\xi) \stackrel{\text{def}}{=} \bigoplus_{1 \leq j \leq k} \omega^{\beta_{M_j}} . \tag{31}$$

Note that this is equivalent to a multiset ordering over the β_{M_j}.

8.3.4. *Termination Argument.* By seeing the KLMST algorithm as constructing a tree with ξ labelling the parent node of ξ' if ξ is imperfect and $\xi' \in \mathrm{dec}(\xi)$, this ranking function shows that the tree has finite height. Since the families Ξ_0 and $\mathrm{dec}(\xi)$ are finite, this tree is also of finite degree, and is therefore finite by Kőnig's Lemma.

9. Fast-Growing Upper Bounds

We establish in this section an $\mathbf{F}_{\omega^3}$ upper bound on the complexity of the KLMST decomposition algorithm, which yields the first upper bound on the complexity of VAS reachability. Without loss of generality, we can assume that the actions in $\boldsymbol{A}$ are in $\{-1,0,1\}^d$.

9.1. **Subrecursive Hierarchies.** As noted early on e.g. by Müller [32], the complexity of the decomposition algorithm of Mayr, Kosaraju, and Lambert is not primitive-recursive. As a consequence, we have to employ some lesser known complexity classes in order to express upper bounds on the running time and space of this algorithm.

9.1.1. *The Hardy Hierarchy.* A convenient tool to this end is found in the *Hardy hierarchy* of functions. Given some monotone increasing function $h\colon \mathbb{N} \to \mathbb{N}$, this is an ordinal-indexed hierarchy of functions $(h^\alpha\colon \mathbb{N} \to \mathbb{N})_\alpha$ defined by transfinite induction by

$$h^0(x) \stackrel{\text{def}}{=} x, \qquad h^{\alpha+1}(x) \stackrel{\text{def}}{=} h(h^\alpha(x)), \qquad h^\lambda(x) \stackrel{\text{def}}{=} h^{\lambda(x)}(x),$$

where λ denotes a limit ordinal and $\lambda(x)$ the xth element of its *fundamental sequence.* The latter is usually defined for limit ordinals below ε_0 by

$$\begin{aligned}(\gamma + \omega^{\beta+1})(x) &\stackrel{\text{def}}{=} \gamma + \omega^\beta \cdot (x+1)\ ,\\ (\gamma + \omega^\lambda)(x) &\stackrel{\text{def}}{=} \gamma + \omega^{\lambda(x)}\ .\end{aligned}$$

Observe that h^k for some finite k is the kth iterate of h. At index ω, $\omega(x) = x+1$ and thus $h^\omega(x) = h^{x+1}(x)$; more generally, h^α is a transfinite iteration of the function h, using a kind of diagonalisation to handle limit ordinals.

Example 9.1. For instance, starting with the successor function $H(x) \stackrel{\text{def}}{=} x+1$, we see that $H^\omega(x) = H^x(x+1) = 2x+1$. The next limit ordinal occurs at $H^{\omega\cdot 2}(x) = H^{\omega+x}(x+1) = H^\omega(2x+1) = 4x+3$. Fast-forwarding a bit, we get for instance a function of exponential growth $H^{\omega^2}(x) = 2^{x+1}(x+1)-1$, and later a non-elementary function H^{ω^3}, an "Ackermannian" non primitive-recursive function H^{ω^ω}, and a "hyper-Ackermannian" non multiply recursive-function $H^{\omega^{\omega^\omega}}$. □

9.1.2. *Complexity Classes.* Although we could derive upper bounds in terms of Hardy functions, it is more convenient to work with coarser-grained complexity classes. For $\alpha > 2$, we define respectively the *fast-growing function* classes $(\mathscr{F}_\alpha)_\alpha$ of Löb and Wainer [30] and the associated *fast-growing complexity* classes $(\mathbf{F}_\alpha)_\alpha$ of [37] by

$$\mathscr{F}_{<\alpha} \stackrel{\text{def}}{=} \bigcup_{\beta<\omega^\alpha} \text{FSPACE}\big(H^\beta(n)\big) , \qquad (32)$$

$$\mathbf{F}_{h,\alpha} \stackrel{\text{def}}{=} \bigcup_{p\in\mathscr{F}_{<\alpha}} \text{SPACE}\big(h^{\omega^\alpha}(p(n))\big) , \quad \mathbf{F}_\alpha \stackrel{\text{def}}{=} \mathbf{F}_{H,\alpha} , \qquad (33)$$

where $\text{FSPACE}(s(n))$ (resp. $\text{SPACE}(s(n))$) denotes the set of functions computable (resp. problems decidable) in space $O(s(n))$ and H is the successor function $H(x) \stackrel{\text{def}}{=} x+1$. This defines for instance $\mathscr{F}_{<\omega}$ as the set of primitive-recursive functions, and $\mathbf{F}_\omega$ as the class of problems that can be solved in Ackermann time of some primitive-recursive function of their input size. Here $\mathbf{F}_{\omega^3}$ is not primitive-recursive, but among the lowest multiply-recursive classes.

9.2. **Length Function Theorems.** Given some wqo $(X, \leq)$, let us posit a norm $|.|_X : X \to \mathbb{N}$ over X such that $X_{\leq n} \stackrel{\text{def}}{=} \{x \in X \mid |x|_X \leq n\}$ is finite for every n. Given a *control function* $g: \mathbb{N} \to \mathbb{N}$ which is monotone increasing and some $n \in \mathbb{N}$, we say that a sequence $x_0, x_1, \ldots$ over X is *controlled* if for all i, $|x_i|_X \leq g^i(n)$ the ith iterate of g. Length function theorems provide upper bounds on the length of bad controlled sequences.

For instance, upper bounds for $(\mathbb{N}^d, \leq)$ with the product ordering can be found in [39, Theorem 2.34], where the norm of a vector $\boldsymbol{x}$ is its infinite norm $\max_{1\leq i\leq d} \boldsymbol{x}(i)$. Another example from [38, Theorem 3.3] is a length function theorem for ordinals below ε_0, where the norm $N(\alpha)$ of an ordinal $\alpha = \omega^{\alpha_1}\cdot c_1 + \cdots + \omega^{\alpha_n}\cdot c_n$ with $\alpha > \alpha_1 > \cdots > \alpha_n \geq 0$ and $\omega > c_1, \ldots, c_n \geq 0$ is the largest constant that appears in it: $N(\alpha) \stackrel{\text{def}}{=} \max_{1\leq i\leq n}\{c_i, N(\alpha_i)\}$.

The upper bounds in [39, 38] are expressed in terms of another hierarchy of functions called the *Cichoń hierarchy* $(h_\alpha{:}\mathbb{N} \to \mathbb{N})_\alpha$. The relation with the Hardy hierarchy is that, if a controlled sequence is of length bounded by some $h_\alpha(x)$ from the Cichoń hierarchy, then the norm of all its elements is bounded by $h^{h_\alpha(x)}(x) = h^\alpha(x) \geq h_\alpha(x)$ in the Hardy hierarchy.

9.3. **Controlling the KLMST Decomposition.** Recall from Section 8.3 that the KLMST algorithm terminates because any descending sequence of ordinals in ω^{ω^3} is finite. As remarked in Example 3.2, descending sequences over an ordinal are bad sequences. From the previous discussion of length function theorems, in order to apply the bounds from [38] on the norms in bad sequences over ω^{ω^3}, we need to find a control function for any sequence

$$r(\xi_0) > r(\xi_1) > \cdots \qquad (34)$$

of ordinals in ω^{ω^3} found along a branch of the tree described in §8.3.4.

9.3.1. *A Measure on Marked Witness Graph Sequences.* Let us write $\|\boldsymbol{v}\| \stackrel{\text{def}}{=} \max_{i \in F} \boldsymbol{v}(i)$ for the infinite norm of partial vectors in $\mathbb{N}_\omega^d$ and $\|\boldsymbol{V}\| \stackrel{\text{def}}{=} \max_{\boldsymbol{v} \in \boldsymbol{V}}(|\boldsymbol{V}|, \|\boldsymbol{v}\|)$ for a set $\boldsymbol{V}$ of partial vectors. Using the norm function N over ε_0 defined above on the ordinals in (30) and (31), we see that $N(f(\xi))$ is bounded by

$$\|\xi\| \stackrel{\text{def}}{=} \max_{0 \leq j \leq k}(2d, k, |E_j|, \|\boldsymbol{s}_j^{\text{in}}\|, \|\boldsymbol{s}_j^{\text{out}}\|, \|\boldsymbol{S}_j\|) \tag{35}$$

for $\xi = M_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_k, M_k$ where M_j is the marked graph $(\boldsymbol{s}_j^{\text{in}}, G_j, \boldsymbol{s}_j^{\text{out}})$ and $G_j = (\boldsymbol{S}_j, E_j, \boldsymbol{s}_j)$. Note that $\|\xi_0\| = \max(2d, 1, |\boldsymbol{A}|)$ initially.

9.3.2. *Controlling Decompositions.* We are going to exhibit a control function g such that $\|\xi_i\| \leq g^i(\|\xi_0\|)$ for all descending sequences (34) and index i, which will therefore also be a control function on (34) for the ordinal norm. It suffices to show that $\|\xi'\| \leq g(\|\xi\|)$ whenever $\xi' \in \text{dec}(\xi)$. Let us analyse how this measure evolves in the different decomposition cases:

(1) In the unpumpable case, the constant c can be bounded by $H^{\omega^{d+1}}(d \cdot |\boldsymbol{S}_j| \cdot \max(\|\boldsymbol{s}_j^{\text{in}}\|, \|\boldsymbol{s}_j^{\text{out}}\|))$, see [39, Eq. 2.76]. The resulting sequences ξ' in $\text{dec}(\xi)$ satisfy therefore $\|\xi'\| \leq H^{\omega^{d+1}}(\|\xi\|^3)$.
(2) In the input/output bounded case, the constant c is at most exponential in the size of the linear system L_ξ, which is of polynomial size in $\|\xi\|$. Thus $\|\xi'\| \leq 2^{p(\|\xi\|)}$ for some fixed polynomial p.
(3) In the edge bounded case, the constant c is similarly at most exponential in the size of L_ξ and again $\|\xi'\| \leq 2^{p(\|\xi\|)}$ for some fixed polynomial p.

Assuming $d \geq 1$, $H^{\omega^{d+1}}(x) > 2^x$, hence we can choose $g(x) \stackrel{\text{def}}{=} H^{\omega^{d+1}}(p(x))$ for some fixed polynomial p as our control function. This is a primitive-recursive function in $\mathscr{F}_{<\omega}$ for any fixed d, and is in $\mathscr{F}_{<\omega+1}$ when d is part of the input.

9.4. **Complexity Bounds.** By [38, Theorem 3.3], the norm of the elements in any sequence (34) controlled by g is at most $g^{\omega^{\omega^3}}(\|\xi_0\|)$. This function can be computed in space $g^{\omega^{\omega^3}}(e(\|\xi_0\|))$ for some elementary function e by [37, Theorem 5.1]. This yields the same bound on the space used by a nondeterministic version of the KLMST decomposition algorithm, which guesses a branch like (34) that leads to a perfect marked witness graph sequence if there is one. Finally, because our function g yields $\mathbf{F}_{g,\omega^3} = \mathbf{F}_{\omega^3}$ by [37, Theorem 4.4], we obtain:

Theorem 9.2. *VAS reachability is in* $\mathbf{F}_{\omega^3}$.

9.5. **A Combinatorial Algorithm.** The bounds in Section 9.4 allow to propose a conceptually simple algorithm for VAS Reachability, based on a *small run property.* If there is a run in $\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$, it must belong to some Ω_ξ for a perfect ξ constructed by the KLMST decomposition. Thus this ξ is of measure $\|\xi\|$ bounded by $g^{\omega^{\omega^3}}(\|\xi_0\|)$. Using Lemma 7.3 we can extract a run of commensurate length ℓ.

The combinatorial algorithm is a nondeterministic algorithm that first computes ℓ and then guesses a run ρ in $\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{y})$ of length at most ℓ. Its space complexity is similar to that of the KLMST decomposition algorithm, in $\mathbf{F}_{\omega^3}$.

10. Conclusion

The KLMST decomposition algorithm of Mayr, Kosaraju, and Lambert is most certainly a stroke of genius, allowing to prove the decidability of reachability in VAS. What was however sorely lacking until now was an explanation for this decomposition that could be adapted and extended in various directions. Far from closing the subject, we expect this demystification to span a whole research programme.

The first natural question is how easily one can use the framework of ideals on runs for various VAS extensions. A good test is the case of VAS with hierarchical zero tests, which were proven to enjoy a decidable reachability problem by Reinhardt [33]. A wqo on runs using nested applications of Higman's Lemma for this extension is defined by Bonnet [6] in his alternative decidability proof using Presburger inductive invariants. Using the algebraic framework of Section 4.2, we see that prerun ideals for this new ordering are essentially nested products, and thus bear at least a superficial resemblance to the structures manipulated by Reinhardt [33]. The framework could also shed new light on reachability in other VAS extensions [25, 36, 24].

A second question is whether we can significantly improve the $\mathbf{F}_{\omega^3}$ upper bound provided in Section 9. The best known lower bound on the running time of the algorithm is Ackermannian, i.e. $\mathbf{F}_{\omega}$, leaving a huge gap on the complexity of the KLMST algorithm, and a gigantic gap on the complexity of VAS reachability, which is only known to ExpSpace-hard.

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Appendix A. Undecidability of Ideal Connectedness

Theorem 5.2. *The connectedness of prerun ideals is already undecidable for ideals of the form* $\downarrow\boldsymbol{x} \times D^* \times \downarrow\boldsymbol{x}$ *for* D *a downward-closed subset of* $\mathrm{Trans}_{\boldsymbol{A}}$ *and* $\boldsymbol{x}$ *in* $\mathbb{N}^d$.

The proof proceeds by a reduction from the *boundedness problem* for *lossy Minsky machines*, which was shown undecidable by Dufourd et al. [10] (see also the survey [40]).

Lossy Minsky machines (LMM) are Minsky machines where counter values might decrease spontaneously at all times. Let us define their syntax and semantics in a style similar to those of VASs. Let d in $\mathbb{N}$ be the dimension of the machine, i.e. its number of counters. A *Minsky action* r is a pair $(Z, \boldsymbol{a})$ where $Z \subseteq \{1, \ldots, d\}$ denotes the components tested for zero, and $\boldsymbol{a}$ is a vector in $\mathbb{Z}^d$ satisfying $\boldsymbol{a}(i) = 0$ for every $i \in Z$. We associate with such a Minsky rule r a transition relation $\xrightarrow{r}$ over the set of configurations $\mathbb{N}^d$ defined by $\boldsymbol{x} \xrightarrow{r} \boldsymbol{y}$ if $\boldsymbol{x}(i) = 0 = \boldsymbol{y}(i)$ for every $i \in Z$ and $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{a}$. A *Minsky machine* is a finite set R of Minsky rules. A Minsky machine R is said to be *lossy* if $(\emptyset, -\boldsymbol{e}_i) \in R$ for every $1 \leq i \leq d$ (where $\boldsymbol{e}_i$ is the unit vector with 1 in coordinate i and 0 everywhere else).

A set $\boldsymbol{X} \subseteq \mathbb{N}^d$ is called a *post-fixpoint* for a Minsky machine R if for every $\boldsymbol{x} \in \boldsymbol{X}$ and $r \in R$ the relation $\boldsymbol{x} \xrightarrow{r} \boldsymbol{y}$ implies $\boldsymbol{y} \in \boldsymbol{X}$. The *reachability set* $\text{Reach}(R, \boldsymbol{x}_{\text{init}})$ of a Minsky machine R from an initial configuration $\boldsymbol{x}_{\text{init}}$ is the minimal post-fixpoint of R that contains the initial configuration.

Problem: LMM Boundedness**.**

> **input:** A d-dimensional LMM R and an initial configuration $\boldsymbol{x}_{\text{init}}$ in $\mathbb{N}^d$.
>
> **question:** Is $\text{Reach}(R, \boldsymbol{x}_{\text{init}})$ finite?

As mentioned earlier this boundedness problem is undecidable [10, 40].

Minimality of Post-Fixpoints. Note that, due to lossiness, any post-fixpoint is downward-closed and has therefore a finite ideal decomposition using vectors in $\mathbb{N}_\omega^d$. The ideal decomposition of $\text{Reach}(R, \boldsymbol{x}_{\text{init}})$ is however not effective—or the boundedness problem would be decidable: the machine is unbounded if and only if some ω-value appears in some coordinate of an ideal from the decomposition of $\text{Reach}(R, \boldsymbol{x}_{\text{init}})$.

Assume we have an oracle to decide whether a post-fixpoint $\boldsymbol{X}$ that contains x_{init} is equal to $\text{Reach}(R, \boldsymbol{x}_{\text{init}})$. Because we can enumerate finite sets of vectors in $\mathbb{N}_\omega^d$ and effectively check whether they define a post-fixpoint $\boldsymbol{X}$ that contains x_{init}, we could use this oracle to construct the ideal decomposition of $\text{Reach}(R, \boldsymbol{x}_{\text{init}})$—and as noted just before, use the latter to decide the boundedness problem. This means that we cannot decide whether a post-fixpoint is equal to $\text{Reach}(R, \boldsymbol{x}_{\text{init}})$—this is similar to [40, Theorem 3.7]:

Problem: Minimality of LMM Post-Fixpoints**.**

> **input:** A d-dimensional LMM R, an initial configuration $\boldsymbol{x}_{\text{init}}$ in $\mathbb{N}^d$, and a post-fixpoint $\boldsymbol{X}$ that contains $\boldsymbol{x}_{\text{init}}$.
>
> **question:** Does $\boldsymbol{X} = \text{Reach}(R, \boldsymbol{x}_{\text{init}})$?

This problem is already undecidable for a slightly restricted class of LMMs: Observe that if $\boldsymbol{x}_{\text{init}} = \boldsymbol{0}$ then the reachability set is infinite if, and only if, there exists $(Z, \boldsymbol{a}) \in R$ for some Z such that $\boldsymbol{a} > \boldsymbol{0}$. So, we can assume in the previous problem that $\boldsymbol{x}_{\text{init}} \neq \boldsymbol{0}$. Observe similarly that if $(Z, \boldsymbol{x}_{\text{init}}) \in R$ for some Z then $n\boldsymbol{x}_{\text{init}}$ is reachable for every $n \in \mathbb{N}$ and by the previous assumption the reachability set is infinite. So we can also assume that for every $(Z, \boldsymbol{a}) \in R$ we have $\boldsymbol{a} \neq \boldsymbol{x}_{\text{init}}$ and retain undecidability.

Proof of Theorem 5.2. We are going to reduce the problem of testing the minimality of LMM post-fixpoints to the ideal connectedness problem for an ideal of the form $\downarrow\boldsymbol{x}_{\text{init}} \times D^* \times \downarrow\boldsymbol{x}_{\text{init}}$ where D is a downward-closed set of transitions. The main intuition is that a downward-closed set of transitions where some maximal transitions have zero components can be used to perform zero tests in a VAS, and simulate the behaviour of a lossy Minsky machine.

Without loss of generality, we assume that $(\emptyset, \boldsymbol{0})$ belongs to R since the reachability set is unchanged by adding this Minsky rule. Let $\boldsymbol{X} \subseteq \mathbb{N}^d$

be a post-fixpoint of R that contains the initial configuration $\boldsymbol{x}_{\text{init}}$. By minimality of $\text{Reach}(R, \boldsymbol{x}_{\text{init}})$ we get $\text{Reach}(R, \boldsymbol{x}_{\text{init}}) \subseteq \boldsymbol{X}$. We define a downward-closed set $D_{\boldsymbol{X}}$ of transitions of some VAS $\boldsymbol{A}$ in such a way that the inclusion $\text{Reach}(R, \boldsymbol{x}_{\text{init}}) \subseteq \boldsymbol{X}$ is an equality if, and only if, the set of preruns $(\boldsymbol{x}_{\text{init}}, w, \boldsymbol{x}_{\text{init}})$ with transition sequence $w \in D_{\boldsymbol{X}}^*$ is an ideal from $\text{Idl}(\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}_{\text{init}}, \boldsymbol{x}_{\text{init}}))$.

Our VAS is defined by

$$\boldsymbol{A} \stackrel{\text{def}}{=} \{\boldsymbol{x}_{\text{init}}\} \cup \{\boldsymbol{a} \mid \exists Z.(Z, \boldsymbol{a}) \in R\} . \tag{36}$$

Our set $D_{\boldsymbol{X}}$ is defined as the set of transitions

$$D_{\boldsymbol{X}} \stackrel{\text{def}}{=} \{(\boldsymbol{0}, \boldsymbol{x}_{\text{init}}, \boldsymbol{x}_{\text{init}})\} \\ \cup \{(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{y}) \in \boldsymbol{X} \times \boldsymbol{A} \times \boldsymbol{X} \mid \exists Z.\exists r = (Z, \boldsymbol{a}) \in R.\boldsymbol{x} \xrightarrow{r} \boldsymbol{y}\} , \tag{37}$$

which is downward-closed because $\boldsymbol{X}$ is, and we let $I_{\boldsymbol{X}}$ denote the following set of preruns using transitions from $D_{\boldsymbol{X}}$, which is an ideal of $\text{PreRuns}_{\boldsymbol{A}}$:

$$I_{\boldsymbol{X}} \stackrel{\text{def}}{=} \downarrow\boldsymbol{x}_{\text{init}} \times D_{\boldsymbol{X}}^* \times \downarrow\boldsymbol{x}_{\text{init}} . \tag{38}$$

Note that a presentation of $I_{\boldsymbol{X}}$ can effectively be computed from a presentation of $\boldsymbol{X}$.

Claim 1. $\text{Reach}(R, \boldsymbol{x}_{\mathit{init}})$ *is the set of configurations* $\boldsymbol{x} \in \mathbb{N}^d$ *such that there exists a run* $(\boldsymbol{x}_{\mathit{init}}, w, \boldsymbol{x})$ *with* $w \in D_{\boldsymbol{X}}^*$.

The proof is by induction on the length of runs $(\boldsymbol{x}_{\text{init}}, w, \boldsymbol{x})$ of $\boldsymbol{A}$ and runs $\boldsymbol{x}_{\text{init}} \xrightarrow{*} \boldsymbol{x}$ of R.

Claim 2. *If* $\boldsymbol{X} = \text{Reach}(R, \boldsymbol{x}_{\mathit{init}})$ *then* $I_{\boldsymbol{X}}$ *is connected.*

Let $t = (\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{y})$ be a transition in $D_{\boldsymbol{X}}$. By definition $\boldsymbol{x} \in \boldsymbol{X} = \text{Reach}(R, \boldsymbol{x}_{\text{init}})$ and we deduce by Claim 1 that there exists a run $(\boldsymbol{x}_{\text{init}}, w_t, \boldsymbol{x})$ with $w_t \in D_{\boldsymbol{X}}^*$. Due to lossiness, there also exists a run with transition sequence w'_t in $D_{\boldsymbol{X}}^*$ from $\boldsymbol{y}$ to $\boldsymbol{0}$ labelled by actions $-\boldsymbol{e}_i$. By definition (37) the transition $t_{\text{init}} \stackrel{\text{def}}{=} (\boldsymbol{0}, \boldsymbol{x}_{\text{init}}, \boldsymbol{x}_{\text{init}})$ belongs to $D_{\boldsymbol{X}}$. Hence for every $t \in D_{\boldsymbol{X}}$ there exists a run with transition sequence $w_t t w'_t t_{\text{init}}$ in $D_{\boldsymbol{X}}^*$ from $\boldsymbol{x}_{\text{init}}$ to $\boldsymbol{x}_{\text{init}}$ along which t occurs.

By concatenating such transition sequences, for every word $w = t_1 \cdots t_k$ of transitions $t_1, \ldots, t_k \in D_{\boldsymbol{X}}$, there exists a run from $\boldsymbol{x}_{\text{init}}$ to $\boldsymbol{x}_{\text{init}}$ with transitions in $D_{\boldsymbol{X}}^*$ and with w as an embedded subsequence. We conclude by noting that these runs form a directed subset of $\text{Runs}_{\boldsymbol{A}}(\boldsymbol{x}_{\text{init}}, \boldsymbol{x}_{\text{init}})$.

Claim 3. *If* $I_{\boldsymbol{X}}$ *is connected then* $\boldsymbol{X} = \text{Reach}(R, \boldsymbol{x}_{\mathit{init}})$.

Assume there exists a directed family Δ of runs with $\downarrow\Delta = I_{\boldsymbol{X}}$. Let $\boldsymbol{x} \in \boldsymbol{X}$; let us show that $\boldsymbol{x} \in \text{Reach}(R, \boldsymbol{x}_{\text{init}})$. The prerun $(\boldsymbol{x}_{\text{init}}, w, \boldsymbol{x}_{\text{init}})$ with

$$w \stackrel{\text{def}}{=} (\boldsymbol{0}, \boldsymbol{x}_{\text{init}}, \boldsymbol{x}_{\text{init}})(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{x}) \tag{39}$$

belongs to $I_{\boldsymbol{X}}$ (recall that we assumed $(\emptyset, \boldsymbol{0}) \in R$). Hence there exists a run $\rho = (\boldsymbol{x}_{\text{init}}, w', \boldsymbol{x}_{\text{init}})$ in Δ with $w \preceq_* w'$ (for the subsequence embedding over $(\mathbb{N}^d \times \boldsymbol{A} \times \mathbb{N}^d)^*$). Thus w' is in $D_{\boldsymbol{X}}^*$ and of the form

$$w' = w_1(\boldsymbol{y}, \boldsymbol{x}_{\text{init}}, \boldsymbol{y} + \boldsymbol{x}_{\text{init}})w_2(\boldsymbol{x} + \boldsymbol{z}, \boldsymbol{0}, \boldsymbol{x} + \boldsymbol{z})w_3 \tag{40}$$

for some vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ in $\mathbb{N}^d$. Because $(Z, \boldsymbol{x}_{\text{init}}) \notin R$ for any Z, $\boldsymbol{y} = \boldsymbol{0}$.

Therefore $(\boldsymbol{x}_{\text{init}}, w_2, \boldsymbol{x} + \boldsymbol{z})$ is a run with transitions in $D_{\boldsymbol{X}}$. Hence by Claim 1, $\boldsymbol{x}+\boldsymbol{z}$ is in $\text{Reach}(R, \boldsymbol{x}_{\text{init}})$, and by lossiness $\boldsymbol{x}$ is also in $\text{Reach}(R, \boldsymbol{x}_{\text{init}})$. This shows $\boldsymbol{X} \subseteq \text{Reach}(R, \boldsymbol{x}_{\text{init}})$ and thus $\text{Reach}(R, \boldsymbol{x}_{\text{init}}) = \boldsymbol{X}$. □

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