Distributed Synthesis of State-Dependent Switching Control*

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Abstract. We present a correct-by-design method of state-dependent control synthesis for linear discrete-time switching systems. Given an objective region R of the state space, the method builds a capture set S and a control which steers any element of S into R. The method works by iterated backward reachability from R. More precisely, S is given as a parametric extension of R, and the maximum value of the parameter is solved by linear programming. The method can also be used to synthesize a stability control which maintains indefinitely within R all the states starting at R. We explain how the synthesis method can be performed in a distributed manner. The method has been implemented and successfully applied to the synthesis of a distributed control of a concrete floor heating system with 11 rooms and $2^{11} = 2048$ switching modes.

1 Introduction

The importance of switched systems has grown up considerably these last years because of their ease of implementation for controlling cyber-physical systems. A switched system is a family of sub-systems, each with its own dynamics characterized by a parameter mode u whose values are in a finite set U (see [12]). However, due to the composition of many switched systems together, the global switched system has a number of modes and dynamics which increases exponentially. Take for example a heating system for a building of 11 rooms (see [9]): each room i has a heater with two modes values $\{off, on\}$. This makes a combination of $2^{11} = 2048$ mode values. If we want to analyze the evolution of a trajectory on a horizon of K units of discrete time, we have to consider the dynamics corresponding to 2^{11K} possible sequences of modes, which is intractable even for small values of K. It is therefore essential to design *compositional* methods in order to obtain control methods of switched systems that give formal guarantees on the correct behavior of the cyber physical systems.

In this paper, we give a symbolic compositional method which allows to synthesize a control of linear discrete-time switched systems that is guaranteed to satisfy *reachability* and *stability* properties. The method starts from an objective

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region R of the state space, which is rectangular (i.e., is a product of closed intervals of reals). It then generates in a backward manner, using linear programming techniques, an increasing sequence of nested rectangles $\{R^{(i)}\}_{i\geq 0}$ such that every trajectory issued from $R^{(i)}$ is guaranteed to reach $R^{(i-1)}$ in a bounded number of time units. Once $R^{(0)} = R$ is reached, the trajectory is also guaranteed to stay in R indefinitely (stability). The method relies on a simple operation of *tiling* of the rectangles $R^{(i)}$ in a finite number of sub-rectangles (tiles), using a standard operation of *bisection*. Although the method works in a backward fashion, it does not require to inverse the linear dynamics of the system (via matrix inversion), and does not compute *predecessors* of symbolic states (tiles), but only *successors* using the forward dynamics. This is useful in order to avoid numerical imprecisions, especially when the dynamics are *contractive*, which happens often in practical systems (see [14]).

Another contribution of this paper is a technique of state *over-approximation* which allows a distributed control synthesis: this over-approximation allows sub-system 1 to infer a correct value for its next local mode u_1 without knowing the exact value of the state of sub-system 2. This distributed synthesis method is computationally efficient, and works in presence of partial observability. This is at the cost of the performance of the control which usually makes the trajectories reach the objective area in more steps than with a centralized approach.

Related Work. In symbolic analysis and control synthesis methods for hybrid systems, the method of backward reachability and the use of polyhedral symbolic states, as used here, is classical (see, e.g., [2,5]). The use of tiling or partitioning the state-space using bisection is also classical (see, e.g., [7,6]). The main original contribution of this paper is to give a simple technique of over-approximation, which allows one component to estimate the symbolic state of the other component, in presence of partial information. This is similar in spirit to an assume-guarantee reasoning where the controller synthesis for each sub-systems assumes that some safety properties are are satisfied by the others [1,13]. In contrast to [3], we do not need, for the mode selection of a sub-system, to explore blindly all the possible mode choices made by the other sub-system. This yields a drastic reduction of the complexity[4]. This approach allows us to treat a real case study which is intractable with a centralized approach. This case study comes from [9], and we use the same decomposition of the system in two parts (rooms 1-5 and rooms 6-11). In contrast to the work of [9] which uses an on-line and heuristic approach with no formal guarantees, we use here an off-line formal method which guarantees reachability and stability properties.

Implementation. The methods of control synthesis both in the centralized context and in the distributed context have been integrated to the tool MINIMATOR [8,4] written in Octave. All the computation times given in the paper have been performed on a 2.80 GHz Intel Core i7-4810MQ CPU with 8 GB of memory.

[4] This separability technique is made possible by the fact that the difference equation $x_1(t+1) = f_1(x_1(t), x_2(t), u_1)$ (see Section 2.1) does not involve the control mode u_2.

Plan. The structure of this paper is as follows. The class of systems considered and some preliminary definitions are given in Section 2. Our symbolic approach, which is based on the tiling of the state space and backward reachability, is explained in Section 3. In Section 4, we present a centralized method to synthesize a controller based on a "generate-and-test" tiling procedure. A distributed approach is then given in Section 5 where we introduce a state over-approximation technique in order to avoid the use of non-local information by the subsystem controllers. For both methods, we provide reachability and stability guarantees on the controlled trajectories of the system. We present a case study in Section 6: the aim of this case is to control the temperature of an eleven rooms house, heated by geothermal energy. We manage to apply our technique, and to synthesize a correct-by-construction control for this example.

2 State-dependent Switching Control

2.1 Control modes

Consider the discrete-time system with *finite control*:

$$x_1(t+1) = f_1(x_1(t), x_2(t), u_1) \qquad x_2(t+1) = f_2(x_1(t), x_2(t), u_2)$$

where x_1 (resp. x_2) is the first (resp. second) component of the state vector variable, which takes its values in $\mathbb{R}^{n_1}$ (resp. $\mathbb{R}^{n_2}$), and u_1 (resp. u_2) is the first (resp. second) component of the control *mode* variable, which takes its values in the *finite* set U_1 (resp. U_2). We will often use x for (x_1, x_2), u for (u_1, u_2), and n for $n_1 + n_2$. We will also abbreviate the set $U_1 \times U_2$ as U. Let N be the cardinal of U, and N_1 (resp. N_2) the cardinal of U_1 (resp. U_2). We have $N = N_1 \cdot N_2$.

More generally, we abbreviate the discrete-time system under the form:

$$x(t+1) = f(x(t), u)$$

where x is a vector state variable which takes its values in $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, u is of the form (u_1, u_2) where u_1 takes its values in U_1 and u_2 in U_2. In this context, we are interested by the following *centralized* control synthesis problem: at each discrete-time t, select the appropriate mode $u \in U$ in order to satisfy a given property. In this paper we focus on *state-dependent* control, which means that, at each time t, the selection of the value of u is done by considering only the values of $x(t)$.

In the *distributed* context, the control synthesis problem consists in concurrently selecting the value of u_1 in U_1 according to the value of $x_1(t)$ *only*, and the value of u_2 in U_2 according to the value of $x_2(t)$ *only*.

The properties that we consider are *reachability* properties: given a set S and a set R, we look for a control which will steer any element of S to R in a bounded number of steps. We will also consider *stability* properties, which means, that once the state x of the system is in R at time t, the control will maintain it in R indefinitely at $t+1$, $t+2$, ... Actually, given a state set R, we will present a method which does not start from a given set S, but *constructs* it, together with a control which steers all the elements of S to R within a bounded number of steps (S can be seen as a "capture set" of R).

In this paper, we consider that R and S are "rectangles" of the state space. More precisely, $R = R_1 \times R_2$ is a rectangle of reals, i.e., R is a product of n closed intervals of reals, and R_1 (resp. R_2) is a product of n_1 (resp. n_2) closed intervals of reals. Likewise, we assume that $S = S_1 \times S_2$ is a rectangular sub-area of the state space.

Example 1. The centralized and distributed approaches will be illustrated by the example of a two rooms apartment, heated by two heaters located in each room (adapted from [6]). In this example, the objective is to control the temperature of the two rooms. There is heat exchange between the two rooms and with the environment. The *continuous* dynamics of the system is given by the equation:

$$\begin{pmatrix} \dot{T_1} \\ T_2 \end{pmatrix} = \begin{pmatrix} -\alpha_{21} - \alpha_{e1} - \alpha_f u_1 & \alpha_{21} \\ \alpha_{12} & -\alpha_{12} - \alpha_{e2} - \alpha_f u_2 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} + \begin{pmatrix} \alpha_{e1} T_e + \alpha_f T_f u_1 \\ \alpha_{e2} T_e + \alpha_f T_f u_2 \end{pmatrix}.$$

Here T_1 and T_2 are the temperatures of the two rooms, and the state of the system corresponds to $T = (T_1, T_2)$. The control mode variable u_1 (respectively u_2) can take the values 0 or 1 depending on whether the heater in room 1 (respectively room 2) is switched off or switched on (hence $U_1 = U_2 = \{0, 1\}$). Hence, here $n_1 = n_2 = 1$, $N_1 = N_2 = 2$ and $n = 2, N = 4$. T_e corresponds to the temperature of the environment, and T_f to the temperature of the heaters. The values of the different parameters are the following: $\alpha_{12} = 5 \times 10^{-2}$, $\alpha_{21} = 5 \times 10^{-2}$, $\alpha_{e1} = 5 \times 10^{-3}$, $\alpha_{e2} = 5 \times 10^{-3}$, $\alpha_f = 8.3 \times 10^{-3}$, $T_e = 10$ and $T_f = 35$.

We suppose that the heaters can be switched periodically at sampling instants $\tau, 2\tau, \ldots$ (here, $\tau = 5s$). By integration of the continuous dynamics between t and $t + \tau$, the system can be easily put under the desired *discrete-time* form:

$$T_1(t+1) = f_1(T_1(t), T_2(t), u_1) \qquad T_2(t+1) = f_2(T_1(t), T_2(t), u_2)$$

where f_1 and f_2 are affine functions.

Given an objective rectangle for $T = (T_1, T_2)$ of the form $R = [18.5, 22] \times [18.5, 22]$, the control synthesis problem is to find a rectangular capture set S as large as possible, from which one can steer the state T to R ("reachability"), then maintain T within R for ever ("stability").

2.2 Control patterns

It is often easier to design a control of the system using several applications of f in a row rather than using just a single application of f at each time. We are thus led to the notion of "macro-step", and "control pattern". A *(control) pattern* $\pi = (\pi_1, \pi_2)$ *of length* k is a sequence of modes defined recursively by:

1. π is of the form $(u_1, u_2) \in U_1 \times U_2$ if $k = 1$,
2. π is of the form $(u_1 \cdot \pi'_1, u_2 \cdot \pi'_2)$, where u_1 (resp. u_2) is in U_1 (resp. U_2), and (π'_1, π'_2) is a (control) pattern of length $k - 1$ if $k \geq 2$.

The set of patterns of length k is denoted by Π^k (for length $k = 1$, $\Pi^1 = U$). Likewise, for $k \geq 1$, we denote by Π_1^k (resp. Π_2^k) the set of sequences of k elements of U_1 (resp. U_2). For a system defined by $x(t+1) = f(x(t), (u_1, u_2))$ and a pattern $\pi = (\pi_1, \pi_2)$ of length k, one can define recursively $x(t + k) = f(x(t), (\pi_1, \pi_2))$ with $(\pi_1, \pi_2) \in \Pi^k$, by:

1. $f(x(t), (\pi_1, \pi_2)) = f(x(t), (u_1, u_2))$, if (π_1, π_2) is a pattern of length $k = 1$ of the form $(u_1, u_2) \in U$,
2. $f(x(t), (\pi_1, \pi_2)) = f(f(x(t), (\pi_1', \pi_2')), (u_1, u_2))$, if (π_1, π_2) is a pattern of length $k \geq 2$ of the form $(u_1 {\cdot} \pi_1', u_2 {\cdot} \pi_2')$ with $(u_1, u_2) \in U$ and $(\pi_1', \pi_2') \in \Pi^{k-1}$.

One defines $(f(x, \pi))_1 \in \mathbb{R}^{n_1}$ and $(f(x, \pi))_2 \in \mathbb{R}^{n_2}$ to be the first and second components of $f(x, \pi) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} = \mathbb{R}^n$, i.e: $f(x, \pi) = ((f(x, \pi))_1, f(x, \pi)_2)$.

In the following, we suppose that $K \in \mathbb{N}$ is an upper bound of the length of patterns. The value of K can be seen as a maximum number of time steps, for which we compute the future behavior of the system ("horizon"). We denote by $\Pi_1^{\leq K}$ (resp. $\Pi_2^{\leq K}$) the expression $\bigcup_{1 \leq k \leq K} \Pi_1^k$ (resp. $\bigcup_{1 \leq k \leq K} \Pi_2^k$). Likewise, we denote by $\Pi^{\leq K}$ the expression $\bigcup_{1 \leq k \leq K} \Pi^k$.

3 Control Synthesis Using Tiling

3.1 Tiling

Let $R = R_1 \times R_2$ be a rectangle. We say that $\mathcal{R}$ is a *(finite rectangular) tiling* of R if $\mathcal{R}$ is of the form $\{r_{i_1,i_2}\}_{i_1 \in I_1, i_2 \in I_2}$, where I_1 and I_2 are given finite sets of positive integers, each r_{i_1,i_2} is a sub-rectangle of R of the form $r_{i_1} \times r_{i_2}$, and r_{i_1}, r_{i_2} are closed sub-intervals of R_1 and R_2 respectively. Besides, we have $\bigcup_{i_1 \in I_1} r_{i_1} = R_1$ and $\bigcup_{i_2 \in I_2} r_{i_2} = R_2$ (hence $R = \bigcup_{i_1 \in I_1, i_2 \in I_2} r_{i_1,i_2}$). We will refer to r_{i_1}, r_{i_2} and r_{i_1,i_2} as "tiles" of R_1, R_2 and R respectively. The same notions hold for rectangle S.

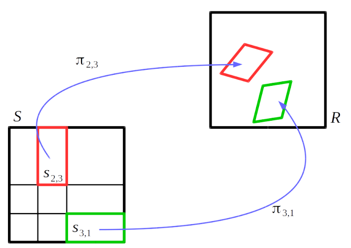


Fig. 1. Mapping of tile $s_{2,3}$ to R via pattern $\pi_{2,3}$, and mapping of tile $s_{3,1}$ via $\pi_{3,1}$.

In the centralized context, given a rectangle R, the *macro-step (backward reachability) control synthesis problem with horizon K* consists in finding a rectangle S and a tiling $\mathcal{S} = \{s_{i_1,i_2}\}_{i_1 \in I_1, i_2 \in I_2}$ of S such that, for each $(i_1, i_2) \in I_1 \times I_2$, there exists $\pi \in \Pi^{\leq K}$ such that: $f(s_{i_1,i_2}, \pi) \subseteq R$ (i.e., for all $x \in s_{i_1,i_2}$, it holds $f(x, \pi) \in R$). This is illustrated in Fig. 1.

3.2 Parametric extension of tiling

In the following, we assume that the set S we are looking for is a *parametric extension* of R, denoted by $R + (a, a)$, which is defined in the following.

Suppose that $R = R_1 \times R_2$ is given as well as a tiling $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 = \{r_{i_1} \times r_{i_2}\}_{i_1 \in I_1, i_2 \in I_2} = \{r_{i_1,i_2}\}_{i_1 \in I_1, i_2 \in I_2}$. R_1 can be seen as a product of n_1 closed intervals of the form $[\ell, m]$. Consider a non negative real parameter a. Let (R_1+a) denote the corresponding product of n_1 intervals of the form $[\ell - a, m + a]$.[5] We define $(R_2 + a)$ similarly. Finally, we define $R + (a, a)$ as $(R_1 + a) \times (R_2 + a)$.

We now consider that S is a (parametric) superset of R of the form $R+(a, a)$. We define a tiling $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ of S of the form $\{s_{i_1} \times s_{i_2}\}_{i_1 \in I_1, i_2 \in I_2}$, which is obtained from $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2 = \{r_{i_1} \times r_{i_2}\}_{i_1 \in I_1, i_2 \in I_2}$ by a simple extension, as follows: a tile r_{i_1} (resp. r_{i_2}) of $\mathcal{R}_1$ (resp. $\mathcal{R}_2$) in "contact" with ∂R_1 (resp. ∂R_2) is prolonged as a tile s_{i_1} (resp. s_{i_2}) in order to be in contact with $\partial(R_1 + a)$ (resp. $\partial(R_2 + a)$); a tile "interior" to R_1 (i.e., with no contact with ∂R_1) is kept unchanged, and coincides with s_{i_1}, and similarly for R_2.

We denote the resulting tiling $\mathcal{S}$ by $\mathcal{R} + (a, a)$. We also denote s_{i_1} (resp. s_{i_2}) as $r_{i_1}+a$ (resp. $r_{i_2} + a$) even if r_{i_1} (resp. r_{i_2}) is "interior" to R_1 (resp. R_2). Likewise, we will denote $s_{i,j}$ as $r_{i,j} + (a, a)$. Note that a tiling of R of index set $I_1 \times I_2$ induces a tiling of $R + (a, a)$ with the same index set $I_1 \times I_2$, hence the same number of tiles as R, for any $a \geq 0$. This is illustrated in Fig. 2, where the tiling of R is represented with black continuous lines, and the extended tiling of $R + (a, a)$ with red dashed lines.

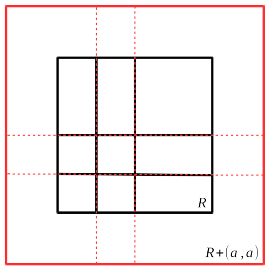


Fig. 2. Tiling of $R + (a, a)$ induced by tiling $\mathcal{R}$ of R.

3.3 Generate and test tilings

By replacing S with $R + (a, a)$ in the notions defined in Section 3.1 the problem of macro-step control synthesis can now be reformulated as finding a tiling $\mathcal{R}$ of R which induces a macro-step control of $R + (a, a)$ towards R, for some $a \geq 0$; besides, if we find such $\mathcal{R}$, we want to compute the *maximum* value of a for which the induced control exists. This problem can be solved by a simple "generate and test" procedure: one *generates* a candidate tiling, then one *tests* if it satisfies the control property (the control test procedure is explained in Section 4.1); if the test fails, one generate another candidate, and so on iteratively.

In practice, the generation of a candidate $\mathcal{R}$ is done, starting from the trivial tiling (made of one tile equal to R), then using successive *bisections* of R until, either the control test succeeds ("success"), or the depth of bisection of the new candidate is greater than a given upper bound D ("failure"). See details of this procedure in [10].

Remark 1. Note that, if the generate-and-test process stops with "success" for a tiling $\mathcal{R}$, then the tiling $\mathcal{R}_{D,uniform}$ also solves the problem, where $\mathcal{R}_{D,uniform}$

[5] Actually, we will consider in the examples that $(R_1 + a)$ is a product of intervals of the form $[\ell - a, m]$ where the interval is extended only at its *lower* end, but the method is strictly identical.

is the "finest" tiling obtained by bisecting D times all the n components of R. Since $\mathcal{R}_{D,uniform}$ has exactly 2^{nD} tiles, it is in general impractical to perform directly the control test on it. From a theoretical point of view however, it is convenient to suppose that $\mathcal{R} = \mathcal{R}_{D,uniform}$ for reducing the *worst case time complexity* of the control synthesis procedure to the complexity of the control test part only (see Section 4.1).

4 Centralized control

4.1 Tiling test procedure

As seen in Section 3.2, the *(macro-step) control synthesis problem with horizon* K consists in finding (the maximum value of) $a \geq 0$, and a tiling $\mathcal{R} = \{r_{i_1,i_2}\}_{i_1 \in I_1, i_2 \in I_2}$ of R such that, for each $(i_1, i_2) \in I_1 \times I_2$, there exists some $\pi \in \Pi^{\leq K}$ with $f(r_{i_1,i_2} + (a,a), \pi) \subseteq R$. In order to *test* if a tiling candidate $\mathcal{R} = \{r_{i_1,i_2}\}_{i_1 \in I_1, i_2 \in I_2}$ of R satisfies the desired property, we define, for each $(i_1, i_2) \in I_1 \times I_2$:

$$\Pi^{\leq K}_{i_1,i_2} = \{\pi \in \Pi^{\leq K} \mid f(r_{i_1,i_2}, \pi) \subseteq R\}.$$

When $\Pi^{\leq K}_{i_1,i_2} \neq \emptyset$, we define $A = \min_{(i_1,i_2) \in I_1 \times I_2}\{a_{i_1,i_2}\}$, where

$$a_{i_1,i_2} = \max_{\pi \in \Pi^{\leq K}_{i_1,i_2}} \max\{a \geq 0 \mid f(r_{i_1,i_2} + (a,a), \pi) \subseteq R\}$$

$$\pi_{i_1,i_2} = \underset{\pi \in \Pi^{\leq K}_{i_1,i_2}}{\text{argmax}} \max\{a \geq 0 \mid f(r_{i_1,i_2} + (a,a), \pi) \subseteq R\}$$

For each tile r_{i_1,i_2} of R and each $\pi \in \Pi^{\leq K}$, the inclusion test $f(r_{i_1,i_2}, \pi) \subseteq R$ can be done in time polynomial in n when f is affine. Hence the test $\Pi^{\leq K}_{i_1,i_2} \neq \emptyset$ can be done in $O(N^K \cdot n^\alpha)$ since $\Pi^{\leq K}$ contains $O(N^K)$ elements. The computation of $\max\{a \geq 0 \mid f(r_{i_1,i_2} + (a,a), \pi) \subseteq R\}$ can be done by *linear programming* in time polynomial in n, the dimension of the state space. The computation time of $\{a_{i_1,i_2}\}_{i_1 \in I, i_2 \in I_2}$, π_{i_1,i_2}, and A is thus in $O(N^K \cdot 2^{nD})$, where D is the maximal depth of bisection. Hence the complexity of testing a candidate tiling $\mathcal{R}$ is in $O(N^K \cdot 2^{nD})$. By Remark 1 above, the complexity of the control synthesis by generate-and-test is also in $O(N^K \cdot 2^{nD})$. We have:

Proposition 1. *Assume that there exists a tiling* $\mathcal{R} = \{r_{i_1,i_2}\}_{i_1 \in I_1, i_2 \in I_2}$ *of* R *such that* $\Pi^{\leq K}_{i_1,i_2} \neq \emptyset$ *for any* $(i_1, i_2) \in I_1 \times I_2$*. Then* $\mathcal{R}$ *induces a macro-step control of horizon* K *of* $R + (A, A)$ *towards* R *with:*

$$\forall (i_1, i_2) \in I_1 \times I_2.\ f(r_{i_1,i_2} + (A, A), \pi_{i_1,i_2}) \subseteq R.$$

Once a candidate tiling $\mathcal{R}$ satisfying the control test property is found, the generate-and-test procedure ends with *success* (see Section 3.3), and a set $S = R + (a^{(1)}, a^{(1)})$ with $a^{(1)} = A$ has been found. One can then *iterate* the "generate and test" procedure in order to construct an increasing sequence of nested rectangles of the form $R + (a^{(1)}, a^{(1)})$, $R + (a^{(1)} + a^{(2)}, a^{(1)} + a^{(2)})$, ..., which can all be driven to R, as explained in [10].

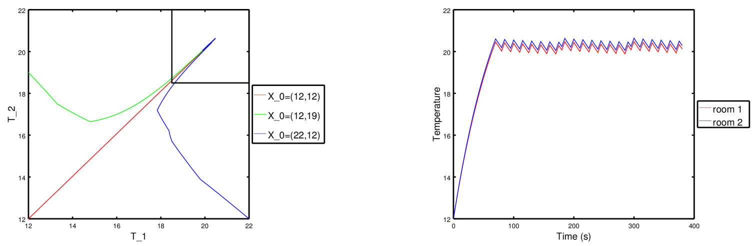


Fig. 3. Simulations of the centralized reachability controller for three different initial conditions plotted in the state space plane (left); simulation of the centralized reachability controller for the initial condition $(12, 12)$ plotted within time (right).

Example 2. Consider the specification of a two-rooms apartment given in Example 1. Set $R = [18.5, 22] \times [18.5, 22]$. Let $D = 1$ (the depth of bisection is at most 1), and $K = 4$ (the maximum length of patterns is 4). We look for a centralized controller which will steer the rectangle $S = [18.5 - a, 22] \times [18.5 - a, 22]$ to R with a as large as possible, and stay in R indefinitely. Using our implementation, the computation of the control synthesis takes 4.14s of CPU time.

The method iterates successfully 15 times the macro-step control synthesis procedure. We find $S = R + (a, a)$ with $a = 53.5$, i.e. $S = [-35, 22] \times [-35, 22]$. This means that any element of S can be driven to R within 15 macro-steps of length (at most) 4, i.e., within $15 \times 4 = 60$ units of time. Since each unit of time is of duration $\tau = 5$s, any trajectory starting from S reaches R within $60 \times 5 = 300$s. Once the trajectory $x(t)$ is in R, it returns in R every macro-step of length (at most) 4, i.e., every $4 \times 5 = 20$s.

These results are consistent with the simulation given in Fig. 3 for the time evolution of (T_1, T_2) starting from $(12, 12)$. Simulations of the control, starting from $(T_1, T_2) = (12, 12)$, $(T_1, T_2) = (12, 19)$ and $(T_1, T_2) = (22, 12)$ are also given in the state space plane in Fig. 3.

4.2 Stability as a special case of reachability

Instead of looking for a set of the form $S = R + (a, a)$ from which R is reachable via a macro-step, let us consider the particular case where $S = R$ (i.e., $a = 0$).

The problem is now to construct a tiling $\mathcal{R} = \{r_{i_1,i_2}\}_{i_1 \in I_1, i_2 \in I_2}$ of R such that, for all $(i_1, i_2) \in I_1 \times I_2$, there exists a pattern $\pi_{i_1,i_2} \in \Pi^{\leq K}$ verifying $f(r_{i_1,i_2}, \pi_{i_1,i_2}) \subseteq R$. If such a tiling $\mathcal{R}$ exists, then $x(t) \in R$ implies $x(t + k) \in R$ for some $k \leq K$.[6] Actually, we can slightly modify the procedure in order to impose, additionally, that $\forall k \leq K \; x(t + k) \in R + \varepsilon$ for some $\varepsilon > 0$ (see Section 5.2). It follows that $R + (\varepsilon, \varepsilon)$ is *stable* under the control induced by $\mathcal{R}$. We can thus treat the stability control of R as a special case of reachability control.

[6] If $x(t) \in R$, then $x(t) \in r_{i,j}$ for some $(i, j) \in I_1 \times I_2$, hence $x(t + k) = f(x, \pi_{i,j}) \in R$ for some $k \leq K$.

5 Distributed control

5.1 Background

In the distributed context, given a set $R = R_1 \times R_2$, the *(macro-step) distributed control synthesis problem with horizon* K consists in finding (the maximum value of) $a \geq 0$, and a tiling $\mathcal{R}_1 = \{r_{i_1}\}_{i_1 \in I_1}$ of R_1 which induces a (macro-step) control on $R_1 + a$, a tiling $\mathcal{R}_2 = \{r_{i_2}\}_{i_2 \in I_2}$ which induces a (macro-step) control on $R_2 + a$.

More precisely, we seek tilings $\mathcal{R}_1$ and $\mathcal{R}_2$ such that: there exists $\ell \in \mathbb{N}$ such that, for each $i_1 \in I_1$ there exists a sequence π_1 of ℓ modes in U_1, and for each $i_2 \in I_2$, a sequence π_2 of ℓ modes in U_2 such that:

$$f((r_{i_1} + a) \times (R_2 + a), (\pi_1, \pi_2))_{|1} \subseteq R_1 \ \wedge \ f((R_1 + a) \times (r_{i_2} + a), (\pi_1, \pi_2))_{|2} \subseteq R_2.$$

In order to synthesize a *distributed* strategy where the control pattern π_1 is determined only by i_1 (regardless of the value of i_2), and the control pattern π_2 only by i_2 (regardless of the value of i_1), we now define an *over-approximation* $X_{i_1}(a, \pi_1)$ for $f((r_{i_1} + a) \times (R_2 + a), (\pi_1, \pi_2))_{|1}$, and an *over-approximation* $X_{i_2}(a, \pi_2)$ for $f((R_1 + a) \times (r_{i_2} + a), (\pi_1, \pi_2))_{|2}$. The correctness of these over-approximations relies on the existence of a fixed positive value for parameter ε. Intuitively, ε represents the width of the additional margin (around $R + (a, a)$) within which all the intermediate states lie when a macro-step is applied to a point of $R + (a, a)$.

5.2 Tiling test procedure

Let π_1^k (resp. π_2^k) denote the prefix of length k of π_1 (resp. π_2), and $\pi_1(k)$ (resp. $\pi_2(k)$) the k-th element of sequence π_1 (resp. π_2).

Definition 1. *Consider an element r_{i_1} (resp. r_{i_2}) of a tiling $\mathcal{R}_1$ (resp. $\mathcal{R}_2$) of R_1 (resp. R_2), and a sequence $\pi_1 \in \Pi_1^{\leq K}$ (resp. $\pi_2 \in \Pi_2^{\leq K}$) of length ℓ_1 (resp. ℓ_2). The* approximate first (resp. second) component sequence $\{X_{i_1}^k(a, \pi_1)\}_{0 \leq k \leq \ell_1}$ *(resp. $\{X_{i_2}^k(a, \pi_2)\}_{0 \leq k \leq \ell_2}$) is defined as follows:*

- $X_{i_1}^0(a, \pi_1) = r_{i_1} + a$ *(resp.* $X_{i_2}^0(a, \pi_2) = r_{i_2} + a$*);*
- $X_{i_1}^k(a, \pi_1) = f_1(X_{i_1}^{k-1}(a, \pi_1), R_2 + a + \varepsilon, \pi_1(k))$ *for* $1 \leq k \leq \ell_1$ *(resp.* $X_{i_2}^k(a, \pi_2) = f_2(R_1 + a + \varepsilon, X_{i_2}^{k-1}(a, \pi_2), \pi_2(k))$ *for* $1 \leq k \leq \ell_2$*).*

We define the property $Prop(a, i_1, \pi_1)$ of $\{X_{i_1}^k(a, \pi_1)\}_{0 \leq k \leq \ell_1}$ as:

$$X_{i_1}^k(a, \pi_1) \subseteq R_1 + a + \varepsilon \text{ for } 1 \leq k \leq \ell_1 - 1, \text{ and } X_{i_1}^{\ell_1}(a, \pi_1) \subseteq R_1.$$

Likewise, we define the property $Prop(a, i_2, \pi_2)$ of $\{X_{i_2}^k(a, \pi_2)\}_{0 \leq k \leq \ell_2}$ as:

$$X_{i_2}^k(a, \pi_2) \subseteq R_2 + a + \varepsilon \text{ for } 1 \leq k \leq \ell_2 - 1, \text{ and } X_{i_2}^{\ell_2}(a, \pi_2) \subseteq R_2.$$

Given a tiling $\mathcal{R}_1 = \{r_{i_1}\}_{i_1 \in I_1}$ of R_1, for each $i_1 \in I_1$, and each $k \in \{1, \ldots, K\}$: we let $\Pi_{i_1}^k = \{\pi_1 \in \Pi_1^k \mid Prop(0, i_1, \pi_1)\}$.

When $\Pi_{i_1}^k \neq \emptyset$, we define:

$$a_{i_1}^k = \max_{\pi_1 \in \Pi_{i_1}^k} \max\{a \geq 0 \mid Prop(a, i_1, \pi_1)\}$$

$$\pi_{i_1}^k = \text{argmax}_{\pi_1 \in \Pi_{i_1}^k} \max\{a \geq 0 \mid Prop(a, i_1, \pi_1)\}$$

Given $\mathcal{R}_2$, we define similarly: $\Pi_{i_2}^k$, $a_{i_2}^k$ and $\pi_{i_2}^k$. Suppose now, that:
(H1) there exists $k_1 \in \{1, \ldots, K\}$ such that $\forall i_1 \in I_1 : \Pi_{i_1}^{k_1} \neq \emptyset$.
(H2) there exists $k_2 \in \{1, \ldots, K\}$ such that $\forall i_2 \in I_2 : \Pi_{i_2}^{k_2} \neq \emptyset$.
Then we define: $a_1^{k_1} = \min_{i_1 \in I_1}\{a_{i_1}^{k_1}\}$, $a_2^{k_2} = \min_{i_2 \in I_2}\{a_{i_2}^{k_2}\}$, $A = \min\{a_1^{k_1}, a_2^{k_2}\}$.

Remark 2. Given a tiling $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$, (H1) means that the points of $R_1 + A$ can be (macro-step) controlled to R_1 using patterns which all have the *same length* k_1; in other terms, all the macro-steps controlling $R_1 + A$ contain the same number k_1 of elementary steps. Symmetrically for (H2).

Remark 3. The determination of an appropriate value for ε is for the moment done by hand, and is the result of a compromise: if ε is too small, then $f_1(r_{i_1} + a, R_2 + a, u_1) \not\subseteq R_1 + a + \varepsilon$; if ε is too large, $f_1(X_{i_1}^{k-1}, R_2 + a + \varepsilon, \pi_1(k)) \not\subseteq R_1 + a$.

Given a tiling $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$ of R and a real $\varepsilon > 0$, the problem of existence and computation of k_1, k_2, $\{\pi_{i_1}^{k_1}\}_{i_1 \in I_1}$, $\{\pi_{i_2}^{k_2}\}_{i_2 \in I_2}$, and A can be solved by *linear programming* since f_1 and f_2 are affine. Using the same kinds of calculation as in the centralized case (see Section 4.1), one can see that the complexity of testing $\Pi_{i_1}^k \neq \emptyset$ and $\Pi_{i_2}^k \neq \emptyset$ for $1 \leq k \leq K$, checking (H1)-(H2), generating k_1, k_2, A and $\{\pi_{i_1}\}_{i_1 \in I_1}$, and $\{\pi_{i_2}\}_{i_2 \in I_2}$ is in $O((\max(N_1, N_2))^K \cdot 2^{\max(n_1,n_2)D})$. Hence the complexity of the control test procedure is also in $O((\max(N_1, N_2))^K \cdot 2^{\max(n_1,n_2)D})$.

Lemma 1. *Consider a tiling $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$ of the form $\{r_{i_1} \times r_{i_2}\}_{(i_1,i_2) \in I_1 \times I_2}$. Let $a \geq 0$. We suppose that (H1) and (H2) hold, and that, for all $i_1 \in I_1$, $Prop(a, i_1, \pi_1)$ holds for some $\pi_1 \in \Pi_1^{k_1}$, and for all $i_2 \in I_2$, $Prop(a, i_2, \pi_2)$ holds for some $\pi_2 \in \Pi_2^{k_2}$, then we have:*

- *in case $k_1 \leq k_2$:*
  $f((r_{i_1} + a) \times (R_2 + a), (\pi_1^k, \pi_2^k))_{|1} \subseteq X_{i_1}^k(a, \pi_1) \subseteq R_1 + a + \varepsilon$ *and*
  $f((R_1 + a) \times (r_{i_2} + a), (\pi_1^k, \pi_2^k))_{|2} \subseteq X_{i_2}^k(a, \pi_2) \subseteq R_2 + a + \varepsilon$,
  for all $1 \leq k \leq k_1$, *and*
  $f((r_{i_1} + a) \times (R_2 + a), (\pi_1^{k_1}, \pi_2^{k_1}))_{|1} \subseteq X_{i_1}^{k_1}(a, \pi_1) \subseteq R_1$,
- *in case $k_2 \leq k_1$:*
  $f((r_{i_1} + a) \times (R_2 + a), (\pi_1^k, \pi_2^k))_{|1} \subseteq X_{i_1}^k(a, \pi_1) \subseteq R_1 + a + \varepsilon$ *and*
  $f((R_1 + a) \times (r_{i_2} + a), (\pi_1^k, \pi_2^k))_{|2} \subseteq X_{i_2}^k(a, \pi_2) \subseteq R_2 + a + \varepsilon$,
  for all $1 \leq k \leq k_2$, *and*
  $f((R_1 + a) \times (r_{i_2} + a), (\pi_1^{k_2}, \pi_2^{k_2}))_{|2} \subseteq X_{i_2}^{k_2}(a, \pi_2) \subseteq R_2$.

At $t = 0$, consider a point $x(0) = (x_1(0), x_2(0))$ of $R+(A, A)$, and let us apply concurrently the strategy induced by $\mathcal{R}_1$ on x_1, and $\mathcal{R}_2$ on x_2. After k_1 steps, by Lemma 1, we obtain a point $x(k_1) = (x_1(k_1), x_2(k_1)) \in R_1 \times (R_2 + A + \varepsilon)$. Then, after k_1 steps, we obtain again a point $x(2k_1) \in R_1 \times (R_2 + A + \varepsilon)$, and so on iteratively. Likewise, we obtain points $x(k_2), x(2k_2), \ldots$ which all belong to $(R_1 + A + \varepsilon) \times R_2$. It follows that, after $\ell = lcm(k_1, k_2)$ steps, we obtain a point $x(\ell)$ which belongs to $R_1 \times R_2 = R$.

Theorem 1. *Assume there are tilings $\mathcal{R}_1 = \{r_{i_1}\}_{i_1 \in I_1}$ of R_1 and $\mathcal{R}_2 = \{r_{i_2}\}_{i_2 \in I_2}$ of R_2, and a positive real ε such that (H1) and (H2) hold, and let k_1, k_2, A be defined as above. Let $\ell = lcm(k_1, k_2)$ with $\ell = \alpha_1 k_1 = \alpha_2 k_2$ for some $\alpha_1, \alpha_2 \in \mathbb{N}$.*

Then $\mathcal{R}_1$ induces a sequence of α_1 macro-steps on $R_1 + A$, and $\mathcal{R}_2$ a sequence of α_2 macro-steps on $R_2 + A$, such that, applied concurrently, we have, for all $i_1 \in I_1$ and $i_2 \in I_2$:

$$f((r_{i_1} + A) \times (R_2 + A), \pi)_{|1} \subseteq R_1 \ \wedge \ f((R_1 + A) \times (r_{i_2} + A), \pi)_{|2} \subseteq R_2,$$

for some $\pi = (\pi_1, \pi_2) \in \Pi^\ell$ where π_1 (resp. π_2) is of the form $\pi_1^1 \cdots \pi_1^{\alpha_1}$ (resp. $\pi_2^1 \cdots \pi_2^{\alpha_2}$) with $\pi_1^i \in \Pi_1^{k_1}$ for all $1 \le i \le \alpha_1$ (resp. $\pi_2^i \in \Pi_2^{k_2}$ for all $1 \le i \le \alpha_2$). Besides, for all prefix π' of π, we have

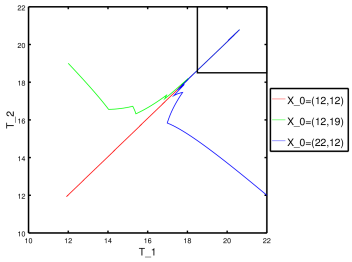
$$f((r_{i_1}+A)\times(R_2+A), \pi')_{|1} \subseteq R_1+A+\varepsilon \wedge f((R_1+A)\times(r_{i_2}+A), \pi')_{|2} \subseteq R_2+A+\varepsilon.$$

If (H1)-(H2) hold, there exists a control that steers $R+(A, A)$ to R in ℓ steps. Letting $R' = R + (A, A)$, it is then possible to iterate the process on R' and, in case of success, generate a rectangle $R'' = R' + (A', A')$ from which R' would be reachable in ℓ' steps, for some $A' \ge 0$ and $\ell' \in \mathbb{N}$. And so on, iteratively, one generates an increasing sequence of nested control rectangles, as in Section 4.1.

Example 3. Consider again the specification of a two-rooms appartment given in Example 1. We consider the distributed control synthesis problem where the first (resp. second) state component corresponds to the temperature of the first (resp. second) room T_1 (resp. T_2), and the first (resp. second) control mode component corresponds to the heater u_1 (resp. u_2) of the the first (resp. second) room.

Set $R = R_1 \times R_2 = [18.5, 22] \times [18.5, 22]$. Let $D = 3$ (the depth of bisection is at most 3), and $K = 10$ (the maximum length of patterns is 10). The parameter ε is set to value $1.5°C$. We look for a distributed controller which steers any temperature state in $S = S_1 \times S_2 = [18.5 - a, 22] \times [18.5 - a, 22]$ to R with a as large as possible, then maintain it in R indefinitely.

Using our implementation, the computation of the control synthesis takes 220s of CPU time. The method iterates 8 times the macro-step control synthesis procedure. We find $S = [18.5 - a, 22] \times [18.5 - a, 22]$ with $a = 6.5$, i.e. $S = [12, 22] \times [12, 22]$. This means that any element of S can be driven to R within 8 macro-steps of length (at most) 10, i.e., within $8 \times 10 = 80$ units of time. Since each unit of time is of duration $\tau = 5$s, any trajectory starting from S reaches R within $80 \times 5 = 400$s. The trajectory is then guaranteed to always stay (at each discrete time t) in $R + (\varepsilon, \varepsilon) = [17, 23.5] \times [17, 23.5]$.



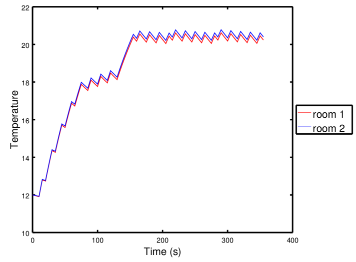


Fig. 4. Simulations of the distributed reachability controller for three different initial conditions plotted in the state space plane (left); simulation of the distributed reachability controller for the initial condition $(12, 12)$ plotted within time (right).

These results are consistent with the simulation given in Fig. 4 showing the time evolution of (T_1, T_2) starting from $(12, 12)$. Simulations of the control are also given in the state space plane, in Fig. 4, for initial states $(T_1, T_2) = (12, 12)$, $(T_1, T_2) = (12, 19)$ and $(T_1, T_2) = (22, 12)$. Not surprisingly, the performance guaranteed by the distributed approach ($a = 6.5$, reachability of R in 400s) are worse than those guaranteed by the centralized approach of Example 2 ($a = 53.5$, reachability of R in 300s). However, unexpectedly, the CPU computation time in the distributed approach (220s) is here worse than the CPU time of the centralized approach (4.14s). This relative inefficiency is due to the small size of the example.

6 Case Study

This case study, proposed by the Danish company Seluxit, aims at controlling the temperature of an eleven rooms house, heated by geothermal energy.

The *continuous* dynamics of the system is the following:

$$\frac{d}{dt}T_i(t) = \sum_{j=1}^{n} A^d_{i,j}(T_j(t) - T_i(t)) + B_i(T_{env}(t) - T_i(t)) + H^v_{i,j}.v_j \tag{1}$$

The temperatures of the rooms are the T_i. The matrix A^d contains the heat transfer coefficients between the rooms, matrix B contains the heat transfer coefficients betweens the rooms and the external temperature, set to $T_{env} = 10°C$ for the computations. The control matrix H^v contains the effects of the control on the room temperatures, and the control variable is here denoted by v_j. We have $v_j = 1$ (resp. $v_j = 0$) if the heater in room j is turned on (resp. turned off). We thus have $n = 11$ and $N = 2^{11} = 2048$ switching modes.

Note that the matrix A^d is parametrized by the open of closed state of the doors in the house. In our case, the average between closed and open matrices was taken for the computations. The exact values of the coefficients are given in [9]. The controller has to select which heater to turn on in the eleven rooms.

Due to a limitation of the capacity supplied by the geothermal device, the 11 heaters cannot be turned on at the same time. In our case, we set to 4 the maximum number of heaters turned on at the same time.

We consider the distributed control synthesis problem where the first (resp. second) state component corresponds to the temperatures of rooms 1 to 5 (resp. 6 to 11), and the first (resp. second) control mode component corresponds to the heaters of rooms 1 to 5 (resp. 6 to 11). Hence $n_1 = 5, n_2 = 6, N_1 = 2^5, N_2 = 2^6$. We impose that at most 2 heaters are switched on at the same time in the first sub-system, and at most 2 in the second sub-system.

Let $D = 1$ (the depth of bisection is at most 1), and $K = 4$ (the maximum length of patterns is 4). The parameter ε is set to value $0.5°C$. The sampling time is $\tau = 15$ min. We look for a distributed controller which steers any temperature state in the rectangle $S = [18 - a, 22]^{11}$ to $R = [18, 22]^{11}$ with a as large as possible, then maintain the temperatures in R indefinitely.

Using our implementation, the computation of the control synthesis takes around 20 hours of CPU time. The method successfully iterates the macro-step control synthesis procedure 15 times. We find $S = [18-a, 22]^{11}$ with $a = 4.2$, i.e. $S = [13.8, 22]^{11}$. This means that any element of S can be driven into R within 15 macro-steps of length (at most) 4, i.e., within $15 \times 4 = 60$ units of time. Since each timeunit has duration $\tau = 15$ min, any trajectory starting from S reaches R within $60 \times 15 = 900$ min. The trajectory is then guaranteed to stay in $R + (\varepsilon, \varepsilon) = [17.5, 22.5]^{11}$. These results are consistent with the simulation of Fig. 5, showing the time evolution of the temperature of the rooms, starting from 14^{11}.

We also performed the same simulations as in Fig. 5, except that the environment temperature is not fixed at 10°C but follows scenarios of soft winter and spring (Fig. 6). The environment temperature is plotted in green in the figures. The spring scenario is taken from [9], and the soft winter scenario is the winter scenario of [9] with 5 additional degrees. We see that our controller, which has been designed for $T_{env} = 10$°C, still satisfies the properties of reachability and stability. These simulations are very close those obtained in [9].

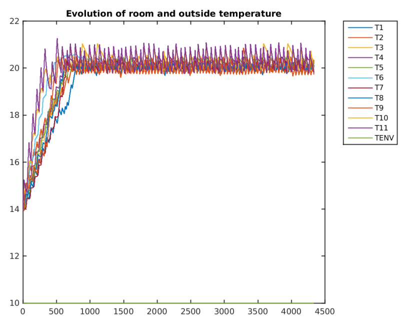


Fig. 5. Simulation of the Seluxit case study plotted with time (in min) for $T_{env} = 10°C$.

7 Final Remarks

In this paper, we have proposed a distributed approach for control synthesis and applied it to a real floor heating system. To our knowledge, this is the first time that reachability and stability properties are guaranteed for a case study of this size. The method can be extended to take into account obstacles and safety constraints. We are currently investigating an extension of the method to systems with non linear dynamics and varying parameters, see [11].

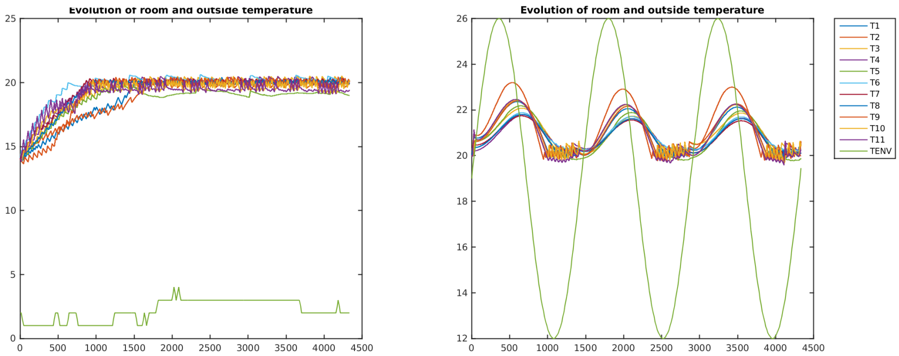


Fig. 6. Simulation of the Seluxit case study in the soft winter scenario (left), and in the spring scenario (right).

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