On the state complexity of closures and interiors of regular languages with subwords

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Abstract

We study the closure of regular languages by taking subwords or superwords, provide exact state complexity in the case of unbounded alphabets, and prove new lower bounds in the case of languages over a two-letter alphabet. We also consider the dual interior sets, for which the nondeterministic state complexity has a doubly-exponential upper bound and for which we prove matching doubly-exponential lower bounds in the case of unbounded alphabets.

Keywords: Finite automata and regular languages; Subwords and superwords; State complexity; Combined operations; Closures and interiors of regular languages.

1. Introduction

Quoting from [1], "*State complexity problems are a fundamental part of automata theory that has a long history. [. . .] However, many very basic questions, which perhaps should have been solved in the sixties and seventies, have not been considered or solved.*"

In this paper, we are concerned with *(scattered) subwords* and the associated operations on regular languages: computing closures and interiors (see definitions in Section 2). Our motivations come from automatic verification of channel systems, see, e.g., [2, 3]. Other applications exist in data processing or bioinformatics [4]. Closures and interiors wrt subwords and superwords are very basic operations, and the above quote certainly applies to them.

It has been known since [5] that $\downarrow L$ and $\uparrow L$, the downward closure and, respectively, the upward closure, of a language $L \subseteq \Sigma^*$, are regular for any L.

In 2009, Gruber *et al.* explicitly raised the issue of the state complexity of downward and upward closures of regular languages [6] (less explicit precursors

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exist, e.g. [7]). Given an n-state automaton A, constructing automata $A^{\downarrow}$ and $A^{\uparrow}$ for $\downarrow L(A)$ and, respectively, $\uparrow L(A)$ can be done by simply adding extra transitions to A. However, when A is a DFA, the resulting automaton is in general not deterministic, and its determinization may entail an exponential blowup in general. Gruber *et al.* proved a $2^{\Omega(\sqrt{n}\log n)}$ lower bound on the number of states of any DFA for $\downarrow L(A)$ or $\uparrow L(A)$, to be compared with the $2^n - 1$ upper bound that comes from the simple closure+determinization algorithm.

Okhotin improved on these results by showing an improved $2^{\frac{n}{2}-2}$ lower bound for $\downarrow L(A)$. He also established the precise state complexity of $\uparrow L(A)$ by showing a $2^{n-2} + 1$ upper bound and proving its tightness [8].

All the above lower bounds assume an unbounded alphabet, and Okhotin showed that his $2^{n-2} + 1$ state complexity for $\uparrow L(A)$ requires $n - 2$ distinct letters: he then considered the case of languages over a *fixed alphabet* with $|\Sigma| = 3$ letters, in which case he demonstrated exponential $2^{\sqrt{2n+30}-6}$ and $\frac{1}{5}4^{\sqrt{n/2}}n^{-\frac{3}{4}}$ lower bounds for $\downarrow L(A)$ and, respectively, $\uparrow L(A)$ [8]. The construction and the proof are quite involved, and they leave open the case where $|\Sigma| = 2$ (the 1-letter case is trivial). It turns out that, in the 2-letter case, Héam had previously proved an $\Omega(r^{\sqrt{n}})$ lower bound for $\uparrow L(A)$, here with $r = (\frac{1+\sqrt{5}}{2})^{\frac{1}{\sqrt{2}}}$ [9]. Regarding $\downarrow L(A)$, the question remains open whether it may require an exponential number of states even when $|\Sigma| = 2$.

Dual to closures are *interiors*. The *upward interior* and *downward interior* of a language L, denoted $\circlearrowleft L$ and $\circlearrowright L$, are the largest upward-closed and, resp., downward-closed, sets included in L. Building closures and interiors are essential operations when reasoning with subwords, e.g., when model-checking lossy channel systems [10]. More generally, one may uses closures as regular overapproximations of more complex languages (as in [11, 12]), and interiors can be used as regular underapproximations.

The state complexity of interiors has not yet been considered in the literature. When working with DFAs, complementation is essentially free so that computing interiors reduces to computing closures, thanks to duality. However, when working with NFAs, the simple complement+closure+complement algorithm only yields a quite large 2^{2^n} upper-bound on the number of states of an NFA for $\circlearrowleft L(A)$ or $\circlearrowright L(A)$ —it actually yields DFAs— and one would like to improve on this, or to prove a matching lower bound.

Our contribution. Regarding closures with DFAs, we prove in Section 3 a tight 2^{n-1} state complexity for downward closure and shows that its tightness requires unbounded alphabets. In Section 4 we prove an exponential lower bound on $\downarrow L(A)$ in the case of a two-letter alphabet, answering the open question raised above.

Regarding interiors on NFAs, we show in Section 5 doubly-exponential lower bounds for downward and upward interiors, assuming an unbounded alphabet. We also provide improved upper bounds, lower than the naive 2^{2^n} but still doubly exponential. Table 1 shows a summary of the known results.

Finally, we provide in Section 6 the computational complexity of basic decision problems for sets of subwords or superwords described by automata.

Table 1: A summary of the results on state complexity for closures and nondeterministic state complexity for interiors, where $\psi(n)$ ($\leqslant 2^{2^n}$) is the nth Dedekind's number, see Section 5.1.

Operation	Unbounded alphabet	Fixed alphabet
$\uparrow L$ (DFA to DFA)	$= 2^{n-2} + 1$ with $\lvert\Sigma\rvert \geqslant n-2$	$2^{\Omega(n^{1/2})}$ for $\lvert\Sigma\rvert = 2$
$\downarrow L$ (DFA to DFA)	$= 2^{n-1}$ with $\lvert\Sigma\rvert \geqslant n-1$	$2^{\Omega(n^{1/3})}$ for $\lvert\Sigma\rvert = 2$
$\circlearrowleft L$ (NFA to NFA)	$\geqslant 2^{2^{\lfloor \frac{n-4}{3} \rfloor}} + 1$ and $\leqslant \psi(n)$	
$\circlearrowright L$ (NFA to NFA)	$\geqslant 2^{2^{\lfloor \frac{n-3}{2} \rfloor}}$ and $\leqslant \psi(n)$	

Related work. We already mentioned previous works on the closure of regular languages: it is also possible to compute closures by subwords or superwords for larger classes like context-free languages or Petri net languages, see [11–13] and the references therein for applications and some results on descriptive complexity.

Interiors and other duals of standard operations have the form "*complement–operation–complement*" and thus can be seen as special cases of the *combined operations* studied in [14] and following papers. Such duals have not yet been considered widely: we are only aware of [15] studying the dual of $L \mapsto \Sigma^* \cdot L$.

2. Basic notions and results

Fix a finite alphabet $\Sigma = \{a, b, \ldots\}$. We say that an ℓ-letter word $x = a_1\, a_2 \cdots a_\ell$ is a subword of y, written $x \sqsubseteq y$, when $y = y_0\, a_1\, y_1 \cdots y_{\ell-1}\, a_\ell\, y_\ell$ for some factors $y_0, \ldots, y_\ell \in \Sigma^*$, i.e., when there are positions $p_1 < p_2 < \cdots < p_\ell$ s.t. $x[i] = y[p_i]$ for all $1 \leqslant i \leqslant \ell = |x|$. For a language $L \subseteq \Sigma^*$, its downward closure is $\downarrow L \stackrel{\text{def}}{=} \{x \in \Sigma^* \mid \exists y \in L : x \sqsubseteq y\}$. Symmetrically, we consider an upward closure operation and we let $\uparrow L \stackrel{\text{def}}{=} \{x \in \Sigma^* \mid \exists y \in L : y \sqsubseteq x\}$. When $x \in \Sigma^*$ is a word, we may write $\downarrow x$ and $\uparrow x$ for $\downarrow\{x\}$, the set of its subwords, and $\uparrow\{x\}$, its superwords, e.g., $\downarrow a\,b\,b = \{\epsilon, a, b, a\,b, b\,b, a\,b\,b\}$. Closures enjoy the following properties:

$$\downarrow\varnothing = \varnothing\,, \quad L \subseteq \downarrow L = \downarrow\downarrow L\,, \quad \downarrow\Big(\bigcup_i L_i\Big) = \bigcup_i \downarrow L_i\,, \quad \downarrow\Big(\bigcap_i \downarrow L_i\Big) = \bigcap_i \downarrow L_i\,,$$

and similarly for upward closures. A language L is *downward-closed* (or *upward-closed*) if $L = \downarrow L$ (respectively, if $L = \uparrow L$). Note that L is downward-closed if, and only if, $\Sigma^* \smallsetminus L$ is upward-closed.

Upward-closed languages are also called *shuffle ideals* since they satisfy $L = L \sqcup\!\sqcup\, \Sigma^*$. They correspond exactly to level $\frac{1}{2}$ of Straubing's hierarchy [16].

Since, by Higman's Lemma, any L has only finitely many minimal elements wrt the subword ordering, one deduces that $\uparrow L$, and then $\downarrow L$, are regular for any L.

Effective construction of a finite-state automaton for $\downarrow L$ or $\uparrow L$ is easy when L is regular (see Section 3), is possible when L is context-free [17, 18], and is not possible in general since this would allow deciding the emptiness of L.

The *upward interior* of L is $\circlearrowleft L \stackrel{\text{def}}{=} \{x \in \Sigma^* \mid \uparrow x \subseteq L\}$. Its *downward interior* is $\circlearrowright L \stackrel{\text{def}}{=} \{x \in \Sigma^* \mid \downarrow x \subseteq L\}$. Alternative characterizations are possible, e.g., by noting that $\circlearrowleft L$ (respectively, $\circlearrowright L$) is the largest upward-closed (respectively, downward-closed) language contained in L, or by using the following dualities:

$$\circlearrowright L = \Sigma^* \smallsetminus \uparrow(\Sigma^* \smallsetminus L)\,, \qquad\qquad \circlearrowleft L = \Sigma^* \smallsetminus \downarrow(\Sigma^* \smallsetminus L)\,. \tag{1}$$

If L is regular, one may compute automata for the interiors of L by combining complementations and closures as in Eq. (1).

State complexity. When considering a finite automaton $A = (\Sigma, Q, \delta, I, F)$, we usually write n for $|Q|$ (the number of states), k for $|\Sigma|$ (the size of the alphabet), and $L(A)$ for the language recognized by A. For a regular language L, $n_{\mathrm{D}}(L)$ and $n_{\mathrm{N}}(L)$ denote the minimum number of states of a DFA (resp., an NFA) that accepts L. (We do not assume that DFAs are complete and this sometimes allows saving one (dead) state.) Obviously $n_{\mathrm{N}}(L) \leqslant n_{\mathrm{D}}(L)$ for any regular language. In cases where $n_{\mathrm{N}}(L) = n_{\mathrm{D}}(L)$ we may use $n_{\mathrm{N\&D}}(L)$ to denote the common value.

We now illustrate a well-known technique for proving lower bounds on $n_{\mathrm{N}}(L)$:

Lemma 2.1 (Extended fooling set technique, [19]) *Let L be a regular language. Suppose that there exists a set of pairs of words $S = \{(x_i, y_i)\}_{1 \leqslant i \leqslant n}$, called a* fooling set*, such that for all i, j, $x_i\, y_i \in L$ and at least one of $x_i\, y_j$ and $x_j\, y_i$ is not in L. Then $n_N(L) \geqslant n$.*

Proof. Let M be an NFA for L. For each i, $x_i y_i \in L$, so M has an accepting path $* \xrightarrow{x_i} q_i \xrightarrow{y_i} *$ starting at some initial state and ending at some accepting state, for some state q_i. The states $q_1, q_2, \ldots, q_n$ are all distinct: indeed, if $q_i = q_j$ for $i \neq j$ then M has accepting paths for both $x_i y_j$ and $x_j y_i$, which contradicts the assumption. □

Lemma 2.2 (An application of the fooling set technique) *Fix a nonempty alphabet Σ and define the following languages:*

$$U_\Sigma \stackrel{\text{def}}{=} \{x \mid \forall a \in \Sigma : \exists i : x[i] = a\}\,, \quad U'_\Sigma \stackrel{\text{def}}{=} \{x \mid \forall a \in \Sigma : \exists i > 1 : x[i] = a\}\,, \tag{2}$$

$$V_\Sigma \stackrel{\text{def}}{=} \{x \mid \forall i \neq j : x[i] \neq x[j]\}\,. \tag{3}$$

Then $n_{\mathrm{N\&D}}(U_\Sigma) = n_{\mathrm{N\&D}}(V_\Sigma) = 2^{|\Sigma|}$ and $n_{\mathrm{N\&D}}(U'_\Sigma) = 2^{|\Sigma|} + 1$.

Note that U_Σ has all words where every letter in Σ appears at least once, U'_Σ has all nonempty words x where every letter in Σ appears at least once in the first suffix $x[2..]$ while V_Σ has all words where no letter appears twice. U_Σ and U'_Σ are upward-closed while V_Σ is downward-closed.

PROOF. It can easily be observed that the upper bounds hold for $n_\mathrm{D}(..)$: one designs DFAs A_U and A_V for, respectively, U_Σ and V_Σ where each state is some subset $\Gamma \subseteq \Sigma$ that corresponds to the set of letters that have been read so far. In both automata, the initial state is $\varnothing$. In A_U, $\delta(\Gamma, a) = \Gamma \cup \{a\}$ and we accept when we reach the state Σ. In A_V, all states are accepting but $\delta(\Gamma, a) = \Gamma \cup \{a\}$ is only defined when $a \notin \Gamma$. A DFA for U'_Σ is obtained from A_U by adding one additional state to indicate that no letter has been read so far.

We now show the lower bounds for $n_\mathrm{N}(U_\Sigma)$ and $n_\mathrm{N}(V_\Sigma)$. With any $\Gamma \subseteq \Sigma$, we associate two words x_Γ and y_Γ, where x_Γ (respectively, y_Γ) has exactly one occurrence of each letter from Γ (respectively, each letter not in Γ). Then $x_\Gamma y_\Gamma$ is in U_Σ and V_Σ, while for any $\Delta \neq \Gamma$ one of $x_\Gamma y_\Delta$ and $x_\Delta y_\Gamma$ is not in U_Σ (and one is not in V_Σ). We may thus let $S = \{(x_\Gamma, y_\Gamma)\}_{\Gamma \subseteq \Sigma}$ be our fooling set and conclude with Lemma 2.1.

For U'_Σ our fooling set will be $S = \{(a\, x_\Gamma, y_\Gamma)\}_{\Gamma \subseteq \Sigma} \cup \{(\epsilon, a\, x_\Sigma)\}$ where a is a fixed letter from Σ. As above, $a\, x_\Gamma y_\Gamma$ is in U'_Σ, while for any $\Delta \neq \Gamma$ one of $a\, x_\Gamma y_\Delta$ and $a\, x_\Delta y_\Gamma$ is not in U'_Σ. Furthermore $\epsilon \cdot a\, x_\Sigma$ is in U'_Σ, while $\epsilon \cdot y_\Gamma$ is not in U'_Σ for any Γ. One concludes again with Lemma 2.1. □

In the following, we use $\Sigma_k \stackrel{\text{def}}{=} \{a_1, \ldots, a_k\}$ to denote a k-letter alphabet, and write U_k and V_k instead of U_{Σ_k} and V_{Σ_k}.

3. State complexity of closures

For a regular language L recognized by an NFA A, one may obtain NFAs for the upward and downward closures of L by simply adding transitions to A, without increasing the number of states. More precisely, an NFA $A^\uparrow$ for $\uparrow L$ is obtained by adding to A self-loops $q \xrightarrow{a} q$ for every state q of A and every letter $a \in \Sigma$. Similarly, an NFA $A^\downarrow$ for $\downarrow L$ is obtained by adding to A epsilon transitions $p \xrightarrow{\epsilon} q$ for every transition $p \to q$ of A (on any letter).

3.1. Deterministic automata for closures

If now L is recognized by a DFA or an NFA A and we want a DFA for $\uparrow L$ or for $\downarrow L$, we can start with the NFA $A^\uparrow$ or $A^\downarrow$ defined above and transform it into a DFA using the powerset construction. This shows that if L has an n-state DFA, then both its upward and downward closures have DFAs with at most $2^n - 1$ states.

It is actually possible to provide tighter upper bounds by taking advantage of specific features of $A^\uparrow$ and $A^\downarrow$.

Proposition 3.1 (State complexity of upward closure) *1. If A is an n-state NFA then $n_D(\uparrow L(A)) \leqslant 2^{n-2} + 1$.*
2. Furthermore, for any $n > 1$ there exists a language L_n with $n_{\text{N\&D}}(L_n) = n$ and $n_D(\uparrow L_n) = 2^{n-2} + 1$.

PROOF. 1. Let $A = (\Sigma, Q, \delta, I, F)$ be an n-state NFA for $L = L(A)$. We assume that $I \cap F = \varnothing$ (and $I \neq \varnothing \neq F$) otherwise L contains ϵ (or is empty) and $\uparrow L$ is trivial.

Since $A^{\uparrow}$ has loops on all its states and for any letter, applying the powerset construction yields a DFA where $P \xrightarrow{a} P'$ implies $P \subseteq P'$, hence any state P reachable from I includes I. Furthermore, if P is accepting (i.e., $P \cap F \neq \varnothing$) and $P \xrightarrow{a} P'$, then P' is accepting too, hence all accepting states recognize exactly Σ^* and are equivalent. Then there can be at most $2^{|Q \smallsetminus (I \cup F)|}$ states in the powerset automaton that are both reachable and not accepting. To this we add 1 for the accepting states since they are all equivalent. Finally $n_{\text{D}}(\uparrow L) \leqslant 2^{n-2} + 1$ since $|I \cup F|$ is at least 2 as we observed.

2. To show that $2^{n-2} + 1$ states are sometimes necessary, we assume $n > 2$ and define $L_n \stackrel{\text{def}}{=} E_{n-2}$ where

$$E_k \stackrel{\text{def}}{=} \{a\,a \mid a \in \Sigma_k\} = \{a_1\,a_1, \ldots, a_k\,a_k\}\,. \tag{4}$$

In other words, L_n contains all words consisting of two identical letters from $\Sigma = \Sigma_{n-2}$. The minimal DFA for L_n has n states, see Fig. 1. Now $\uparrow L_n = \{x \in \Sigma^{\geqslant 2} \mid \exists j > i : x[i] = x[j]\} = \bigcup_{a \in \Sigma} \Sigma^* \cdot a \cdot \Sigma^* \cdot a \cdot \Sigma^*$, i.e., $\uparrow L_n$ has all words in Σ^* where *some letter reappears*, i.e., $\uparrow L_n$ is the complement of V_{n-2} from Lemma 2.2. A DFA for $\uparrow L_n$ has to record all letters previously read in

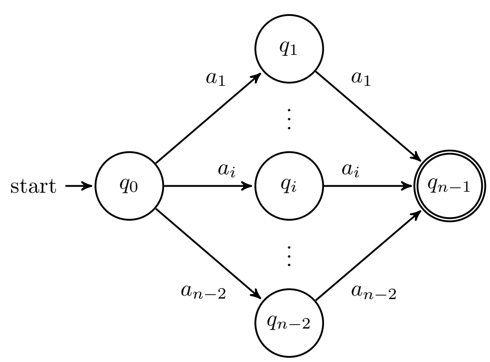


Figure 1: n-state DFA for $L_n = E_{n-2} = \{a_1\,a_1, a_2\,a_2, \ldots, a_{n-2}\,a_{n-2}\}$.

its non-accepting states, and has one accepting state: the minimal DFA has $2^{|\Sigma|} + 1 = 2^{n-2} + 1$ states. Note that we can infer $n_{\text{N}}(L_n) = n$ just from $n_{\text{D}}(\uparrow L_n) = 2^{n-2} + 1$ and the first part of the lemma.

When $n = 2$, taking $L_2 = \{a\}$ over a 1-letter alphabet witnesses both $n_{\text{D}}(L_n) = n = 2$ and $n_{\text{D}}(\uparrow L_n) = 2^{n-2} + 1 = 2$. Finally, the $2^{n-2} + 1$ bound is tight even for the upward closure of DFAs. □

Remark 3.2 *The above Proposition essentially reproduces Lemma 4.3 from [8] except that we do not assume that A is a DFA.*

Proposition 3.3 (State complexity of downward closure) *1. If A is an n-state NFA with only one initial state (in particular when A is a DFA) then $n_D(\downarrow L(A)) \leqslant 2^{n-1}$.*
2. Furthermore, for any $n > 1$ there exists a language L'_n with $n_D(L'_n) = n$ and $n_D(\downarrow L'_n) = 2^{n-1}$.

Proof. 1. We assume, w.l.o.g., that all states in $A = (\Sigma, Q, \delta, \{q_{\text{init}}\}, F)$ are reachable from the single initial state. From A one derives an NFA $A^{\downarrow}$ for $\downarrow L(A)$ by adding ϵ-transitions to A. With these ϵ-transitions, the language recognized from a state $q \in Q$ is a subset of the language recognized from q_{init}. Hence, in the powerset automaton obtained by determinizing $A^{\downarrow}$, all states $P \subseteq Q$ that contain q_{init} are equivalent and recognize exactly $\downarrow L(A)$. There also are 2^{n-1} states in 2^Q that do not contain q_{init}. Thus $2^{n-1} + 1$ bounds the number of non-equivalent states in the powerset automaton of $A^{\downarrow}$, and this includes a sink state (namely $\varnothing \in 2^Q$) that will be omitted in the canonical minimal DFA for $\downarrow L(A)$.

2. To show that 2^{n-1} states are sometimes necessary, we assume $n > 1$ and let $L'_n \stackrel{\text{def}}{=} D_{n-1}$ where

$$D_k \stackrel{\text{def}}{=} \{x \in \Sigma_k^+ \mid \forall i > 1 : x[i] \neq x[1]\} = \bigcup_{a \in \Sigma_k} a \cdot \left(\Sigma_k \smallsetminus \{a\}\right)^* . \tag{5}$$

In other words, L'_n has all words in Σ_{n-1}^+ where *the first letter does not reappear.*

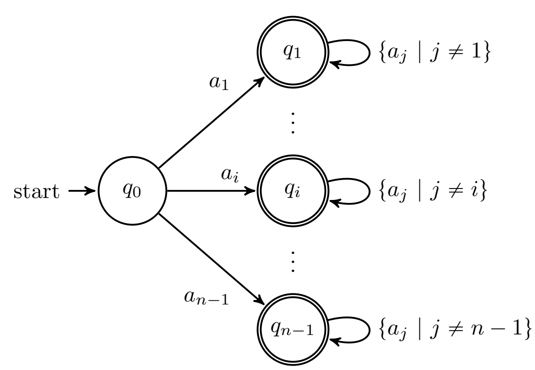


Figure 2: n-state DFA for $L'_n = D_{n-1} = \bigcup_{a \in \Sigma} a \cdot (\Sigma - \{a\})^*$ with $|\Sigma| = n - 1$.

The minimal DFA for L'_n has n states, see Fig. 2. Now $\downarrow L'_n = \{x \mid \exists a \in \Sigma_{n-1} : \forall i \geqslant 2 : x[i] \neq a\}$, i.e., $\downarrow L'_n$ has all words x such that the first suffix $x[2..]$ does not use all letters. Equivalently $x \in \downarrow L'_n$ iff $x \in L'_n$ or x does not use all letters, i.e., $\downarrow L'_n$ is the union of L'_n and the complement of U_{n-1} from Lemma 2.2. The minimal DFA for $\downarrow L'_n$ just reads a first letter and then records all letters encountered after the first, hence needs exactly $2^{|\Sigma|}$ states. Thus 2^{n-1} states may be required for a DFA recognizing the downward closure of an n-state DFA. □

Remark 3.4 *The condition of a single initial state in Prop. 3.3 cannot be lifted. In general $2^n - 1$ states may be required for a DFA recognizing the downward closure of an n-state NFA: the (downward-closed) language $\Sigma_n^* \smallsetminus U_n$ of all words that do not use all letters is recognized by an n-state NFA (see Fig. 3) but its minimal DFA has $2^n - 1$ states.*

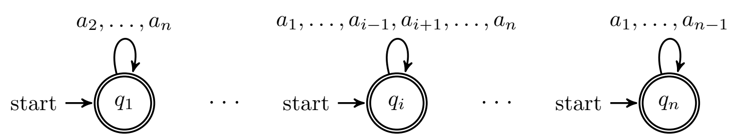


Figure 3: n-state NFA for $\Sigma_n^* \smallsetminus U_n$.

3.2. State complexity of closures for languages over small alphabets

The language families $(L_n)_{n\in\mathbb{N}}$ and $(L'_n)_{n\in\mathbb{N}}$ used to prove that the upper bounds given in Propositions 3.1 and 3.3 are tight use linear-sized alphabets.

It is indeed known that the size of the alphabets matter for the state complexity of closure operations. In fact the automata witnessing tightness in Figs. 1 and 2 use the smallest possible alphabets. For example, Okhotin showed that the $2^{n-2} + 1$ state complexity for $\uparrow L$ cannot be achieved with an alphabet of size smaller than $n - 2$, see [8, Lemma 4.4].

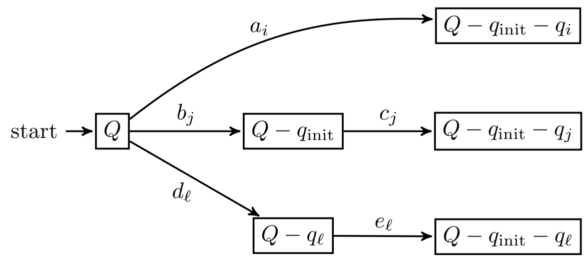
We now prove a similar result for downward closures:

Lemma 3.5 *For $n > 2$ let $A = (\Sigma, Q, \delta, \{q_{init}\}, F)$ be a n-state NFA with a single initial state. If $|\Sigma| < n - 1$ then $n_D(\downarrow L(A)) < 2^{n-1}$.*

Proof. We assume that $n_{\mathrm{D}}(\downarrow L(A)) = 2^{n-1}$ and deduce that $|\Sigma| \geqslant n - 1$.

We write $Q = \{q_{\text{init}}, q_1, \ldots, q_{n-1}\}$ to denote the states of A. As we saw with the proof of the first part of Proposition 3.3, the powerset automaton of $A^{\downarrow}$ can only have 2^{n-1} non-equivalent reachable states if all non-empty subsets of $Q \smallsetminus \{q_{\text{init}}\}$ (written $Q - q_{\text{init}}$ for short) are reachable. Since $A^{\downarrow}$ has ϵ-transitions doubling all transitions from A, it is possible to construct the powerset automaton of $A^{\downarrow}$ with Q as its initial state. Then all edges $P \xrightarrow{a} P'$ in the powerset automaton satisfy $P \supseteq P'$. As a consequence, if $P \xrightarrow{x} P'$ for some $x \in \Sigma^*$ then in particular one can pick x with $|x| \leqslant |P \smallsetminus P'|$.

Since every non-empty subset of $Q - q_{\text{init}}$ is reachable from Q there is in particular, for every $i = 1, \ldots, n-1$, some x_i of length 1 or 2 such that $Q \xrightarrow{x_i} Q - q_{\text{init}} - q_i$. If we pick x_i of minimal length then, for a given i, there are three possible cases (see picture): $x_i = a_i$ is a single letter (type 1), or x_i is



some $b_i\,c_i$ with $Q \xrightarrow{b_i} Q - q_{\text{init}} \xrightarrow{c_i} Q - q_{\text{init}} - q_i$ (type 2), or x_i is some $d_i\,e_i$ with $Q \xrightarrow{d_i} Q - q_i \xrightarrow{e_i} Q - q_{\text{init}} - q_i$.

We now claim that the a_i's for type-1 states, the c_i's for type-2 states and the d_i's for type-3 states are all distinct, hence $|\Sigma| \geqslant n-1$.

Clearly the a_i's and the d_i's are pairwise distinct since they take Q to different states in the deterministic powerset automaton. Similarly, the c_i's are pairwise distinct, taking $Q - q_{\text{init}}$ to different states.

Assume now that $a_i = c_j$ for a type-1 q_i and a type-2 q_j. Then $Q - q_{\text{init}} \xrightarrow{c_j} Q - q_{\text{init}} - q_j$ and $Q \xrightarrow{a_i(=c_j)} Q - q_{\text{init}} - q_i$ contradict the monotonicity of $P \mapsto \delta(P, x)$ in the powerset automaton. Similarly, assuming $d_\ell = c_j$ leads to $Q - q_{\text{init}} \xrightarrow{c_j} Q - q_{\text{init}} - q_j$ and $Q \xrightarrow{c_j(=d_\ell)} Q - q_\ell$, again contradicting monotonicity. Thus we can associate a distinct letter with each state $q_1, \ldots, q_{n-1}$, which concludes the proof. □

In view of the above results, the main question is whether, *in the case of a fixed alphabet*, exponential lower bounds still apply for the state complexity of upward and downward closures with DFAs as both input and output. The 1-letter case is degenerate since then both $n_{\text{D}}(\uparrow L)$ and $n_{\text{D}}(\downarrow L)$ are $\leqslant n_{\text{D}}(L)$. In the 3-letter case, exponential lower bounds for upward and downward closures were shown by Okhotin [8].

In the critical 2-letter case, say $\Sigma = \{a, b\}$, an exponential lower bound for upward closure was shown by Héam with the following witness: For $n > 0$, let $L''_n = \{a^i b\, a^{2j} b\, a^i \mid i + j + 1 = n\}$. Then $n_{\text{D}}(L''_n) = (n+1)^2$, while $n_{\text{D}}(\uparrow L''_n) \geqslant \frac{1}{7}(\frac{1+\sqrt{5}}{2})^n$ for $n \geqslant 4$ [9, Prop. 5.11]. Regarding downward closures for languages over a binary alphabet, the question was left open and we answer it in the next section.

4. Exponential state complexity of closures in the 2-letter case

In this section we show an exponential lower bound for the state complexity of downward closure in the case of a two-letter alphabet. Interestingly, the same languages can also serve as hard case for upward closure (but it gives weaker bounds than in [9]).

Theorem 4.1 (State complexity of closures with $|\Sigma| = 2$) *The state complexity of downward closure for languages over a binary alphabet is in $2^{\Omega(n^{1/3})}$. The same result holds for upward closure.*

We now prove the theorem. Fix a positive integer n. Let

$$H = \{n, n+1, \ldots, 2n\} ,$$

and define morphisms $c, d : H^* \to \{a, b\}^*$ with, for any $i \in H$:

$$c(i) \stackrel{\text{def}}{=} a^i\, b^{3n-i} , \qquad\qquad d(i) \stackrel{\text{def}}{=} c(i)\, c(i) .$$

Note that $c(i)$ always has length $3n$, begins with at least n a's, and ends with at least n b's. If we now let

$$L_n \stackrel{\text{def}}{=} \{c(i)^n \mid i \in H\} ,$$

L_n is a finite language of $n+1$ words, each of length $3n^2$ so that $n_D(L_n)$ is in $3n^3 + O(n^2)$. (In fact, $n_D(L_n) = 3n^3 + 1$.) In the rest of this section we show that both $n_D(\uparrow L_n)$ and $n_D(\downarrow L_n)$ are in $2^{\Omega(n)}$.

Lemma 4.2 *For $i, j \in H$, the longest prefix of $c(i)^\omega$ that embeds in $d(j) = c(j)\, c(j)$ is $c(i)$ if $i \neq j$ and $c(i)\, c(i)$ if $i = j$.*

Proof (Sketch). The case $i = j$ is clear. Fig. 4 displays the leftmost embedding of $c(i)^\omega$ in $d(j)$ in a case where $i > j$. The remaining case, $i < j$, is similar. □

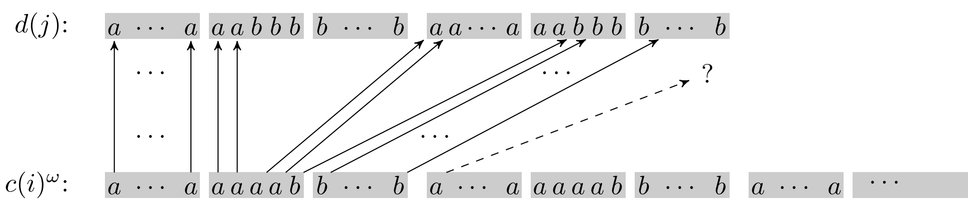


Figure 4: Case "$i > j$" in Lemma 4.2: here $i = n + 4$ and $j = n + 2$ for $n = 5$.

For each $i \in H$, let the morphisms $\eta_i, \theta_i : H^* \to (\mathbb{N}, +)$ be defined by

$$\eta_i(j) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } i \neq j , \\ 2 & \text{if } i = j , \end{cases} \qquad\qquad \theta_i(j) \stackrel{\text{def}}{=} \begin{cases} 2 & \text{if } i \neq j , \\ 1 & \text{if } i = j . \end{cases}$$

Thus for $\sigma = p_1\, p_2 \cdots p_s \in H^*$, $\eta_i(\sigma)$ is s plus the number of occurrences of i in σ, while $\theta_i(\sigma)$ is $2s$ minus the number of these occurrences of i.

Lemma 4.3 *Let $\sigma \in H^*$. The smallest ℓ such that $c(\sigma)$ embeds in $c(i)^\ell$ is $\theta_i(\sigma)$.*

PROOF. We write $\sigma = p_1\, p_2 \cdots p_s$ and prove the result by induction on s. The case where $s = 0$ is trivial. The case where $s = 1$ follows from Lemma 4.2, since for any p_1 and i, $c(p_1) \sqsubseteq c(i)$ iff $p_1 = i$, and $c(p_1) \sqsubseteq d(i) = c(i)^2$ always.

Assume now $s > 1$, write $\sigma = \sigma' p_s$ and let $\ell' = \theta_i(\sigma')$. By the induction hypothesis, $c(\sigma') \not\sqsubseteq c(i)^{\ell'-1}$ and $c(\sigma') \sqsubseteq c(i)^{\ell'} = c(i)^{\ell'-1} a^i b^{3n-i}$. Write now $c(i)^{\ell'} = w\, v$ where w is the shortest prefix of $c(i)^{\ell'}$ with $c(\sigma') \sqsubseteq w$. Since $c(\sigma')$ ends with some b that only embeds in the $a^i b^{3n-i}$ suffix of $c(i)^{\ell'}$, v is necessarily b^r for some r. So, for all $z \in \{a,b\}^*$, $c(p_s) \sqsubseteq z$ if and only if $c(p_s) \sqsubseteq v\, z$. We have $c(p_s) \sqsubseteq c(i)^{\theta_i(p_s)}$ and $c(p_s) \not\sqsubseteq v\, c(i)^{\theta_i(p_s)-1}$. Noting that $\sigma = \sigma' p_s$, we get $c(\sigma) \sqsubseteq c(i)^{\theta_i(\sigma)}$ and $c(\sigma) \not\sqsubseteq c(i)^{\theta_i(\sigma)-1}$. □

We now derive a lower bound on $n_{\mathrm{D}}(\downarrow L_n)$. For every subset X of H of size $n/2$ (assume n is even), let $w_X \in \{a,b\}^*$ be defined as follows: let the elements of X be $p_1 < p_2 < \cdots < p_{n/2}$ and let

$$w_X \stackrel{\text{def}}{=} c(p_1 p_2 \cdots p_{n/2}) \,.$$

Note that $\theta_i(p_1 p_2 \cdots p_{n/2}) = n$ if $i \notin X$ and $\theta_i(p_1 p_2 \cdots p_{n/2}) = n - 1$ if $i \in X$.

Lemma 4.4 *Let X and Y be subsets of H of size $n/2$ with $X \neq Y$. There exists a word $v \in \{a,b\}^*$ such that $w_X v \in \downarrow L_n$ and $w_Y v \notin \downarrow L_n$.*

PROOF. Let $i \in X \smallsetminus Y$. Let $v = c(i)$. Then

- By Lemma 4.3, $w_X \sqsubseteq c(i)^{n-1}$, and so $w_X v \sqsubseteq c(i)^n$. So $w_X v \in \downarrow L_n$.
- By Lemma 4.3, the smallest ℓ such that $w_Y v \sqsubseteq c(i)^\ell$ is $n+1$. Similarly, for $j \neq i$, the smallest ℓ such that $w_Y v \sqsubseteq c(j)^\ell$ is at least $n-1+2 = n+1$ (the w_Y contributes at least $n-1$ and the v contributes 2). So $w_Y v \notin \downarrow L_n$. □

This shows that for any DFA A recognizing $\downarrow L_n$, the state of A reached from the start state by every word in $\{w_X \mid X \subseteq H, |X| = n/2\}$ is distinct. Thus A has at least $\binom{n+1}{n/2}$ states, which is $\approx \frac{2^{n+3/2}}{\sqrt{\pi n}}$.

For $n_{\mathrm{D}}(\uparrow L_n)$, the reasoning is similar:

Lemma 4.5 *Let $\sigma \in H^*$. For all $i \in H$, the longest prefix of $c(i)^\omega$ that embeds in $d(\sigma)$ is $c(i)^{\eta_i(\sigma)}$.*

PROOF. By induction on the length of σ and applying Lemma 4.2. □

For every subset X of H of size $n/2$ (assume n is even), let $w'_X \in \{a,b\}^*$ be defined as follows: let the elements of X be $p_1 < p_2 < \cdots < p_{n/2}$ and let

$$w'_X \stackrel{\text{def}}{=} d(p_1 p_2 \cdots p_{n/2}) = c(p_1 p_1 p_2 p_2 \cdots p_{n/2} p_{n/2}) \,.$$

Lemma 4.6 *Let X and Y be subsets of H of size $n/2$ with $X \neq Y$. There exists a word $v \in \{a,b\}^*$ such that $w'_X v \in \uparrow L_n$ and $w'_Y v \notin \uparrow L_n$.*

PROOF. Let $i \in X \smallsetminus Y$. Let $v = c(i)^{n-(n/2+1)} = c(i)^{n/2-1}$.

- By Lemma 4.5, $c(i)^{n/2+1} \sqsubseteq w'_X$, thus $c(i)^n \sqsubseteq w'_X v$, hence $w'_X v \in \uparrow L_n$.
- By Lemma 4.5, the longest prefix of $c(i)^n$ that embeds in $w'_Y v$ is at most $c(i)^\ell$ where $\ell = n/2 + n/2 - 1 = n - 1$. The longest prefix of $c(j)^n$ that embeds in $w'_Y v$ for $j \neq i$ is at most $c(j)^\ell$ where
$$\ell = \frac{n}{2} + 1 + \left\lceil \frac{n/2 - 1}{2} \right\rceil \leqslant n - 1.$$
Therefore $c(j)^n \not\sqsubseteq w'_Y v$ when $j = i$ and also when $j \neq i$. Thus $w'_Y v \notin \uparrow L_n$.
□

With Lemma 4.6 we reason exactly as we did for $n_{\mathrm{D}}(\downarrow L_n)$ after Lemma 4.4 and conclude that $n_{\mathrm{D}}(\uparrow L_n) \geqslant \binom{n+1}{n/2}$ here too.

5. State complexity of interiors

Recall Eq. (1) that expresses interiors with closures and complements. Since complementation of DFAs does not increase the number of states, the state complexity of interiors, seen as DFA to DFA operations, is the same as the state complexity of closures (modulo swapping of up and down).

The remaining question is the *nondeterministic state complexity* of interiors, now seen as NFA to NFA operations. For this, Eq. (1) provides an obvious 2^{2^n} upper bound on the nondeterministic state complexity of both upward and downward interiors, simply by combining the powerset construction for complementation and the results of Section 3. Note that this procedure yields DFAs for the interiors while we are happy to accept NFAs it it improves the state complexity.

In the rest of this section, we prove that the nondeterministic state complexity of $\circlearrowleft L$ and $\circlearrowright L$ are in $2^{2^{\Theta(n)}}$.

5.1. Upper bounds for interiors and the approximation problem

A generic argument lets us improve slightly on the 2^{2^n} upper bound:

Proposition 5.1 *The (deterministic) state complexity of both the upward interior and the downward interior is $< \psi(n)$.*

Here $\psi(n)$ is the Dedekind number that counts the number of antichains in the lattice of subsets of an n-element set, ordered by inclusion. Kahn [20, Coro. 1.4] shows
$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \leqslant \log_2 \psi(n) \leqslant \left(1 + \frac{2\log(n+1)}{n}\right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Note that the $\psi(n)$ upper bound is still in $2^{2^{\Theta(n)}}$, or "doubly exponential".

We prove Prop. 5.1 in a uniform way for both interiors. For this we adapt a technique already present in [21, Theo. 6.1]: the state complexity of a language that is a positive Boolean combination of left-quotients of some regular L is $\leqslant \psi(n_{\mathrm{N}}(L))$.

Let K_0 and $K_1, \ldots, K_p$ be arbitrary languages in Σ^* (these need not be regular). With the K_i's we associate an alphabet $\Gamma = \{b_1, \ldots, b_p\}$ and a substitution σ given inductively by $\sigma(\epsilon) \stackrel{\text{def}}{=} K_0$ and $\sigma(w\, b_i) \stackrel{\text{def}}{=} \sigma(w) \cdot K_i$. With a language $L \subseteq \Sigma^*$, we associate the language $W \subseteq \Gamma^*$ defined by

$$W \stackrel{\text{def}}{=} \{x \in \Gamma^* \mid \sigma(x) \subseteq L\} .$$

Remark 5.2 *In the classical setting there is no K_0 and $\sigma(\epsilon) = \{\epsilon\}$, see [22, Chapter 6] and [23, Section 6]. There W is the best under-approximation of L by sums of products of K_i's and it is known that if L is regular then W is too. We allowed $\sigma(\epsilon) = K_0$ to account directly for upward interiors.*

Proposition 5.3 (State complexity of approximations) *If L is regular then W is regular. Furthermore $n_D(W) < \psi(n_N(L))$.*

PROOF. Assume $A_1 = (\Sigma, Q, \delta_1, I_1, F_1)$ is an n-state NFA for L. Using the powerset construction, one obtains a DFA $A_2 = (\Sigma, Q_2, \delta_2, i_2, F_2)$ for L. We have as usual $Q_2 = 2^Q$, with typical elements $S, S', \ldots$, δ_2 given by $\delta_2(S, a) = \bigcup_{q \in S} \delta_1(q, a)$, $i_2 = I_1$, and $F_2 = \{S \mid S \cap F_1 \neq \varnothing\}$.

We now use A_2 to get a DFA $A_3 = (\Gamma, Q_3, \delta_3, i_3, F_3)$ for W, where

- $Q_3 = 2^{Q_2}$, with typical elements $U, U', \ldots$;
- $\delta_3(U, b_j) = \{\delta_2(S, z) \mid S \in U, z \in K_j\}$;
- $i_3 = \{\delta_2(i_2, z) \mid z \in K_0\}$;
- $F_3 = 2^{F_2} = \{U \mid U \subseteq F_2\}$.

Claim 5.4 *For all words $w \in \Gamma^*$, $\delta_3(i_3, w) = \{\delta_2(i_2, z) \mid z \in \sigma(w)\}$.*

PROOF. By induction on w. For the base case, one has $\delta_3(i_3, \epsilon) = i_3 = \{\delta_2(i_2, z) \mid z \in K_0\}$ by definition, and $\sigma(\epsilon) = K_0$. For the inductive case, one has

$$\begin{aligned}
&\delta_3(i_3, w\, b_j) \\
&= \delta_3(\delta_3(i_3, w), b_j) \\
&= \delta_3(\{\delta_2(i_2, z) \mid z \in \sigma(w)\}, b_j) && \text{(ind. hypothesis)} \\
&= \big\{\delta_2(S, z') \mid S \in \{\delta_2(i_2, z) \mid z \in \sigma(w)\}, z' \in K_j\big\} && \text{(defn. of } \delta_3) \\
&= \{\delta_2(\delta_2(i_2, z), z') \mid z \in \sigma(w), z' \in \sigma(b_j)\} && \text{(rearrange, use } \sigma(b_j) = K_j) \\
&= \{\delta_2(i_2, z'') \mid z'' \in \sigma(w\, b_j)\} .
\end{aligned}$$

□

Corollary 5.5 *The language accepted by A_3 is W.*

Proof. For all $w \in \Gamma^*$, $w \in W$ iff $\sigma(w) \subseteq L$ iff $\forall z \in \sigma(w)[\delta_2(i_2, z) \in F_2]$ iff $\{\delta_2(i_2, z) \mid z \in \sigma(w)\} \subseteq F_2$ iff $\delta_3(i_3, w) \in F_3$ iff w is accepted by A_3. □

So far, we have a DFA A_3 for W, with $|Q_3| = 2^{2^n}$ states. We now examine the construction more closely to detect equivalent states. Observe that the powerset construction for A_2 in terms of A_1 is "existential", that is, a state of A_2 accepts if and only if at least one of its constituent states from A_1 accepts. In contrast, the powerset construction for A_3 in terms of A_2 is "universal", that is, a state of A_3 accepts if and only if all of its constituent states from A_2 accept. Suppose $B \subseteq C \in Q_2$ are two states of A_2. Then if some word is accepted by A_2 starting from B, it is also accepted starting from C. If a state of A_3 contains both B and C, then B already imposes a stronger constraint than C, and so C can be eliminated. We make this precise below:

Define an equivalence relation $\equiv$ on Q_3 as follows:

$$U \equiv U' \overset{\text{def}}{\Leftrightarrow} (\forall S \in U : \exists S' \in U' : S' \subseteq S) \wedge (\forall S' \in U' : \exists S \in U : S \subseteq S') .$$

A state $U \in Q_3$ is called an *antichain* if it does not contain some $S, S' \in Q_2$ with $S \subsetneq S'$. It is easy to see that every state $U \in Q_3$ is $\equiv$-equivalent to the antichain $U_{\min}$ obtained by retaining only the minimal-by-inclusion elements of U. Further, no two distinct antichain states can be $\equiv$-equivalent.

We now claim that, in A_3, $\equiv$-equivalent states accept the same language. First $U \equiv V$ and $U \in F_3$ imply $V \in F_3$ (proof: for any $S \in V$, there is a $S' \in U$ which is a subset, and since $S' \in F_2$, also $S \in F_2$). Furthermore, for each b_j, $\delta(U, b_j) \equiv \delta(V, b_j)$ (proof: an arbitrary element of $\delta_3(U, b_j)$ is $\delta_2(S, z)$ for some $S \in U$ and $z \in K_j$. There exists $S' \in V$ such that $S' \subseteq S$, and then $\delta_2(S', z)$ belongs to $\delta_3(V, b_j)$ and is a subset of $\delta_2(S, z)$ because δ_2 is monotone in its first argument. The reasoning in the reverse direction is similar).

Thus we can quotient the DFA A_3 by $\equiv$ to get an equivalent DFA for W. The number of states of $A_3/\equiv$ is exactly the number of subsets of 2^Q which are antichains, and this is the Dedekind number $\psi(n)$. Further, we can remove (the equivalence class of) the sink state $\{\varnothing\}$. □

We instantiate the above for the upward and downward interiors to conclude that the nondeterministic state complexity of both is $< \psi(n)$: Choose alphabets $\Sigma = \Gamma = \{b_1, \ldots, b_k\}$. For the upward interior, define $K_0 = \Sigma^*$ and $K_i = \Sigma^* b_i \Sigma^*$, and apply Prop. 5.3. For the downward interior, define $K_0 = \{\epsilon\}$ and $K_i = \{b_i, \epsilon\}$, and apply Prop. 5.3. This completes the proof of Prop. 5.1.

5.2. Lower bound for downward interiors

We first establish a doubly-exponential lower bound for downward interiors:

Proposition 5.6 *The nondeterministic state complexity of the downward interior is* $\geqslant 2^{2^{\lfloor \frac{n-3}{2} \rfloor}}$.

Let ℓ be a positive integer, and let $\Sigma = \{0, 1, 2, \ldots, 2^\ell - 1\}$, so that $|\Sigma| = 2^\ell$. Let

$$L \stackrel{\text{def}}{=} \Sigma^* \smallsetminus \{a\,a \mid a \in \Sigma\} = \{w \mid |w| \neq 2\} \cup \{a\,b \mid a, b \in \Sigma, a \neq b\}\,.$$

Then $\circlearrowleft L$ consists of all words where every letter is distinct (equivalently, no letter appears more than once), a language called V_Σ in Lemma 2.2, showing $n_{\mathrm{N}}(\circlearrowleft L) \geqslant 2^{|\Sigma|} = 2^{2^\ell}$.

Claim 5.7 $n_N(L) \leqslant 2\ell + 3$.

Proof. Two letters in Σ, viewed as ℓ-bit sequences, are distinct if and only if they differ in at least one bit. An NFA can check this by guessing the position in which they differ and checking that the letters indeed differ in this position. Fig. 5 shows an NFA for $\{a\,b \mid a \neq b\}$ with $2\ell + 2$ states.

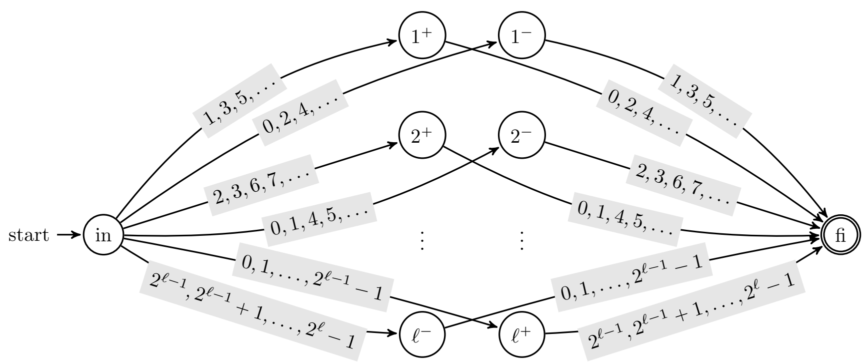


Figure 5: NFA for $\{a\,b \mid a, b \in \Sigma_{2^\ell}, a \neq b\}$ with $2\ell + 2$ states.

We need to modify this NFA to accept all words whose length is not 2. For this, we add a new state Z, and add the following transitions, on all letters: from each i^+ and i^- to Z, from Z to fi, and from fi to itself. We declare all states other than Z accepting. L is accepted by this resulting NFA, with $2\ell + 3$ states. □

Finally, combining $n_{\mathrm{N}}(L) \leqslant 2\ell + 3$ with the previously observed $n_{\mathrm{N}}(\circlearrowleft L) \geqslant 2^{2^\ell}$ concludes the proof of Prop. 5.6.

5.3. Lower bound for upward interiors

We now establish the following doubly-exponential lower bound:

Proposition 5.8 *The nondeterministic state complexity of the upward interior is* $\geqslant 2^{2^{\lfloor \frac{n-4}{3} \rfloor}} + 1$.

Our parameter is $\ell \in \mathbb{N}$ and we let $\Gamma \stackrel{\text{def}}{=} \{0, 1, \ldots, 2^\ell - 1\}$, $\Upsilon \stackrel{\text{def}}{=} \{1, \ldots, \ell\}$ and $\Sigma \stackrel{\text{def}}{=} \Gamma \cup \Upsilon$. The symbols in Γ, denoted $x, y, \ldots$ are disjoint from the symbols in Υ, denoted $k, k', \ldots$ (e.g., we can imagine that they have different colors) and one has $|\Sigma| = 2^\ell + \ell$.

For $x, y \in \Gamma$ and $k \in \Upsilon$, we write $x =_k y$ when x and y, viewed as ℓ-bit sequences, have the same kth bit. We consider the following languages:

$$L_1 \stackrel{\text{def}}{=} \{x\, w\, y\, k\, w' \in \Gamma \cdot \Sigma^* \cdot \Gamma \cdot \Upsilon \cdot \Sigma^* \mid x =_k y\}\,,$$
$$L_2 \stackrel{\text{def}}{=} \Gamma \cdot (\Gamma \cdot \Upsilon)^*\,,$$
$$L \stackrel{\text{def}}{=} L_1 \cup (\Sigma^* \smallsetminus L_2)\,.$$

L_1 contains all words such that the initial letter $x \in \Gamma$ has one common bit with a later $y \in \Gamma$ and this bit is indicated by the $k \in \Upsilon$ that immediately follows the occurrence of y. Fig. 6 displays an NFA for L_1: it reads the first letter x, nondeterministically guesses k, and switches to a state r_k^+ or r_k^- depending on what is x's kth bit. From there it waits nondeterministically for the appearance of a factor $y\,k$ with $x =_k y$ before accepting. This uses $3\ell + 2$ states. Combining

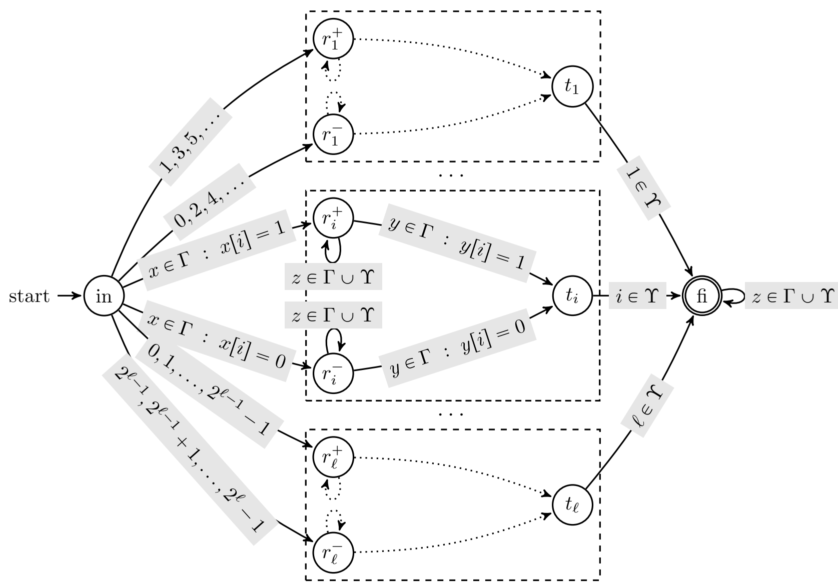


Figure 6: NFA for L_1 with $3\ell + 2$ states.

with an NFA for $\Sigma^* \smallsetminus L_2$, we see that $n_{\mathrm{N}}(L) \leqslant 3\ell + 4$.

We consider the upward interior of L. As in Lemma 2.2, let $U_\Gamma \subseteq \Gamma^*$ be the language that contains all words over Γ where every letter appears at least once and $U'_\Gamma = \Gamma \cdot U_\Gamma$ be the language that has all words where the first suffix $w[2..]$ is in U_Γ.

Claim 5.9 $(\circlearrowleft L) \cap \Gamma^* = U'_\Gamma$.

Proof. We first show $(\circlearrowleft L) \cap \Gamma^* \subseteq U'_\Gamma$, by showing the contrapositive. Let $w \in \Gamma^* \smallsetminus U'_\Gamma$. If $w = \epsilon$, then clearly $w \notin \circlearrowleft L$. Otherwise, $w = z\, z_1 \cdots z_p$, where $z, z_i \in \Gamma$. Since $z_1 \cdots z_p$ is not in U_Γ, there is some $x \in \Gamma$ that differs from all the z_i's. Pick $k_1, \ldots, k_p$ witnessing this, i.e., such that $x \neq_{k_i} z_i$ for all i. If $x = z$ we let $w' \stackrel{\text{def}}{=} z\, z_1\, k_1 \cdots z_p\, k_p$ so that $w' \in L_2$ and $w' \notin L_1$, i.e., $w' \notin L$. If $x \neq z$ we let $w' \stackrel{\text{def}}{=} x\, z\, k\, z_1\, k_1 \cdots z_p\, k_p \notin L$ for some k witnessing $x \neq z$, so that $w' \notin L$. In both cases $w \sqsubseteq w' \notin L$ and we deduce $w \notin \circlearrowleft L$.

We now show $U'_\Gamma \subseteq (\circlearrowleft L) \cap \Gamma^*$. Let $w = z\, z_1 \cdots z_p \in U'_\Gamma$. We show that $w \in \circlearrowleft L$ by showing that $w' \in L$ for every w' such that $w \sqsubseteq w'$. If $w' \notin L_2$, then $w' \in L$. So assume $w' = x\, y_1\, k_1 \cdots y_n\, k_n \in L_2$. There is some i such that $x = z_i$ (since $w \in U'_\Gamma$) and some j such that $z_i = y_j$ (since $w \sqsubseteq w'$). We then have $x =_{k_j} y_j$ (this does not depend on the actual value of k_j). Hence $w' \in L_1 \subseteq L$. Thus $w \in \circlearrowleft L$. □

Corollary 5.10 $n_N(\circlearrowleft L) \geqslant 2^{2^\ell} + 1$.

Proof. From Lemma 2.2 we know that $n_{\mathrm{N}}(U'_\Gamma) = 2^{2^\ell} + 1$ and it is easily observed that $n_{\mathrm{N}}(\circlearrowleft L \cap \Gamma^*) \leqslant n_{\mathrm{N}}(\circlearrowleft L)$. □

Finally, combining $n_{\mathrm{N}}(\circlearrowleft L) \geqslant 2^{2^\ell} + 1$ with the previously observed $n_{\mathrm{N}}(L) \leqslant 3\ell + 4$ concludes the proof of Prop. 5.8.

5.4. On interiors of languages over a fixed alphabet

The doubly-exponential lower bounds exhibited in Props. 5.6 and 5.8 rely on alphabets of exponential size. It is an open question whether, in the case of a fixed alphabet, the nondeterministic state complexity of downward and/or upward interior is still doubly-exponential.

At some point we considered the language

$$P_n = \Sigma^* \smallsetminus \{w \# w \mid |w| = n \text{ and } w \text{ has no } \#\}$$

over an alphabet of the form $\Sigma = \{\#\} \cup \{a_1, \ldots, a_k\}$ for some fixed k. The point is that P_n avoid words that are "squares" $w \# w$ of words w of length n, separated by the special symbol $\#$. As a consequence, an arbitrary $u \# v$ in $\Sigma_k^* \# \Sigma_k^*$ is in $\circlearrowright P_n$ iff $\downarrow_{=n} u$ and $\downarrow_{=n} v$ are disjoint, where $\downarrow_{=n} x$ denotes the set $\Sigma^n \cap \downarrow x$ of subwords of x having length exactly n.

Let us write $u \sim_n v$ when $\downarrow_{=n} u = \downarrow_{=n} v$ and $\mathcal{C}_k(n)$ for the number of $\sim_n$-equivalence classes in Σ_k^*: obviously $\mathcal{C}_k(n) \leqslant 2^{k^n}$.

Claim 5.11 $n_N(P_n) \leqslant (k+1)n + k + 3$ *and* $n_D(\circlearrowright P_n) \geqslant \mathcal{C}_k(n)$.

Proof (Sketch). One can recognize all words w with $w[i] \neq w[i+n+1]$ for some position i using a NFA with $k(n+1)+2$ states. With $n+1$ extra states, the NFA also recognizes the words that are not of the form $\Sigma_k^n \# \Sigma_k^n$.

For $n_{\mathrm{D}}(\circlearrowright P_n) \geqslant \mathcal{C}_k(n)$, we claim that if $\mathcal{C}(u) \neq \mathcal{C}(v)$ then $\delta(q_{\mathrm{init}}, u) \neq \delta(q_{\mathrm{init}}, v)$ in any DFA for $\circlearrowright P_n$. Indeed, pick some x in $\mathcal{C}(u) \smallsetminus \mathcal{C}(v)$ (interchanging u and v if necessary) and note that $u \# x \notin \circlearrowright P_n \ni v \# x$. □

Thus if $\mathcal{C}_k(n)$ is doubly-exponential in n (for some k), P_n witnesses a doubly-exponential lower bound for downward interiors over a fixed alphabet, at least if we accept an NFA as input and DFA as output, which would still be an improvement over existing results.

Estimating $\mathcal{C}_k(n)$ was an open problem raised by Sakarovitch and Simon more than thirty years ago in [24, p. 110] and no doubly-exponential lower bound was known. The connection with the state complexity of $\wp P_n$ spurred two of us on to look at this question, and eventually solve it by showing that $\mathcal{C}_k(n)$ is in $2^{O(n^k)}$ [25], hence not doubly exponential as hoped.

At the moment we can only demonstrate a $2^{2^{\Omega(\sqrt{n})}}$ lower bound for the nondeterministic state complexity of *restricted interiors* over a 3-letter alphabet: this relies on a notion of "restricted" subwords where the alphabet is partitioned in two set: letters than can be omitted (as usual) when building subwords, and letters that must be retained, see [26, Theo. 4.3] for details.

6. Complexity of decision problems on subwords

In automata-based procedures for logic and verification, the state complexity of automata constructions is not always the best measure of computational complexity. In this section we gather some elementary results on the complexity of subword-related decision problems on automata: deciding whether the languages they describe are downward (or upward) closed, and deciding whether they describe the same language modulo downward (or upward) closure. This is in the spirit of the work done in [27, 28] for closures by prefixes, suffixes, and factors. Some of the results we give are already known but they remain scattered in the literature.

6.1. Deciding closedness

Deciding whether $L(A)$ is upward-closed, or downward-closed, is unsurprisingly PSPACE-complete for NFAs, and NL-complete for DFAs. (For upward-closedness, this is already shown in [9], and quadratic-time algorithms that decide upward-closedness of $L(A)$ for a DFA A already appear in [16, 29].)

Proposition 6.1 *Deciding whether $L(A)$ is upward-closed or downward-closed is* PSPACE*-complete when A is an NFA, even in the 2-letter alphabet case.*

Proof (Sketch). A PSPACE algorithm simply tests for inclusion between two automata, A and $A^{\uparrow}$ (or $A^{\downarrow}$). PSPACE-hardness can be shown by adapting the proof for hardness of universality. Let R be a length-preserving semi-Thue system and x, x' two strings of same length. It is PSPACE-hard to say whether $x \xrightarrow{*}_R x'$, even for a fixed R over a 2-letter alphabet Σ. We reduce (the negation of) this question to our problem.

Fix x and x' of length $n > 1$: a word $x_1\, x_2 \cdots x_m$ of length $n \times m$ encodes a derivation if $x_1 = x$, $x_m = x'$, and $x_i \to_R x_{i+1}$ for all $i = 1, \ldots, m-1$. The language L of words that do *not* encode a derivation from x to x' is regular and recognized by an NFA with $O(n)$ states. Now, there is a derivation $x \xrightarrow{*}_R x'$ iff $L \neq \Sigma^*$. Since L contains all words of length not divisible by $n > 1$, it is upward-closed, or downward-closed, iff $L = \Sigma^*$, iff $\neg(x \xrightarrow{*}_R x')$. □

Proposition 6.2 *Deciding whether $L(A)$ is upward-closed or downward-closed is* NL*-complete when A is a DFA, even in the 2-letter alphabet case.*

Proof. Since L is downward-closed if, and only if, $\Sigma^* \setminus L$ is upward-closed, and since one easily builds a DFA for the complement of $L(A)$, it is sufficient to prove the result for upward-closedness.

We rely on the following easy lemma: L is upward-closed iff for all $u, v \in \Sigma^*$, $u\,v \in L$ implies $u\,a\,v \in L$ for all $a \in \Sigma$. Therefore, $L(A)$ is not upward-closed —for $A = (\Sigma, Q, \delta, \{q_{\text{init}}\}, F)$— iff there are states $p, q \in Q$, a letter a, and words u, v such that $\delta(q_{\text{init}}, u) = p$, $\delta(p, a) = q$, $\delta(p, v) \in F$ and $\delta(q, v) \notin F$. If such words exist, in particular one can take u and v of length $< n = |Q|$ and respectively $< n^2$. Hence testing (the negation of) upward-closedness can be done in nondeterministic logarithmic space by guessing u, a, and v within the above length bounds, finding p and q by running u and then a from q_{init}, then running v from both p and q.

For hardness, one may reduce from vacuity of DFAs, a well-known NL-hard problem that is essentially equivalent to GAP, the Graph Accessibility Problem. Note that for any DFA A (in fact any NFA) the following holds:

$$L(A) = \varnothing \quad \text{iff} \quad L(A) \cap \Sigma^{<n} = \varnothing \quad \text{iff} \quad L(A) \cap \Sigma^{<n} \text{ is upward-closed,}$$

where n is the number of states of A. This provides the required reduction since, given a DFA A, one easily builds a DFA for $L(A) \cap \Sigma^{<n}$. □

6.2. Deciding equivalence modulo closure

The question whether $\downarrow L(A) = \downarrow L(B)$ or, similarly, whether $\uparrow L(A) = \uparrow L(B)$, is relevant in some settings where closures are used to build regular overapproximations of more complex languages.

Bachmeier *et al.* recently showed that the above two questions are coNP-complete when A and B are NFAs [12, Section 5], hence "easier" than deciding whether $L(A) = L(B)$. Here we give an improved version of their result.

Proposition 6.3 (after [12]) *1. Deciding whether $\downarrow L(A) \subseteq \downarrow L(B)$ or whether $\uparrow L(A) \subseteq \uparrow L(B)$ is* coNP*-complete when A and B are NFAs.*

2. Deciding $\downarrow L(A) = \downarrow L(B)$ or $\uparrow L(A) = \uparrow L(B)$ is coNP*-hard even when A and B are DFAs over a two-letter alphabet.*

3. These problems are NL*-complete when restricting to NFAs over a 1-letter alphabet.*

Proof. 1. Let $B = (\Sigma, Q, \delta, I, F)$ and $n_B = |Q|$. Assume that $\downarrow L(A) \not\subseteq \downarrow L(B)$ and pick a shortest witness $x = x_1 \cdots x_\ell \in \Sigma^*$ with $x \in \downarrow L(A)$ and $x \notin \downarrow L(B)$. We claim that $|x| < n_B$: indeed in the powerset automaton obtained by determinizing $B^{\downarrow}$, the (unique) run $Q = S_0 \xrightarrow{x_1} S_1 \xrightarrow{x_2} \ldots \xrightarrow{x_\ell} S_\ell$ of x is such that

$S_0 \supseteq S_1 \supseteq S_2 \cdots \supseteq S_\ell$ (recall the proof of Lemma 3.5). If $S_{i-1} = S_i$ for some i, a shorter witness is obtained by omitting the ith letter in x (this does not affect membership in $\downarrow L(A)$ since this language is downward-closed). One concludes that the S_i have strictly diminishing size, hence $\ell < n_B$. This provides an NP algorithm deciding $\downarrow L(A) \nsubseteq \downarrow L(B)$: guess x in $\Sigma^{<n_B}$ and check in polynomial time that it is accepted by $A^\downarrow$ and not by $B^\downarrow$.

For upward closure the reasoning is even simpler and now a shortest witness has length $|x| < n_A$: if x is longer, we can find a subword x' that is still in $A^\uparrow$ (e.g., with pumping lemma), and this x' is not in $\uparrow L(B)$ since x is not.

2. coNP-hardness is shown by reduction from validity of DNF-formulae. Consider an arbitrary DNF formula $\phi = C_1 \vee C_2 \vee \ldots \vee C_m$ made of m disjunctive clauses and using k Boolean variables $v_1, \ldots, v_k$, e.g., $\phi = (v_1 \wedge \neg v_2 \wedge v_4) \vee (v_2 \wedge \cdots) \cdots$: it is easy to list all the valuations (seen as words in $\{0,1\}^k$) that make ϕ hold true. In order to recognize them with a DFA A_ϕ, we prefix each

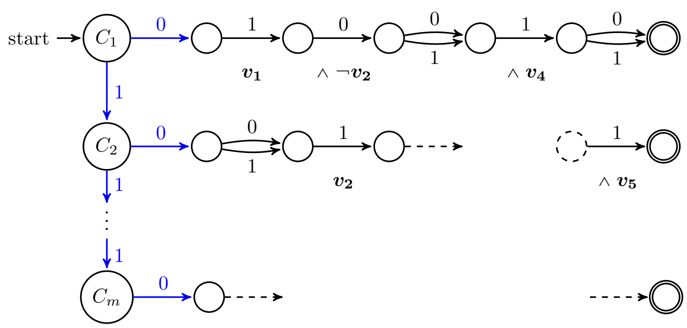


Figure 7: DFA A_ϕ for $\phi = (v_1 \wedge \neg v_2 \wedge v_4) \vee (v_2 \wedge \cdots \wedge v_5) \vee \cdots \vee C_m$ with $k = 5$ variables.

valuation by a string $1^\ell 0$ where ℓ witnesses the index of a clause $C_{\ell+1}$ made true by the valuation: see Fig. 7 where the prefix $1^\ell 0$ uses blue letters while the valuation uses black letters.

Now A_ϕ has $m(k+2)$ states and accepts all words $1^\ell 0\, x_1 \cdots x_k$ in $\{0,1\}^*$ such that $x_1 \cdots x_k$ is (the code for) a valuation that makes $C_{\ell+1}$ true. Let now B_ϕ be a DFA for $L(A_\phi) \cup 1^m 0(0+1)^k$, i.e., where all valuations are allowed after the $1^m 0$ prefix. It is clear that $\uparrow L(A_\phi) = \uparrow L(B_\phi)$ iff all words $1^m 0\, x_1 \cdots x_k$ appear in $\uparrow L(A)$ iff all valuations make ϕ true iff ϕ is valid, which completes the reduction for equality of upward closures.

For downward closures, we modify A_ϕ by adding a transition $C_m \xrightarrow{1} C_1$ so that now A_ϕ accepts all words $1^\ell 0\, x_1 \cdots x_k$ such that $x_1 \cdots x_k$ makes $C_{(\ell+1)\%m}$ true. For B we now take a DFA for $1^*0(0+1)^k$ and see that $\downarrow L(A_\phi) = \downarrow L(B)$ iff all valuations make ϕ true.

3. In the 1-letter case, comparing upward or downward closures amounts to comparing the length of the shortest (resp., longest) word accepted by the automata, which is easily done in nondeterministic logspace. And since $\uparrow L(A) =$

$\downarrow L(A) = \varnothing$ iff $L(A) = \varnothing$, NL-hardness is shown by reduction from emptiness of NFAs, i.e., a question "is there a path from I to F" that is just another version of GAP, the Graph Accessibility Problem. □

A special case of language comparison is universality. The question whether $\uparrow L(A) = \Sigma^*$ is trivial since it amounts to asking whether ϵ is accepted by A. For downward closures one has the following:

Proposition 6.4 (after [28]) *Deciding whether $\downarrow L(A) = \Sigma^*$ when A is a NFA over Σ is NL-complete.*

PROOF. Rampersad *et al.* show that the problem can be solved in linear time [28, Section 4.4]. Actually the characterization they use, namely $\downarrow L(A) = \Sigma^*$ iff $A = (\Sigma, Q, \delta, I, F)$ has a state $q \in Q$ with $I \xrightarrow{*} q \xrightarrow{*} F$ and such that for any $a \in \Sigma$ there is a path of the form $q \xrightarrow{*}\xrightarrow{a}\xrightarrow{*} q$ from q to itself, is a $\mathsf{FO}+\mathsf{TC}$ sentence on A seen as a labeled graph, hence can be checked in NL [30]. NL-hardness can be shown by reduction from emptiness of NFAs, e.g., by adding loops $p \xrightarrow{a} p$ on any accepting state $p \in F$ and for every $a \in \Sigma$. □

7. Concluding remarks

For words ordered by the (scattered) subword relation, we considered the state complexity of computing closures and interiors, both upward and downward, of regular languages given by finite-state automata. These operations are essential when reasoning with subwords, e.g., in symbolic model checking for lossy channel systems, see [10, Section 6]. We completed the known results on closures by providing exact state complexities in the case of unbounded alphabets, and by demonstrating an exponential lower bound on downward closures even in the case of a two-letter alphabet.

The nondeterministic state complexity of interiors is a new problem that we introduced in this paper and for which we could show doubly-exponential upper and lower bounds.

These results contribute to a more general research agenda: what are the right data structures and algorithms for reasoning with subwords and superwords? The algorithmics of subwords and superwords has mainly been developed in string matching and combinatorics [4, 31] but other applications exist that require handling sets of strings rather than individual strings, e.g., model-checking and constraint solving [32]. When reasoning about sets of strings, there are many different ways of representing closed sets and automata-based representation are not always the preferred option, see, e.g., the SREs used for downward-closed languages in [2]. The existing trade-offs between all the available options are not yet well understood and certainly deserve scrutiny. In this direction, let us mention [7, Theo. 2.1(3)] showing that if $n_{\mathrm{D}}(L) = n$ then $\min(L) \stackrel{\text{def}}{=} \{x \in L \mid \forall y \in L : y \sqsubseteq x \implies y = x\} = L \smallsetminus (L \sqcup \Sigma)$ may have $n_{\mathrm{N}}(\min(L)) = (n-2)2^{n-3} + 2$, which suggests that it is more efficient to represent $\uparrow L$ directly than by its minimal elements.

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