Distributed control synthesis using Euler's method

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Abstract. In a previous work, we explained how Euler's method for computing approximate solutions of systems of ordinary differential equations can be used to synthesize *safety controllers* for sampled switched systems. We continue here this line of research by showing how Euler's method can also be used for synthesizing safety controllers in a *distributed* manner. The global system is seen as an *interconnection* of two (or more) sub-systems where, for each component, the sub-state corresponding to the other component is seen as an "input"; the method exploits (a variant of) the notions of *incremental input-to-state stability (δ-ISS)* and *ISS Lyapunov function.* We illustrate this distributed control synthesis method on a building ventilation example.

1 Introduction

The computation of reachable sets for continuous-time dynamical systems has been intensively studied during the last decades. Most of the methods to compute the reachable set start from an *initial value problem* for a system of *ordinary differential equations* (ODE) defined by

$$\dot{x}(t) = f(t, x(t)) \quad \text{with} \quad x(0) \in X_0 \subset \mathbb{R}^n \quad \text{and} \quad t \in [0, t_{\text{end}}] \ . \qquad (1)$$

As an analytical solution of Equation (1) is usually not computable, numerical approaches have been considered. A numerical method to solve Equation (1), when X_0 is reduced to one value, produces a discretization of time, such that $t_0 \leqslant \cdots \leqslant t_N = t_{\text{end}}$, and a sequence of states $x_0, \ldots, x_N$ based on an integration method which starts from an initial value x_0 at time t_0 and a finite time horizon h (the step-size), produces an approximation x_{k+1} at time $t_{k+1} = t_k + h$, of the exact solution $x(t_{k+1})$, for all $k = 0, \ldots, N-1$. The simplest numerical method is Euler's method in which $t_{k+1} = t_k + h$ for some step-size h and $x_{k+1} = x_k + hf(t_k, x_k)$; so the derivative of x at time t_k, $f(t_k, x_k)$, is used as an approximation of the derivative on the whole time interval.

The global error $error(t)$ at $t = t_0 + kh$ is equal to $\|x(t) - x_k\|$. In case $n = 1$, if the solution x has a bounded second derivative and f is Lipschitz continuous in its second argument, then it satisfies:

$$error(t) \leqslant \frac{hM}{2L}(e^{L(t-t_0)} - 1) \tag{2}$$

where M is an upper bound on the second derivative of x on the given interval and L is the Lipschitz constant of f [3]. [3]

In [14], we gave an upper bound on the global error $error(t)$, which is more precise than (2). This upper bound makes use of the notion of *One-Sided Lipschitz (OSL)* constant. This notion has been used for the first time by [7] in order to treat "stiff" systems of differential equations for which the explicit Euler method is numerically "unstable" (unless the step size is taken to be extremely small). Unlike Lipschitz constants, OSL constants can be *negative*, which express a form of contractivity of the system dynamics. Even if the OSL constant is positive, it is in practice much lower than the Lipschitz constant [5]. The use of OSL thus allows us to obtain a much more precise upper bound for the global error. We also explained in [14] how such a precise estimation of the global error can be used to synthesize *safety controllers* for a special form hybrid systems, called "sampled switched systems".

In this paper, we explain how such an Euler-based method can be extended to synthesize safety controllers in a *distributed* manner. This allows us to control separately a component using only partial information on the other components. It also allows us to scale up the size of the global systems for which a control can be synthesized. In order to perform such a distributed synthesis, we will see the components of the global systems as being *interconnected* (see, *e.g.*, [18]), and use (a variant of) the notions of *incremental input-to-state stability (δ-ISS)* and *ISS Lyapunov functions* [11] instead of the notion of OSL used in the centralized framework.

The plan of the paper is as follows: In Section 2, we recall the results of [14] obtained in the centralized framework; in Section 3 we extend these results to the framework of distributed systems; we then apply the distributed synthesis method to a nontrivial example (Section 4), and conclude in Section 5.

2 Euler's method applied to control synthesis

In this Section, we recall the results obtained in [14]. We first give results concerning a system governed by a single ODE system (Section 2.1), then consider results for a switched system composed of several ODEs (Section 2.2).

[3] Such a bound has been used in hybridization methods: $error(t) = \frac{E_D}{L}(e^{Lt} - 1)$ [2, 4], where E_D gives the maximum difference of the derivatives of the original and approximated systems.

2.1 ODE systems

We make the following hypothesis:

$$(H0) \quad f \text{ is a locally Lipschitz continuous map.}$$

We make the assumption that the vector field f is such that the solutions of the differential equation (7) are defined. We will denote by $\phi(t; x^0)$ the solution at time t of the system:

$$\begin{aligned} \dot{x}(t) &= f(x(t)), \\ x(0) &= x^0. \end{aligned} \tag{3}$$

Consider a compact and convex set $S \subset \mathbb{R}^n$, called "safety set". We denote by T a compact overapproximation of the image by ϕ of S for $0 \leqslant t \leqslant \tau$, *i.e.*, T is such that

$$T \supseteq \{\phi(t; x^0) \mid 0 \leqslant t \leqslant \tau, x^0 \in S\}.$$

The existence of T is guaranteed by assumption $(H0)$. We know furthermore by $(H0)$ that there exists a constant $L > 0$ such that:

$$\|f(y) - f(x)\| \leqslant L \|y - x\| \quad \forall x, y \in S. \tag{4}$$

Let us define C:

$$C = \sup_{x \in S} L \|f(x)\|. \tag{5}$$

We make the additional hypothesis that the mapping f is *one-sided Lipschitz* (OSL) [7]. Formally:

$$(H1) \quad \text{There exists a constant } \lambda \in \mathbb{R} \text{ such that}$$

$$\langle f(y) - f(x), y - x \rangle \leqslant \lambda \|y - x\|^2 \quad \forall x, y \in T,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of two vectors of $\mathbb{R}^n$.

Remark 1. Constants λ, L and C can be computed using constrained optimization algorithms, namely, the 'sqp' function from GNU Octave [8].

Given an initial point $\tilde{x}^0 \in S$, we define the following "linear approximate solution" $\tilde{\phi}(t; \tilde{x}^0)$ for t on $[0, \tau]$ by:

$$\tilde{\phi}(t; \tilde{x}^0) = \tilde{x}^0 + t f(\tilde{x}^0). \tag{6}$$

We define the closed ball of center $x \in \mathbb{R}^n$ and radius $r > 0$, denoted $B(x, r)$, as the set $\{x' \in \mathbb{R}^n \mid \|x' - x\| \leqslant r\}$.

Given a positive real δ^0, we now define the expression $\delta(t)$ which, as we will see in Theorem 1, represents (an upper bound on) the error associated to $\tilde{\phi}(t; \tilde{x}^0)$ (*i.e.*, $\|\tilde{\phi}(t; \tilde{x}^0) - \phi(t; x^0)\|$).

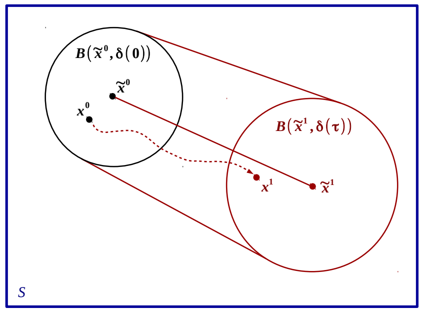


Fig. 1. Illustration of Corollary 1, with $\tilde{x}_1 = \tilde{\phi}(\tau; \tilde{x}^0)$ and $x_1 = \phi(\tau; x^0)$.

Definition 1. *Let δ^0 be a positive constant. Let us define, for all $0 \leqslant t \leqslant \tau$, $\delta(t)$ as follows:*

– *if $\lambda < 0$:*

$$\delta(t) = \left((\delta^0)^2 e^{\lambda t} + \frac{C^2}{\lambda^2} \left(t^2 + \frac{2t}{\lambda} + \frac{2}{\lambda^2} \left(1 - e^{\lambda t}\right) \right) \right)^{\frac{1}{2}}$$

– *if $\lambda = 0$:*

$$\delta(t) = \left((\delta^0)^2 e^t + C^2(-t^2 - 2t + 2(e^t - 1)) \right)^{\frac{1}{2}}$$

– *if $\lambda > 0$:*

$$\delta(t) = \left((\delta^0)^2 e^{3\lambda t} + \frac{C^2}{3\lambda^2} \left(-t^2 - \frac{2t}{3\lambda} + \frac{2}{9\lambda^2} \left(e^{3\lambda t} - 1\right) \right) \right)^{\frac{1}{2}}$$

Note that $\delta(t) = \delta^0$ for $t = 0$. The function $\delta(\cdot)$ depends implicitly on parameter: $\delta^0 \in \mathbb{R}$. In Section 2.2, we will use the notation $\delta'(\cdot)$ where the parameter is denoted by $(\delta')^0$.

Theorem 1. *Given an ODE system satisfying (H0-H1), consider a point $\tilde{x}^0$ and a positive real δ^0. We have, for all $x^0 \in B(\tilde{x}^0, \delta^0)$, $t \in [0, \tau]$:*

$$\phi(t; x^0) \in B(\tilde{\phi}(t; \tilde{x}^0), \delta(t)).$$

Corollary 1. *Given an ODE system satisfying (H0-H1), consider a point $\tilde{x}^0 \in S$ and a real $\delta^0 > 0$ such that:*

1. *$B(\tilde{x}^0, \delta^0) \subseteq S$,*
2. *$B(\tilde{\phi}(\tau; \tilde{x}^0), \delta(\tau)) \subseteq S$, and*
3. *$\frac{d^2(\delta(t))}{dt^2} > 0$ for all $t \in [0, \tau]$.*

Then we have, for all $x^0 \in B(\tilde{x}^0, \delta^0)$ and $t \in [0, \tau]$: $\phi(t; x^0) \in S$.

2.2 Sampled switched systems

Let us consider the nonlinear switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) \tag{7}$$

defined for all $t \geqslant 0$, where $x(t) \in \mathbb{R}^n$ is the state of the system, $\sigma(\cdot) : \mathbb{R}^+ \longrightarrow U$ is the switching rule. The finite set $U = \{1, \dots, N\}$ is the set of switching *modes* of the system. We focus on *sampled switched systems*: given a sampling period $\tau > 0$, switchings will occur at times τ, 2τ, $\dots$ The switching rule $\sigma(\cdot)$ is thus constant on the time interval $[(k-1)\tau, k\tau)$ for $k \geqslant 1$. For all $j \in U$, f_j is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$.

We will denote by $\phi_\sigma(t; x^0)$ the solution at time t of the system:

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}(x(t)), \\ x(0) &= x^0. \end{aligned} \tag{8}$$

Often, we will consider $\phi_\sigma(t; x^0)$ on the interval $0 \leqslant t < \tau$ for which $\sigma(t)$ is equal to a constant, say $j \in U$. In this case, we will abbreviate $\phi_\sigma(t; x^0)$ as $\phi_j(t; x^0)$. We will also consider $\phi_\sigma(t; x^0)$ on the interval $0 \leqslant t < k\tau$ where k is a positive integer, and $\sigma(t)$ is equal to a constant, say $j_{k'}$, on each interval $[(k'-1)\tau, k'\tau)$ with $1 \leqslant k' \leqslant k$; in this case, we will abbreviate $\phi_\sigma(t; x^0)$ as $\phi_\pi(t; x^0)$, where π is a sequence of k modes (or "pattern") of the form $\pi = j_1 \cdot j_2 \cdot \dots \cdot j_k$.

We will assume that ϕ_σ is *continuous* at time $k\tau$ for all positive integer k. This means that there is no "reset" at time $k'\tau$ $(1 \leqslant k' \leqslant k)$; the value of $\phi_\sigma(t, x^0)$ for $t \in [(k'-1)\tau, k\tau]$ corresponds to the solution of $\dot{x}(u) = f_{j_{k'}}(x(u))$ for $u \in [0, \tau]$ with initial value $\phi_\sigma((k'-1)\tau; x^0)$.

More generally, given an initial point $\tilde{x}^0 \in S$ and pattern π of U^k, we can define a "(piecewise linear) approximate solution" $\tilde{\phi}_\pi(t; \tilde{x}^0)$ of ϕ_π at time $t \in [0, k\tau]$ as follows:

- $\tilde{\phi}_\pi(t; \tilde{x}^0) = t f_j(\tilde{x}^0) + \tilde{x}^0$ if $\pi = j \in U$, $k = 1$ and $t \in [0, \tau]$, and
- $\tilde{\phi}_\pi(k\tau + t; \tilde{x}^0) = t f_j(\tilde{z}) + \tilde{z}$ with $\tilde{z} = \tilde{\phi}_{\pi'}((k-1)\tau; \tilde{x}^0)$, if $k \geqslant 2$, $t \in [0, \tau]$, $\pi = j \cdot \pi'$ for some $j \in U$ and $\pi' \in U^{k-1}$.

We wish to synthesize a safety control σ for ϕ_σ using the approximate functions $\tilde{\phi}_\pi$. Hypotheses (H0) and (H1), as defined in Section 2.1, are naturally extended to every mode j of U, as well as definition of T, constants L, C and λ, definitions of $\tilde{\phi}_j$ and δ^0 (see [14]). From a notation point of view, we will assign an index j to symbols $\lambda, L, C, \dots$ in order to relate them to the dynamics of mode j.

Consider a point $\tilde{x}^0 \in S$, a positive real δ^0 and a pattern π of length k. Let $\pi(k')$ denote the k'-th element (mode) of π for $1 \leqslant k' \leqslant k$. Let us abbreviate the k'-th approximate point $\tilde{\phi}_\pi(k'\tau; \tilde{x}^0)$ as $\tilde{x}_\pi^{k'}$ for $k' = 1, ..., k$, and let $\tilde{x}_\pi^{k'} = \tilde{x}^0$ for

$k' = 0$. It is easy to show that $\tilde{x}_\pi^{k'}$ can be defined recursively for $k' = 1, ..., k$, by: $\tilde{x}_\pi^{k'} = \tilde{x}_\pi^{k'-1} + \tau f_j(\tilde{x}_\pi^{k'-1})$ with $j = \pi(k')$.

Let us now define the expression $\delta_\pi^{k'}$ as follows: For $k' = 0$: $\delta_\pi^{k'} = \delta^0$, and for $1 \leqslant k' \leqslant k$: $\delta_\pi^{k'} = \delta'_j(\tau)$ where $(\delta')^0$ denotes $\delta_\pi^{k'-1}$, and j denotes $\pi(k')$. Likewise, for $0 \leqslant t \leqslant k\tau$, let us define the expression $\delta_\pi(t)$ as follows:

- for $t = 0$: $\delta_\pi(t) = \delta^0$,
- for $0 < t \leqslant k\tau$: $\delta_\pi(t) = \delta'_j(t')$ with $(\delta')^0 = \delta_\pi^\ell$, $j = \pi(\ell)$, $t' = t - \ell\tau$ and $\ell = \lfloor \frac{t}{\tau} \rfloor$.

Note that, for $0 \leqslant k' \leqslant k$, we have: $\delta_\pi(k'\tau) = \delta_\pi^{k'}$. We have

Theorem 2. *Given a sampled switched system satisfying (H0-H1), consider a point $\tilde{x}^0 \in S$, a positive real δ^0 and a pattern π of length k such that, for all $1 \leqslant k' \leqslant k$:*

1. *$B(\tilde{x}_\pi^{k'}, \delta_\pi^{k'}) \subseteq S$ and*
2. *$\frac{d^2(\delta'_j(t))}{dt^2} > 0$ for all $t \in [0, \tau]$, with $j = \pi(k')$ and $(\delta')^0 = \delta_\pi^{k'-1}$.*

Then we have, for all $x^0 \in B(\tilde{x}^0, \delta^0)$ and $t \in [0, k\tau]$: $\phi_\pi(t; x^0) \in S$.

Remark 2. In Theorem 2, we have supposed that the step size h used in Euler's method was equal to the sampling period τ of the switching system. Actually, in order to have better approximations, it is often convenient to take a *fraction* of τ as for h (*e.g.*, $h = \frac{\tau}{10}$). Such a splitting is called "sub-sampling" in numerical methods.

Consider now a compact set R, called "recurrence set", contained in the safety set $S \subset \mathbb{R}^n$ ($R \subseteq S$). We are interested in the synthesis of a control such that: starting from any initial point $x \in R$, the controlled trajectory always returns to R within a bounded time while never leaving S.

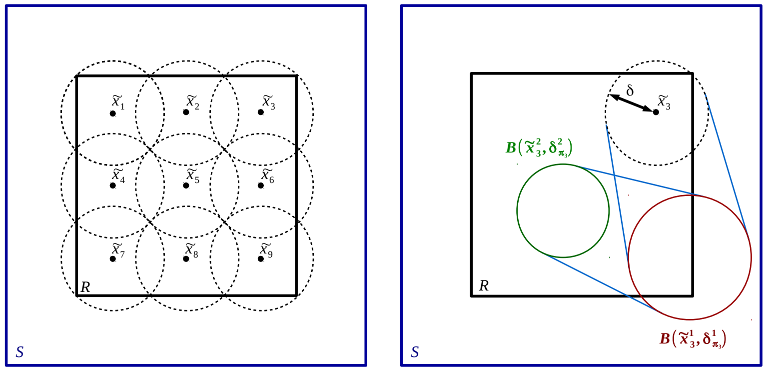
Corollary 2. *Given a switched system satisfying (H0-H1), consider a positive real δ^0 and a finite set of points $\tilde{x}_1, \ldots \tilde{x}_m$ of S such that all the balls $B(\tilde{x}_i, \delta^0)$ cover R and are included into S (i.e., $R \subseteq \bigcup_{i=1}^m B(\tilde{x}_i, \delta^0) \subseteq S$).*

Suppose furthermore that, for all $1 \leqslant i \leqslant m$, there exists a pattern π_i of length k_i such that:

1. *$B((\tilde{x}_i)_{\pi_i}^{k'}, \delta_{\pi_i}^{k'}) \subseteq S$, for all $k' = 1, \ldots, k_i - 1$*
2. *$B((\tilde{x}_i)_{\pi_i}^{k_i}, \delta_{\pi_i}^{k_i}) \subseteq R$.*
3. *$\frac{d^2(\delta'_j(t))}{dt^2} > 0$ with $j = \pi_i(k')$ and $(\delta')^0 = \delta_{\pi_i}^{k'-1}$, for all $k' \in \{1, ..., k_i\}$ and $t \in [0, \tau]$.*

These properties induce a control σ[4] which guarantees

[4] Given an initial point $x \in R$, the induced control σ corresponds to a sequence of patterns $\pi_{i_1}, \pi_{i_2}, \ldots$ defined as follows: Since $x \in R$, there exists a a point $\tilde{x}_{i_1}$ with $1 \leqslant i_1 \leqslant m$ such that $x \in B(\tilde{x}_{i_1}, \delta^0)$; then using pattern π_{i_1}, one has: $\phi_{\pi_{i_1}}(k_{i_1}\tau; x) \in R$. Let $x' = \phi_{\pi_{i_1}}(k_{i_1}\tau; x)$; there exists a point $\tilde{x}_{i_2}$ with $1 \leqslant i_2 \leqslant m$ such that $x' \in B(\tilde{x}_{i_2}, \delta^0)$, etc.



(a) (b)

Fig. 2. (a): A set of balls covering R and contained in S. (b): Control of ball $B(\tilde{x}_3, \delta^0)$ with Euler-based method.

– *(safety): if* $x \in R$*, then* $\phi_\sigma(t; x) \in S$ *for all* $t \geqslant 0$*, and*
– *(recurrence): if* $x \in R$ *then* $\phi_\sigma(k\tau; x) \in R$ *for some* $k \in \{k_1, \ldots, k_m\}$.

Corollary 2 gives the theoretical foundations of the following method for synthesizing σ ensuring recurrence in R and safety in S:

– we (pre-)compute λ_j, L_j, C_j for all $j \in U$;
– we find m points $\tilde{x}_1, \ldots \tilde{x}_m$ of S and $\delta^0 > 0$ such that $R \subseteq \bigcup_{i=1}^m B(\tilde{x}_i, \delta^0) \subseteq S$;
– we find m patterns π_i $(i = 1, ..., m)$ such that conditions 1-2-3 of Corollary 2 are satisfied.

A covering of R with balls as stated in Corollary 2 is illustrated in Figure 2 (a). The control synthesis method based on Corollary 2 is illustrated in Figure 2 (b).

For the sake of simplicity, we will suppose in the following that R is a *rectangle*, i.e., the Cartesian product of n closed real intervals, and we will denote its center by c. We will also assume that T is a ball of centre c and radius Δ (i.e., $T = B(c, \Delta)$).

3 Distributed synthesis

The goal is to split the system into two (or more) sub-systems and synthesize controllers for the sub-systems independently. The allows to break the exponential complexity (curse of dimensionality) of the method w.r.t. the dimension of the system, as well as the dimension of the control input.

We consider the distributed control system

$$\dot{x}_1 = f^1_{j_1}(x_1, x_2) \quad (9)$$

$$\dot{x}_2 = f^2_{j_2}(x_1, x_2) \quad (10)$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$, with $n_1 + n_2 = n$. Furthermore, $j_1 \in U_1$ and $j_2 \in U_2$ and $U = U_1 \times U_2$.

Note that the system (9-10) can be seen as the *interconnection* of sub-system (9) where x_2 plays the role of an "input" given by (10), with sub-system (10) where x_1 is an "input" given by (9).

Let: $R = R_1 \times R_2$, $S = S_1 \times S_2$, and $c = (c_1, c_2)$[5]. We denote by $L^1_{j_1}$ the Lipschitz constant for sub-system 1 under mode j_1 on S:

$$\|f^1_{j_1}(x_1, x_2) - f^1_{j_1}(y_1, y_2)\| \leqslant L^1_{j_1} \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|$$

We then introduce the constant:

$$C^1_{j_1} = \sup_{x_1 \in S_1} L^1_{j_1} \|f^1_{j_1}(x_1, c_2)\|$$

Similarly, we define the constants for sub-system 2:

$$\|f^2_{j_2}(x_1, x_2) - f^2_{j_2}(y_1, y_2)\| \leqslant L^2_{j_2} \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|$$

and

$$C^2_{j_2} = \sup_{x_2 \in S_2} L^2_{j_2} \|f^2_{j_2}(c_1, x_2)\|$$

Let us now make additional assumptions on the coupled sub-systems, closely related to the notion of (incremental) input-to-state stability.

(H2) For every mode $j_1 \in U_1$, there exists constants $\lambda^1_{j_1} \in \mathbb{R}$ and $\gamma^1_{j_1} \in \mathbb{R}_{\geqslant 0}$ such that $\forall x, x' \in T_1$ and $\forall y, y' \in T_2$, the following expression holds

$$\langle f^1_{j_1}(x, y) - f^1_{j_1}(x', y'), x - x' \rangle \leqslant \lambda^1_{j_1} \|x - x'\|^2 + \gamma^1_{j_1} \|x - x'\| \|y - y'\|.$$

(H3) For every mode $j_2 \in U_2$, there exists constants $\lambda^2_{j_2} \in \mathbb{R}$ and $\gamma^2_{j_2} \in \mathbb{R}_{\geqslant 0}$ such that $\forall x, x' \in T_1$ and $\forall y, y' \in T_2$, the following expression holds

$$\langle f^2_{j_2}(x, y) - f^2_{j_2}(x', y'), y - y' \rangle \leqslant \lambda^2_{j_2} \|y - y'\|^2 + \gamma^2_{j_2} \|x - x'\| \|y - y'\|.$$

These assumptions express (a variant of) the fact that the function $V(x, x') = \|x - x'\|^2$ is an *ISS-Lyapunov function* (see, *e.g.*, [1, 9]). Note that all the constants defined above can be numerically computed using constrained optimization algorithms.

Let us define the distributed Euler scheme:

$$\tilde{x}_1(\tau) = \tilde{x}_1(0) + \tau f^1_{j_1}(\tilde{x}_1(0), c_2) \quad (11)$$

$$\tilde{x}_2(\tau) = \tilde{x}_2(0) + \tau f^2_{j_2}(c_1, \tilde{x}_2(0)) \quad (12)$$

The exact trajectory is now denoted, for all $t \in [0, \tau]$, by $\phi_{(j_1, j_2)}(t; x^0)$ for an initial condition $x^0 = \left(x^0_1 \; x^0_2\right)^\top$, and when sub-system 1 is in mode $j_1 \in U_1$, and sub-system 2 is in mode $j_2 \in U_2$.

[5] So $T = T_1 \times T_2$ with: $T_1 = B(c_1, \varDelta)$, $T_2 = B(c_2, \varDelta)$.

We define the approximate trajectory computed with the distributed Euler scheme by $\tilde{\phi}^1_{j_1}(t;\tilde{x}^0_1) = \tilde{x}^0_1 + tf^1_{j_1}(\tilde{x}^0_1, c_2)$ for $t \in [0,\tau]$, when sub-system 1 is in mode j_1 and with an initial condition $\tilde{x}^0_1$. Similarly, for sub-system 2, $\tilde{\phi}^2_{j_2}(t;\tilde{x}^0_2) = \tilde{x}^0_2 + tf^2_{j_2}(c_1, \tilde{x}^0_2)$ when sub-system 2 is in mode j_2 and with an initial condition $\tilde{x}^0_2$.

We now give a distributed version of Theorem 1.

Theorem 3. *Given a distributed sampled switched system, a positive real δ^0 and a point $\tilde{x}^0_1 \in S_1$, suppose that sub-system 1 satisfies (H2) and $\tilde{\phi}^1_{j_1}(t;\tilde{x}^0_1)$ belongs to S_1 for all $t \in [0,\tau]$. We have, for all $x^0_1 \in B(\tilde{x}^0_1, \delta^0)$, $x^0_2 \in S_2$, $t \in [0,\tau]$, $j_1 \in U_1$, $j_2 \in U_2$:*

$$\phi_{(j_1,j_2)}(t;x^0)_{|1} \in B(\tilde{\phi}^1_{j_1}(t;\tilde{x}^0_1), \delta_{j_1}(t)).$$

with $x^0 = \begin{pmatrix} x^0_1 & x^0_2 \end{pmatrix}^\top$ and

– *if $\lambda^1_{j_1} < 0$,*

$$\begin{aligned}\delta_{j_1}(t) = \Bigg(&\frac{(C^1_{j_1})^2}{-(\lambda^1_{j_1})^4}\left(-(\lambda^1_{j_1})^2 t^2 - 2\lambda^1_{j_1} t + 2e^{\lambda^1_{j_1} t} - 2\right) \\ &+ \frac{2}{(\lambda^1_{j_1})^2}\Bigg(\frac{C^1_{j_1}\gamma^1_{j_1}\mathit{\Delta}}{-\lambda^1_{j_1}}\left(-\lambda^1_{j_1} t + e^{\lambda^1_{j_1} t} - 1\right) \\ &+ \lambda^1_{j_1}\Bigg(\frac{(\gamma^1_{j_1})^2\mathit{\Delta}^2}{-\lambda^1_{j_1}}(e^{\lambda^1_{j_1} t} - 1) + \lambda^1_{j_1}(\delta^0)^2 e^{\lambda^1_{j_1} t}\Bigg)\Bigg)\Bigg)^{1/2}\end{aligned} \tag{13}$$

– *if $\lambda^1_{j_1} > 0$,*

$$\begin{aligned}\delta_{j_1}(t) = \frac{1}{(3\lambda^1_{j_1})^{3/2}} \Bigg(&\frac{C_1^2}{\lambda^1_{j_1}}\left(-9(\lambda^1_{j_1})^2 t^2 - 6\lambda^1_{j_1} t + 2e^{3\lambda^1_{j_1} t} - 2\right) \\ &+ 6\lambda^1_{j_1}\Bigg(\frac{C_1\gamma^1_{j_1}\mathit{\Delta}}{\lambda^1_{j_1}}\left(-3\lambda^1_{j_1} t + e^{3\lambda^1_{j_1} t} - 1\right) \\ &+ 3\lambda^1_{j_1}\Bigg(\frac{(\gamma^1_{j_1})^2\mathit{\Delta}^2}{\lambda^1_{j_1}}(e^{3\lambda^1_{j_1} t} - 1) + 3\lambda^1_{j_1}(\delta^0)^2 e^{3\lambda^1_{j_1} t}\Bigg)\Bigg)\Bigg)^{1/2}\end{aligned} \tag{14}$$

– *if $\lambda^1_{j_1} = 0$,*

$$\begin{aligned}\delta_{j_1}(t) = \big(&(C^1_{j_1})^2\left(-t^2 - 2t + 2e^t - 2\right) \\ &+ \left(2C^1_{j_1}\gamma^1_{j_1}\mathit{\Delta}\left(-t + e^t - 1\right)\right. \\ &+ \left.\left((\gamma^1_{j_1})^2\mathit{\Delta}^2(e^t - 1) + (\delta^0)^2 e^t\right)\right)\big)^{1/2}\end{aligned} \tag{15}$$

A similar result can be established for sub-system 2, permitting to perform a distributed control synthesis.

Proof. Consider on $t \in [0, \tau]$ the differential system (9-10) with initial conditions $x_1(0) \in B(\tilde{x}_1(0), \delta^0)$, $x_2(0) \in S_2$, and the system (11-12) with initial conditions $\tilde{x}_1(0) \in S_1$, $\tilde{x}_2(0) \in S_2$. We will abbreviate $\phi_{j_1}(t; x_1(0))$ as x_1, $\phi_{j_2}(t; x_2(0))$ as x_2, and $\tilde{\phi}_{j_1}(t; x_1(0))$ as $\tilde{x}_1$. In order to simplify the notation, we omit the mode j_1 and write $f^1_{j_1} \equiv f_1$, $L^1_{j_1} \equiv L_1$, $C^1_{j_1} \equiv C_1$, $\lambda^1_{j_1} \equiv \lambda_1$. Since, $\frac{d(x_1 - \tilde{x}_1)}{dt} = f_1(x_1, x_2) - f_1(\tilde{x}_1(0), c_2)$, we have, using the facts $\tilde{x}_1 \in S_1$ and $c_2 \in S_2$:

$$
\begin{aligned}
&\frac{1}{2}\frac{d(\|x_1 - \tilde{x}_1\|^2)}{dt} = \langle f_1(x_1, x_2) - f_1(\tilde{x}_1(0), c_2), x_1 - \tilde{x}_1 \rangle \\
&= \langle f_1(x_1, x_2) - f_1(\tilde{x}_1, c_2) + f_1(\tilde{x}_1, c_2) - f_1(\tilde{x}_1(0), c_2), x_1 - \tilde{x}_1 \rangle \\
&\leqslant \langle f_1(x_1, x_2) - f_1(\tilde{x}_1, c_2), x_1 - \tilde{x}_1 \rangle + \langle f_1(\tilde{x}_1, c_2) - f_1(\tilde{x}_1(0), c_2), x_1 - \tilde{x}_1 \rangle \\
&\leqslant \langle f_1(x_1, x_2) - f_1(\tilde{x}_1, c_2), x_1 - \tilde{x}_1 \rangle + \|f_1(\tilde{x}_1, c_2) - f_1(\tilde{x}_1(0), c_2)\| \|x_1 - \tilde{x}_1\| \\
&\leqslant \langle f_1(x_1, x_2) - f_1(\tilde{x}_1, c_2), x_1 - \tilde{x}_1 \rangle + L_1 \left\| \begin{pmatrix} \tilde{x}_1 \\ c_2 \end{pmatrix} - \begin{pmatrix} \tilde{x}_1(0) \\ c_2 \end{pmatrix} \right\| \|x_1 - \tilde{x}_1\| \\
&\leqslant \lambda_1 \|x_1 - \tilde{x}_1\|^2 + \gamma_1 \|x_2 - c_2\| \|x_1 - \tilde{x}_1\| + L_1 t \, \|f_1(\tilde{x}_1(0), c_2)\| \, \|x_1 - \tilde{x}_1\| \\
&\leqslant \lambda_1 \|x_1 - \tilde{x}_1\|^2 + (\gamma_1 \varDelta + C_1 t) \, \|x_1 - \tilde{x}_1\|
\end{aligned}
$$

Using the fact that $\|x_1 - \tilde{x}_1\| \leqslant \frac{1}{2}(\alpha \|x_1 - \tilde{x}_1\|^2 + \frac{1}{\alpha})$ for any $\alpha > 0$, we can write three formulas following the sign of λ_1.

- if $\lambda_1 < 0$, we can choose $\alpha = \frac{-\lambda_1}{C_1 t + \gamma_1 \varDelta}$, and we get the differential inequality:

$$\frac{d(\|x_1 - \tilde{x}_1\|^2)}{dt} \leqslant \lambda_1 \|x_1 - \tilde{x}_1\|^2 + \frac{C_1^2}{-\lambda_1} t^2 + \frac{2C_1 \gamma_1 \varDelta}{-\lambda_1} t + \frac{\gamma_1^2 \varDelta^2}{-\lambda_1}$$

- if $\lambda_1 > 0$, we can choose $\alpha = \frac{\lambda_1}{C_1 t + \gamma_1 \varDelta}$, and we get the differential inequality:

$$\frac{d(\|x_1 - \tilde{x}_1\|^2)}{dt} \leqslant 3\lambda_1 \|x_1 - \tilde{x}_1\|^2 + \frac{C_1^2}{\lambda_1} t^2 + \frac{2C_1 \gamma_1 \varDelta}{\lambda_1} t + \frac{\gamma_1^2 \varDelta^2}{\lambda_1}$$

- if $\lambda_1 = 0$, we can choose $\alpha = \frac{1}{C_1 t + \gamma_1 \varDelta}$, and we get the differential inequality:

$$\frac{d(\|x_1 - \tilde{x}_1\|^2)}{dt} \leqslant \|x_1 - \tilde{x}_1\|^2 + C_1^2 t^2 + 2C_1 \gamma_1 \varDelta t + \gamma_1^2 \varDelta^2$$

In every case, the differential inequalities can be integrated to obtain the formulas of the theorem.

□

It then follows a distributed version of Corollary 2.

Corollary 3. *Given a positive real δ^0, consider two sets of points $\tilde{x}^1_1, \ldots, \tilde{x}^1_{m_1}$ and $\tilde{x}^2_1, \ldots, \tilde{x}^2_{m_2}$ such that all the balls $B(\tilde{x}^1_{i_1}, \delta^0)$ and $B(\tilde{x}^2_{i_2}, \delta^0)$, for $1 \leqslant i_1 \leqslant m_1$ and $1 \leqslant i_2 \leqslant m_2$, cover R_1 and R_2. Suppose that there exists patterns $\pi^1_{i_1}$ and $\pi^2_{i_2}$ of length k_{i_1} and k_{i_2} such that :*

1. $B((\tilde{x}^1_{i_1})^{k'}_{\pi^1_{i_1}}, \delta^{k'}_{\pi^1_{i_1}}) \subseteq S_1$, *for all* $k' = 1, \ldots, k_{i_1} - 1$
2. $B((\tilde{x}^1_{i_1})^{k_{i_1}}_{\pi^1_{i_1}}, \delta^{k_{i_1}}_{\pi^1_{i_1}}) \subseteq R_1$.
3. $\frac{d^2(\delta'_{j_1}(t))}{dt^2} > 0$ *with* $j_1 = \pi^1_{i_1}(k')$ *and* $(\delta')^0 = \delta^{k'-1}_{\pi^1_{i_1}}$, *for all* $k' \in \{1, ..., k_{i_1}\}$ *and* $t \in [0, \tau]$.

1. $B((\tilde{x}^2_{i_2})^{k'}_{\pi^2_{i_2}}, \delta^{k'}_{\pi^2_{i_2}}) \subseteq S_2$, *for all* $k' = 1, \ldots, k_{i_2} - 1$
2. $B((\tilde{x}^2_{i_2})^{k_{i_2}}_{\pi^2_{i_2}}, \delta^{k_{i_2}}_{\pi^2_{i_2}}) \subseteq R_2$.
3. $\frac{d^2(\delta'_{j_2}(t))}{dt^2} > 0$ *with* $j_2 = \pi^2_{i_2}(k')$ *and* $(\delta')^0 = \delta^{k'-1}_{\pi^2_{i_2}}$, *for all* $k' \in \{1, ..., k_{i_2}\}$ *and* $t \in [0, \tau]$.

The above properties induce a distributed control $\sigma = (\sigma_1, \sigma_2)$ *guaranteeing (non simultaneous) recurrence in* R *and safety in* S. *I.e.*

- *if* $x \in R$, *then* $\phi_\sigma(t; x) \in S$ *for all* $t \geqslant 0$
- *if* $x \in R$, *then* $\phi_\sigma(k_1\tau; x)_{|1} \in R_1$ *for some* $k_1 \in \{k_{i_1}, \ldots, k_{i_{m_1}}\}$, *and symmetrically* $\phi_\sigma(k_2\tau; x)_{|2} \in R_2$ *for some* $k_2 \in \{k_{i_2}, \ldots, k_{i_{m_2}}\}$

4 Application

We demonstrate the feasibility of our approach on a (linearized) building ventilation application adapted from [16]. The system is a four-room apartment subject to heat transfer between the rooms, with the external environment and with the underfloor. The dynamics of the system is given by the following equation:

$$\frac{dT_i}{dt} = \sum_{j \in \mathcal{N}^* \setminus \{i\}} a_{ij}(T_j - T_i) + c_i \max\left(0, \frac{V_i - V_i^*}{\bar{V}_i - V_i^*}\right)(T_u - T_i). \tag{16}$$

The state of the system is given by the temperatures in the rooms T_i, for $i \in \mathcal{N} = \{1, \ldots, 4\}$. Room i is subject to heat exchange with different entities stated by the indexes $\mathcal{N}^* = \{1, 2, 3, 4, u, o, c\}$. The heat transfer between the rooms is given by the coefficients a_{ij} for $i, j \in \mathcal{N}^2$, and the different perturbations are the following:

- The external environment: it has an effect on room i with the coefficient a_{io} and the outside temperature T_o, set to $30°C$.
- The heat transfer through the ceiling: it has an effect on room i with the coefficient a_{ic} and the ceiling temperature T_c, set to $30°C$.
- The heat transfer with the underfloor: it is given by the coefficient a_{iu} and the underfloor temperature T_u, set to $17°C$ (T_u is constant, regulated by a PID controller).

The control V_i, $i \in \mathcal{N}$, is applied through the term $c_i \max(0, \frac{V_i - V_i^*}{\bar{V}_i - V_i^*})(T_u - T_i)$. A voltage V_i is applied to force ventilation from the underfloor to room i, and the

Table 1. Numerical results for centralized four-room example.

	Centralized
R	$[20, 22]^4$
S	$[19, 23]^4$
τ	30
Time subsampling	$\tau/20$
Complete control	Yes
Error parameters	$\max_{j=1,\dots,16} \lambda_j = -6.30 \times 10^{-3}$ $\max_{j=1,\dots,16} C_j = 4.18 \times 10^{-6}$
Number of balls/tiles	256
Pattern length	2
CPU time	48 seconds

command of an underfloor fan is subject to a dry friction. Because we work in a switching control framework, V_i can take only discrete values, which removes the problem of dealing with a "max" function in interval analysis. In the experiment, V_1 and V_4 can take the values 0V or 3.5V, and V_2 and V_3 can take the values 0V or 3V. This leads to a system of the form (8) with $\sigma(t) \in U = \{1, \dots, 16\}$, the 16 switching modes corresponding to the different possible combinations of voltages V_i. The system can be decomposed in sub-systems of the form (9)-(10). The sampling period is $\tau = 30$s. The parameters V_i^*, $\bar{V}_i$, a_{ij}, b_i, c_i are given in [16] and have been identified with a proper identification procedure detailed in [17].

The main difficulty of this example is the large number of modes in the switching system, which induces a combinatorial issue. The centralized controller was obtained with 256 balls in 48 seconds, the distributed controller was obtained with 16 + 16 balls in less than a second. In both cases, patterns of length 2 are used. A sub-sampling of $h = \tau/20$ is required to obtain a controller with the centralized approach. For the distributed approach, no sub-sampling is required for the first sub-system, while the second one requires a sub-sampling of $h = \tau/10$. Simulations of the centralized and distributed controllers are given in Figure 3, where the control objective is to stabilize the temperature in $[20, 22]^4$ while never going out of $[19, 23]^4$.

5 Final remarks and future work

We have given a new distributed control synthesis method based on Euler's method. The method makes use of the notions of δ-ISS-stability and ISS Lyapunov functions. From a certain point of view, this method is along the lines of [6] and [12] which are inspired by small-gain theorems of control theory (see, *e.g.*, [10]). In the future, we plan to apply our distributed Euler-based method to significant examples such as the 11-room example treated in [13, 15].

Table 2. Numerical results for the distributed four-room example.

	Sub-system 1	Sub-system 2
R S	$[20, 22]^2 \times [20, 22]^2$ $[19, 23]^2 \times [19, 23]^2$	
τ	30	
Time subsampling	No	$\tau/10$
Complete control Error parameters	Yes $\max_{j_1=1,\dots,4} \lambda^1_{j_1} = -1.39 \times 10^{-3}$ $\max_{j_1=1,\dots,4} \gamma^1_{j_1} = 1.79 \times 10^{-4}$ $\max_{j_1=1,\dots,4} C^1_{j_1} = 4.15 \times 10^{-4}$	Yes $\max_{j_2=1,\dots,4} \lambda^2_{j_2} = -1.42 \times 10^{-3}$ $\max_{j_2=1,\dots,4} \gamma^2_{j_2} = 2.47 \times 10^{-4}$ $\max_{j_2=1,\dots,4} C^2_{j_2} = 5.75 \times 10^{-4}$
Number of balls/tiles Pattern length	16 2	16 2
CPU time	< 1 second	< 1 second

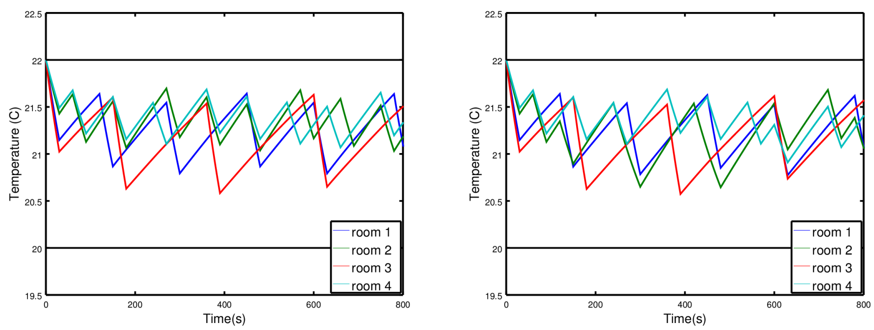


Fig. 3. Simulation of the centralized (left) and distributed (right) controllers from the initial condition $(22, 22, 22, 22)$.

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