FIXED-DIMENSIONAL ENERGY GAMES ARE IN PSEUDO-POLYNOMIAL TIME

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Abstract. We generalise the hyperplane separation technique (Chatterjee and Velner, 2013) from multi-dimensional mean-payoff to energy games, and achieve an algorithm for solving the latter whose running time is exponential only in the dimension, but not in the number of vertices of the game graph. This answers an open question whether energy games with arbitrary initial credit can be solved in pseudo-polynomial time for fixed dimensions 3 or larger (Chaloupka, 2013). It also improves the complexity of solving multi-dimensional energy games with given initial credit from non-elementary (Brázdil, Jančar, and Kučera, 2010) to 2EXPTIME, thus establishing their 2EXPTIME-completeness.

Key Words. Energy game, bounding game, first-cycle game, vector addition system with states

1. Introduction

Multi-Dimensional Energy Games are played turn-by-turn by two players on a finite *multi-weighted* game graph, whose edges are labelled with integer vectors modelling discrete energy consumption and refuelling. Player 1's objective is to keep the accumulated energy non-negative in every component along infinite plays. This setting is relevant to the synthesis of resource-sensitive controllers balancing the usage of various resources like fuel, time, money, or items in stock, and finding optimal trade-offs; see [3, 11, 4, 13] for some examples. Maybe more importantly, energy games are the key ingredient in the study of several related resource-conscious games, notably multi-dimensional mean-payoff games [7] and games played on vector addition systems with states (VASS) [3, 1, 9].

The main open question about these games is to pinpoint the complexity of deciding whether Player 1 has a winning strategy when starting from a particular vertex and given an initial energy vector as part of the input. This particular *given initial credit* variant of energy games is also known as *Z-reachability* VASS games [3, 5]. The problem is also equivalent via logarithmic-space reductions to deciding *single-sided* VASS games with a non-termination objective [1], and to deciding whether a given VASS (or, equivalently, a Petri net) simulates a given finite state system [12, 14, 9, 2]. As shown by Brázdil, Jančar, and Kučera [3], all these problems can be solved in $(d-1)$EXPTIME where $d \geq 2$ is the number of energy components, i.e. a TOWER of exponentials when d is part of the input. The best known lower bound for this problem is 2EXPTIME-hardness [9], leaving a substantial complexity gap. So far, the only tight complexity bounds are for the case

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$d = 2$: Chaloupka [5] shows the problem to be P-complete when using unit updates, i.e. when the energy levels can only vary by -1, 0, or 1. However, quoting Chaloupka, 'since the presented results about 2-dimensional VASS are relatively complicated, we suspect this [general] problem is difficult.'

When inspecting the upper bound proof of Brázdil et al. [3], it turns out that the main obstacle to closing the gap and proving 2EXPTIME-completeness lies in the complexity upper bounds for energy games with an *arbitrary initial credit*—which is actually the variant commonly assumed when talking about energy games. Given a multi-weighted game graph and an initial vertex v, we now wish to decide whether there exists an initial energy vector $\mathbf{b}$ such that Player 1 has a winning strategy starting from the pair $(v, \mathbf{b})$. As shown by Chatterjee, Doyen, Henzinger, and Raskin [7], this variant is simpler: it is coNP-complete. However, the parameterised complexity bounds in the literature [3, 8] for this simpler problem involve an exponential dependency on the number $|V|$ of vertices in the input game graph, which translates into a tower of exponentials when solving the given initial credit variant.

Contributions. We show in this paper that the arbitrary initial credit problem for d-dimensional energy games can be solved in time $(|V| \cdot \|E\|)^{O(d^4)}$ where $|V|$ is the number of vertices of the input multi-weighted game graph and $\|E\|$ the maximal value that labels its edges, i.e. in pseudo-polynomial time (see Theorem 3.3). We then deduce that the given initial credit problem for general multi-dimensional energy games is 2EXPTIME-complete, and also in pseudo-polynomial time when the dimension is fixed (see Theorem 3.4), thus closing the gap left open in [3, 9]. Our parameterised bounds are of practical interest because typical instances of energy games would have small dimension but might have a large number of vertices.

By the results of Chatterjee et al. [7], another consequence is that we can decide the existence of a *finite-memory* winning strategy for fixed-dimensional *mean-payoff* games in pseudo-polynomial time. The existence of a finite-memory winning strategy is the most relevant problem for controller synthesis, but until now, solving fixed-dimensional mean-payoff games in pseudo-polynomial time required infinite memory strategies [6].

Overview. We prove our upper bounds on the complexity of the arbitrary initial credit problem for d-dimensional energy games by reducing them to *bounding games*, where Player 1 additionally seeks to prevent arbitrarily high energy levels (Section 2.3). We further show these games to be equivalent to *first cycle bounding games* in Section 6, where the total effect of the first simple cycle defined by the two players determines the winner. More precisely, first-cycle bounding games rely on a hierarchically-defined colouring of the game graph by *perfect half-spaces* (see Section 5), and the two players strive respectively to avoid or produce cycles in those perfect half-spaces.

First-cycle bounding games coloured with perfect half-spaces can be seen as generalising quite significantly both

- the 'local strategy' approach of Chaloupka [5] for 2-dimensional energy games, and

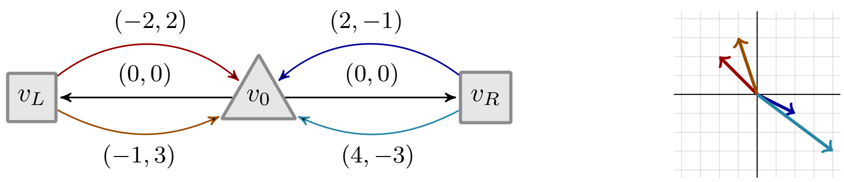

FIGURE 1. A 2-dimensional multi-weighted game graph.

- the 'separating hyperplane technique' of Chatterjee and Velner [6] for multi-dimensional mean-payoff games; see Section 4 for an overview of the latter approach.

The reduction to first cycle bounding games has several important corollaries: the *determinacy* of bounding games, and the existence of a *small hypercube property*, which in turn allow to derive the announced complexity bounds on energy games (see Section 3). In fact, we found with first-cycle bounding games a highly versatile tool, which we use extensively in our proofs on energy games.

We start by presenting the necessary background on energy and bounding games in Section 2. Some omitted material on linear algebra can be found in Appendix A.

2. Multi-Weighted Games

We define in this section the various games we consider in this work. We start by defining multi-weighted game graphs, which provide a finite representation for the infinite arenas over which our games are played. We then define energy games in Section 2.2, and their generalisation as bounding games in Section 2.3.

2.1. Multi-Weighted Game Graphs.

We consider game graphs whose edges are labelled by vectors of integers. They are tuples of the form (V, E, d), where d is the dimension in $\mathbb{N}$, $V \stackrel{\text{def}}{=} V_1 \uplus V_2$ is a finite set of vertices, which is partitioned into Player 1 vertices (V_1) and Player 2 vertices (V_2), and E is a finite set of edges included in $V \times \mathbb{Z}^d \times V$, and such that every vertex has at least one outgoing edge; we call the edgelabels in $\mathbb{Z}^d$ 'weights'.

Example 2.1. Figure 1 shows on its left an example of a 2-dimensional multi-weighted game graph. Throughout this paper, Player 1 vertices are depicted as triangles and Player 2 vertices as squares. □

2.1.1. *Norms.* For a vector $\mathbf{a}$, we denote the maximum absolute value of its entries by $\|\mathbf{a}\| \stackrel{\text{def}}{=} \max_{1\leq i\leq d} |\mathbf{a}(i)|$, and we call it the *norm* of $\mathbf{a}$. By extension, for a set of edges E, we let $\|E\| \stackrel{\text{def}}{=} \max_{(v,\mathbf{u},v')\in E} \|\mathbf{u}\|$. We assume, without loss of generality, that $\|E\| > 0$ in our multi-weighted game graphs. Regarding complexity, we encode vectors of integers in binary, hence $\|E\|$ may be exponential in the size of the multi-weighted game graph.

2.1.2. *Paths and Cycles.* Given a multi-weighted game graph (V, E, d), a *configuration* is a pair $(v, \mathbf{a})$ with v in V and $\mathbf{a}$ in $\mathbb{Z}^d$. A *path* is a finite sequence of configurations $\pi = (v_0, \mathbf{a}_0)(v_1, \mathbf{a}_1)\cdots(v_n, \mathbf{a}_n)$ in $(V \times \mathbb{Z}^d)^*$ such that for every $0 \leq j < n$ there exists an edge $(v_j, \mathbf{a}_{j+1} - \mathbf{a}_j, v_{j+1})$ in E (where addition is performed componentwise). The *total weight* of such a path π is $w(\pi) \stackrel{\text{def}}{=} \sum_{0 \leq j < n} \mathbf{a}_{j+1} - \mathbf{a}_j = \mathbf{a}_n - \mathbf{a}_0$.

A *cycle* is a path $(v_0, \mathbf{a}_0)(v_1, \mathbf{a}_1)\cdots(v_n, \mathbf{a}_n)$ with $v_0 = v_n$. Such a cycle is *simple* if whenever $v_j = v_{j'}$ for some $0 \leq j < j' \leq n$, then $j = 0$ and $j' = n$. We assume, without loss of generality, that every cycle contains at least one Player 1 vertex. We often identify simple cycles with their respective weights; the weights of the four simple cycles of the game graph in Figure 1 are displayed on its right.

Proposition 2.2. *In any game graph (V, E, d), the total weight of any simple cycle has norm at most $|V| \cdot \|E\|$.*

2.1.3. *Strategies and Plays.* Let v_0 be a vertex from V. A *play from v_0* is an infinite configuration sequence $\rho = (v_0, \mathbf{a}_0)(v_1, \mathbf{a}_1)\cdots$ such that $\mathbf{a}_0 = \mathbf{0}$ is the null vector and every finite prefix $\rho|_n \stackrel{\text{def}}{=} (v_0, \mathbf{a}_0)\cdots(v_n, \mathbf{a}_n)$ is a path. Note that, because $\mathbf{a}_0 = \mathbf{0}$, the total weight of this prefix is $w(\rho|_n) = \mathbf{a}_n$. We define the *norm* of a play ρ as the supremum of the norms of total weights of its prefixes: $\|\rho\| \stackrel{\text{def}}{=} \sup_n \|w(\rho|_n)\|$.

A *strategy* for Player p, $p \in \{1, 2\}$, is a function σ_p taking as input a non-empty path $\pi \cdot (v, \mathbf{a})$ ending in a Player p vertex $v \in V_p$, and returning an edge $\sigma_p(\pi \cdot (v, \mathbf{a})) = (v, \mathbf{u}, v')$ from E. A play $\rho = (v_0, \mathbf{a}_0)(v_1, \mathbf{a}_1)\cdots$ is *consistent* with a strategy σ_p for Player p if whenever v_n is a Player p vertex in V_p, then $\sigma_p(\rho|_n) = (v_n, \mathbf{a}_{n+1} - \mathbf{a}_n, v_{n+1})$. Given two strategies σ_1 and σ_2 for Player 1 and Player 2 respectively, and an initial vertex v_0 in V, observe that there is a unique play $\rho_{v_0, \sigma_1, \sigma_2}$ for v_0 which is consistent with both σ_1 and σ_2.

Example 2.1 (continued)*.* For instance, in the game graph depicted in Figure 1, a strategy for Player 1 could be to move to v_L whenever the current energy level on the first coordinate is non-negative, and to v_R otherwise:

$$\sigma_1(\pi \cdot (v_0, \mathbf{a})) \stackrel{\text{def}}{=} \begin{cases} (v_0, (0,0), v_L) & \text{if } \mathbf{a}(1) \geq 0, \\ (v_0, (0,0), v_R) & \text{otherwise,} \end{cases} \tag{1}$$

and one for Player 2 could be to always select one particular edge in every vertex, regardless of the current energy vector—this is called a *counterless* strategy [3]—:

$$\sigma_2(\pi \cdot (v, \mathbf{a})) \stackrel{\text{def}}{=} \begin{cases} (v_L, (-2,2), v_0) & \text{if } v = v_L \\ (v_R, (4,-3), v_0) & \text{otherwise.} \end{cases} \tag{2}$$

These strategies define a unique consistent play for v_0, which starts with

$$(v_0, 0, 0)(v_L, 0, 0)(v_0, -2, 2)(v_R, -2, 2)(v_0, 2, -1)(v_L, 2, -1)(v_0, 0, 1)\,. \tag{3}$$

□

In the following we consider several different winning conditions on plays, which define different games played on multi-weighted game graphs.

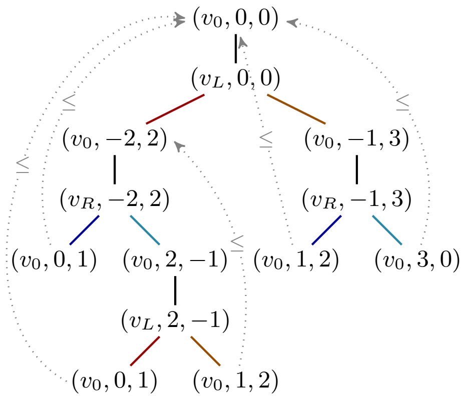

FIGURE 2. Self-covering strategy tree for Player 1 in the energy game of Figure 1.

2.2. **Multi-Dimensional Energy Games.** Suppose (V, E, d) is a multi-weighted game graph, v_0 an initial vertex, and $\mathbf{b}$ is a vector from $\mathbb{N}^d$. A play ρ from v_0 is *winning* for Player 1 in the *energy game* $\Delta_{\mathbf{b}}(V, E, d)$ with *initial credit* $\mathbf{b}$ if, for all n, $\mathbf{b} + w(\rho|_n) \geq \mathbf{0}$, using the product ordering over $\mathbb{Z}^d$. Otherwise, Player 2 wins the play. As usual, this means that Player 1 wins the energy game $\Delta_{\mathbf{b}}(V, E, d)$ in v_0 if there exists a *winning strategy* σ_1 for Player 1, i.e. σ_1 is such that for all strategies σ_2 for Player 2 the play $\rho_{v_0,\sigma_1,\sigma_2}$ is winning for Player 1. An immediate property of energy games is *monotonicity*: if σ_1 is winning for Player 1 with some initial credit $\mathbf{b}$, and $\mathbf{b}' \geq \mathbf{b}$, then it is also winning for Player 1 with initial credit $\mathbf{b}'$.

Example 2.1 (continued). For example, one may observe that the strategy (1) for Player 1 is winning for the game graph of Figure 1 with initial credit $(2, 1)$ (or larger). A geometric intuition comes from the directions of the total weights of simple cycles in Figure 1: by choosing alternatively edges to v_L or v_R, Player 1 is able to balance the energy levels above the '$x + y = 0$' line. One way to see this more formally is to build the corresponding *self-covering strategy tree* up to the first time when a configuration is greater or equal to another configuration higher in the tree [3]. By monotonicity of the game, Player 1 can then repeat the same actions from those leaves. See Figure 2 for our example.

Strategy σ_1 uses the comparison of $\mathbf{a}(1)$ with 0 as a *soft bound* to trigger a change of strategy and attempt to forbid cycles with a negative effect on the first coordinate. Note that the energy level $\mathbf{a}(1)$ might nevertheless become less than 0, but will remain ≥ -2 at all times; we call this the *hard bound*. This follows the general scheme of Chaloupka [5]—and also ours—for Player 1 strategies. □

Example 2.3. As a rather different example, consider the multi-weighted game graph of Figure 3. Although Player 2 does not control any vertex, he wins the associated energy game for any initial credit $\mathbf{b}$.

To see this, consider any infinite path and let ℓ_n, r_n and c_n denote the numbers of times the left-hand-side self-loop, the right-hand-side self-loop, and the central simple loop are closed, respectively, in the first n steps. Note

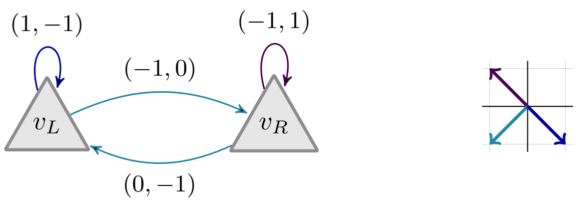

FIGURE 3. A 2-dimensional game graph with only Player 1 vertices.

that the energy levels after the first n steps are $(r_n - \ell_n) - c_n$ in the first component and $-(r_n - \ell_n) - c_n$ in the second component. If c_n is bounded then clearly either $r_n - \ell_n$ or $-(r_n - \ell_n)$ are not bounded from below, because one of ℓ_n and r_n is bounded and the other is not. Consider the other case—when c_n is not bounded—and for the sake of contradiction assume that both energy levels are bounded from below, and hence there is an initial credit $\mathbf{b}$ such that $(r_n - \ell_n) - c_n \geq \mathbf{b}(1)$ and $-(r_n - \ell_n) - c_n \geq \mathbf{b}(2)$. This is equivalent to $r_n - \ell_n \geq c_n + \mathbf{b}(1)$ and $-(r_n - \ell_n) \geq c_n + \mathbf{b}(2)$. Since c_n is not bounded, it follows that eventually both $r_n - \ell_n$ and $-(r_n - \ell_n)$ are always positive, which is absurd. □

2.3. **Multi-Dimensional Bounding Games.** A generalisation of energy games sometimes considered in the literature is to further impose a maximal *capacity* $\mathbf{c} \in \mathbb{N}^d$ (also called an upper bound) on the energy levels during the play [11, 13]. Player 1 then wins a play ρ if $0 \leq \mathbf{b} + w(\rho|_n) \leq \mathbf{c}$ for all n.

In the spirit of the arbitrary initial credit variant of energy games, we also quantify $\mathbf{c}$ existentially. This defines the *bounding game* $\Gamma(V, E, d)$ over a multi-weighted game graph (V, E, d), where a play ρ is winning for Player 1 if its norm $\|\rho\|$ is finite, i.e. if the set $\{\|w(\rho|_n)\| : n \in \mathbb{N}\}$ of norms of total weights of all finite prefixes of ρ is bounded, and Player 2 wins otherwise, if it is unbounded. In other words, Player 1 strives to contain the current vector within some d-dimensional hypercube, while Player 2 attempts to escape.

For example, note that Player 2 is now winning the bounding game defined by the game graph of Figure 1, because he can ensure that the total energy $\sum_{1 \leq i \leq d} \mathbf{a}_n(i)$ increases with n along any play $\rho = (v_0, \mathbf{a}_0)(v_1, \mathbf{a}_1) \cdots$, hence the norm $\|\rho\|$ is unbounded. Regarding the game graph of Figure 3, since Player 2 wins the energy game for all initial credits, he has a uniform counterless winnning strategy, which is winning for all initial credits [3, Lemma 19], and thus also winning in the bounding game.

2.3.1. *Lossy Game Graphs.* As hinted above, if Player 1 wins the bounding game, then there exists some initial credit for which she also wins the energy game. For a converse, let $\mathsf{lossy}(V, E, d)$ denote the *lossy* multi-weighted game graph obtained from (V, E, d) by inserting, at each Player 1 vertex and for each $1 \leq i \leq d$, a self-loop labelled by the negative unit vector $-\mathbf{e}_i$. In a bounding game played over a lossy game graph, it turns out that Player 1 can always bound the current vector from above by playing these unit decrements, hence she only has to ensure that the current vector remains bounded from below, i.e. she has to win the energy game for some initial credit. Formally:

Proposition 2.4. *From any vertex in any multi-weighted game graph* (V, E, d)*:*

(1) *Player* 1 *wins the energy game* $\Delta_{\mathbf{b}}(V, E, d)$ *for some* $\mathbf{b} \in \mathbb{N}^d$ *if and only if Player* 1 *wins the bounding game* $\Gamma(\mathsf{lossy}(V, E, d))$*.*

(2) *Player* 2 *wins the energy game* $\Delta_{\mathbf{b}}(V, E, d)$ *for all* $\mathbf{b} \in \mathbb{N}^d$ *if and only if Player* 2 *wins the bounding game* $\Gamma(\mathsf{lossy}(V, E, d))$*.*

Proof sketch. We rely for this proof on results proven later in Section 6. By the determinacy of single-cycle bounding games and their equivalence with infinite bounding games, it suffices to prove the two 'only if' directions.

For item 1, assume that Player 1 has a winning strategy σ_1 in the bounding game on $\mathsf{lossy}(V, E, d)$. In particular, there exists a bound B such that Player 1 can maintain $\|\rho\| \leq B$ on any play ρ consistent with σ_1 (an explicit such B is provided by Lemma 3.1). Observe that along any consistent play, σ_1 must always eventually play an edge which is not a decrementing self-loop, or the play would eventually become unbounded from below. Hence the following strategy σ_1' which simply skips all the decrement actions is well-defined:

$$\sigma_1'(\pi(v, \mathbf{a})) \stackrel{\text{def}}{=} \begin{cases} (v, \mathbf{u}, v') & \text{if } \sigma_1(\pi(v, \mathbf{a})) = (v, \mathbf{u}, v') \text{ with } v \neq v' \\ & \text{or } \forall 1 \leq i \leq d \,.\, \mathbf{u} \neq -\mathbf{e}_i, \\ \sigma_1'(\pi(v, \mathbf{a})(v, \mathbf{a} - \mathbf{e}_i)) & \text{otherwise, i.e. if there is } 1 \leq i \leq d \\ & \text{s.t. } \sigma_1(\pi(v, \mathbf{a})) = (v, -\mathbf{e}_i, v). \end{cases} \tag{4}$$

By monotonicity of energy games, this strategy is winning in the energy game on (V, E, d) with initial credit B on all components.

For item 2, assume that Player 2 has a winning strategy in the infinite bounding game on $\mathsf{lossy}(V, E, d)$. Then he has a winning strategy σ_2 in the first-cycle bounding game on $\mathsf{lossy}(V, E, D)$, which only plays colours that are disjoint from the non-negative orthant $\mathsf{cone}(\mathbf{e}_1, \ldots, \mathbf{e}_d)$, because Player 1 could otherwise use a decrementing self-loop to create a cycle outside this colour. Observe that, for any open half-subspace H that is disjoint from the non-negative orthant, and for any bound B, there exists a bound on the distances between the boundary of H and vectors in H whose every component is at least $-B$. It follows that the corresponding winning strategy $\widetilde{\sigma_2}$ of Player 2 in the infinite bounding game (cf. Lemma 6.2) is also winning in the energy game $\Delta_{\mathbf{b}}(V, E, d)$, for all initial credits $\mathbf{b}$ (cf. Claim 6.3). ☐

Our task in the following will be therefore to prove an upper bound on the time complexity required to solve bounding games.

3. Complexity Upper Bounds

Our main results are new parameterised complexity upper bounds for deciding whether Player 1 has a winning strategy in a given energy game. In turn, we rely for these results on a *small hypercube property* of bounding games, which we introduce next, and which will be a consequence of the study of first cycle bounding games in Section 6.

3.1. **Small Hypercube Property.** In a bounding game, if Player 1 is winning, then by definition she has a winning strategy σ_1 such that for all plays ρ consistent with σ_1 there exists some bound B_ρ with $\|\rho\| \leq B_\rho$. We considerably strengthen this statement in Section 6 where we construct an explicit winning strategy, which yields an explicit *uniform* bound B for all consistent plays:

Lemma 3.1. *Let (V, E, d) be a multi-weighted game graph. If Player 1 wins the bounding game $\Gamma(V, E, d)$, then she has a winning strategy which ensures*

$$\|\rho\| \leq (4|V| \cdot \|E\|)^{2(d+2)^3}$$

for all consistent plays ρ.

Note that our bound is polynomial in $|V|$ the number of vertices, unlike the bounds found in comparable statements by Brázdil et al. [3, Lemma 7] and Chatterjee et al. [8, Lemma 3], which incur an *exponential* dependence on $|V|$. This entails pseudo polynomial complexity bounds when d is fixed:

Corollary 3.2. *Bounding games on multi-weighted graphs (V, E, d) are solvable in deterministic time $(|V| \cdot \|E\|)^{O(d^4)}$.*

Proof. By Lemma 3.1, the bounding game is equivalent to a reachability game where Player 2 attempts to see the norm of the total weight exceed $B \stackrel{\text{def}}{=} (4|V| \cdot \|E\|)^{2(d+2)^3}$. This can be played within a finite arena of size $(B + 1)^d$ and solved in time linear in that size using the usual attractor computation algorithm. □

3.2. **Energy Games with Arbitrary Initial Credit.** The *arbitrary initial credit problem* for energy games takes as input a multi-weighted game graph and an initial vertex v_0 and asks whether there exists a vector $\mathbf{b}$ in $\mathbb{N}^d$ such that Player 1 wins $\Delta_{\mathbf{b}}(V, E, d)$ in v_0:

Theorem 3.3. *The arbitrary initial credit problem for energy games on multi-weighted game graphs (V, E, d) is solvable in deterministic time $(|V| \cdot \|E\|)^{O(d^4)}$.*

Proof. This follows from Proposition 2.4, and Corollary 3.2 applied to the game graph $\Gamma(\mathsf{lossy}(V, E, d))$. □

3.3. **Energy Games with Given Initial Credit.** The *given initial credit problem* for energy games takes as input a multi-weighted game graph (V, E, d), an initial vertex v_0, and a credit $\mathbf{b}$ in $\mathbb{N}^d$ and asks whether Player 1 wins the energy game $\Delta_{\mathbf{b}}(V, E, d)$ in v_0:

Theorem 3.4. *The given initial credit problem with credit $\mathbf{b}$ for energy games on multi-weighted game graphs (V, E, d) is solvable in deterministic time $(|V| \cdot \|E\| \cdot (\|\mathbf{b}\| + 1))^{2^{O(d \cdot \log d)}}$.*

This is again a pseudo-polynomial upper bound when d is fixed, and encompasses Chaloupka's P upper bound in dimension $d = 2$ with unit updates, i.e. with $\|E\| = 1$. Because the given initial credit problem for energy games of fixed dimension $d \geq 4$ is $\mathsf{EXPTIME}$-hard [9], the bound in terms of $\|E\|$ in Theorem 3.4 cannot be improved.

Thanks to Lemma 3.1, a proof of Theorem 3.4 could be obtained by bounding on the height of self-covering strategy trees, using the work of Brázdil et al. [3], and more generally the techniques of Rackoff [15]. We sketch an arguably simpler proof next, which does not explicitly mention self-covering trees. Throughout the remainder of this section, we let (V, E, d) denote some multi-weighted game graph, v be in V and $\mathbf{b}$ in $\mathbb{N}^d$.

3.3.1. *Fenced Games.* The main difference between energy games with a given initial credit $\mathbf{b}$ and those with arbitrary initial credit is that Player 2 wins the former as soon as the current total weight goes below $-\mathbf{b}(i)$ on some coordinate i. We can enforce a similar condition on bounding games. Given $I \subseteq \{1, \ldots, d\}$, we say that Player 2 wins a play ρ in the *fenced* bounding game $\overline{\Gamma}^I_{\mathbf{b}}(V, E, d)$ if

- there exists n in $\mathbb{N}$ and i in I such that $(\mathbf{b} + w(\rho|_n))(i) < 0$, or
- $\|\rho\| = \omega$ as in the usual bounding game,

and Player 1 wins otherwise. Observe that the fenced bounding game $\overline{\Gamma}^{\emptyset}_{\mathbf{b}}(V, E, d)$ is simply the bounding game $\Gamma(V, E, d)$. Using the fact that fenced games are equivalent to bounding games played on double exponentially larger arenas (as we shall see next in §3.3.2), we can essentially repeat the arguments of the proof of Proposition 2.4 and obtain:

Claim 3.5. *For both $p \in \{1, 2\}$, Player p wins the energy game $\Delta_{\mathbf{b}}(V, E, d)$ from vertex v iff Player p wins the fenced bounding game $\Gamma^{\{1,\ldots,d\}}_{\mathbf{b}}(\mathsf{lossy}(V, E, d))$ from vertex v.*

3.3.2. *Capped Game Graphs.* Fenced bounding games allow to enforce the given initial credit of an energy game through semantic means, by modifying the winning conditions of bounding games. We now show that the same effect can be obtained syntactically, by considering a bounding game played over a different game graph.

Consider for this a coordinate $1 \leq i \leq d$. The idea in the following is to track the current value of $w(\rho|_n)(i)$ in the vertices of the game graph, and go to a Player 2-winning vertex $v_\perp$ if it ever goes below $-\mathbf{b}(i)$. The issue with that plan is that $w(\rho|_n)(i)$ might grow unboundedly high during the course of a play. However, by Lemma 3.1, we know that if Player 2 manages to force this value above

$$B \stackrel{\text{def}}{=} (4|V| \cdot \|E\|)^{2(d+2)^3} , \tag{5}$$

then he also wins the game.

We construct accordingly a *capped* d-dimensional multi-weighted game graph $\mathsf{capped}^i_{\mathbf{b}}(V, E, d)$ from (V, E, d) as follows:

- its set of Player 1 vertices is $\{v_\perp\} \uplus V_1 \times \{-\mathbf{b}(i), \ldots, B\}$ and its set of Player 2 vertices is $V_2 \times \{-\mathbf{b}(i), \ldots, B\}$;
- for any vertex (v, a), and for any edge $(v, \mathbf{u}, v')$ in E with $-\mathbf{b}(i) \leq a + \mathbf{u}(i) \leq B$, it has an edge $((v, a), \mathbf{u}, (v', a + \mathbf{u}(i)))$,
- for any vertex (v, a), if there is an edge $(v, \mathbf{u}, v')$ in E with $a + \mathbf{u}(i) < -\mathbf{b}(i)$ or $a + \mathbf{u}(i) > B$, it has an edge $((v, a), \mathbf{u}, v_\perp)$;
- finally $v_\perp$ has only a self-loop labelled by some unit vector, and is therefore losing for Player 1.

Claim 3.6. *For any $I \subseteq \{1, \ldots, d\}$, $i \notin I$, and all $p \in \{1, 2\}$, Player p wins the fenced bounding game $\overline{\Gamma}_{\mathbf{b}}^{I \uplus \{i\}}(V, E, d)$ from vertex v iff Player p wins the fenced bounding game $\overline{\Gamma}_{\mathbf{b}}^{I}(\mathsf{capped}_{\mathbf{b}}^{i}(V, E, d))$ from vertex $(v, 0)$.*

Proof of Theorem 3.4. For any $v \in V$ and $p \in \{1, 2\}$, using Claim 3.5 and using Claim 3.6 successively for $i = 1, \ldots, d$, Player p wins the energy game $\Delta_{\mathbf{b}}(V, E, d)$ from vertex v iff Player p wins the fenced bounding game

$$\overline{\Gamma}_{\mathbf{b}}^{\emptyset}\Big(\mathsf{capped}_{\mathbf{b}}^{d}\big(\cdots \mathsf{capped}_{\mathbf{b}}^{1}(\mathsf{lossy}(V, E, d)) \cdots\big)\Big) \tag{6}$$

from $(v, \mathbf{0})$, which is the same as the bounding game

$$\Gamma\Big(\mathsf{capped}_{\mathbf{b}}^{d}\big(\cdots \mathsf{capped}_{\mathbf{b}}^{1}(\mathsf{lossy}(V, E, d)) \cdots\big)\Big)\ . \tag{7}$$

As capping preserves the norm of edge labels and the dimension, all we need to do is compute the number of vertices in that game graph. Let A_i be the number of vertices in $\mathsf{capped}_{\mathbf{b}}^{i}\big(\cdots \mathsf{capped}_{\mathbf{b}}^{1}(\mathsf{lossy}(V, E, d)) \cdots\big)$ and $A_0 \stackrel{\text{def}}{=} |V|$; by (5), we see that

$$A_{i+1} = 1 + A_i\left(\mathbf{b}(i+1) + (4A_i\|E\|)^{2(d+2)^3}\right) \leq (4A_i\|E\|(\|\mathbf{b}\| + 1))^{2(d+2)^3+1}\ . \tag{8}$$

We show by induction over i that

$$4A_i\|E\|(\|\mathbf{b}\| + 1) \leq (4|V| \cdot \|E\|(\|\mathbf{b}\| + 1))^{2^i(d+3)^{3i}}\ . \tag{9}$$

This holds for $i = 0$, and for the induction step,

$$\begin{aligned}
4A_{i+1}\|E\|(\|\mathbf{b}\| + 1) &\leq 4\,(4A_i\|E\|(\|\mathbf{b}\| + 1))^{2(d+2)^3+1}\,\|E\|(\|\mathbf{b}\| + 1) && \text{by (8)}\\
&\leq 4(4|V| \cdot \|E\|(\|\mathbf{b}\| + 1))^{2^i(d+3)^{3i}\cdot(2(d+2)^3+1)}\|E\|(\|\mathbf{b}\| + 1)\\
&\leq (4|V| \cdot \|E\|(\|\mathbf{b}\| + 1))^{2^i(d+3)^{3i}\cdot(2(d+2)^3+1)+1}\\
&\leq (4|V| \cdot \|E\|(\|\mathbf{b}\| + 1))^{2^{i+1}(d+3)^{3(i+1)}}\ .
\end{aligned}$$

We conclude using Corollary 3.2. □

Remark 3.7 (Pareto Limit). Another consequence of our bounds is that we can effectively compute the set of minimal initial credits that allow Player 1 to win a given energy game, sometimes called the *Pareto limit* of the game. By Proposition 2.4 and Lemma 3.1, these vectors have a norm bounded by an exponential value in d but polynomial in $|V|$ and $\|E\|$, and we can use Theorem 3.4 to test for each such vector whether it suffices for Player 1 to win the game. This algorithm works in double exponential time when d is part of the input but pseudo-polynomial time when d is fixed. □

4. Multi-Dimensional Mean-Payoff Games

This section summarises the technique for solving multi-dimensional mean-payoff games proposed by Chatterjee and Velner [6], which relies on *open half-spaces*. The rest of the paper does not rely formally on this section and it may be omitted by a reader eager to get on with our new 'perfect half-spaces' technique for solving multi-dimensional bounding games. We believe, however, that starting here helps put our work in context, appreciate similarities and differences between the two techniques, and understand the conceptual and some of the technical challenges we had to overcome.

4.1. **Multi-Dimensional Mean-Payoff Games.** Given a play ρ over a multi-weighted game graph (V, E, d), we define its *long-term average* in $\mathbb{Q}^d$ as $\mathsf{avg}(\rho) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{w(\rho|_n)}{n}$. We say that ρ is winning for Player 1 in the *mean-payoff game* $M(V, E, d)$ if $\mathsf{avg}(\rho) \geq \mathbf{0}$. Otherwise, i.e. if there is a coordinate $1 \leq i \leq d$ such that $\mathsf{avg}(\rho)(i) < 0$, the play ρ is winning for Player 2. As shown by Chatterjee et al. [7], determining the winner in multi-dimensional mean-payoff games is coNP-complete, and in pseudo-polynomial time when the dimension is fixed [6, Theorem 1].

4.2. **Energy Versus Mean-Payoff.** In a one-dimensional arbitrary-initial-credit energy game, the goal of Player 1 is to keep the energy level bounded from below. It is folklore that Player 1 has a winning strategy in such a game if and only if she has a strategy in the mean-payoff game on the same game graph that guarantees a non-negative long-term average.

4.2.1. *Infinite Memory Strategies for Player 1.* This relationship between energy games and mean-payoff games does not generalise to multi-dimensional games. We illustrate this on the example of a 2-dimensional game graph from Figure 3. In Example 2.3 we have argued that Player 2 has a winning strategy in the bounding game (and hence also in the arbitrary-initial-credit energy game). On the other hand, we argue that Player 1 has a strategy to guarantee that the long-term average in both dimensions is non-negative. Indeed, consider a strategy in which in stage m—for all $m = 1, 2, 3, \ldots$— Player 1 performs one of the self-loops m times, then she moves to the other vertex where she performs the other self-loop m times, and then finally returns to the starting vertex. After m stages, the energy level in both dimensions is $-m$ and the number of steps performed is $\Theta(m^2)$, hence the long-term average in the infinite play is 0 in both dimensions, because $\lim_{m\to\infty} -\frac{m}{m^2} = 0$.

Note that this strategy for Player 1 in the game graph of Figure 3 is infinite-memory, since the actions depend on the stage m. Multi-dimensional mean-payoff games might require infinite memory in order to be won, as shown by Chatterjee et al. [7, Lemma 4]—their proof can be used to show that the game of Figure 3 actually requires infinite memory.

4.2.2. *Finite Memory Strategies for Player 1.* In the multi-dimensional case, there is nevertheless a strong relation between energy and mean-payoff games. Call a strategy σ *finite memory* if there exists an equivalence relation $\sim$ with finite index over $(V \times \mathbb{Z}^d)^+$ such that, whenever $\pi \sim \pi'$ for some non-empty paths π and π' in the domain of σ, then $\sigma(\pi) = \sigma(\pi')$ (such strategies are typically described using *Moore machines*).

Fact 4.1 (Chatterjee et al. [7]). *Let (V, E, d) be a multi-weighted game graph. There exists an initial credit $\mathbf{b}$ such that Player 1 wins the energy game $\Delta_{\mathbf{b}}(V, E, d)$ if and only if Player 1 has a finite memory winning strategy in the mean-payoff game $M(V, E, d)$.*

Hence our complexity bounds in Theorem 3.3 on multi-dimensional energy games also yield a pseudo-polynomial time algorithm to find a winning finite-memory strategy for Player 1 in a given fixed-dimensional mean-payoff game.

4.3. **The Open Half-Space Technique for Mean-Payoff Games.** For technical convenience, we follow Chatterjee and Velner in considering mean-payoff games on lossy game graphs (see Section 2.3 for a formal definition of lossy game graphs). In this context, the goal of Player 1 is to achieve a long-term average of 0 in all dimensions, and the goal of Player 2 is to achieve a negative long-term average in at least one dimension.

4.3.1. *Winning Strategies for Player 2.* The first key observation that underpins the solution of (lossy) multi-dimensional mean-payoff games by Chatterjee and Velner is the following sufficient condition for the existence of a winning strategy for Player 2 from some vertex in the game graph: there is a vertex v_0, an open half-space $H \subseteq \mathbb{R}^d$ and a strategy for Player 2 that guarantees all simple cycles formed along a play from v_0 to be in H. One can then argue that if Player 2 uses such a strategy indefinitely then the norms of the energy level vectors grow linearly in the number of steps performed, and hence the long-term average is non-zero in at least one dimension.

Observe that every open half-space can be determined by a non-zero vector $\mathbf{n} \in \mathbb{R}^d$ that is normal to the hyperplane on the boundary of the half-space:

$$H_{\mathbf{n}} = \{\mathbf{v} \in \mathbb{R}^d \ : \ \mathbf{n} \cdot \mathbf{v} < 0\} \ .$$

Chatterjee and Velner [6, Lemma 1] crucially point out that for every vector $\mathbf{n} \in \mathbb{R}^d$, one can check whether Player 2 has a strategy that guarantees all simple cycles formed to be in $H_{\mathbf{n}}$ from v_0 by solving a one-dimensional mean-payoff game on the multi-weighted game graph with every weight $\mathbf{u}$ replaced by the dot-product $\mathbf{n} \cdot \mathbf{u}$.

4.3.2. *Winning Strategies for Player 1.* The second key insight of Chatterjee and Velner is that the above sufficient condition for the existence of a winning strategy for Player 2 in a lossy multi-dimensional mean-payoff game is necessary. Indeed, by (positional) determinacy of mean-payoff games [10], it follows that, if the sufficient condition described above does not hold for any open half-space $H_{\mathbf{n}}$ and any initial vertex v_0, then for all non-zero vectors $\mathbf{n}$ and all vertices v_0, Player 1 has a (positional) strategy to block simple cycles in $H_{\mathbf{n}}$ along any play from v_0, or in other words to force all simple cycles formed to be in

$$\mathbb{R}^d \setminus H_{\mathbf{n}} = \overline{H_{-\mathbf{n}}} = \{\mathbf{v} \in \mathbb{R}^d \ : \ \mathbf{n} \cdot \mathbf{v} \geq 0\} \ .$$

In such a case, Chatterjee and Velner [6, Lemma 2] show that such strategies of Player 1, which force simple cycles formed to be in any closed half-space, can be carefully combined to ensure that the long-term average is 0 in every dimension. The main idea in the construction of the strategy for Player 1 is to proceed in stages $m = 1, 2, 3, \ldots$, to monitor the energy-level vector at the beginning of stage m of the game, say $\mathbf{g}_m$, and to 'counteract' its further growth in the direction of $\mathbf{g}_m$ throughout stage m by using the strategy that blocks simple cycles in the open half-space $H_{\mathbf{g}_m}$, i.e., that forces all the formed simple cycles to be in the closed half-space $\overline{H_{-\mathbf{g}_m}}$. In the winning strategy we described for Player 1 for the mean-payoff game over the graph of Figure 3, Player 1 can avoid cycles in $H_{(-1,1)}$ by playing the self-loop on v_L, and she can avoid cycles in $H_{(1,-1)}$ by playing the self-loop on v_R.

Moving from one stage to another, and hence switching between such counteracting strategies to force simple cycles in different half-spaces, cannot be done too often because as a result of switching from one strategy to another a bounded number of unfavourable simple cycles may be formed. This is the case in our example, since switching between v_L and v_R closes a cycle with effect $(-1,-1)$ resulting in a drift away from the non-negative orthant.

The strategy for Player 1 proposed by Chatterjee and Velner overcomes this complication by increasing the number of steps made in every stage; in particular, they proposed making $s(m) \stackrel{\text{def}}{=} m$ steps in stage m before proceeding to stage $m+1$. The purpose is to make the drift grow slower than the number of steps in the play. This, as can be deduced from their analysis, gives a bound of $O(n^{3/4})$ for the norm of the energy-level vector after n steps, and hence the long-term average is 0 in all dimensions because $\lim_{n\to\infty} \frac{n^{3/4}}{n} = 0$. One may observe that more generally, if we set $s(m) \stackrel{\text{def}}{=} m^{\varepsilon}$, for any $\varepsilon > 0$, then the norm of the energy-level vector after n steps is $O(n^{1/2+\varepsilon/2(1+\varepsilon)})$. Hence, the best upper bounds on the norm of the energy-level vectors after m steps that can be guaranteed by Player 1—when using a strategy similar to that constructed by Chatterjee and Velner—are in $\omega(\sqrt{n})$. Let us point out that such strategies require infinite memory because they need to 'keep the count' of the stage they are in and of the number of steps they need to perform in the current stage, both of which are unbounded.

5. Perfect Half-Spaces

We recall in this section the definition of subsets of $\mathbb{Q}^d$ called *perfect half-spaces*, which can also be characterised as the *maximal* salient blunt cones in $\mathbb{Q}^d$. They will be used next in Section 6 to define a condition for Player 2 to win bounding games, which relies on Player 2's ability to force cycles inside perfect half-spaces. This can be understood as a generalisation of Chatterjee and Velner's approach for solving multi-dimensional mean-payoff games, which relies on a similar ability to force cycles inside open half-spaces. We employ perfect half-spaces in Section 6 to colour the edges in *first-cycle bounding games*, which determine the winner using both the colours and the weight of the first cycle formed along a play.

5.1. **Definitions from Linear Algebra.** Given a subset $\mathbf{A}$ of $\mathbb{Q}^d$, we write $\mathsf{span}(\mathbf{A})$ (resp., $\mathsf{cone}(\mathbf{A})$) for the *vector space* (resp., the *cone*) *generated* by $\mathbf{A}$, i.e., the closure of $\mathbf{A}$ under addition and under multiplication by all (resp., nonnegative) rationals.

Observe that the sufficient condition for existence of a winning strategy for Player 2 in a lossy multi-dimensional mean-payoff game is also a sufficient condition for him to have a winning strategy in a bounding game. Unlike for multi-dimensional mean-payoff games and as witnessed with the game on Figure 3, however, this condition is not necessary. In order to formulate a new more powerful sufficient condition, we use instead perfect half-spaces: a *k-perfect half-space* of $\mathbb{Q}^d$, where $k \in \{1, 2, \ldots, d\}$, is a (necessarily disjoint) union $H_d \cup \cdots \cup H_k$ such that:

- H_d is an open half-space of $\mathbb{Q}^d$;
- for all $j \in \{k, \ldots, d-1\}$, $H_j \subseteq \mathbb{Q}^d$ is an open half-space of the boundary of H_{j+1}.

Whenever we write a k-perfect half-space in form $H_d \cup \cdots \cup H_k$, we assume that each H_j is j-dimensional. We additionally define the $(d+1)$-perfect half-space as the empty set; a *partially-perfect half-space* is then a k-perfect half-space for some k in $\{1, \ldots, d+1\}$. A *perfect half-space* is a 1-perfect half-space. Observe that a partially-perfect half-space is always a cone, which is *blunt*, i.e., does not contain $\mathbf{0}$, and *salient*, i.e., if it contains a vector $\mathbf{v}$ then it does not contain its opposite $-\mathbf{v}$. Moreover, a perfect half-space is a *maximal* blunt and salient cone.

5.2. **Generated Perfect Half-Spaces.** In order to pursue effective and parsimonious strategy constructions, we consider perfect half-spaces generated by particular sets of vectors, which will correspond to the total weights of simple cycles in multi-weighted game graphs. Given a norm M in $\mathbb{N}$, we say that an open half-space H is *M-generated* if its boundary equals $\mathsf{span}(\mathbf{B})$ for some set $\mathbf{B}$ of vectors of norm at most M. By extension, a partially-perfect half-space is *M-generated* if each of its open half-spaces is M-generated.

Proposition 5.1. *Any M-generated k-dimensional vector space of $\mathbb{Q}^d$ has at most $\mathcal{L}(k) \stackrel{\text{def}}{=} 2(2M+1)^{d(k-1)}$ open half-spaces that are M-generated.*

Example 5.2. In the game graph of Figure 3, there are three 1-generated open half-spaces of interest: the half-plane $H_2 \stackrel{\text{def}}{=} \{(x, y) : x + y < 0\}$ with boundary $\mathsf{span}((-1, 1), (1, -1))$ and containing $(-1, -1)$, and the two half-lines $H_1 \stackrel{\text{def}}{=} \{(x, y) : x + y = 0 \wedge x < 0\}$ and $H_1' \stackrel{\text{def}}{=} \{(x, y) : x + y = 0 \wedge x > 0\}$ with boundary $\mathsf{span}(\mathbf{0})$ and containing respectively $(-1, 1)$ and $(1, -1)$. In turn, those three open half-spaces define two perfect half-spaces: $H_2 \cup H_1$ and $H_2 \cup H_1'$. □

5.3. **Hierarchy of Perfect Half-Spaces.** Finally, we fix a ranked tree-like structure on all M-generated partially-perfect half-spaces, which provide a scaffolding on which we will build strategies in multi-dimensional bounding games. Observe that an M-generated partially-perfect half-space $H_d \cup \cdots \cup H_k$ for $k > 1$ can be extended using any of the M-generated open half-spaces H of the boundary of H_k; note that this boundary then equals $\mathsf{span}(H)$. In Example 5.2, H_2 can be extended using H_1 or H_1', and $\mathsf{span}(H_1) = \mathsf{span}(H_1') = \{(x, y) : x + y = 0\}$.

The set of M-generated perfect half-spaces can be totally ordered by positing a linear ordering $<$ between all M-generated open half-spaces. We write $\prec$ for the lexicographically induced linear ordering between all M-generated perfect half-spaces of $\mathbb{Q}^d$: if $\mathcal{H} = H_d \cup \cdots \cup H_1$ and $\mathcal{H}' = H_d' \cup \cdots \cup H_1'$, we define $\mathcal{H} \prec \mathcal{H}'$ to hold iff $H_j = H_j'$ for all $j \in \{k+1, \ldots, d\}$ and $H_k < H_k'$ for some $k \in \{1, 2, \ldots, d\}$.

6. First Cycle Bounding Games

We define in this section *first-cycle bounding games*, which provide the key technical arguments for most of our results. Such games end as soon as a cycle is formed along a play, and the weight of this cycle determines the winner, along with a colouring information chosen by Player 2. In sections 6.2 and 6.3, we are going to show that first-cycle bounding games and infinite bounding games are equivalent, by translating winning strategies for each Player p, $p \in \{2, 1\}$, from first-cycle bounding games to bounding games. This yields in particular the small hypercube property of Lemma 3.1.

6.1. Definition.

We define the *first-cycle bounding game* $G(V, E, d)$ on a multi-weighted game graph (V, E, d):

- at any Player-1 vertex, Player 2 chooses a $|V| \cdot \|E\|$-generated perfect half-space $\mathcal{H}$ of $\mathbb{Q}^d$, and then Player 1 chooses an outgoing edge, whose occurrence in the play becomes coloured by $\mathcal{H}$;
- at any Player-2 vertex, he chooses an outgoing edge;
- the game finishes as soon as a vertex is visited twice, which produces a simple cycle C with coloured Player-1 edges;
- Player 2 wins if $w(C)$, the total weight of the cycle, is in the largest partially-perfect half-space of $\mathbb{Q}^d$ that is contained in all the colours in C, or in other words the least common ancestor of all the colours in C;
- Player 1 wins otherwise.

Example 6.1. Player 2 wins the first-cycle bounding game played on Figure 3. Indeed, he can choose the colour $H_2 \cup H_1$ in v_L and the colour $H_2 \cup H_1'$ in v_R. Then Player 1 cannot avoid forming a simple cycle in either $H_2 \cup H_1$ (if cycling on v_L), in $H_2 \cup H_1'$ (if cycling on v_R), or in H_2 (if cycling between v_L and v_R). □

Observe that first-cycle bounding games are finite perfect information games, and are thus *determined*, i.e. Player 1 wins from a vertex iff Player 2 loses from the same vertex and vice versa.

6.2. Winning Strategies for Player 2.

Player 2 might not have the power to choose a perfect half-space and then force all simple cycles formed to be in that perfect half-space; the bounding game on Figure 3 is a glaring example of this phenomenon. Nevertheless, Player 2 can very nearly achieve this: if he has a winning strategy, then he can win by colouring every edge taken in a play with a perfect half-space, and by forcing every simple cycle formed to be in the *largest* partially-perfect half-space that is contained in all the colours of edge occurring in the cycle.

Suppose σ is a strategy of Player 2 from a vertex v_0 in a first-cycle bounding game $G(V, E, d)$. Let $\widetilde{\sigma}$ be the following strategy of Player 2 in the infinite bounding game $\Gamma(V, E, d)$:

- at any Player-2 vertex, $\widetilde{\sigma}$ chooses the edge specified by σ;
- whenever a cycle is formed, $\widetilde{\sigma}$ cuts it out of its memory, and continues playing according to σ.

Lemma 6.2. *If σ is winning for Player 2 in $G(V, E, d)$ from some vertex v_0, then $\widetilde{\sigma}$ is winning for Player 2 in $\Gamma(V, E, d)$ from the same vertex v_0.*

Proof. Consider any infinite play ρ consistent with $\widetilde{\sigma}$, and let:

- $\rho^\sharp$ be obtained from ρ by colouring all Player 1's edges with the $|V| \cdot \|E\|$-generated perfect half-spaces of $\mathbb{Q}^d$ as specified by σ;
- $C_1, C_2, \ldots$ be the cycle decomposition of $\rho^\sharp$;
- $\mathcal{H}_n$ be the largest partially-perfect half-space of $\mathbb{Q}^d$ that is contained in all the colours in C_n, for each n.

First observe that, for any n, either $\mathcal{H}_n$ contains $\mathcal{H}_{n+1}$ or vice-versa. Thus the set of all $|V|\cdot\|E\|$-generated partially-perfect half-spaces of $\mathbb{Q}^d$ that occur infinitely often in the sequence $\mathcal{H}_1, \mathcal{H}_2, \ldots$ has an element that is contained in all the others. We then conclude using the following Claim 6.3. □

Claim 6.3. *Suppose $\mathcal{H} = H_d \cup \cdots \cup H_k$ is a partially-perfect half-space of $\mathbb{Q}^d$ and $\mathbf{a}_1, \mathbf{a}_2, \ldots$ is an infinite sequence of vectors such that:*

- *the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots\}$ is finite;*
- *for each n, there exists a partially-perfect half-space of $\mathbb{Q}^d$ that contains $\mathcal{H}$ and $\mathbf{a}_n$;*
- *we have $\mathbf{a}_n \in \mathcal{H}$ for infinitely many n.*

Then there exist $k' \in \{d, \ldots, k\}$ and $N > 0$ such that

- *for each $n \geq N$, $\mathbf{a}_N + \cdots + \mathbf{a}_n$ belongs to $\overline{H_{k'}}$ the topological closure of $H_{k'}$, and*
- *the set of all distances of $\mathbf{a}_N + \cdots + \mathbf{a}_n$ from the boundary of $H_{k'}$ is unbounded.*

In particular, the set of all norms $\|\mathbf{a}_1 + \cdots + \mathbf{a}_n\|$ is unbounded.

Proof. We have that $\mathcal{H}$ is of the form $H_d \cup \cdots \cup H_k$. Let $k' \in \{d, \ldots, k\}$ be maximal such that $\mathbf{a}_n \in H_{k'}$ for infinitely many n. Observe that $\{1, 2, \ldots\}$, the set of all positive integers, can be partitioned into three:

(1) The set of all n such that $\mathbf{a}_n \in H_d \cup \cdots \cup H_{k'+1}$, which is finite by definition of k'. We let N be larger than the index of the last such $\mathbf{a}_n$; then the vectors $\mathbf{a}_n$ for $n \geq N$ belong to $\overline{H_{k'}}$.
(2) The set of all n such that $\mathbf{a}_n \in H_{k'}$, which is infinite by definition of k'. Since $H_{k'}$ is open and the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots\}$ is finite, there is a positive minimal distance of those $\mathbf{a}_n$ from the boundary of $H_{k'}$. These vectors bring the sums $\mathbf{a}_N + \cdots + \mathbf{a}_n$ for $n \geq N$ unboundedly far from the boundary of $H_{k'}$.
(3) The set of all n such that $\mathbf{a}_n$ is contained in the boundary of $H_{k'}$. These vectors have no effect on the distance between the sums $\mathbf{a}_N + \cdots + \mathbf{a}_n$ for $n \geq N$ and the boundary of $H_{k'}$. □

6.3. **Winning Strategies for Player 1.** If there is no winning strategy for Player 2 in the first-cycle bounding game $G(V, E, d)$ from a vertex v_0, i.e. no colouring by perfect half spaces that allows him to win, then by determinacy of first-cycle bounding games, there is a winning strategy σ for Player 1 in $G(V, E, d)$ from v_0 against all possible colourings. This is our main technical result. It consists in constructing a finite-memory winning strategy $\widetilde{\sigma}$ for Player 1 in the infinite bounding game $\Gamma(V, E, d)$ from the same vertex v_0, which balances her various 'perfect half-space avoidance strategies' in order to ensure the small hypercube property stated in Lemma 3.1.

The memory of $\widetilde{\sigma}$ consists of:

a simple path: γ from the initial vertex v_0 to the current vertex v, in which Player 1's edges are coloured by $|V| \cdot \|E\|$-generated perfect half-spaces of $\mathbb{Q}^d$;

a colour: i.e. a $|V| \cdot \|E\|$-generated perfect half-space $\mathcal{H} = H_d \cup \cdots \cup H_1$ of $\mathbb{Q}^d$;

counters: $\mathsf{c}(k, W)$ for every $k \in \{1, 2, \ldots, d\}$ and for every nonzero total weight W of a simple cycle, which are natural numbers.

Strategy $\widetilde{\sigma}$ copies its moves from strategy σ for the first-cycle bounding game, based on the coloured simple path and the colour it has in its memory.

6.3.1. *Simple Path Updates.* The simple path is maintained in a straightforward manner: whenever a cycle is formed it is removed from the simple path; otherwise the current vertex is added to the simple path.

6.3.2. *Colour Updates.* The perfect half-space $\mathcal{H}$ in the memory is used to avoid cycles with weights in $\mathcal{H}$, relying for this on the strategy σ. It is initially the $\prec$-minimal $|V| \cdot \|E\|$-generated perfect half-space. During the play, it is either left unchanged, or updated if the current energy level threatens to become unbounded. These updates can take two forms:

- a *k-shift* to $H'_k > H_k$ moves to the $\prec$-minimal perfect half-space of the form $H_d \cup \cdots \cup H_{k+1} \cup H'_k \cup \cdots \cup H'_1$, and
- a *k-cancellation* moves to the $\prec$-minimal perfect half-space of the form $H_d \cup \cdots \cup H_{k+1} \cup H'_k \cup \cdots \cup H'_1$.

The colour updates define the main phases of the strategy $\widetilde{\sigma}$. A *k-event* is either a k-shift or a k-cancellation. By a *k-month* we mean a maximal period with only $<k$-events. By a *k-year* we mean a maximal period with only $<k$-cancellations and $\leq k$-shifts. This hierarchy of k-events mirrors in some sense the hierarchical structure of $|V| \cdot \|E\|$-generated perfect half-spaces.

6.3.3. *Counters Updates.* The purpose of the counters is to determine when to perform the k-events, and to choose the appropriate H'_ks for k-shifts. Together with the current path, they also provide the current energy level, which equals $w(\gamma) + \sum_W \mathsf{c}(d, W) \cdot W$ throughout the play, where W ranges over all cycle weights, and help the strategy contain this energy within a small hypercube. The strategy thus aims at bounding these counters.

Initially, all the counters $\mathsf{c}(k, W)$ have value 0. Each such counter keeps track of how many simple cycles with weight W have been formed since the latest $>k$-event, where it was reset to 0. The behaviour in presence of $\leq k$-shifts and $<k$-cancellations is very transparent: their values are incremented each time a cycle of weight $W \neq \mathbf{0}$ is formed, and they are never decreased during a k-year. As we are going to see, the k-cancellation operation breaks down the straightforward link between the counter values and numbers of occurrences of cycle weights, by subtracting positive numbers from counters. This preserves the invariant on the current energy level: counter decreases performed in a k-cancellation always correspond to removing a (non-simple) cycle whose total weight is 0.

Let us define for all $1 \le k \le d$ the *k-soft bound*

$$\mathcal{U}(k) \stackrel{\text{def}}{=} (4|V| \cdot \|E\|)^{2k(d+2)^2} . \tag{10}$$

We say that the k-soft bound *holds* for a cycle weight W if $\mathsf{c}(k, W) < \mathcal{U}(k)$. Whenever the k-soft bound fails for a cycle weight W in $\widehat{H_k} \stackrel{\text{def}}{=} \mathsf{span}(H_k) \setminus \overline{H_k}$, the strategy $\widetilde{\sigma}$ should avoid any further increases, and attempts to perform a k-shift to some H'_k. The aim is for every cycle weight W with $\mathsf{c}(k, W) \ge \mathcal{U}(k)$ to belong to $\widehat{H'_k}$.

If no such H'_k exists, then, by standard results in integer linear algebra (see Appendix A), all the counters $\mathsf{c}(k', W)$ for $k' \ge k$ s.t. $\mathsf{c}(k, W) \ge \mathcal{U}(k)$ can be cancelled out by subtracting an appropriate multiple of a small positive integral solution of a system of linear equations, so that the k-soft upper bounds are re-established.

This strategy shows a statement dual to Lemma 6.2, and thereby entails both the equivalence of infinite bounding games with first-cycle bounding games and the small hypercube property of Lemma 3.1:

Lemma 6.4. *If σ is winning for Player 1 in $G(V, E, d)$ from some vertex v_0, then $\widetilde{\sigma}$ is winning for Player 1 in $\Gamma(V, W, d)$ from v_0, and ensures energy levels of norm at most $(4|V| \cdot \|E\|)^{2(d+2)^3}$.*

6.3.4. $\widetilde{\sigma}$ *Summarised.* Let us first summarise the definition of $\widetilde{\sigma}$. At any Player-1 vertex, $\widetilde{\sigma}$ chooses the edge that σ specifies for history γ and perfect half-space $\mathcal{H}$. After any move that leads to a vertex not occurring in γ, the memory of $\widetilde{\sigma}$ is updated only by extending γ. Otherwise, a cycle C is formed, and the memory is updated as follows:

- Cycle C is cut out of γ. For all $k \in \{1, 2, \dots, d\}$, counters $\mathsf{c}(k, w(C))$ are incremented, unless $w(C) = \mathbf{0}$.
- If all soft upper bounds hold, that is if for all $k \in \{1, 2, \dots, d\}$ and all simple-cycle weights $W \in \widehat{H_k}$ we have $\mathsf{c}(k, W) < \mathcal{U}(k)$, then the memory update is finished.
- Otherwise, let $k \in \{1, 2, \dots, d\}$ be the largest for which the k-soft upper bound $\mathsf{c}(k, W) < \mathcal{U}(k)$ fails for some $W \in \widehat{H_k}$.
- (k-shift) If there is a $|V| \cdot \|E\|$-generated open half-space H of $\mathsf{span}(H_k)$ such that the k-soft upper bound holds for all simple-cycle weights $W \in \widehat{H}$, then denoting by H'_k the $<$-minimal such H, $\mathcal{H}$ is replaced by the $\prec$-minimal perfect half-space of form $H_d \cup \dots \cup H_{k+1} \cup H'_k \cup \dots \cup H'_1$. All counters $\mathsf{c}(k', W)$, where $k' \in \{1, 2, \dots, k-1\}$ and W is a simple-cycle weight, are reset to 0.
- (k-cancellation) Otherwise, let $W_1, W_2, \dots, W_n$ be all the non-zero simple-cycle weights in $\mathsf{span}(H_k)$ that fail the k-soft upper bound, and let $\mathbf{A}$ be the matrix whose columns are the vectors $W_1, W_2, \dots, W_n$. Then by duality and existence of small positive integer solutions of systems of linear equations (Lemma A.2, Lemma A.4 and Proposition 2.2), it follows that $\mathbf{Ax} = \mathbf{0}$ has a solution in positive integers bounded by $\mathcal{S}(k) \stackrel{\text{def}}{=} (2(|V| \cdot \|E\| + 1))^{(k+2)^2}$. The perfect half-space $\mathcal{H}$ is replaced by the $\prec$-minimal $|V| \cdot \|E\|$-generated perfect half-space

of the form $H_d \cup \cdots \cup H_{k+1} \cup H'_k \cup \cdots \cup H'_1$. Let

$$u(k) \stackrel{\text{def}}{=} (4|V| \cdot \|E\|)^{(2k-1)(d+2)^2} \tag{11}$$

define the *k-hard* bound

$$\mathcal{U}(k) + u(k) \ . \tag{12}$$

For every W_i and every $k' \in \{k, k+1, \ldots, d\}$, the value of $\mathsf{c}(k', W_i)$ is replaced by

$$\mathsf{c}(k', W_i) - u(k) \cdot \mathbf{x}(i) \ . \tag{13}$$

All counters $\mathsf{c}(k', W)$, where $k' \in \{1, 2, \ldots, k-1\}$ and W is a simple-cycle weight, are reset to 0. A k-cancellation is *well-defined* if all the differences in (13) are non-negative.

We prove Lemma 6.4 through a sequence of claims. The first claim shows that k-cancellations are always well-defined:

Claim 6.5. *Every k-cancellation is well-defined.*

Proof. We need to show that, just before the k-cancellation, for every cycle weight W_i in $\mathsf{span}(H_k)$ that fails the k-soft upper bound, and all $k' \geq k$, $\mathsf{c}(k', W_i) \geq u(k) \cdot \mathbf{x}(i)$. Indeed,

$$\begin{aligned}
\mathsf{c}(k', W_i) &\geq \mathsf{c}(k, W_i) && \text{by monotonicity of the counters for } k, \\
&\geq \mathcal{U}(k) && \text{since } W_i \text{ fails the } k\text{-soft upper bound}, \\
&= (4|V| \cdot \|E\|)^{(2k-1)(d+2)^2} \cdot (4|V| \cdot \|E\|)^{(d+2)^2} \\
&\geq u(k) \cdot \mathcal{S}(k) && \text{since } \mathcal{S}(k) \stackrel{\text{def}}{=} (2(|V| \cdot \|E\| + 1))^{(k+2)^2}, \\
&\geq u(k) \cdot \mathbf{x}(i) && \text{by Lemma A.4.}
\end{aligned}$$

□

As explained before, the k-soft bound $\mathcal{U}(k)$ in (10) is employed by $\widetilde{\sigma}$ to trigger a k-event and a change of strategy to avoid cycles with weight inside some perfect half-spaces. However, this change of strategy might allow a few more instances of those cycles to be formed—but, crucially, no more than $u(k)$ further instances. The k-hard bound in (12) is therefore enforced.

This informal argument is proven formally in the following Claim 6.6. It entails in particular that the d-hard bound is *always* enforced, since $\mathsf{span}(H_d)$ is the whole space $\mathbb{Q}^d$:

Claim 6.6. *For all $k \in \{1, 2, \ldots, d\}$ and for all cycle weights W in $\mathsf{span}(H_k)$,*

(1) *(k-soft bound) at the beginning of every k-year, $\mathsf{c}(k, W) < \mathcal{U}(k)$, and*
(2) *(k-hard bound) $\mathsf{c}(k, W) < \mathcal{U}(k) + u(k)$.*

Proof. We prove the two statements by nested induction, first on k and second on the sequence of k-years seen so far.

Let us start with (1). For the initial k-year, and for k-years that begin just after a $>k$-shift or a $>k$-cancellation, since then $\mathsf{c}(k, W) = 0$, (1) holds trivially. We are left with the case of a k-year that begins just after a k-cancellation. We can assume using the secondary induction hypothesis that (2) holds at the end of the previous k-year. Consider then some W_i in $\mathsf{span}(H_k)$ that fails the k-soft upper bound just before that k-cancellation. At that time, since $\mathsf{span}(H_k)$ was not changed by the k-cancellation, (2) applies and $\mathsf{c}(k, W_i) < \mathcal{U}(k) + u(k)$. Therefore, at the beginning of the k-year, $\mathsf{c}(k, W_i) < \mathcal{U}(k) + u(k) - u(k) \cdot \mathbf{x}(i)$, and thus $\mathsf{c}(k, W_i) < \mathcal{U}(k)$ since $\mathbf{x}(i) > 0$.

By the secondary induction, it remains to establish (2) for every k-year such that (1) held at its beginning—this is the heart of the proof. Let $H_k^1, H_k^2, \ldots, H_k^N$ be the sequence of k-dimensional open half-spaces considered during the N k-months spanned by the current k-year so far, where $N \leq \mathcal{L}(k)$ by Proposition 5.1. We know that all these open half-spaces define the same vector space $\mathsf{span}(H_k^1) = \mathsf{span}(H_k^2) = \cdots = \mathsf{span}(H_k^N)$; let W belong to that space.

If W satisfies the soft bound, there is nothing to be done. Otherwise, by the construction of $\widetilde{\sigma}$ and the assumption of (1) at the beginning of the k-year, there exists a *first* k-month in this sequence, say the Lth for some $1 \leq L < N$, after which W fails the soft bound onward. Then, during the k-months $1, \ldots, L$, $\mathsf{c}(k, W) < \mathcal{U}(k)$, and for all $n \in \{L+1, \ldots, N\}$, we know that w belongs to the closure $\overline{H_k^n}$.

Consider the nth k-month for $n \in \{L+1, \ldots, N\}$ in the current k-year; we want to bound the increase on $\mathsf{c}(k, W)$ during that k-month. There are two cases:

If $W \in H_k^n$: then the k-dimensional space in the current colour is left unchanged during the nth k-month. In turn, this means that no vertex of the game graph can be visited twice during that k-month while forming a cycle of weight W, as otherwise σ would allow to form a cycle with effect inside $H_d \cup \cdots \cup H_k^n$ and Player 1 would lose. Therefore, cycles with weight W that are closed during the nth k-month can only be formed by consuming edges from the simple path at the beginning of the k-month. Hence, $\mathsf{c}(k, W)$ can be increased by at most $|V|$.

Otherwise: W belongs to the boundary $\overline{H_k^n} \setminus H_k^n$ of H_k^n and thus $k > 1$. In this case, during the nth k-month, $\mathsf{c}(k, W)$ can only be increased by at most the maximal value of $\mathsf{c}(k-1, W)$ during the same k-month. This is because $\mathsf{c}(k-1, W)$ is 0 at the beginning of the nth k-month, and thereafter it can only decrease through $<k$-cancellations (or that k-month would have ended), which decrease $\mathsf{c}(k, W)$ by the same value. By the main induction hypothesis for (1) with $W \in \mathsf{span}(H_{k-1}) \subseteq (\overline{H_k^n} \setminus H_k^n)$, $\mathsf{c}(k-1, W)$ is less than $\mathcal{U}(k-1) + u(k-1)$.

We conclude that, during the current k-year, since $N - L \leq \mathcal{L}(k)$,

- if $k = 1$, $\mathsf{c}(k, W) - \mathcal{U}(k)$ is less than $\mathcal{L}(1) \cdot |V| = 2|V| < u(1)$, and
- if $k > 1$,

$$\begin{aligned}
\mathsf{c}(k, W) - \mathcal{U}(k) &< \mathcal{L}(k) \cdot \max(|V|, \mathcal{U}(k-1) + u(k-1)) \\
&< \mathcal{L}(k) \cdot 2\mathcal{U}(k-1) \\
&< (4|V| \cdot \|E\|)^{(d+2)^2}/2 \cdot 2(4|V| \cdot \|E\|)^{2(k-1)(d+2)^2} \\
&= u(k) \ . \qquad \square
\end{aligned}$$

Proof of Lemma 6.4. By the previous claims, $\widetilde{\sigma}$ is winning for Player 1 in the infinite bounding game, and thanks to the d-hard bound, it ensures that the norm of the current energy is bounded by $\|w(\gamma)\| + \sum_W (\mathcal{U}(d) + u(d)) \cdot \|W\|$

where W ranges over the total weights of simple cycles in the game graph, and $w(\gamma)$ is the weight of a simple path. Hence the norms $\|w(\gamma)\|$ and $\|W\|$ are bounded by $(|V| \cdot \|E\|)^d$. Finally, there are at most $(2(|V|\|E\|)^d + 1)^d$ different total weights of simple cycles W. ☐

7. Concluding Remarks

In this paper, we have shown in Theorem 3.3 and Theorem 3.4 that fixed-dimensional energy games can be solved in pseudo-polynomial time, regardless of whether the initial credit is arbitrary or fixed. For the variant with given initial credit, this closes a large complexity gap between the TOWER upper bounds of Brázdil, Jančar, and Kučera [3] and the lower bounds of Courtois and Schmitz [9], and also settles the complexity of simulation problems between VASS and finite state systems [9]:

Corollary 7.1. *The given initial credit problem for energy games is* 2EXPTIME-*complete, and* EXPTIME-*complete in fixed dimension* $d \geq 4$.

The main direction for extending these results is to consider a *parity* condition on top of the energy condition. Abdulla, Mayr, Sangnier, and Sproston [1] show that multi-dimensional energy parity games with given initial credit are decidable. They do not provide any complexity upper bounds—although one might be able to show TOWER upper bounds from the memory bounds on winning strategies shown by Chatterjee et al. [8, Lemma 3]—, leaving a large complexity gap with 2EXPTIME-hardness. This gap also impacts the complexity of *weak simulation* games between VASS and finite state systems [1].

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Appendix A. Linear Algebra

An Alternatives Lemma. Given a norm M in $\mathbb{N}$, we write $\mathbb{Z}_M^{\pm}$ for the set of integers $\{-M, \ldots, M\}$. We say that a vector space, cone, or half-space in $\mathbb{Q}^d$ is *M-generated* iff it can be generated by vectors in $(\mathbb{Z}_M^{\pm})^d$. We use *Weyl's Theorem*:[1]

Theorem A.1 (Weyl [19], Theorem 1)**.** *Any d-dimensional cone generated by a finite set $\mathbf{A}$ in $\mathbb{Q}^d$ is the intersection of a finite number of closed half-spaces, where the boundary of each half-space contains $d-1$ linearly independent vectors from $\mathbf{A}$.*

Lemma A.2. *Suppose $\mathbf{A} \subseteq (\mathbb{Z}_M^{\pm})^d$ is contained in an M-generated subspace S of $\mathbb{Q}^d$. Either $\mathbf{A}$ is contained in some M-generated closed half-space of S, or $\sum_{\mathbf{a}\in\mathbf{A}} \mathbb{Q}_{>0}\mathbf{a}$ contains the zero vector.*

Proof. If the $\mathsf{cone}(\mathbf{A})$ is not the whole space $\mathsf{span}(\mathbf{A})$, then by Theorem A.1, it is contained in an M-generated closed half-space $\overline{H}$ of $\mathsf{span}(\mathbf{A})$. Since S is M-generated, it is easy to obtain from $\overline{H}$ an M-generated closed half-space of S that contains $\mathsf{cone}(\mathbf{A}) \supseteq \mathbf{A}$.

[1]'In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain.' [18]

Otherwise, if $\mathsf{cone}(\mathbf{A})$ is the whole space $\mathsf{span}(\mathbf{A})$, it contains in particular the vectors $-\sum_{\mathbf{a}\in\mathbf{A}}\mathbf{a}$ (from $\mathsf{span}(\mathbf{A})$) and $\sum_{\mathbf{a}\in\mathbf{A}}\mathbf{a}$ (from $\mathsf{cone}(\mathbf{A})$), and thus $\sum_{\mathbf{a}\in\mathbf{A}}\mathbb{Q}_{>0}\mathbf{a}$ contains the zero vector. □

Small Solutions. We also use a lemma that bounds the positive integral solutions on systems of linear equations. The lemma is a corollary of the following result of von zur Gathen and Sieveking [16]:

Theorem A.3 (von zur Gathen and Sieveking [16])**.** *Let* $\mathbf{A}$*,* $\mathbf{b}$*,* $\mathbf{C}$*,* $\mathbf{d}$ *be* $m\times n$*-,* $m\times 1$*-,* $p\times n$*-,* $p\times 1$*-matrices respectively with integer entries. The rank of* $\mathbf{A}$ *is* r*, and* s *is the rank of the* $(m+p)\times n$*-matrix* $\left(\begin{smallmatrix}\mathbf{A}\\\mathbf{C}\end{smallmatrix}\right)$*. Let* M *be an upper bound on the absolute values of those* $(s-1)\times(s-1)$*- or* $s\times s$*-subdeterminants of the* $(m+p)\times(n+1)$*-matrix* $\left(\begin{smallmatrix}\mathbf{A}&\mathbf{b}\\\mathbf{C}&\mathbf{d}\end{smallmatrix}\right)$*, which are formed with at least* r *rows from* $(\mathbf{A},\mathbf{b})$*. If* $\mathbf{Ax}=\mathbf{b}$ *and* $\mathbf{Cx}\geq\mathbf{d}$ *have a common integer solution, then they have one with coefficients bounded by* $(n+1)M$*.*

Lemma A.4 (Small Solutions Lemma)**.** *Suppose* $\mathbf{A}$ *is a* $d\times n$*-matrix with entries from* $\mathbb{Z}_M^{\pm}$ *and mutually distinct columns. If* $\mathbf{Ax}=\mathbf{0}$ *has a solution in positive rationals, then it has a solution in positive integers bounded by* $(2(M+1))^{(r+2)^2}$*, where* r *is the rank of* $\mathbf{A}$*.*

Proof. We can assume that $d=r$. Apply Theorem A.3 with $\mathbf{b}$ the d-dimensional zero vector, $\mathbf{C}$ the n-dimensional identity matrix and $\mathbf{d}$ the n-dimensional vector of ones. The absolute value of any subdeterminant of $\left(\begin{smallmatrix}\mathbf{A}&\mathbf{0}\\\mathrm{Id}&\mathbf{1}\end{smallmatrix}\right)$ is at most $n^{d+1}M^d$. Since $n\leq(2M+1)^d$, we have that

$$(n+1)n^{d+1}M^d\leq 2(2(M+1))^d(2(M+1))^{d(d+1)}M^d=\\ 2^{(d+1)^2}(M+1)^{d(d+3)}\leq(2(M+1))^{(d+2)^2}.\quad\square$$

References

[1] P. A. Abdulla, R. Mayr, A. Sangnier, and J. Sproston. Solving parity games on integer vectors. In *Concur 2013*, volume 8052 of *LNCS*, pages 106–120. Springer, 2013. doi:10.1007/978-3-642-40184-8_9.

[2] P. A. Abdulla, M. F. Atig, P. Hofman, R. Mayr, K. N. Kumar, and P. Totzke. Infinite-state energy games. In *CSL-LICS 2014*. ACM, 2014. doi:10.1145/2603088.2603100.

[3] T. Brázdil, P. Jančar, and A. Kučera. Reachability games on extended vector addition systems with states. In *ICALP 2010*, volume 6199 of *LNCS*, pages 478–489. Springer, 2010. doi:10.1007/978-3-642-14162-1_40.

[4] T. Brázdil, K. Chatterjee, A. Kučera, and P. Novotný. Efficient controller synthesis for consumption games with multiple resource types. In *CAV 2012*, volume 7358 of *LNCS*, pages 23–38. Springer, 2012. doi:10.1007/978-3-642-31424-7_8.

[5] J. Chaloupka. Z-reachability problem for games on 2-dimensional vector addition systems with states is in P. *Fund. Inform.*, 123(1):15–42, 2013. doi:10.3233/FI-2013-798.

[6] K. Chatterjee and Y. Velner. Hyperplane separation technique for multidimensional mean-payoff games. In *Concur 2013*, volume 8052 of *LNCS*, pages 500–515. Springer, 2013. doi:10.1007/978-3-642-40184-8_35.

[7] K. Chatterjee, L. Doyen, T. A. Henzinger, and J.-F. Raskin. Generalized mean-payoff and energy games. In *FSTTCS 2010*, volume 8 of *LIPIcs*, pages 505–516. LZI, 2010. doi:10.4230/LIPIcs.FSTTCS.2010.505.

[8] K. Chatterjee, M. Randour, and J.-F. Raskin. Strategy synthesis for multi-dimensional quantitative objectives. *Acta Inf.*, 51(3–4):129–163, 2014. doi:10.1007/s00236-013-0182-6.

[9] J. Courtois and S. Schmitz. Alternating vector addition systems with states. In *MFCS 2014, Part I*, volume 8634 of *LNCS*, pages 220–231. Springer, 2014. doi:10.1007/978-3-662-44522-8_19.

[10] A. Ehrenfeucht and J. Mycielski. Positional strategies for mean payoff games. *Int. J. Game Theor.*, 8(2):109–113, 1979. doi:10.1007/BF01768705.

[11] U. Fahrenberg, L. Juhl, K. G. Larsen, and J. Srba. Energy games in multiweighted automata. In *ICTAC 2011*, volume 6916 of *LNCS*, pages 95–115. Springer, 2011. doi:10.1007/978-3-642-23283-1_9.

[12] P. Jančar and F. Moller. Checking regular properties of Petri nets. In *Concur '95*, volume 962 of *LNCS*, pages 348–362. Springer, 1995. doi:10.1007/3-540-60218-6_26.

[13] L. Juhl, K. G. Larsen, and J. Raskin. Optimal bounds for multiweighted and parametrised energy games. In *Theories of Programming and Formal Methods—Essays Dedicated to Jifeng He on the Occasion of his 70th Birthday*, volume 8051 of *LNCS*, pages 244–255. Springer, 2013. doi:10.1007/978-3-642-39698-4_15.

[14] S. Lasota. EXPSPACE lower bounds for the simulation preorder between a communication-free Petri net and a finite-state system. *Inf. Process. Lett.*, 109(15): 850–855, 2009. doi:10.1016/j.ipl.2009.04.003.

[15] C. Rackoff. The covering and boundedness problems for vector addition systems. *Theor. Comput. Sci.*, 6(2):223–231, 1978. doi:10.1016/0304-3975(78)90036-1.

[16] J. von zur Gathen and M. Sieveking. A bound on solutions of linear integer equalities and inequalities. *Proc. Amer. Math. Soc.*, 72(1):155–158, 1978. doi:10.1090/S0002-9939-1978-0500555-0.

[17] H. Weyl. Elementare Theorie der konvexen Polyeder. *Comm. Math. Helv.*, 7:290–306, 1934–35.

[18] H. Weyl. Invariants. *Duke Math. J.*, 5(3):489–502, 1939. doi:10.1215/S0012-7094-39-00540-5.

[19] H. Weyl. The elementary theory of convex polyhedra. In *Contributions to the Theory of Games*, volume 24 of *Ann. Math. Stud.*, pages 3–18. Princeton University Press, 1950. Translation from the German original [17].

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