Adequacy and Complete Axiomatization for Timed Modal Logic

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Abstract

In this paper we develop the metatheory for Timed Modal Logic (TML), which is the modal logic used for the analysis of timed transition systems (TTSs). We solve a series of long-standing open problems related to TML. Firstly, we prove that TML enjoys the Hennessy-Milner property and solve one of the open questions in the field. Secondly, we prove that the set of validities are not recursively enumerable. Nevertheless, we develop a strongly-complete proof system for TML. Since the logic is not compact, the proof system contains infinitary rules, but only with countable sets of instances. Thus, we can involve topological results regarding Stone spaces, such as the Rasiowa-Sikorski lemma, to complete the proofs.

Keywords: Non-compact modal logics, complete axiomatization, timed modal logic.

1 Introduction

In the areas of embedded and cyber-physical systems, more than two decades of research have been dedicated to developing quantitative modeling and specification formalisms that allow for the construction of systems with guaranteed functional and non-functional properties. In particular, many embedded systems are highly safety critical, with hard constraints on real-time behaviors being essential. Here, the notion of *timed transition system* and *timed automata* [4] have proven extremely convenient for modeling purposes and are now routinely used for the analysis of real-time communication protocols and control programs.

Accompanying the timed transition systems, a variety of *timed temporal logics* have been introduced as convenient ways of capturing requirements to real-time systems. These logics include MTL [14], MITL [5], TPTL [7, 8] and ECL [23] all providing timed extensions of LTL. Similarly TCTL [2], T_μ [13] and L_ν [17] provide timed extensions of CTL and the modal μ-calculus. Emphasis has been on detailed investigation of decidability and complexity of model checking and satisfiability checking, identifying the importance of (absence of) punctual constraints in the logic [5, 16] as well as of the choice of semantics of timed automata (point-wise or continuous) [19–21]. Also, with the purpose of gaining decidability, satisfiability checking given constraints on number of clocks and size of constants in a satisfying timed automata has been considered [17]. In all of the aforementioned

logics, quantitative requirements are either obtained by decorating modalities with timing constraints, or by using clocks in formulas.

The presence of time in all these settings makes it difficult to address fundamental meta-theoretical questions regarding the timed logics, such as *adequacy* [1] or the development of complete proof systems. Regarding the axiomatization, there exists a series of results proved in very restricted settings, such as for TPTL [11] that only looks to models with discrete time, or for ECTL [24] under the restriction of models with finite variability – only finitely many state changes can occur in a finite amount of time. Other attempts, such as [9], failed already in achieving soundness results, as argued in [24].

In this paper we take the challenge of developing a strongly-complete proof system for the most basic Timed Modal Logic (TML) defined for the most general model of real-time systems, without any restriction on the nature of time. Our models are timed labelled transition systems (TTSs), which generalize timed automata [1]: their transitions are labeled with actions or time delays (real numbers). Our logic is the non-recursive fragment of L_ν [17], which generalizes the logic of [1]. In these settings we solve a series of open problems.

Firstly, we prove that TML is adequate not only for timed automata, but in general, for entire class of TTSs. This settles an open problem and disproves a belief often found in the literature, e.g., [1] – that such a logic is not sufficiently expressive to characterize timed-bisimulation. In proving this result, we use a novel exploitation of formulas with free clock variables.

Secondly, we prove that the satisfiability problem for TML is undecidable despite its restrictive expressive power, thus generalizing the known undecidability result of satisfiability for TCTL [2]. Moreover, we show that the set of TML-validities is not recursively-enumerable. This implicitly means that any complete proof system will generate a non-recursively enumerable set of provable formulas, and makes one wonder whether this logic can, in fact, be axiomatized at all - see e.g., the discussion in [6].

We prove that TML can be axiomatized and we develop a proof system for it that is *strongly-complete* [2] for the TTS-semantics. However, TML is not *compact*: there exists an infinite set of formulas that admits no model while all its finite subsets have models. For this reason our axiomatization must contains infinitary proof rules; they reflect Archimedean properties of the rational numbers used to interpret clocks. Due to these infinitary rules, the proofs of our system cannot be enumerated. Nevertheless, these rules have countable sets of instances.

Our axioms provide a set of sufficient conditions that characterizes the concept of time in models like TTSs and timed automata. They reflect properties such as persistence, continuity, linearity, density of time, synchronization, or the fact that all clocks measure the same time flow. Most of these axioms are similar to axioms seen in other modal logics and these relations open interesting further questions.

An other important contribution of the paper is the construction of the canonical model, which generalizes for the case of a (non-compact) higher-order modal logic the classic

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[1] A logic is adequate when its semantic equivalence over the class of models coincides with bisimilarity.

[2] Strong completeness means that [$\Phi \vdash \phi$ iff $\Phi \models \phi$], where $\Phi \models \phi$ denotes that all the models of the set Φ of formulas are also models of the formula ϕ; and $\Phi \vdash \phi$ denotes that ϕ is provable from Φ.

filtration construction usually used for propositional modal logics. This follows the line opened by the second and the third author in collaboration with Panangaden and Kozen in [15]; it involves complex topological and model-theoretical arguments, such as the Rasiowa-Sikorski lemma, which are essential in achieving the main results and are pointing to a general methodology for constructing canonical models for non-compact modal logics.

This paper does not aim at addressing problems related to timed automata, nor to model verification, model construction or model checking. Our purpose is to clarify the open problems of model theory for TML. The interesting questions regarding timed automata-semantics will be addressed in a future work.

2 Preliminaries

Hereafter we fix the notation used in the paper.

Kleene equality. Given a partial function $f : X \to Y$, we write $f(x) \stackrel{\sim}{=} f(x')$ for $x, x' \in X$ to denote the fact that $f(x)$ and $f(x')$ are simultaneously well-defined and whenever they are well-defined, they are equal.

Orders on reals. We use $\trianglelefteq$ and $\trianglerighteq$ to range over the set $\{\leq, \geq\}$ such that $\{\trianglelefteq, \trianglerighteq\} = \{\leq, \geq\}$; this means that $\trianglelefteq$ can either be interpreted as $\leq$ or as $\geq$; if $\trianglelefteq$ represents one of the two, then $\trianglerighteq$ denotes the other. Similarly, we use $\triangleleft$ and $\triangleright$ to range over the set $\{<, >\}$ such that $\{\triangleleft, \triangleright\} = \{<, >\}$. Moreover, $x \triangleleft y$ means $[x \trianglelefteq y$ and $x \neq y]$, and similarly for $\triangleright$ and $\trianglerighteq$. We use $\bowtie$ to denote to an arbitrary element of the set $\{\leq, \geq, <, >\}$.

Interpretations on reals. Given a set $\mathcal{K}$, an interpretation of $\mathcal{K}$ (on non-negative reals) is a function $i : \mathcal{K} \to \mathbb{R}_{\geq 0}$; if $x \in \mathcal{K}$ and $r \in \mathbb{R}_{\geq 0}$, we denote by $i[x \mapsto r]$ the interpretation j of $\mathcal{K}$ such that $j(x) = r$ and $j(y) = i(y)$ for $y \neq x$. The arithmetic operations on interpretations are defined pointwise; 0 is used for the constant null interpretation and $[x \mapsto r]$ denotes $0[x \mapsto r]$.

The Rasiowa-Sikorski Lemma [10, 22] is a result with important applications in logic.

Definition 2.1 Let $\mathcal{B}$ be a Boolean algebra and let $T \subseteq \mathcal{B}$ such that T has a greatest lower bound $\bigwedge T$ in $\mathcal{B}$. An ultrafilter (maximal filter) U is said to *respect* T if $T \subseteq U$ implies that $\bigwedge T \in U$.

If $\mathcal{T}$ is a family of subsets of $\mathcal{B}$, we say that an ultrafilter U respects $\mathcal{T}$ if it respects every member of $\mathcal{T}$.

Lemma 2.2 (Rasiowa–Sikorski [22]) *For any Boolean algebra $\mathcal{B}$ and any countable family $\mathcal{T}$ of subsets of $\mathcal{B}$, each member of which has a meet in $\mathcal{B}$, and for any nonzero $x \in \mathcal{B}$, there exists an ultrafilter in $\mathcal{B}$ that contains x and respects $\mathcal{T}$.*

3 Timed Transition Systems

A *timed transition system* (TTS) [1] is a labeled transition system that uses both actions and time delays as transition labels. The delay transitions describe the time flow and con-

sequently are continuous and deterministic. Here we propose an equivalent definition for TTS that encodes the time in an algebraic format and simplifies our future developments.

Definition 3.1 [Timed Transition System] A *timed labeled transition system* (TTS) is a tuple $\mathcal{W} = (M, \Sigma, \theta, \oplus)$ where M is a non-empty set of *states*, Σ a non-empty set of *actions*, $\theta : M \times \Sigma \to 2^M$ is the *action-labelling function* and $\oplus : \mathbb{R}_{\geq 0} \times M \rightharpoonup M$ is a partial function that encodes the *delay transitions*; for arbitrary $m \in M$ and $d, d' \in \mathbb{R}_{\geq 0}$,
1. $0 \oplus m = m$;
2. $d \oplus (d' \oplus m) \stackrel{\sim}{=} (d + d') \oplus m$.

Whenever it is defined, $d \oplus m$ denotes a time delay d applied to the state m. Condition 1 guarantees that a zero-delay is always well-defined and it does not change the state of the system; and condition 2 expresses that time is both *additive* and *deterministic*. As usual, instead of $m' \in \theta(m, a)$ we will write $m \stackrel{a}{\to} m'$.

In the rest of this paper we fix the set Σ and omit it in the description of TTSs.

A *timed bisimulation* is a relation that equates states of a TTS with identical behaviours.

Definition 3.2 [Timed Bisimulation] Given a TTS $\mathcal{W} = (M, \theta, \oplus)$, a *timed bisimulation* is an equivalence relation $R \subseteq M \times M$ such that whenever $(m, n) \in R$, for all $a \in \Sigma$ and $d \in \mathbb{R}_{\geq 0}$,
- if $m \stackrel{a}{\to} m'$, then there exists $n' \in M$ s.t. $n \stackrel{a}{\to} n'$ and $(m', n') \in R$;
- if $d \oplus m$ is well-defined, then $d \oplus n$ is well-defined and $(d \oplus m, d \oplus n) \in R$.

As for the other types of bisimulation, the previous definition can be extended to define the time bisimulation between distinct TTSs by considering bisimulation relations on their disjoint union. *Time bisimilarity* is the largest time-bisimulation relation; if $\mathcal{W}_i = (M_i, \theta_i, \oplus_i)$, $m_i \in M_i$ for $i = 1, 2$ and m_1 and m_2 are bisimilar, we write $(m_1, \mathcal{W}_1) \sim (m_2, \mathcal{W}_2)$.

4 Timed Modal Logic

In this section we introduce the Timed Modal Logic (TML) that encodes properties of TTSs. It is defined for a countable set $\mathcal{K}$ of *clocks* that we consider fixed in what follows. It contains Henessy-Milner operators $[a]\phi$ for the actions $a \in \Sigma$, where Σ is the fixed set of actions for which we have defined TTSs in the previous section. In addition, it is endowed with *time inequalities* of type $x \trianglelefteq r$ for rational values r that evaluate the clock $x \in \mathcal{K}$ at the current state; with *delay quantifiers* $\mathbb{W}\phi$ that predicate properties for any time-delay of the current state; and *clock quantifiers* $\forall x.\phi$ that predicate properties for any interpretation of the clock x at the current state.

Definition 4.1 [Syntax] For arbitrary $r \in \mathbb{Q}_{\geq 0}$ and $a \in \Sigma$,

$$\mathcal{L} : \quad \phi := \bot \mid x \trianglelefteq r \mid \phi \to \phi \mid [a]\phi \mid \mathbb{W}\phi \mid \forall x.\phi.$$

Let $\mathfrak{I}(\mathcal{K})$ be the set of interpretations of $\mathcal{K}$. The semantics of TML is defined for an arbitrary TTS $M = (M, \theta, \oplus)$, $m \in M$ and $i \in \mathfrak{I}(\mathcal{K})$ as follows.

- $M, m, i \models \bot$ – never;
- $M, m, i \models x \trianglelefteq r$ if $i(x) \trianglelefteq r$;
- $M, m, i \models \phi \rightarrow \psi$ if $M, m, i \models \psi$ whenever $M, m, i \models \phi$;
- $M, m, i \models [a]\phi$ if [for any $m' \in M$ s.t. $m \xrightarrow{a} m'$, $M, m', i \models \phi$];
- $M, m, i \models \mathbb{W}\phi$ if [for any $d \in \mathbb{R}_{\geq 0}$ s.t. $d \oplus m$ is well-defined, $M, d \oplus m, i + d \models \phi$];
- $M, m, i \models \forall x.\phi$ if [for any $t \in \mathbb{R}_{\geq 0}, M, m, i[x \mapsto t] \models \phi$].

We also use, in addition, all the boolean operators defined as usual and the De Morgan duals of the modal operators: $\langle a \rangle \phi \overset{df}{=} \neg[a]\neg\phi$, $x \triangleright r \overset{df}{=} \neg(x \trianglelefteq r)$, $\exists\!\!\!\exists \phi \overset{df}{=} \neg(\mathbb{W}\neg\phi)$, and $\exists x.\phi \overset{df}{=} \neg(\forall x.\neg\phi)$. Other derived operators used in what follows are:
$(x = r) \overset{df}{=} (x \leq r) \wedge (x \geq r)$ and $\forall(x \bowtie r).\phi \overset{df}{=} \forall x.(x \bowtie r \rightarrow \phi)$, $\bowtie \in \{\leq, \geq, <, >\}$.

In TML we can express the reset operator used in [1, 13, 17] by
$$x \text{ in } \phi \overset{df}{=} \forall x.(x = 0 \rightarrow \phi).$$
Whenever it is not the case that $M, m, i \models \phi$, we write $M, m, i \not\models \phi$. We say that a formula ϕ is *satisfiable* if there exists at least one TTS that satisfies it in one of its states under at least one interpretation; ϕ is a *validity* if it is satisfied in all states of any TTS under any interpretation - in this case we write $\models \phi$. For an arbitrary set $\Phi \subseteq \mathcal{L}$, we write $\Phi \models \phi$ if all the models of all the formulas in Φ are also models of ϕ.

4.1 Undecidability of TML.

Individual formulae of TML express properties which only depend on the behaviour of a TTS up to a finite action-depth, thus making TML significantly less expressive than TCTL [3]. In a number of papers it has been shown how recursive extensions of TML – e.g. extensions with the ability to define logical properties recursively – enable the encoding of TCTL, while maintaining decidability of model-checking.

As stated in the theorem below, despite its limited expressive power the question of satisfiability for TML is (highly) undecidable, as is the case of TCTL. Formally, given an arbitrary TML formula $\phi \in \mathcal{L}$, it is undecidable whether there exists a TTS $(M, \theta, \oplus)$ with $m \in M$ and interpretation $i \in \mathfrak{I}(\mathcal{K})$ such that $M, m, i \models \phi$.

Theorem 4.2 (Undecidability of TML) *The satisfiability question for TML is Σ_1^1-hard, hence undecidable.*

Proof. We show that we can reduce the TML satisfiability question into the question as to whether a non-deterministic 2-counter machines has a computation with the initial location being visited infinitely often. This last question is known to be Σ_1^1-hard. Our proof is similar to the one in [3] and for this reason it is presented in the Appendix. ∎

The undecidability of satisfiability for TML implies, as usual, the undecidability of validity for TML. In fact, Theorem 4.2 proves that the set of validities is not recursively enumerable.

5 Adequacy of TML

The Hennessy-Milner property (H-Mp), which states for a logic that bisimilarity of the models coincides with the semantic equivalence induced by the logic, is currently an open problem for timed logic. In [1] it was proven that the closed formulas (without free clock variables) cannot characterize bisimilarity. In this section we prove the H-Mp for TML, therefore we solve the adequacy problem.

Before proceeding with the proof, observe the essential role of interpretations in the semantics of TML. Consider the two TTSs depicted in Figure 1, where the horizontal lines represent the time flow from the initial states m and m' respectively. The two systems in the initial states can delay forever and they can both take an a-transition to states that satisfy *END* after each delay $2 - \frac{2}{2^n}$ and $2 + \frac{2}{2^n}$ for each integer $n > 0$. However, the two systems differ: m can take an a-transition after the delay 2, while m' cannot.

If we consider an interpretation $i \in \mathfrak{I}(\mathcal{K})$ s.t. $i(x) \notin \mathbb{Q}_{\geq 0}$ for any $x \in \mathcal{K}$, one can notice that m and m' satisfy exactly the same formulas. However, this is not true if we consider an interpretation i' s.t. $i'(x) = 0$ for some $x \in \mathcal{K}$, since $m, i' \models \exists (x = 2 \wedge \langle a \rangle \top)$ and $m', i' \models \neg(\exists(x = 2 \wedge \langle a \rangle \top))$.
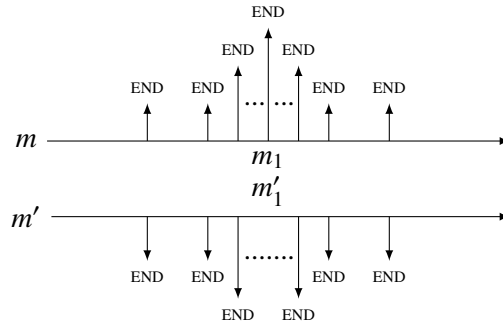

Fig. 1. non-bisimilar TTSs

To clarify this situation, we start from analysing how the formulas satisfied by a model under a certain interpretation change when we change the interpretation.

It is useful in what follows to identify the set $\mathcal{K}(\phi)$ of the free clocks in a formula $\phi \in \mathcal{L}$, defined by: $\mathcal{K}(\bot) = \emptyset, \mathcal{K}(x \bowtie r) = \{x\}, \mathcal{K}(\phi \rightarrow \psi) = \mathcal{K}(\phi) \cup \mathcal{K}(\psi), \mathcal{K}([a]\phi) = \mathcal{K}(\mathbb{W}\phi) = \mathcal{K}(\phi), \mathcal{K}(\forall x.\phi) = \mathcal{K}(\phi) \setminus \{x\}$.

For a clock variable $y \in \mathcal{K}$ that does not appear in the syntax of ϕ and $x \in \mathcal{K}(\phi)$, we denote by $\phi\{y/x\}$ the formula obtained by uniformly substituting all the occurrences of x in ϕ by y.

Definition 5.1 Given two rational interpretations $f_-, f_+ \colon \mathcal{K} \rightarrow \mathbb{Q}$ and a bijection $\sigma : \mathcal{K} \rightarrow \mathcal{K}$, for any formula $\phi \in \mathcal{L}$ let $\phi +_\sigma {}^{f_-}/_{f_+}$ be defined as follows, where $x \bowtie t$ for $t < 0$ should be read as $x \geq 0$:

6

$$\bot +_\sigma {}^{f_-}/_{f_+} \overset{\text{df}}{=} \bot \qquad\qquad (\phi \wedge \psi) +_\sigma {}^{f_-}/_{f_+} \overset{\text{df}}{=} (\phi +_\sigma {}^{f_-}/_{f_+}) \wedge (\psi +_\sigma {}^{f_-}/_{f_+})$$

$$(x \le r) +_\sigma {}^{f_-}/_{f_+} \overset{\text{df}}{=} \sigma(x) \le (r + f_+(x)) \qquad (x \ge r) +_\sigma {}^{f_-}/_{f_+} \overset{\text{df}}{=} \sigma(x) \ge (r + f_-(x))$$

$$(\neg\phi) +_\sigma {}^{f_-}/_{f_+} \overset{\text{df}}{=} \neg(\phi +_\sigma {}^{f_+}/_{f_-}) \qquad ([a]\phi) +_\sigma {}^{f_-}/_{f_+} \overset{\text{df}}{=} [a](\phi +_\sigma {}^{f_-}/_{f_+})$$

$$(\mathbb{W}\phi) +_\sigma {}^{f_-}/_{f_+} \overset{\text{df}}{=} \mathbb{W}(\phi +_\sigma {}^{f_-}/_{f_+}) \qquad (\forall x.\phi) +_\sigma {}^{f_-}/_{f_+} \overset{\text{df}}{=} \forall \sigma(x).(\phi +_\sigma {}^{f_-[x \mapsto 0]}/_{f_+[x \mapsto 0]})$$

Whenever $f_- = f_+ = f$, we write $+_\sigma f$; whenever σ is the identity on $\mathcal{K}$, we write $+{}^{f_-}/_{f_+}$.

The following lemma, which can be proved by induction over the structure of formulas, and its corollaries characterize the relationships between the formulas satisfied by the same model under different interpretations.

Lemma 5.2 *Let $M = (M, \theta, \oplus)$ be a TTS, $\sigma : \mathcal{K} \to \mathcal{K}$, $\delta : \mathcal{K} \to \mathbb{R}$ and $f_-, f_+ : \mathcal{K} \to \mathbb{Q}$ s.t. $f_- \le \delta \le f_+$. Then for any $m \in M$, $\phi \in \mathcal{L}$ and $i \in \mathfrak{I}(\mathcal{K})$, if $i + \delta \ge 0$ then:*
$$M, m, i \models \phi \implies M, m, (i + \delta) \circ \sigma^{-1} \models \phi +_\sigma {}^{f_-}/_{f_+}.$$

The implication from right to left is not always true, since f_- and f_+ are approximations of δ. However we have equivalences in some concrete cases.

Corollary 5.3 *Let $M = (M, \theta, \oplus)$ be a TTS and $f : \mathcal{K} \to \mathbb{Q}$. Then for any $m \in M$, $\phi \in \mathcal{L}$ and $i \in \mathfrak{I}(\mathcal{K})$, if $i + f \ge 0$ then:*
$$M, m, i \models \phi \iff M, m, i + f \models \phi + f.$$

Corollary 5.4 *Let $M = (M, \theta, \oplus)$ be a TTS and $\delta : \mathcal{K} \to \mathbb{R}$. Then for any $m \in M$, $\phi \in \mathcal{L}$ and $i \in \mathfrak{I}(\mathcal{K})$, if $i + \delta \ge 0$ and for any $x \in \mathcal{K}(\phi)$, $\delta(x) = 0$, then:*
$$M, m, i \models \phi \iff M, m, i + \delta \models \phi.$$

With these, we can proceed with the proof of the H-Mp. We say that a TTS $M = (M, \theta, \oplus)$ has the *finite image property* if for any action $a \in \Sigma$ and any $m \in M$, $\theta(m, a)$ is finite.

Theorem 5.5 (Hennessy-Milner Theorem) *Consider a TTS $M = (M, \theta, \oplus)$ satisfying the finite image property. Then for any $m, n \in M$:*
$$m \sim n \quad \textit{iff} \quad \forall i \in \mathfrak{I}(\mathcal{K}), \phi \in \mathcal{L}, M, m, i \models \phi \Leftrightarrow M, n, i \models \phi.$$

Proof. ($\Leftarrow$): We prove that R is a bisimulation
$$R = \{(m, n) \mid \forall i, \forall \phi, \ M, m, i \models \phi \Leftrightarrow M, n, i \models \phi\}.$$

Supp. that $m \xrightarrow{a} m'$. If there exists no $n' \in M$ s.t $n \xrightarrow{a} n'$, then $M, n, i \models [a]\bot$, implying $M, m, i \models [a]\bot$ - contradicting the assumption. Let $F = \{n_k \mid n \xrightarrow{a} n_k\}$, which is finite since the TTS is image finite. Suppose that $(m', n_k) \notin R$ for any k. Then, there exists $i_k \in \mathfrak{I}(\mathcal{K})$ and $\phi_k \in \mathcal{L}$ s.t. $M, m', i_k \models \phi_k$ and $M, n_k, i_k \models \neg\phi_k$, for any k. For every $x \in \mathcal{K}(\phi_k)$, consider a new variable x_k distinct from all the other variables. Let $\phi'_k = \phi_k\{x_k/x\}$, for every k and let $i' \in \mathfrak{I}(\mathcal{K})$ s.t. $i'(x_k) = i_k(x)$ for any k. We have: $M, m', i' \models \bigwedge_k \phi'_k$ and $M, n_k, i' \models \neg\phi'_k$ by Corollary 5.4. Then $M, m, i' \models \langle a \rangle \bigwedge_k \phi'_k$ and $M, n, i' \models [a] \bigvee \neg\phi'_k$ - contradiction.

Supp. that $m' = d \oplus m$. For $r \in \mathbb{Q}_{\ge 0}$, $r \ge d$, $M, m, i[x \mapsto r - d] \models \exists(x = r)$, which implies $M, n, i[x \mapsto r - d] \models \exists(x = r)$. Hence, there exists $n' \in M$ s.t. $n' = d \oplus n$. We prove that $(m', n') \in R$. For any $i \in \mathfrak{I}(\mathcal{K})$ and $\phi \in \mathcal{L}$, $M, m', i \models \phi$ implies, using Corollary 5.4, that $M, m', i[x \mapsto 0] \models \phi$ for any $x \notin \mathcal{K}(\phi)$; further, applying Corollary 5.3,

$M, m', i[x \mapsto 0] + r \models \phi + r$. Then, $M, m, i[x \mapsto 0] + r - d \models \exists\!\!\!\!\exists(x = r \wedge (\phi + r))$ and $M, n, i[x \mapsto 0] + r - d \models \exists\!\!\!\!\exists(x = r \wedge (\phi + r))$. Consequently, $M, n, i[x \mapsto 0] + r - d \models \mathbb{W}(x = r \rightarrow (\phi + r))$. This implies that $M, n', i[x \mapsto 0] + r \models x = r \rightarrow (\phi + \lceil d \rceil)$ ($\lceil d \rceil$ denotes the smallest natural number bigger than d), which implies $M, n', i[x \mapsto 0] + r \models (\phi + r)$. So $M, n', i + r \models \phi + r$, and using Corollary 5.3, $M, n', i \models \phi$.

The symmetry of R proves the other cases. ∎

6 Metatheory for TML

In this section we develop a proof system for TML. We prove that TML is not compact and consequently requires infinitary rules. However, we demonstrate that our proof system is strongly-complete for the TTS-semantics. The completeness proofs consist of the construction of a canonical model. Being the role of interpretations in the semantics of TTS, the canonical model is not constructed from maximal consistent sets, as for other modal logics. This is because a maximal consistent set does not identify a state of a TTS; it identifies a set in the presence of a fixed interpretation. Moreover, the same maximal consistent set might be satisfied by non-bisimilar models under different interpretations. To cope with all this complex situation we propose a new method for constructing canonical models.

6.1 Axiomatization for TML

Modal prefixes are words $w \in Mod^*$ over the alphabet
$$Mod = \{[a] \mid a \in \Sigma\} \cup \{\mathbb{W}\} \cup \{\forall x. \mid x \in \mathcal{K}\},$$
e.g., $\forall x.[a]\mathbb{W}[b][c], \forall x.\mathbb{W}, [a], \varepsilon \in Mod^*$, where ε is the empty word. A *context* is a word formed by a modal prefix $w \in Mod^*$ concatenated with the metavariable X; e.g., $[a]X$, $\forall x. \mathbb{W} X$, $\forall x.[a] \mathbb{W} [b][c]X$ are contexts. To emphasize the presence of the metavariable we will use the functional representation of type $C[X]$ for contexts; this will allow us to instantiate the metavariable with elements from $\mathcal{L}$. For example, if $C[X] = \forall x.[a]\mathbb{W}[b][c]X$ is a context, then $C[(x \geq r)] = \forall x.[a]\mathbb{W}[b][c](x \geq r) \in \mathcal{L}$. Also $\varepsilon[X]$ is a context - the empty one - and for $\phi \in \mathcal{L}$, $\varepsilon[\phi] = \phi$.

The axiomatic system of TML includes, in addition to the axioms and the rules of propositional logic, the axioms and the rules in Table 1. They are stated for arbitrary $\phi, \psi \in \mathcal{L}$, $r, s, t \in \mathbb{Q}_{\geq 0}$, $a \in \Sigma$, $\square \in Mod$ and arbitrary context $C[X]$.

A formula ϕ is *provable*, denoted by $\vdash \phi$, if it can be proven from the given axioms and rules. We say that ϕ is *consistent*, if $\phi \rightarrow \bot$ is not provable. Given a set Φ of formulas, we say that Φ *proves* ϕ, $\Phi \vdash \phi$, if from the formulas of Φ and from the axioms one can prove ϕ, eventually using Boolean or infinitary-Boolean reasoning. In other words, we assume that the provability is closed under the rule
$$\Phi \cup \{\phi\} \vdash \psi \text{ iff } \Phi \vdash \phi \rightarrow \psi,$$
for arbitrary (possibly infinite) sets $\Phi \subseteq \mathcal{L}$. Φ is *consistent* if it is not the case that $\Phi \vdash \bot$.

The axioms (A1)-(A3) state simple facts about the clock values. The axiom (A4) reflects the fact that action-transitions in a TTS happen instantaneously.

(A1): $\vdash x \geq 0$	(A11): $\vdash x \trianglelefteq r \wedge y \trianglelefteq s \rightarrow \mathbb{W}(x \trianglelefteq r + t \rightarrow y \trianglelefteq s + t)$
(A2): $\vdash (x \geq r) \vee (x \leq r)$	(A12): $\vdash \forall x.\phi \rightarrow \phi + {}^{[x \mapsto r]}/_{[x \mapsto s]},\ r \leq s$
(A3): $\vdash x \leq r \rightarrow \neg(x \geq s),\ r < s$	(A13): $\vdash \forall(x \leq r).\mathbb{W}(x = s \rightarrow \phi) \rightarrow$
(A4): $\vdash x \trianglelefteq r \rightarrow [a](x \trianglelefteq r)$	$\quad\quad \forall(x = r).\mathbb{W}(s \leq x \leq s + r \rightarrow \phi)$
(A5): $\vdash \mathbb{W}\phi \rightarrow \phi \wedge \mathbb{W}\mathbb{W}\phi$	(A14): $\vdash \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$
(A6): $\vdash \mathbb{W}\phi \rightarrow \mathbb{W}(r \leq x \leq s \rightarrow \phi),\ r \leq s$	(R1): If $\vdash \phi$, then $\vdash \Box\phi$
(A7): $\vdash x \geq r \rightarrow \mathbb{W}(x \geq r)$	(R2): $\{C[x \trianglelefteq r] \mid r \triangleright s\} \vdash C[x \trianglelefteq s]$
(A8): $\vdash \exists(x = r \wedge \phi) \rightarrow \mathbb{W}(x = r \rightarrow \phi)$	(R3): $\{C[x \geq r] \mid r \in \mathbb{Q}_{\geq 0}\} \vdash C[\bot]$
(A9): $\vdash \exists(x \leq r \wedge \mathbb{W}\phi) \rightarrow \mathbb{W}(x \geq r \rightarrow \phi)$	(R4): $\{C[\phi + {}^{[x \mapsto r]}/_{[x \mapsto s]}] \mid r \leq s\} \vdash C[\forall x.\phi]$
(A10): $\vdash \exists(x \leq r) \wedge \exists(x \geq r) \rightarrow \exists(x = r)$	(R5): $\{C[\mathbb{W}(x \leq s \rightarrow \phi)] \mid s \in \mathbb{Q}_{\geq 0}\} \vdash C[\mathbb{W}\phi]$

Table 1
The Axiomatic System of TML

The axioms (A5)-(A10) describe the nature of time in TTSs. Thus, (A5) states that the time is linear and 0-delays do not perturb the system; (A6) that the time is persistent; (A7) that the flow of time is unidirectional (the past and the future are disjoint); (A8) that the time is deterministic; (A9) and (A10) that the time is continuous. The axiom (A11) guarantees that all the clocks measure the same time flow.

The axiom (A12) together with the infinitary rule (R4) describe the fact that the information provided by a clock variable x in a formula prefixed by $\forall x$ is superfluous. The role of (A13) is to characterize the interaction between the two types of universal quantifiers.

The axiom (A14) and the rule (R1) state that all the box-like operators of TML are normal.

The rules (R2)-(R5) are infinitary and have instances for any possible context. For instance, the formulas below are instances of the rules (R2) and (R3) respectively.
$$\{[a](x \geq r) \mid r < s\} \vdash [a](x \geq s),$$
$$\{\mathbb{W}[a](x \geq r) \mid r \in \mathbb{Q}_{\geq 0}\} \vdash \mathbb{W}[a]\bot.$$
The rule (R2) reflects the Archimedean property of rationals. (R3) guarantees that the value of any clock is finite in any model under any interpretation.

By induction on the structure of possible proofs, we prove the soundness.

Theorem 6.1 (Soundness) The axiomatic system in Table 1 is sound with respect to the TTS-semantics, i.e., for arbitrary $\Phi \subseteq \phi$ and $\phi \in \mathcal{L}$,
$$\Phi \vdash \phi \text{ implies } \Phi \models \phi.$$

6.2 Non-Compactness of TML

We have seen in Section 4 that the set of validities of TML are not recursively enumerable. This means that any complete axiomatization of TML must be infinitary. Otherwise, we could enumerate all the proofs and the set of provable formulas, which in a complete logic coincides with the set of validities, is recursively enumerable - contradiction!

There is also a model theoretic result that ensures us the necessity of having infinitary rules:

9

Theorem 6.2 (Non-Compactness of TML) *TML with the TTS-semantics is not compact, i.e., there exists an infinite set $\Phi \subseteq \mathcal{L}$ such that each finite subset of Φ admits a model but Φ does not admit any model.*

Proof. The result derives from the soundness of the infinitary rules and each rule can be used to produce examples of such sets. Consider, for example, $s \in \mathbb{Q}_{\geq 0}$ and the set
$$\Phi = \{x \geq r \mid r < s\} \cup \{x < s\}.$$
Since for any $i \in \mathfrak{I}(\mathcal{K})$, $i(x) \geq r$ for each $r < s$ implies $i(x) \geq s$, Φ does not admit any model. However, it is not difficult to construct a model for just any finite subset of Φ. ∎

6.3 Canonical Model and Completeness

In this section we prove that the axiomatic system of TML is not only sound, but also complete for the TTS-semantics, meaning that for arbitrary $\Phi \subseteq \mathcal{L}$ and $\phi \in \mathcal{L}$, $\Phi \models \phi$ iff $\Phi \vdash \phi$. To complete this proof it is sufficient to show that any consistent formula has a model. In the following we construct a canonical model, which is a TTS such that each consistent formula is satisfied at some state under some interpretation. In modal logics such a construction is usually done using maximally consistent sets of formulas as states.

For some set $\Lambda \subseteq \mathcal{L}$, we say that $\Phi \subseteq \mathcal{L}$ is Λ-*maximally consistent* if Φ is consistent and no formula of Λ can be added to Φ without making it inconsistent. Φ is *maximally-consistent* if it is $\mathcal{L}$-maximally-consistent.

The aforementioned technique to construct canonical models cannot be applied directly for TML because to the same state of a given TTS corresponds different maximally-consistent sets of formulas under different interpretations. We generalize this construction to cope with the complexity of TML. To the best of our knowledge, the following construction is original.

For the beginning, we observe that given a maximally-consistent set of formulas, the information contained about a given clock is complete.

Let Ω be the set of $\mathcal{L}$-maximally consistent sets.

Lemma 6.3 *For arbitrary $\Lambda \in \Omega$ and $x \in \mathcal{K}$,*
$$\sup\{r \in \mathbb{Q}^+ \mid x \geq r \in \Lambda\} = \inf\{r \in \mathbb{Q}^+ \mid x \leq r \in \Lambda\} \in \mathbb{R}_{\geq 0}.$$

Proof. Let $A = \{r \in \mathbb{Q}^+ \mid x \geq r \in \Lambda\}$ and $B = \{r \in \mathbb{Q}^+ \mid x \leq r \in \Lambda\}$. (A1) guarantees that $A \neq \emptyset$ and if $B = \emptyset$, we can derive a contradiction from (R3) for $C[X] = X$.

Since the two sets are non-empty, the sup and inf exist. Moreover, (R3) can also be used to prove that $\sup A < \infty$. Let $\sup A = u$ and $\inf B = v$. If $u < v$, there exists $r \in \mathbb{Q}^+$ such that $u < r < v$. Hence, $x \leq r \in \Lambda$, which contradicts $r \leq v$. If $v < u$, there exists $r_1, r_2 \in \mathbb{Q}^+$ such that $v < r_1 < r_2 < u$. Hence, $x \leq r_i, x \geq r_i \in \Lambda$ for $i = 1, 2$. Since $r_2 - r_1 > 0$, (A3)$\vdash x \geq r_2 \to \neg(x \leq r_1)$, which proves the inconsistency of Λ - contradiction. ∎

The previous lemma demonstrates that to each maximally-consistent set corresponds a unique interpretation of clocks that we will identify in what follows using the function $\mathscr{I} : \Omega \longrightarrow \mathfrak{I}(\mathcal{K})$ defined for arbitrary $\Lambda \in \Omega$ and $x \in \mathcal{K}$ by

$$\mathscr{I}(\Lambda)(x) = \sup\{r \in \mathbb{Q}^+ \mid x \geq r \in \Lambda\} \in \mathbb{R}_{\geq 0}.$$

Since $\mathscr{I}(\Lambda)$ synthesize only the information in Λ regarding the clocks, there exist disjoint sets $\Lambda_1, \Lambda_2 \in \Omega$ s.t. $\mathscr{I}(\Lambda_1) = \mathscr{I}(\Lambda_2)$; this equality defines an equivalence relation on Ω and the equivalence classes are in one to one correspondence with the interpretations in $\mathfrak{I}(\mathcal{K})$.

Observe that any state in a model of TML corresponds to a function from $\mathfrak{I}(\mathcal{K})$ to Ω: given a model, each interpretation identifies a maximally-consistent set of formulas satisfied by that model under the given interpretation. Consequently, to construct the canonical model we will have to take as states not maximally-consistent sets of formulas, but functions from interpretations to maximally-consistent sets. However, not just any function $\gamma : \mathfrak{I}(\mathcal{K}) \rightarrow \Omega$ is a good candidate for a model, because between the maximally-consistent sets associated to a model under different interpretation there are certain coherence conditions as the ones described in Lemma 5.2 and corollaries 5.3 and 5.4. These coherences are formally described in the next definition.

Definition 6.4 A function $\gamma : \mathfrak{I}(\mathcal{K}) \rightarrow \Omega$ is *coherent*, if for any $i \in \mathfrak{I}(\mathcal{K})$, any bijection σ on $\mathcal{K}$, any $\delta : \mathcal{K} \rightarrow \mathbb{R}$ s.t. $i + \delta \geq 0$, and any $f_-, f_+ : \mathcal{K} \rightarrow \mathbb{Q}$ s.t. $f_- \leq \delta \leq f_+$,
1. $(\mathscr{I} \circ \gamma)(i) = i$;
2. if $\phi \in \gamma(i)$, then $\phi +_\sigma {}^{f_-}/_{f_+} \in \gamma((i + \delta) \circ \sigma^{-1})$.

The first fundamental result that we prove is that any maximally-consistent set Λ belongs to the image $\gamma(\mathfrak{I}(\mathcal{K}))$ of some coherent function γ. After constructing the canonical model on the set of coherent functions, this result will guarantee that any maximally-consistent set is satisfied by some model under some interpretation.

Lemma 6.5 *For any $\Lambda \in \Omega$, there exists a coherent function γ s.t. $\gamma(\mathscr{I}(\Lambda)) = \Lambda$.*

Proof. We say that a set $S \subseteq \Omega$ is *coherent* if there exists a set $T \subseteq \mathfrak{I}(\mathcal{K})$ and a bijection $\gamma : T \rightarrow S$ that satisfies the two conditions of Definition 6.4. S is *maximally-coherent* if $T = \mathfrak{I}(\mathcal{K})$. Observe that the bijection that defines a maximally-coherent set is a coherent function in the sense of Definition 6.4.

We prove, using a transfinite induction, that any coherent set has a maximally-coherent extension. And this proves the existence of γ, because $\{\Lambda\}$ is coherent.

I. Firstly, observe that $S \subseteq \Omega$ is coherent iff for any $\Lambda_1, \Lambda_2 \in \Omega$, with $i_s = \mathscr{I}(\Lambda_s)$, $s = 1, 2$,

$$\Lambda_1 + (i_2 - i_1) \subseteq \Lambda_2 \text{ and } \Lambda_2 + (i_1 - i_2) \subseteq \Lambda_1,$$

where for arbitrary $\Lambda' \in \Omega$ and $\delta : \mathcal{K} \rightarrow \mathbb{R}$,

$$\Lambda' + \delta = \{\phi + {}^{f_-}/_{f_+} \mid f_-, f_+ : \mathcal{K} \rightarrow \mathbb{Q}, f_- \leq \delta \leq f_+\}.$$

Moreover, $\Lambda_1 + (i_2 - i_1) \subseteq \Lambda_2$ iff $\Lambda_2 + (i_1 - i_2) \subseteq \Lambda_1$.

II. If $i = \mathscr{I}(\Lambda)$ and $i' \in \mathfrak{I}(\mathcal{K})$, then there exists $\Lambda' \in \Omega$ s.t. $\mathscr{I}(\Lambda') = i'$ and $\{\Lambda, \Lambda'\}$ is coherent. To prove this, we firstly use Definition 5.1 and the axioms to prove that $\Lambda + (i' - i)$ is consistent. Then, we observe that all the infinitary rules have countable sets of instances, which allows us to apply Rasiowa-Sikorski lemma to conclude that $\Lambda + (i' - i)$ must have

11

a maximal-consistent extension Λ'. Since $\Lambda + (i' - i) \subseteq \Lambda'$, we also have $\Lambda' + (i - i') \subseteq \Lambda$. Hence, $\{\Lambda, \Lambda'\}$ is coherent.

III. If $S = \{\Lambda_0, \Lambda_1, ..\Lambda_k\}$ is a finite coherent set, $i_s = \mathscr{I}(\Lambda_s)$, $s = \overline{1..k}$ and $j \in \mathfrak{I}(\mathcal{K})$, then there exists $\Lambda_j \in \Omega$ s.t. $\mathscr{I}(\Lambda_j) = j$ and $S \cup \{\Lambda_j\}$ is coherent. Supp. $\Lambda = \Lambda_0$. It is sufficient to show that the following set is consistent, which is not a difficult task:
$$X = \bigcup_{s=\overline{0..k}} (\Lambda_s + (j - i_s)).$$

IV. If $S = \{\Lambda_s \mid s \in \mathbb{N}\}$ is a countable coherent set (supp. $\Lambda = \Lambda_0$) and $j \in \mathfrak{I}(\mathcal{K})$. There exists $\Lambda_j \in \Omega$ s.t. $S \cup \{\Lambda_j\}$ is coherent. As before, we need to prove that
$$X = \bigcup_{s \in \mathbb{N}} (\Lambda_s + (j - i_s))$$
is consistent. Let $X_t = \bigcup_{s \leq t} (\Lambda_s + (j - i_s))$ for $t \in \mathbb{N}$. We have previously proved that X_t is consistent for any $t \in \mathbb{N}$. Moreover, observe that $X = \bigcup_{t \in \mathbb{N}} X_t$.

Suppose that X is not consistent. If the contradiction can be proven without involving the infinitary rules, then there must exist one X_t that contains all the formulas in the proof, which contradicts the consistency of X_t. Consequently, if X is inconsistent it is because, when t goes to infinity, the sets $X_t \setminus X_{t-1}$ collect the left-hand side of some of our infinitary rules.

If for arbitrary contexts C, formulas $\phi \in \mathcal{L}$, $r \in \mathbb{Q}_{\geq 0}$ and $x \in \mathcal{K}$ we denote by
$\mathcal{L}_1(C, x) = \{C[x \leq s], C[x \geq s] \mid s \in \mathbb{Q}_{\geq 0}\}$
$\mathcal{L}_2(C, x, \phi) = \{C[\phi + {}^{[x \mapsto s]}/_{[x \mapsto s']}] \mid s, s' \in \mathbb{Q}, s \leq s'\}$,
$\mathcal{L}_3(C, x, \phi, r) = \{C[\mathbb{W}(r \leq x \leq s \rightarrow \phi)] \mid r, s \in \mathbb{Q}_{\geq 0}, r \leq s\}$,
one can prove that for each $t \in \mathbb{N}$, X_t is $\mathcal{L}_1(C, x)$-maximally consistent and for each L of type $\mathcal{L}_2(C, x, \phi)$ or $\mathcal{L}_3(C, x, \phi, r)\}$, either $X_t \cap L = \emptyset$ or X_t is L-maximally consistent. This demonstrates that if in X we can use some infinitary rule, it must be used within some X_t for some $t \in \mathbb{N}$. Consequently, the consistency of X_t proves the consistency of X.

V. If $S \subseteq \Omega$ is an uncountable coherent set and $j \in \mathfrak{I}(\mathcal{K})$, then there exists $\Lambda_j \in \Omega$ s.t. $S \cup \{j\}$ is coherent. Supp. $S = \{\Lambda_s \mid s \in \mathbb{R}\}$, $\Lambda = \Lambda_0$ and $i_s = \mathscr{I}(\Lambda_s)$. As in the other cases it is sufficient to prove that the following set is consistent:
$$Y = \bigcup_{s \in \mathbb{R}} (\Lambda_s + (j - i_s)).$$
Since $Y \subseteq \mathcal{L}$ and $\mathcal{L}$ is countable, Y is countable and consequently we can assume that
$$Y = \bigcup_{s \in \mathbb{N}} (\Lambda_s + (j - i_s)).$$
Further the result is a consequence of IV. ∎

In what follows we define a TTS using the set
$$\Gamma = \{\gamma : \mathfrak{I}(\mathcal{K}) \rightarrow \Omega \mid \gamma \text{ is a coherent function}\}$$
as the support-set of the model and the structure
- $\gamma \xrightarrow{a} \gamma'$ if $[\forall i \in \mathfrak{I}(\mathcal{K}), [a]\phi \in \gamma(i) \Rightarrow \phi \in \gamma'(i)]$;
- $\gamma' = d \oplus \gamma$ if $[\forall i, \mathbb{W}\phi \in \gamma(i) \Rightarrow \phi \in \gamma'(i + d)]$.

Theorem 6.6 (Canonical Model) *The tuple* $\Gamma = (\Gamma, \theta, \oplus)$ *defined above is a TTS.*

Proof. Firstly, we prove that $\gamma_1 = d \oplus \gamma$ and $\gamma_2 = d \oplus \gamma$ implies $\gamma_1 = \gamma_2$.

For any $\forall i \in \mathfrak{I}(\mathcal{K})$ and $x \in \mathcal{K} \setminus \mathcal{K}(\phi)$, $\phi \in \gamma_1(i)$ implies $x = \lceil d \rceil \wedge \phi \in \gamma_1(i[x \mapsto \lceil d \rceil])$, where $\lceil d \rceil$ denote the smallest integer larger than d. Since $\gamma_1 = d \oplus \gamma$,
$$\exists(x = \lceil d \rceil \wedge \phi) \in \gamma(i[x \mapsto \lceil d \rceil] - d),$$
which further implies, due to (A8), that
$$\forall(x = \lceil d \rceil \to \phi) \in \gamma(i[x \mapsto \lceil d \rceil] - d).$$
$x = \lceil d \rceil \to \phi \in \gamma_2(i[x \mapsto \lceil d \rceil])$ since $\gamma_2 = d \oplus \gamma$. Then $\phi \in \gamma_2(i[x \mapsto \lceil d \rceil])$, which implies $\phi \in \gamma_2(i)$ because $x \notin \mathcal{K}(\phi)$. Hence for any $i \in \mathfrak{I}(\mathcal{K})$, $\gamma_1(i) \subseteq \gamma_2(i)$. Similarly, $\gamma_2(i) \subseteq \gamma_2(i)$.

Observe now that due to (A5), $\forall \phi \in \gamma(i)$ implies $\phi \in \gamma(i)$, for any $\gamma \in \Gamma$ and $i \in \mathfrak{I}(\mathcal{K})$. Hence, $0 \oplus \gamma = \gamma$.

Now we prove that $d \oplus (d' \oplus m) \tilde{=} (d + d') \oplus m$, i.e. $\exists \gamma_1, \gamma_2$ s.t. $\gamma_1 = d' \oplus \gamma$, $\gamma_2 = d \oplus \gamma_1$ iff $\exists \gamma_3$ s.t. $\gamma_3 = (d + d') \oplus \gamma$.

($\Rightarrow$) Suppose $\exists \gamma_1, \gamma_2$ s.t. $\gamma_1 = d \oplus \gamma$, $\gamma_2 = d' \oplus \gamma_1$. From (A5), $\forall \phi \in \gamma(i)$ implies $\forall \forall \phi \in \gamma(i)$ and since $\gamma_1 = d \oplus \gamma$, $\forall \phi \in \gamma_1(i + d)$. This implies $\phi \in \gamma_2(i + d + d')$ because $\gamma_2 = d' \oplus \gamma_1$. Hence, there exists $\gamma_3 = (d + d') \oplus \gamma$ and, in fact, $\gamma_3 = \gamma_2$.

($\Leftarrow$) Supp. that $\exists \gamma_3$ s.t. $\gamma_3 = (d + d') \oplus \gamma$.

Firstly, we prove that there exists γ_1 s.t. $\gamma_1 = d' \oplus \gamma$ by constructing it. For each $i \in \mathfrak{I}(\mathcal{K})$, $\gamma(i) +_\sigma d' \subseteq \gamma_1(i + d')$. It is not difficult to verify that for arbitrary $i, j \in \mathfrak{I}(\mathcal{K})$, $\gamma(i) +_\sigma d' + (j - i) \subseteq \gamma(j) +_\sigma d'$ and
$$\bigcup_{x \in \mathcal{K}} \{x \trianglelefteq r \mid r \in \mathbb{Q}^+, i(x) \trianglelefteq r\} \subseteq \gamma(i) +_\sigma d'.$$
Now, we can use a similar transfinite construction as in Lemma 6.5 to construct entire γ_1.

Secondly, we need to prove that there exists $d \oplus \gamma_1$. For any $i \in \mathfrak{I}(\mathcal{K})$, $\forall \phi \in \gamma_1(i)$ implies $x \leq r \wedge \forall \phi \in \gamma_1(i)$ for $r \in \mathbb{Q}^+$ s.t. $i(x) \leq r \leq i(x) + d$. Then,
$$x \leq r + \lceil d' \rceil \wedge \forall(\phi + \lceil d' \rceil) \in \gamma_1(i + \lceil d' \rceil).$$
Now because $\gamma_1 = d' \oplus \gamma$, we obtain
$$\exists(x \leq r + \lceil d' \rceil \wedge \forall(\phi + \lceil d' \rceil)) \in \gamma(i + \lceil d' \rceil - d').$$
From this we get, by applying (A9), that
$$\forall(x \geq r + \lceil d' \rceil \to \forall(\phi + \lceil d' \rceil)) \in \gamma(i + \lceil d' \rceil - d').$$
Since $\gamma_3 = (d + d') \oplus \gamma$,
$$x \geq r + \lceil d' \rceil \to \forall(\phi + \lceil d' \rceil) \in \gamma_3(i + \lceil d' \rceil + d).$$
And because $r \leq i(x) + d$, $x \geq r + \lceil d' \rceil \in \gamma_3(i + \lceil d' \rceil + d)$. We then get that $\forall(\phi + \lceil d' \rceil) \in \gamma_3(i + \lceil d' \rceil + d)$, implying $\forall \phi \in \gamma_3(i + d)$, which using (A5) gives $\phi \in \gamma_3(i + d)$. Hence, there exists $d \oplus \gamma_1 = \gamma_3$. ∎

Lemma 6.7 (Truth Lemma) *For any $\psi \in \mathcal{L}$, $i \in \mathfrak{I}(\mathcal{K})$ and $\gamma \in \Gamma$,*
$$\Gamma, \gamma, i \models \psi \text{ iff } \psi \in \gamma(i).$$

Proof. Induction on ψ.

[The case $\psi = \forall x.\phi$]: $\Gamma, \gamma, i \models \forall x.\phi$ iff $\Gamma, \gamma, i[x \mapsto u] \models \phi$ for any $u \in \mathbb{R}_{\geq 0}$, i.e., $\phi \in \gamma(i[x \mapsto u])$ for all $u \in \mathbb{R}_{\geq 0}$.
($\Longrightarrow$) $\phi \in \gamma(i[x \mapsto u])$ implies $\phi + {}^{[x \mapsto r]}/_{[x \mapsto s]} \in \gamma(i)$ for all $r, s \in \mathbb{Q}$ s.t $r \leq u - i(x) \leq s$. This must happen for any $u \in \mathbb{R}_{\geq 0}$, so for any $r, s \in \mathbb{Q}$, $\phi + {}^{[x \mapsto r]}/_{[x \mapsto s]} \in \gamma(i)$. Hence, $\forall x.\phi \in \gamma(i)$ by rule (R4).
($\Longleftarrow$) Using (A12), $\forall x.\phi \in \gamma(i)$ implies $\phi + {}^{[x \mapsto r]}/_{[x \mapsto s]} \in \gamma(i[x \mapsto u)$ for any $u \in \mathbb{R}_{\geq 0}$ and

$r, s \in \mathbb{Q}$ s.t. $r \le u - i(x) \le s$. Consequently, $\forall x.\phi \in \gamma(i[x \mapsto u])$ by (R4). And (A12), $\phi \in \gamma(i[x \mapsto u])$ for all $u \in \mathbb{R}_{\ge 0}$.

[The case $\psi = \mathbb{W}\phi$]: $\Gamma, \gamma, i \models \mathbb{W}\phi$ iff [for any $d \in \mathbb{R}_{\ge 0}$ and $\gamma' \in \Gamma$ s.t. $\gamma' = d \oplus \gamma$, $\Gamma, \gamma', i + d \models \phi$] iff $\phi \in \gamma'(i + d)$ by the inductive hypothesis.

($\Longrightarrow$) $\phi \in \gamma'(i+d)$ implies $(x = r) \wedge \phi \in \gamma'((i+d)[x \mapsto r])$, for any $r \in \mathbb{Q}^+, r \ge d$. Applying (A8), $\mathbb{W}(x = r \to \phi) \in \gamma(i[x \mapsto r - d])$, for any $r \in \mathbb{Q}^+, r \ge d$. This implies $x \le r \to \mathbb{W}(x = r \to \phi) \in \gamma(i[x \mapsto u])$ for any $u \in \mathbb{R}_{\ge 0}$. Then, $\forall(x \le r).\mathbb{W}(x = r \to \phi) \in \gamma(i)$. Applying (A13), $\forall(x = r).\mathbb{W}(r \le x \le 2r \to \phi) \in \gamma(i)$, which further implies $\forall(x = 0).\mathbb{W}(0 \le x \le r \to \phi) \in \gamma(i)$ by (A12) and (R4). Using (R5), $\forall(x = 0).\mathbb{W}(x \ge 0 \to \phi) \in \gamma(i)$, which implies $\forall(x = 0).\mathbb{W}(x \ge 0 \to \phi) \in \gamma(i[x \mapsto 0])$. So, $x = 0 \to \mathbb{W}(x \ge 0 \to \phi) \in \gamma(i[x \mapsto 0])$, which implies $\mathbb{W}(x \ge 0 \to \phi) \in \gamma(i[x \mapsto 0])$. Using (A1), $\mathbb{W}\phi \in \gamma(i[x \mapsto 0])$ and further, $\mathbb{W}\phi \in \gamma(i)$.

($\Longleftarrow$) derives from the definition of $\oplus$.

[The case $\psi = [a]\phi$]: $\Gamma, \gamma, i \models [a]\phi$ iff for any $\gamma' \in \Gamma$ s.t. $\gamma \xrightarrow{a} \gamma'$, $\Gamma, \gamma', i \models \phi$, iff $\phi \in \gamma'(i)$ by induction.

($\Longrightarrow$) Supp. $\langle a\rangle\neg\phi \in \gamma(i)$. Let $A = \{\gamma' \mid \gamma \xrightarrow{a} \gamma'\}$, $B_i = \{\neg\phi\} \cup \{\psi \mid [a]\psi \in \gamma(i)\} \cup \Delta_i$ and $B_j = \{\psi \mid [a]\psi \in \gamma(j)\} \cup \Delta_i$ for any $j \ne i$, where $\Delta_k = \bigcup_{x \in K}\{x \le r \mid r \ge i(x)\} \cup \{x \ge r \mid r \le i(x)\}$.

$\{\psi \mid [a]\psi \in \gamma(i)\} \cup \Delta_i$ and B_j, $j \ne i$ are consistent.

Suppose that B_i is inconsistent. Then, there exists a set $F \subseteq B_i$ s.t. $F \vdash \phi$. If F is finite, (R1) guarantees that $[a]F \vdash [a]\phi$, where $[a]F = \{[a]\rho \mid \rho \in F\}$. Otherwise, $F \vdash \phi$ is (modulo Boolean reasoning possible involving infinite meets) an instance of one of the rules (R2)-(R5); in all these cases, $[a]F \vdash [a]\phi$ is an instance of the same rule for the context $C[X] = [a]X$. Since $F \subseteq B_i$, $[a]F \subseteq \gamma(i)$ implying $[a]\phi \in \gamma(i)$, which contradicts the consistency of $\gamma(i)$. Hence, B_i is consistent.

Now we prove that for any $j, j' \in \mathfrak{I}(K)$, B_j and $B_{j'}$ are such that $B_j + (j' - j) \subseteq B_{j'}$. If $j \ne i$, then for arbitrary $\rho \in B_j$ either $[a]\rho \in \gamma(j)$, or $\rho = x \trianglelefteq r$. In the first case, $[a]\rho +_\sigma {}^{f_-}/_{f_+} \in \gamma(j')$, for all $f_- \le j' - j \le f_+$. So, $\rho +_\sigma {}^{f_-}/_{f_+} \in B_{j'}$. In the second case, since $\rho = x \trianglelefteq r$ is closed under any interpretation transformation, for any $f_- \le j' - j \le f_+$, $\rho +_\sigma {}^{f_-}/_{f_+} \in B_{j'}$.

If $j = i$, consider an arbitrary $\rho \in B_j$. If $\rho \ne \neg\phi$, we get a similar case as above. Otherwise, $\langle a\rangle\rho \in \gamma(i)$, which implies $\langle a\rangle\rho +_\sigma {}^{f_-}/_{f_+} \in \gamma(j)$ for all $f_- \le j - i \le f_+$. So, $\rho +_\sigma {}^{f_-}/_{f_+} \in B_j$.

At this point we can use a similar strategy as in Theorem 6.5 to prove that there exists $\gamma'' \in \Gamma$ s.t. for any $j \in \mathfrak{I}(K)$, $B_j \subseteq \gamma''(j)$. But then, $\gamma'' \in A$, which implies $\phi \in \gamma''(i)$ - contradiction!

Hence, $[a]\phi \in \gamma(i)$.

($\Longleftarrow$) derives from the definition of θ. ∎

Corollary 6.8 *If $\Phi \subseteq \mathcal{L}$ is consistent, there exists $\gamma \in \Gamma$ and $i \in \mathfrak{I}(\mathcal{K})$ s.t. $\Gamma, \gamma, i \models \Phi$.*

Proof. Because the infinitary rules of TML have countable sets of instances, the Rasiowa-Sikorski lemma guarantees the existence of some $\Lambda \in \Omega$ s.t. $\Lambda \supseteq \Phi$. Lemma 6.5 guarantees that there exists $\gamma \in \Gamma$ s.t. $\Lambda = \gamma(\mathscr{I}(\Lambda))$. Hence $\Gamma, \gamma, \mathscr{I}(\Lambda) \models \Phi$ by applying Lemma 6.7. ∎

Corollary 6.8 is a well known equivalent formulation of the strong completeness theorem.

Theorem 6.9 (Strong Completeness) *TML is strongly-complete with respect to the TTS-semantics, i.e., for arbitrary $\Phi \subseteq \mathcal{L}$ and $\phi \in \mathcal{L}$,*

$$\Phi \models \phi \ \text{ implies } \Phi \vdash \phi.$$

7   Conclusions

In this paper we addressed and solved a series of open problems regarding the timed logics and real-time systems. We develop the metatheory for the Timed Modal Logic (TML), which is the most basic logic used in practice for specifying and analyzing timed transition systems (TTSs).

In this paper we demonstrate that TML is adequate for the TTS semantics. We show that its satisfiability problem is undecidable and the set of the validities is not recursively enumerable. Despite this, we develop a strongly-complete proof system for TML. Because TML is not compact, the proof system contains necessarily infinitary rules and these rules also explain why the set of validities is not recursively enumerable. Our axioms characterize the concept of time used in the definition of TTS. Our completeness proof is based on a novel method that generalizes the classic filtration technique used in modal logics for the construction of canonical models. Essential in the proof is the use of the Rasiowa-Sikorski lemma.

All these results open new perspective on real-time systems and on their analysis and reveal new research directions. The proof system contains similar axioms to those of other well-known logics, which makes one think about other possible connections and research perspectives.

References

[1] Luca Aceto, Anna Ingólfsdóttir, Kim Guldstrand Larsen, and Jiri Srba. *Reactive Systems: modelling, specification and verification*. Cambridge University Press, 2007.

[2] Rajeev Alur, Costas Courcoubetis, and David L. Dill. Model-checking in dense real-time. *Information and Computation*, 104(1):2–34, May 1993.

[3] Rajeev Alur, Costas Courcoubetis, and David L. Dill. Model-checking in dense real-time. *Inf. Comput.*, 104(1):2–34, 1993.

[4] Rajeev Alur and David L. Dill. A theory of timed automata. *Theoretical Computer Science*, 126(2):183–235, April 1994.

[5] Rajeev Alur, Tómas Feder, and Thomas A. Henzinger. The benefits of relaxing punctuality. *Journal of the ACM*, 43(1):116–146, January 1996.

[6] Rajeev Alur and Thomas A. Henzinger. Logics and models of real time: A survey. In *REX Workshop*, volume 600, pages 74–106. Springer, 1991.

[7] Rajeev Alur and Thomas A. Henzinger. Real-time logics: Complexity and expressiveness. *Information and Computation*, 104(1):35–77, May 1993.

[8] Rajeev Alur and Thomas A. Henzinger. A really temporal logic. *Journal of the ACM*, 41(1):181–203, January 1994.

[9] Howard Barringer, Ruurd Kuiper, and Amir Pnueli. A really abstract concurrent model and its temporal logic. In *POPL*, pages 173–183. ACM Press, 1986.

[10] R. Goldblatt. On the role of the Baire category theorem in the foundations of logic. *Journal of Symbolic logic*, pages 412–422, 1985.

[11] Thomas A. Henzinger. Half-order modal logic: How to prove real-time properties. In Cynthia Dwork, editor, *PODC*, pages 281–296. ACM, 1990.

[12] Thomas A. Henzinger, Pei-Hsin Ho, and Howard Wong-Toi. A user guide to HyTech. In *TACAS'95*, volume 1019, pages 41–71. Springer-Verlag, 1995.

[13] Thomas A. Henzinger, Xavier Nicollin, Joseph Sifakis, and Sergio Yovine. Symbolic model checking for real-time systems. In *LICS*, pages 394–406, 1992.

[14] Ron Koymans. Specifying real-time properties with metric temporal logic. *Real-Time Systems*, 2(4):255–299, 1990.

[15] Dexter Kozen, Kim G. Larsen, Radu Mardare, and Prakash Panangaden. Stone duality for markov processes. In *LICS*, pages 321–330. IEEE Computer Society, 2013.

[16] Salvatore La Torre and Margherita Napoli. A decidable dense branching-time temporal logic. In *FSTTCS'00*, volume 1974, pages 139–150. Springer-Verlag, 2000.

[17] François Laroussinie, Kim Guldstrand Larsen, and Carsten Weise. From timed automata to logic - and back. In *MFCS*, volume 969, pages 529–539. Springer, 1995.

[18] Kim Gulstrand Larsen, Paul Pettersson, and Wang Yi. UPPAAL in a nutshell. *International Journal on Software Tools for Technology Transfer*, 1(1-2):134–152, October 1997.

[19] Joël Ouaknine and James Worrell. On the decidability of metric temporal logic. In *LICS'05*, pages 188–197. IEEE Comp. Soc. Press, 2005.

[20] Joël Ouaknine and James Worrell. On metric temporal logic and faulty Turing machines. In *FoSSaCS'06*, volume 3921, pages 217–230. Springer-Verlag, 2006.

[21] Joël Ouaknine and James Worrell. On the decidability and complexity of metric temporal logic over finite words. *Logical Methods in Computer Science*, 3(1), March 2007.

[22] H. Rasiowa and R. Sikorski. A proof of the completeness theorem of Gödel. *Fund. Math*, 37:193–200, 1950.

[23] Jean-François Raskin and Pierre-Yves Schobbens. Real-time logics: Fictitious clock as an abstraction of dense time. In *TACAS*, volume 1217, pages 165–182. Springer, 1997.

[24] Pierre-Yves Schobbens, Jean-François Raskin, and Thomas A. Henzinger. Axioms for real-time logics. *Theor. Comput. Sci.*, 274(1-2):151–182, 2002.

[25] Farn Wang. Efficient model-checking of dense-time systems with time-convexity analysis. *Theor. Comput. Sci.*, 467:89–108, 2013.

[26] Sergio Yovine. Kronos: A verification tool for real-time systems. *International Journal on Software Tools for Technology Transfer*, 1(1-2):123–133, October 1997.

Appendix

This appendix contains the details of the proofs of the major results presented in the paper.

Proof. [Proof of Theorem 4.2] Our reduction is similar to the one in [3]. The question as to whether a non-deterministic 2-counter machine has a computation with the initial location being visited infinitely often is Σ_1^1-hard. We show how to reduce this problem into a TML satisfiability question.

Let M be a 2-counter machine with counters X and Y and with $n + 1$ program instructions $\ell_0, \ldots, \ell_n$. The instruction ℓ_n represents termination, and each instruction ℓ_i ($i < n$) is either an increment of the form $[\ell_i : X := X + 1; \text{goto } \ell_j]$ or a decrement of the form $[\ell_i : \text{if } X \neq 0 \text{ then } X := X - 1; \text{goto } \ell_j \text{ else goto } \ell_k]$ or a non-deterministic jump of the form $[\ell_i : \text{goto } \ell_j \text{ or } \ell_k]$. A configuration of M is a triple $\langle \ell_i, x, y \rangle$, where x and y are natural numbers representing the current values of X and Y. A computation of M is a "valid" sequence of configurations starting in $\langle \ell_0, 0, 0 \rangle$ and ending in some configuration of the type $\langle \ell_n, x, y \rangle$.

We encode the computation of M in TML using the actions $\Sigma = \{\ell_0, \ldots, \ell_n, X, Y\}$. We say that a state m of a given TTS $\mathcal{T}$ encodes the configuration $\langle \ell_i, x, y \rangle$ in the interval $[a, b]$, with $a, b \in \mathbb{R}^+$ and $a < b$, iff the following holds:

- $(d \oplus m) \xrightarrow{X}$ for exactly x distinct time-points d in (a, b);

- $(d \oplus m) \xrightarrow{Y}$ for exactly y distinct time-points d in (a, b);

- $(d \oplus m) \xrightarrow{\ell_i}$ for $d = a$;

- $(d \oplus m) \not\longrightarrow \ell_j$ for all time-points d in (a, b) and for all j.

Let $\{\langle \ell_{i(j)}, x_j, y_j \rangle : j \geq 0\}$ be a computation of M (i.e. $i(0) = 0$). We may construct a (closed) TML formula ϕ^M, such that for any TTS $\mathcal{T}$ and any state m, $\mathcal{T}, m \models \phi$ if and only if m encodes the j'th configuration over the interval $[j, j + 1)$ for all $j \geq 0$. Also ϕ^M will ensure that m encodes a computation where ℓ_0 is visited infinitely often. The formula ϕ^M is obtained as the conjunction of a formula expressing the initial configuration, a formula expressing infinite repetition of the initial location and a formula for each instruction of M ensuring that states $(d \oplus m)$ of $\mathcal{T}$ being separated by a delay of 1 correctly encodes the given instruction. The following formula ensures that instructions of M are unique:

(i) $\mathbb{W}(\langle \ell_i \rangle \top \to [\ell_j] \bot)$, for all $i \neq j$ (uniqueness);

The initial configuration may be encoded as the conjunction of the following TML formulae:

(i) $\langle \ell_0 \rangle \top$;

(ii) x in $\mathbb{W}(0 < x < 1 \to [\ell_i] \bot)$, for all $i \geq 0$;

(iii) x in $\mathbb{W}(0 < x < 1 \to ([X] \bot \wedge [Y] \bot))$.

An increment statement of the form $[\ell_i : X := X+1; \text{goto } \ell_j]$ is reflected by the conjunction of the following formulae relating the behaviour in a unit-interval $\{(d \oplus m) : d \in [j, j + 1)\}$ with $(j \oplus m) \xrightarrow{\ell_i}$ with the successor unit-interval:

(i) $\mathbb{W}(\langle\ell_i\rangle\top \to x \text{ in } \exists(x = 1 \land \langle\ell_j\rangle)\top)$;

(ii) $\mathbb{W}(\langle\ell_i\rangle\top \to x \text{ in } \mathbb{W}(0 < x < 1 \to [\ell_j] \perp)$, for all i, j;

(iii) $\mathbb{W}[\langle\ell_i\rangle\top \to x \text{ in } \mathbb{W}(x < 1 \land \langle X\rangle\top \to y \text{ in } \exists(y = 1 \land \langle X\rangle\top))]$;

(iv) $\mathbb{W}[\langle\ell_i\rangle\top \to$
$\quad x \text{ in } \mathbb{W}(x < 1 \land \langle X\rangle\top \land y \text{ in } \mathbb{W}(y > 0 \land x < 1 \to [X] \perp)$
$\qquad \to$
$\qquad\quad y \text{ in } (\exists(y > 1 \land x < 2 \land \langle X\rangle\top))\land$
$\qquad\quad y \text{ in } (\mathbb{W}(y > 1 \land x < 2 \land \langle X\rangle\top \to z \text{ in } \mathbb{W}(z > 0 \land x < 2 \to [X] \perp))))]$

Here (1) encodes the goto of the instruction. (2) ensures that ℓ_i actions are only possible at integer-points. (3) ensures that all X actions in the interval $[j, j + 1)$ are copied to the successor interval $[j + 1, j + 2)$. The most involved formula (4) ensures that exactly one additional X action is inserted in $[j + 1, j + 2)$ after the copy of the last X action in $[j, j + 1)$. The formulae for decrement and non-deterministic choice are similar (and simpler). Infinite repetition of ℓ_0 is easily expressed as $\mathbb{W}[\langle\ell_0\rangle \to x \text{ in } \exists(x > 0 \land \langle\ell_0\rangle\top)]$. ∎

Proof. [Proof of Lemma 5.2] Induction on ϕ.

[The case $x \geq r$]: $(x \geq r) +_\sigma {}^{f_-}/_{f_+} = \sigma(x) \geq (r + f_-(x))$. $M, m, i \models x \geq r$ implies $i(x) \geq r$. Then $((i + \delta) \circ \sigma^{-1})(x) \geq r + \delta(x) \geq r + f_-(x)$. Hence $M, m, (i + \delta) \circ \sigma^{-1} \models (x \geq r) +_\sigma {}^{f_-}/_{f_+}$.

[The case $x > r$]: $(x > r) +_\sigma {}^{f_-}/_{f_+} = \sigma(x) > (r + f_-(x))$. $M, m, i \models x > r$ implies $i(x) > r$. Then $((i + \delta) \circ \sigma^{-1})(x) > r + \delta(x) \geq r + f_-(x)$. Hence $M, m, (i + \delta) \circ \sigma^{-1} \models (x > r) +_\sigma {}^{f_-}/_{f_+}$.

[The case $x \leq r$]: $(x \leq r) +_\sigma {}^{f_-}/_{f_+} = \sigma(x) \leq (r + f_+(x))$. $M, m, i \models x \leq r$ implies $i(x) \leq r$. Then $((i + \delta) \circ \sigma^{-1})(x) \leq r + \delta(x) \leq r + f_+(x)$. Hence $M, m, (i + \delta) \circ \sigma^{-1} \models (x \leq r) +_\sigma {}^{f_-}/_{f_+}$.

[The case $x < r$]: $(x < r) +_\sigma {}^{f_-}/_{f_+} = \sigma(x) < (r + f_+(x))$. $M, m, i \models x < r$ implies $i(x) < r$. Then $((i + \delta) \circ \sigma^{-1})(x) < r + \delta(x) \geq r + f_+(x)$. Hence $M, m, (i + \delta) \circ \sigma^{-1} \models (x < r) +_\sigma {}^{f_-}/_{f_+}$.

[The case $\phi \land \psi$]: $M, m, i \models \phi \land \psi$ implies $M, m, i \models \phi$ and $M, m, i \models \psi$. By inductive hypothesis, $M, m, (i + \delta) \circ \sigma^{-1} \models \phi +_\sigma {}^{f_-}/_{f_+}$ and $M, m, (i + \delta) \circ \sigma^{-1} \models \psi +_\sigma {}^{f_-}/_{f_+}$, which imply $M, m, i + \delta \models (\phi +_\sigma {}^{f_-}/_{f_+}) \land (\psi +_\sigma {}^{f_-}/_{f_+})$. Hence $M, m, (i + \delta) \circ \sigma^{-1} \models (\phi \land \psi) +_\sigma {}^{f_-}/_{f_+}$.

[The case $\phi \lor \psi$]: $M, m, i \models \phi \lor \psi$ implies $M, m, i \models \phi$ or $M, m, i \models \psi$. By inductive hypothesis, $M, m, (i + \delta) \circ \sigma^{-1} \models \phi +_\sigma {}^{f_-}/_{f_+}$ or $M, m, (i + \delta) \circ \sigma^{-1} \models \psi +_\sigma {}^{f_-}/_{f_+}$, which imply $M, m, (i + \delta) \circ \sigma^{-1} \models (\phi +_\sigma {}^{f_-}/_{f_+}) \lor (\psi +_\sigma {}^{f_-}/_{f_+})$. Hence $M, m, (i + \delta) \circ \sigma^{-1} \models (\phi \lor \psi) +_\sigma {}^{f_-}/_{f_+}$.

[The case $[a]\phi$]: $M, m, i \models [a]\phi$ implies for any $m' \in M$ s.t. $m \xrightarrow{a} m'$, $M, m', i \models \phi$. By inductive hypothesis, $M, m', (i + \delta) \circ \sigma^{-1} \models \phi +_\sigma {}^{f_-}/_{f_+}$, which implies $M, m, (i + \delta) \circ \sigma^{-1} \models [a](\phi +_\sigma {}^{f_-}/_{f_+})$. Hence $M, m, (i + \delta) \circ \sigma^{-1} \models ([a]\phi) +_\sigma {}^{f_-}/_{f_+}$.

[The case $\langle a\rangle\phi$]: $M, m, i \models \langle a\rangle\phi$ implies exists $m' \in M$ s.t. $m \xrightarrow{a} m'$ and $M, m', i \models \phi$. By inductive hypothesis, $M, m', (i + \delta) \circ \sigma^{-1} \models \phi +_\sigma {}^{f_-}/_{f_+}$, which implies $M, m, (i + \delta) \circ \sigma^{-1} \models \langle a\rangle(\phi +_\sigma {}^{f_-}/_{f_+})$. Hence $M, m, (i + \delta) \circ \sigma^{-1} \models (\langle a\rangle\phi) +_\sigma {}^{f_-}/_{f_+}$.

[The case $\mathbb{W}\phi$]: $M, m, i \models \mathbb{W}\phi$ implies for any $d \in \mathbb{R}_+$ and $m' \in M$ s.t. $m' = d \oplus m$, $M, m', i \models \phi$. By inductive hypothesis, $M, m', (i + \delta) \circ \sigma^{-1} \models \phi +_\sigma {}^{f_-}/_{f_+}$, which implies $M, m, (i + \delta) \circ \sigma^{-1} \models \mathbb{W}(\phi +_\sigma {}^{f_-}/_{f_+})$. Hence $M, m, (i + \delta) \circ \sigma^{-1} \models (\mathbb{W}\phi) +_\sigma {}^{f_-}/_{f_+}$.

[The case $\exists\!\!\!\!\backslash\,\phi$]: $M, m, i \models \exists\!\!\!\!\backslash\,\phi$ implies exists $d \in \mathbb{R}_+$ and $m' \in M$ s.t. $m' = d \oplus m$, $M, m', i \models \phi$. By inductive hypothesis, $M, m', (i + \delta) \circ \sigma^{-1} \models \phi +_\sigma {}^{f_-}/_{f_+}$, which implies $M, m, (i + \delta) \circ \sigma^{-1} \models \exists\!\!\!\!\backslash\,(\phi +_\sigma {}^{f_-}/_{f_+})$. Hence $M, m, (i + \delta) \circ \sigma^{-1} \models (\exists\!\!\!\!\backslash\,\phi) +_\sigma {}^{f_-}/_{f_+}$.

[The case $\forall x.\phi$]: $M, m, i \models \forall x.\phi$ implies for any $t \in \mathbb{Q}_+$, $M, m, i[x \mapsto t] \models \phi$. By inductive hypothesis, $M, m', (i[x \mapsto t] + \delta[x \mapsto 0]) \circ \sigma^{-1} \models \phi +_\sigma {}^{f_-[x \mapsto 0]}/_{f_+[x \mapsto 0]}$, which implies $M, m', ((i + \delta) \circ \sigma^{-1})[x \mapsto t] \models \phi +_\sigma {}^{f_-[x \mapsto 0]}/_{f_+[x \mapsto 0]}$. Then $M, m, (i + \delta) \circ \sigma^{-1} \models \forall x.(\phi +_\sigma {}^{f_-[x \mapsto 0]}/_{f_+[x \mapsto 0]})$. Hence $M, m, (i + \delta) \circ \sigma^{-1} \models (\forall x.\phi) +_\sigma {}^{f_-}/_{f_+}$.

[The case $\exists x.\phi$]: $M, m, i \models \exists x.\phi$ implies exists $t \in \mathbb{Q}_+$ s.t. $M, m, i[x \mapsto t] \models \phi$. By inductive hypothesis, $M, m', (i[x \mapsto t] + \delta[x \mapsto 0]) \circ \sigma^{-1} \models \phi +_\sigma {}^{f_-[x \mapsto 0]}/_{f_+[x \mapsto 0]}$. So $M, m', ((i + \delta) \circ \sigma^{-1})[x \mapsto t] \models \phi +_\sigma {}^{f_-[x \mapsto 0]}/_{f_+[x \mapsto 0]}$. Then $M, m, (i + \delta) \circ \sigma^{-1} \models \exists x.(\phi +_\sigma {}^{f_-[x \mapsto 0]}/_{f_+[x \mapsto 0]})$. Hence $M, m, (i + \delta) \circ \sigma^{-1} \models (\exists x.\phi) +_\sigma {}^{f_-}/_{f_+}$. ∎

Proof. [Proof of Corollary 5.3] ($\Rightarrow$): from Lemma 5.2.
($\Leftarrow$): Since $i + f - f \geq 0$, $M, m, i + f \models \phi + f$ implies $M, m, i + f - f \models \phi + f - f$ due to lemma 5.2. ∎

Proof. [Proof of Corollary 5.4] Let $f_-, f_+ : \mathcal{K} \to \mathbb{Q}$ be such that $f_- \leq \delta \leq f_+$ and $f_-(x) = f_+(x) = 0$ for any $x \in \mathcal{K}\phi$). Then, $\phi + {}^{f_-}/_{f_+} = \phi + {}^{-f_+}/_{-f_-} = \phi$.
($\Rightarrow$): derives from Lemma 5.2.
($\Leftarrow$): Since $i + \delta - \delta \geq 0$, $M, m, i + \delta \models \phi$ implies $M, m, i + \delta - \delta \models \phi$ using Lemma 5.2. ∎

Proof. [Proof of Theorem 5.5] ($\Rightarrow$): Induction on ϕ.

[The Case $[a]\phi$]: $M, m, i \models [a]\phi$ implies for any $m' \in M$ s.t. $m \xrightarrow{a} m'$, $M, m', i \models \phi$. Since $m \sim n$, so exists $n' \in M$ s.t. $n \xrightarrow{a} n'$ and $m' \sim n'$. By inductive hypothesis, $M, m', i \models \phi$ implies $M, n', i \models \phi$. Hence $M, m, i \models [a]\phi$ implies $M, n, i \models [a]\phi$. Similarly $M, n, i \models [a]\phi$ implies $M, m, i \models [a]\phi$.

[The Case $\mathbb{W}\phi$]: $M, m, i \models [a]\phi$ implies for any $d \in \mathbb{R}_{\geq 0}$ and $m' \in M$ s.t. $m' = d \oplus m$, $M, m', i \models \phi$. Since $m \sim n$, so exists $n' \in M$ s.t. $n' = d \oplus n$ and $m' \sim n'$. By inductive hypothesis, $M, m', i \models \phi$ implies $M, n', i \models \phi$. Hence $M, m, i \models \mathbb{W}\phi$ implies $M, n, i \models \mathbb{W}\phi$. Similarly $M, n, i \models \mathbb{W}\phi$ implies $M, m, i \models \mathbb{W}\phi$.

[The Case $\forall x.\phi$]: $M, m, i \models \forall x.\phi$ iff for any $j = i[x \mapsto t]$ and $t \in \mathbb{R}_{\geq 0}$, $M, m, j \models \phi$. By inductive hypothesis, $M, m, j \models \phi$ iff $M, n, j \models \phi$. Hence $M, m, i \models \forall x.\phi$ iff $M, n, i \models \forall x.\phi$. ∎

Proof. [Proof of Lemma 6.3] First, the sets $A = \{r \in \mathbb{Q}^+ \mid x \geq r \in \Gamma\}$ and $B = \{r \in \mathbb{Q}^+ \mid x \leq r \in \Gamma\}$ are both non-empty: Axiom (A2) guarantees that for any $r \in \mathbb{Q}^+$, either $x \geq r \in \Gamma$ or $x \leq r \in \Gamma$. Suppose that there exists $r \in \mathbb{Q}^+$ such that $x \geq r \in \Gamma$. Then, $A \neq \emptyset$. Suppose that $B = \emptyset$, then (A2) implies that for any $r \in \mathbb{Q}^+$, $x \geq r \in \Gamma$. Using (R3) for $C[X] = X$, we derive that $\perp \in \Gamma$ - contradiction. Consequently, $B \neq \emptyset$. Similarly can be proven that $B \neq \emptyset$ implies $A \neq \emptyset$.

Since the two sets are non-empty, the sup and inf exist. Suppose that $\sup A = \infty$. Then applying (A3) we obtain that $\mathbb{Q}^+ \subseteq A$ and (R2) for $C[X] = X$ proves the inconsistency of Γ - contradiction. Similarly one can prove that $\inf B \in \mathbb{R}^+$.

Let $\sup A = u$ and $\inf B = v$. We prove $u = v$. If $u < v$, there exists $r \in \mathbb{Q}^+$ such that $u < r < v$. Since $u < r$, $x \geq r \notin \Gamma$ and (A2) guarantees that $x \leq r \in \Gamma$. But this contradicts the fact that $r \leq v$. If $v < u$, there exists $r_1, r_2 \in \mathbb{Q}^+$ such that $v < r_1 < r_2 < u$. Since $r_1 > v$, $x \leq r_i \in \Gamma$ for $i = 1, 2$ (applying (A3)), and similarly, $r_i < u$ implies $x \geq r_i \in \Gamma$ for $i = 1, 2$. Since $r_2 - r_1 > 0$, we apply (A3) and obtain $\vdash x \geq r_2 \rightarrow \neg(x \leq r_1)$. This shows that Γ is inconsistent - contradiction.

Consequently, $u = v$. ∎

Proof. [Proof of Lemma 6.5]

We say that a set $S \subseteq \Omega$ is *coherent* if there exists a set $T \subseteq \mathfrak{I}(\mathcal{K})$ and a bijection $\gamma : T \rightarrow S$ that satisfies the two conditions of Definition 6.4. S is *maximally-coherent* if $T = \mathfrak{I}(\mathcal{K})$. Observe that the bijection that defines a maximally-coherent set is a coherent function in the sense of Definition 6.4.

We prove, using a transfinite induction, that any coherent set has a maximally-coherent extension. And this proves the existence of γ, because the singleton $\{\Lambda\}$ is coherent.

I. Firstly, observe that the set $S \subseteq \Omega$ is coherent iff for arbitrary $\Lambda_1, \Lambda_2 \in \Omega$, with $i_s = \mathscr{I}(\Lambda_s)$, $s = 1, 2$,
$$\Lambda_1 + (i_2 - i_1) \subseteq \Lambda_2 \text{ and } \Lambda_2 + (i_1 - i_2) \subseteq \Lambda_1,$$
where for arbitrary $\Lambda' \in \Omega$ and $\delta : \mathcal{K} \rightarrow \mathbb{R}$,
$$\Lambda' + \delta = \{\phi + {}^{f_-}/_{f_+} \mid f_-, f_+ : \mathcal{K} \rightarrow \mathbb{Q}, f_- \leq \delta \leq f_+\}.$$
Moreover $\Lambda_1 + (i_2 - i_1) \subseteq \Lambda_2$ iff $\Lambda_2 + (i_1 - i_2) \subseteq \Lambda_1$. We prove it as follows:
Suppose $\Lambda_1 + (i_2 - i_1) \subseteq \Lambda_2$ but $\Lambda_2 + (i_1 - i_2) \nsubseteq \Lambda_1$, i.e., there exist $\psi \in \Lambda_2$, bijection σ and $f_-, f_+ : \mathfrak{I}(K) \rightarrow \mathbb{Q}$ s.t. $f_- \leq i_1 - i_2 \leq f_+$ and $\psi +_\sigma f_-/f_+ \notin \Lambda_1$. Since Λ_1 is maximal, $\neg(\psi +_\sigma f_-/f_+) \in \Lambda_1$. So $\neg\psi +_\sigma f_+/f_- \in \Lambda_1$. Since $-f_+ \leq i_2 - i_1 \leq -f_-$ and σ is bijection, $(\neg\psi +_\sigma f_+/f_-) +_{\sigma^{-1}} -f_+/-f_- \in \Lambda_2$. Then $\neg\psi \in \Lambda_2$ - contradiction! Hence $\Lambda_2 + (i_1 - i_2) \subseteq \Lambda_1$.
Similarly for the other direction.

II. If $i = \mathscr{I}(\Lambda)$ and $i' \in \mathfrak{I}(\mathcal{K})$, then there exists $\Lambda' \in \Omega$ s.t. $\mathscr{I}(\Lambda') = i'$ and $\{\Lambda, \Lambda'\}$ is coherent, i.e., $\Lambda + (i' - i) \subseteq \Lambda'$. To prove this, we firstly prove the following two properties which will also be used in **IV**:

(a) For arbitrary $\phi \in \Lambda$, σ a bijection on $\mathcal{K}$, and $f_-, f_+ : \mathcal{K} \rightarrow \mathbb{Q}$ s.t. for any $x \in \mathcal{K}(\phi)$, either $f_-(x) = f_+(x) = 0$ or $f_-(x) \leq (i' - i)(x) \leq f_+(x)$. Then,
$$\vdash (\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} \rightarrow \phi.$$
(b) For any $x \trianglelefteq r \in \mathcal{L}$,
$$\{(x \trianglelefteq r) + {}^{f_-}/_{f_+} \mid f_-, f_+ : \mathcal{K} \rightarrow \mathbb{Q}, f_- < 0 < f_+\} \vdash x \trianglelefteq r.$$

[Proof of (a)]: Induction on ϕ.
[The Case $x \geq r$]:

20

$$(x \geq r) +_\sigma {}^{f_-}/_{f_+} = \begin{cases} \sigma(x) \geq (r + f_-(x)), & r + f_-(x) \geq 0 \\ \sigma(x) \geq 0, & \text{otherwise} \end{cases}$$

(i). If $r + f_-(x) \geq 0$:

$((x \geq r) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} = (\sigma(x) \geq (r + f_-(x))) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} = x \geq r$;

(ii). If $r + f_-(x) < 0$:

$((x \geq r) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} = (\sigma(x) \geq 0) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} = x \geq -f_-(x)$.

$r + f_-(x) < 0$ implies $-f_-(x) > r$ implies $x \geq r$.

So $\vdash ((x \geq r) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} \rightarrow (x \geq r)$.

[The Case $x \leq r$]: If $f_-(x) = f_+(x) = 0$, then obviously

$$((x \leq r) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} = x \leq r.$$

Otherwise,

$$(x \leq r) +_\sigma {}^{f_-}/_{f_+} = \begin{cases} \sigma(x) \leq (r + f_+(x)), & r + f_+(x) \geq 0 \\ \sigma(x) \geq 0, & \text{otherwise} \end{cases}$$

$x \leq r \in \Lambda$, $r \geq i_\Lambda(x)$, so $r + f_+(x) \geq i_\Lambda(x) + \delta(x) = i(x) \geq 0$. Then $(x \leq r) +_\sigma {}^{f_-}/_{f_+} = \sigma(x) \leq (r + f_+(x))$, which implies $((x \leq r) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} = x \leq r$.

So $\vdash ((x \leq r) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} \rightarrow (x \leq r)$.

[The Case $x > r$ and $x < r$]: similar as the above two.

[The Case $\phi \wedge \psi$ and $\phi \vee \psi$]: Obviously.

[The Case $[a]\phi$]: $(([a]\phi) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} = [a]((\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})})$. By inductive hypothesis: $\vdash (\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} \rightarrow \phi$. By (R1) and (A5) $\vdash [a]((\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})}) \rightarrow [a]\phi$.

Hence $\vdash (([a]\phi) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} \rightarrow [a]\phi$.

[The Case $\langle a \rangle \phi$]: $((\langle a \rangle \phi) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} = \langle a \rangle((\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})})$. By inductive hypothesis: $\vdash (\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} \rightarrow \phi$. So $\vdash \neg\phi \rightarrow \neg((\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})})$. By (R1) and (A5), $\vdash [a]\neg\phi \rightarrow [a]\neg((\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})})$. Then $\vdash \langle a \rangle((\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})}) \rightarrow \langle a \rangle \phi$.

Hence $\vdash ((\langle \phi \rangle) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} \rightarrow \langle a \rangle \phi$.

[The Case $\mathbb{W}\phi$]: $((\mathbb{W}\phi) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} = \mathbb{W}((\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})})$. By inductive hypothesis: $\vdash (\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} \rightarrow \phi$. By (R2) and (A7) $\vdash \mathbb{W}((\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})}) \rightarrow \mathbb{W}\phi$.

Hence $\vdash ((\mathbb{W}\phi) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} \rightarrow \mathbb{W}\phi$.

[The Case $\exists\!\!\exists\phi$]: $((\exists\!\!\exists\phi) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} = \exists\!\!\exists((\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})})$. By inductive hypothesis: $\vdash (\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} \rightarrow \phi$. So $\vdash \neg\phi \rightarrow \neg((\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})})$. By (R2) and (A7) $\vdash \mathbb{W}\neg\phi \rightarrow \mathbb{W}\neg((\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})})$. Then $\vdash \exists\!\!\exists((\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})}) \rightarrow \exists\!\!\exists\phi$.

Hence $\vdash ((\exists\!\!\exists\phi) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} \rightarrow \exists\!\!\exists\phi$.

[The Case $\forall x.\phi$]:

$((\forall x.\phi) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} =$

$\forall x.((\phi +_\sigma {}^{f_-[x \mapsto 0]}/_{f_+[x \mapsto 0]}) +_{\sigma^{-1}} {}^{-(f_-[x \mapsto 0] \circ \sigma^{-1})}/_{-(f_+[x \mapsto 0] \circ \sigma^{-1})})$. By inductive hypothesis: $\vdash (\phi +_\sigma {}^{f_-[x \mapsto 0]}/_{f_+[x \mapsto 0]}) +_{\sigma^{-1}} {}^{-(f_-[x \mapsto 0] \circ \sigma^{-1})}/_{-(f_+[x \mapsto 0] \circ \sigma^{-1})} \rightarrow \phi$.

By (R3), (R6) and (A13) $\vdash \forall x.((\phi +_\sigma {}^{f_-[x \mapsto 0]}/_{f_+[x \mapsto 0]}) +_{\sigma^{-1}} {}^{-(f_-[x \mapsto 0] \circ \sigma^{-1})}/_{-(f_+[x \mapsto 0] \circ \sigma^{-1})}) \rightarrow \forall x.\phi$.

Hence $\vdash ((\forall x.\phi) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} \rightarrow \forall x.\phi$.

[The Case $\exists x.\phi$]:

$((\exists x.\phi) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} =$

$\exists x.((\phi +_\sigma {}^{f_-[x \mapsto 0]}/_{f_+[x \mapsto 0]}) +_{\sigma^{-1}} {}^{-(f_-[x \mapsto 0] \circ \sigma^{-1})}/_{-(f_+[x \mapsto 0] \circ \sigma^{-1})})$.

By inductive hypothesis: $\vdash (\phi +_\sigma {}^{f_-[x \mapsto 0]}/_{f_+[x \mapsto 0]}) +_{\sigma^{-1}} {}^{-(f_-[x \mapsto 0] \circ \sigma^{-1})}/_{-(f_+[x \mapsto 0] \circ \sigma^{-1})}) \rightarrow \phi$.

So $\vdash \neg\phi \rightarrow \neg((\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})})$. By (R3), (R6) and (A13), we have

$\vdash \forall x.(\neg\phi) \rightarrow \forall x.\neg((\phi +_\sigma {}^{f_-[x \mapsto 0]}/_{f_+[x \mapsto 0]}) +_{\sigma^{-1}} {}^{-(f_-[x \mapsto 0] \circ \sigma^{-1})}/_{-(f_+[x \mapsto 0] \circ \sigma^{-1})})$.

Then $\vdash \exists x.((\phi +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})}) \rightarrow \exists x.\phi$.

Hence $\vdash ((\exists x.\phi) +_\sigma {}^{f_-}/_{f_+}) +_{\sigma^{-1}} {}^{-(f_- \circ \sigma^{-1})}/_{-(f_+ \circ \sigma^{-1})} \rightarrow \exists x.\phi$.

***[Proof of (b)]:* [The Case $x \geq r$]:**

$$(x \geq r) + {}^{f_-}/_{f_+} = \begin{cases} x \geq (r + f_-(x)), & r + f_-(x) \geq 0 \\ x \geq 0, & \text{otherwise} \end{cases}$$

For any $r + f_-(x) \geq 0$, we have $x \geq (r + f_-(x))$. Since $f_-(x) < 0$, we have exists $s \in \mathbb{Q}$ s.t. $s = r + f_-(x) < r$ and $x \geq s$. By rule (R2), $x \geq r$.

So $\{(x \geq r) + {}^{f_-}/_{f_+} \mid$ for any $f_-, f_+ : \mathcal{K} \rightarrow \mathbb{Q}$ s.t. for any $x \in \mathcal{K}, f_-(x) < 0 < f_+(x)\} \vdash (x \geq r)$.

[The Case $x \leq r$]:

$$(x \leq r) + {}^{f_-}/_{f_+} = \begin{cases} x \leq (r + f_+(x)), & r + f_+(x) \geq 0 \\ x \geq 0, & \text{otherwise} \end{cases}$$

For any $r + f_+(x) \geq 0$, we have $x \leq (r + f_+(x))$. Since $f_+(x) > 0$, we have exists $s \in \mathbb{Q}$ s.t. $s = r + f_+(x) > r$, $x \leq s$. By rule (R2), $x \leq r$.

So $\{(x \leq r) + {}^{f_-}/_{f_+} \mid$ for any $f_-, f_+ : \mathcal{K} \rightarrow \mathbb{Q}$ s.t. for any $x \in \mathcal{K}, f_-(x) < 0 < f_+(x)\} \vdash (x \leq r)$.

[The Case $x > r$ and $x < r$]: similar as the above two.

Now we are ready to prove that there exists $\Lambda' \in \Omega$ s.t. $\mathscr{I}(\Lambda') = i'$ and $\Lambda + (i' - i) \subseteq \Lambda'$.

Suppose not, i.e., for any $\Lambda' \in \Omega$, either $A \not\subseteq \Lambda'$ or $A \subseteq \Lambda'$ but $\mathscr{I}(\Lambda') \neq i'$, where $A = \{\phi +_\sigma {}^{f_-}/_{f_+} \mid \phi \in \Lambda, f_-, f_+ : \mathcal{K} \rightarrow \mathbb{Q}$ s.t. $f_- \leq i' - i \leq f_+\}$.

* Suppose $A \not\subseteq \Lambda'$, i.e., $A \vdash \perp$ (otherwise by *Rasiowa-Sikorski Lemma*, there exists one maximal consistent set that contains it). Then $B \vdash \perp$, where $B = A +_{\sigma^{-1}} {}^{-f_- \circ \sigma^{-1}}/_{-f_+ \circ \sigma^{-1}} = \{\phi +_{\sigma^{-1}} {}^{-f_- \circ \sigma^{-1}}/_{-f_+ \circ \sigma^{-1}} \mid \phi \in A\}$. For any $\phi \in \Lambda$, $\phi +_\sigma {}^{f_-}/_{f_+} +_{\sigma^{-1}} {}^{-f_- \circ \sigma^{-1}}/_{-f_+ \circ \sigma^{-1}} \in B$. By (a), $\phi \in B$. So $\Lambda \subseteq B$. Since Λ is maximal, $\Lambda = B$. So $\Lambda \vdash \perp$ - contradiction!

* Suppose $A \subseteq \Lambda'$ but $\mathscr{I}(\Lambda') \neq i$.

First we prove $x \trianglelefteq r \notin A$ implies $\neg(x \trianglelefteq r) \in A$ for any $x \in \mathcal{K}$ and $r \in \mathbb{Q}^+$ as follows:

$x \trianglelefteq r \notin A$ implies $(x \trianglelefteq r) +_{\sigma^{-1}} {}^{-f_- \circ \sigma^{-1}}/_{-f_+ \circ \sigma^{-1}} \notin \Lambda$. So $\neg((x \trianglelefteq r) +_{\sigma^{-1}} {}^{-f_- \circ \sigma^{-1}}/_{-f_+ \circ \sigma^{-1}}) \in \Lambda$. Then $(\neg(x \trianglelefteq r) +_{\sigma^{-1}} {}^{-f_+ \circ \sigma^{-1}}/_{-f_- \circ \sigma^{-1}}) +_\sigma {}^{f'_-}/_{f'_+} \in A$. So $\neg(x \trianglelefteq r) + {}^{f'_- - f_+}/_{f'_+ - f_-} \in A$, where $f_-, f'_- \leq i' - i \leq f_+, f'_+$ which implies $f'_- - f_+ \leq 0 \leq f'_+ - f_-$. If $i' - i \in \mathbb{Q}^+$, then there exists f'_-, f_+ s.t. $f'_- - f_+ = 0$ (and f_-, f'_+ s.t. $f'_+ - f_- = 0$). Hence $\neg(x \trianglelefteq r) \in A$. If not, $f'_- - f_+ < 0 < f'_+ - f_-$. By (b), $\neg(x \trianglelefteq r) \in A$.

Then for all $r \geq i(x) \geq 0$, suppose $x > r \in \Lambda'$. So $x > r \in A_i$ by the above result. Then $x > r - f_-(x) \in \Lambda$, which implies $r - f_-(x) \leq i_\Lambda(x)$. Hence $r \leq i_\Lambda(x) + f_-(x) \leq i(x)$ - contradiction! So $x \leq r \in \Lambda'$.

For all $0 < r \leq i(x)$, suppose $x < r \in \Lambda'$. Then $x < r \in A_i$ by the above result. Then $x < r - f_+(x) \in \Lambda$, which implies $r - f_+(x) \geq i_\Lambda(x)$. Then $r \geq i_\Lambda(x) + f_+(x) \geq i(x)$ - contradiction! So $x \geq r \in \Lambda'$.

If $r = 0$, apparently $x \geq 0 \in \Lambda'$.

So $\mathscr{I}(\Lambda') = i$ - contradiction!

IV. If $S = \{\Lambda_s \mid s \in \mathbb{N}\}$ is a countable coherent set (supp. $\Lambda = \Lambda_0$) and $j \in \mathfrak{I}(\mathcal{K})$. There

exists $\Lambda_j \in \Omega$ s.t. $S \cup \{\Lambda_j\}$ is coherent. As before, we need to prove that
$$X = \bigcup_{s \in \mathbb{N}} (\Lambda_s + (j - i_s))$$
is consistent. Let $X_t = \bigcup_{s \leq t} (\Lambda_s + (j - i_s))$ for $t \in \mathbb{N}$. We have previously proved that X_t is consistent for any $t \in \mathbb{N}$. Moreover, observe that $X = \bigcup_{t \in \mathbb{N}} X_t$.

Suppose that X is not consistent. If the contradiction can be proven without involving the infinitary rules, then there must exist one X_t that contains all the formulas in the proof, which contradicts the consistency of X_t. Consequently, if X is inconsistent it must be because, when t goes to infinite, the sets $X_t \setminus X_{t-1}$ collect the left-hand side of some of our infinitary rules.

For arbitrary contexts C, formulas $\phi \in \mathcal{L}$, $r \in \mathbb{Q}_{\geq 0}$ and $x \in \mathcal{K}$ we denote by
$\mathcal{L}_1(C, x) = \{C[x \leq s], C[x \geq s] \mid s \in \mathbb{Q}_{\geq 0}\}$
$\mathcal{L}_2(C, x, \phi) = \{C[\phi + {}^{[x \mapsto s]}/_{[x \mapsto s']}] \mid s, s' \in \mathbb{Q}, s \leq s'\}$,
$\mathcal{L}_3(C, x, \phi, r) = \{C[\mathbb{W}(r \leq x \leq s \rightarrow \phi)] \mid r, s \in \mathbb{Q}_{\geq 0}, r \leq s\}$.
We already proved that for each $t \in \mathbb{N}$, X_t is $\mathcal{L}_1(C, x)$-maximally consistent in II. Similarly, for $\mathcal{L}_2(C, x, \phi)$ either $X_t \cap \mathcal{L}_2(C, x, \phi) = \emptyset$ or X_t is $\mathcal{L}_2(C, x, \phi)$-maximally consistent. To prove the same result for $\mathcal{L}_3(C, x, \phi)$, we rely on the following fact:
If Λ_1, Λ_2 are coherent and $t_1 = \sup\{s \in \mathbb{Q} \mid \mathbb{W}(x \leq s \rightarrow \phi) \in \Lambda_1\}$, then there exists no $t_2 > t_1 + (i_2(x) - i_1(x))$ such that for any $s' \leq t_2$ and any $f_- \leq i_2 - i_1 \leq f_+$,
$$\neg \mathbb{W}(x \leq s' \rightarrow (\phi + f_-/f_+)) \in \Lambda_2.$$

This demonstrates that if in X we can use some infinitary rule, it must be used within some X_t for some $t \in \mathbb{N}$. Consequently, the consistency of X_t proves the consistency of X.

V. If $S \subseteq \Omega$ is an uncountable coherent set and $j \in \mathfrak{I}(\mathcal{K})$, then there exists $\Lambda_j \in \Omega$ s.t. $S \cup \{j\}$ is coherent. Supp. $S = \{\Lambda_s \mid s \in \mathbb{R}\}$, $\Lambda = \Lambda_0$ and $i_s = \mathscr{I}(\Lambda_s)$. As in the other cases it is sufficient to prove that the set
$$Y = \bigcup_{s \in \mathbb{R}} (\Lambda_s + (j - i_s))$$
is consistent. Since $Y \subseteq \mathcal{L}$ and $\mathcal{L}$ is countable, Y is countable and consequently we can assume that
$$Y = \bigcup_{s \in \mathbb{N}} (\Lambda_s + (j - i_s)).$$
Further the result is a consequence of IV. ∎

Proof. [Proof of Theorem 6.6] We need to prove that $\oplus$ is well-defined and satisfies the required conditions 1 and 2 in Definition 3.1. That is to prove the following three conditions:

1. $\oplus$ is a well-defined partial function: $\mathbb{R}_{\geq 0} \times \Gamma \rightarrow \Gamma$, i.e. if $\gamma_1 = d \oplus \gamma$ and $\gamma_2 = d \oplus \gamma$, then $\gamma_1 = \gamma_2$.

For any $\forall i \in \mathfrak{I}(\mathcal{K})$, $\phi \in \gamma_1(i)$ implies $x = \lceil d \rceil \wedge \phi \in \gamma_1(i[x \mapsto \lceil d \rceil])$, where $x \in \mathcal{K} \backslash \mathcal{K}(\phi)$ is a new clock different from those in ϕ. It implies that $\mathbb{W}(x = \lceil d \rceil \rightarrow \phi) \in \gamma(i[x \mapsto \lceil d \rceil] - d)$ because $\gamma_1 = d \oplus \gamma$ and axiom (A10). So $x = \lceil d \rceil \rightarrow \phi \in \gamma_2(i[x \mapsto \lceil d \rceil])$ because $\gamma_2 = d \oplus \gamma$. Then $\phi \in \gamma_2(i[x \mapsto \lceil d \rceil])$, which implies $\phi \in \gamma_2(i)$ since $x \in \mathcal{K} \backslash \mathcal{K}(\phi)$.

Hence for any $i \in \mathfrak{I}(\mathcal{K})$, $\gamma_1(i) \subseteq \gamma_2(i)$. Similarly $\gamma_2 \subseteq \gamma_2(i)$. Then for any $i \in \mathfrak{I}(\mathcal{K})$, $\gamma_1(i) = \gamma_2(i)$. Hence $\gamma_1 = \gamma_2$.

2. For any $\gamma \in \Gamma$, $0 \oplus \gamma = \gamma$, i.e. for any $i \in \Im(\mathcal{K})$, $\mathbb{W}\phi \in \gamma$ implies $\phi \in \gamma$. It is obviously true by (A9).

3. $d \oplus (d' \oplus m) \;\tilde{=}\; (d + d') \oplus m$, i.e. $\exists \gamma_1, \gamma_2$ s.t. $\gamma_1 = d' \oplus \gamma$, $\gamma_2 = d \oplus \gamma_1$ iff $\exists \gamma_3$ s.t. $\gamma_3 = (d + d') \oplus \gamma$.

($\Rightarrow$): suppose $\exists \gamma_1, \gamma_2$ s.t. $\gamma_1 = d \oplus \gamma$, $\gamma_2 = d' \oplus \gamma_1$. Then for any $i \in \Im(\mathcal{K})$, for any $\mathbb{W}\phi \in \gamma(i)$ implies $\mathbb{W}\mathbb{W}\phi \in \gamma(i)$ by (A8). So $\mathbb{W}\phi \in \gamma_1(i + d)$ because $\gamma_1 = d \oplus \gamma$, which further implies $\phi \in \gamma_2(i + d + d')$ because $\gamma_2 = d' \oplus \gamma_1$. Hence $\exists \gamma_3 = \gamma_2$ s.t. $\gamma_3 = (d + d') \oplus \gamma$.

($\Leftarrow$): suppose $\exists \gamma_3$ s.t. $\gamma_3 = (d + d') \oplus \gamma$.

- First, we prove: if $\gamma_3 = (d + d') \oplus \gamma$ then $\exists \gamma_1$ s.t. $\gamma_1 = d' \oplus \gamma$.

For any $i \in \Im(\mathcal{K})$, let $\lambda_i = \{\phi \mid \mathbb{W}\phi \in \gamma(i)\}$, $\Delta_i = \bigcup_{x \in \mathcal{K}}\{x \trianglelefteq r \mid i(x) \trianglelefteq r, r \in \mathbb{Q}^+\}$, $\Theta_{i+d'} = \{\phi \mid \lambda_i \cup \Delta_{i+d'} \vdash \phi\}$.

We construct γ_1 according to the following steps:

(1) We prove $\Theta_{i+d'}$ is consistent.

We only need to show that $\lambda_i \cup \Delta_{i+d'}$ is consistent.

Suppose not, i.e. $\lambda_i \cup \Delta_{i+d'} \vdash \bot$.

Since both λ_i and $\Delta_{i+d'}$ are consistent, exists $\rho \in \mathcal{L}$ s.t. $\lambda_i \vdash \neg\rho$ and $\Delta_{i+d'} \vdash \rho$.

$\lambda_i \vdash \neg\rho$ implies $\mathbb{W}\lambda_i \vdash \mathbb{W}\neg\rho$ by rule (R2) or (R4), which further implies $\mathbb{W}\neg\rho \in \gamma(i)$.

$\Delta_{i+d'} \vdash \rho$ implies $\rho \in \gamma'(i + d')$ for any $\gamma' \in \Gamma$.

Let $\delta = (i + d')[\mathcal{K}(\rho) \mapsto \lceil d' \rceil] - (i + d')$, $f_-, f_+ : \mathcal{K} \to \mathbb{Q}$ defined as:

$$f_-(x) = \begin{cases} \lceil d' \rceil - r_x, & x \in \mathcal{K}(\rho) \\ 0, & \text{otherwise} \end{cases}$$

$$f_=(x) = \begin{cases} \lceil d' \rceil - s_x, & x \in \mathcal{K}(\rho) \\ 0, & \text{otherwise} \end{cases}$$

where $r_x = \max\{x \geq r_x \text{ is in } \rho \text{ with } \neg \text{ on atoms}\}$, $s_x = \min\{x \leq s_x \text{ is in } \rho \text{ with } \neg \text{ on atoms}\}$.

Then $\rho + {}^{f_-}/_{f_+} \in \gamma'((i + d')[\mathcal{K}(\rho) \mapsto \lceil d' \rceil])$, which implies $\rho + {}^{f_-}/_{f_+} \in \gamma'(i'[\mathcal{K}(\rho) \mapsto \lceil d' \rceil])$ for any $i' \in \Im(\mathcal{K})$. Then $\bigwedge_{x \in \mathcal{K}(\rho)}(x = \lceil d' \rceil) \to \rho + {}^{f_-}/_{f_+} \in \gamma'(i')$ for any γ' and i'. By rule (R2) $\mathbb{W}(\bigwedge_{x \in \mathcal{K}(\rho)}(x = \lceil d' \rceil) \to \rho + {}^{f_-}/_{f_+}) \in \gamma'(i')$.

$x \leq \lceil d' \rceil \in \gamma(i[\mathcal{K}(\rho) \mapsto \lceil d' \rceil - d'])$ for any $x \in \mathcal{K}(\rho)$. Then $\bigvee_{x \in \mathcal{K}(\rho)}(x \leq \lceil d' \rceil) \in \gamma(i[\mathcal{K}(\rho) \mapsto \lceil d' \rceil - d'])$. And similarly $\bigvee_{x \in \mathcal{K}(\rho)}(x \geq \lceil d' \rceil) \in \gamma(i[\mathcal{K}(\rho) \mapsto \lceil d' \rceil - d' + (d + d'))$, which implies $\boxminus(\bigvee_{x \in \mathcal{K}(\rho)}(x \geq \lceil d' \rceil)) \in \gamma(i[\mathcal{K}(\rho) \mapsto \lceil d' \rceil - d'])$. So $\bigvee_{x \in \mathcal{K}(\rho)}(x \leq \lceil d' \rceil) \wedge \boxminus(\bigvee_{x \in \mathcal{K}(\rho)}(x \geq \lceil d' \rceil)) \in \gamma(i[\mathcal{K}(\rho) \mapsto \lceil d' \rceil - d'])$. By axiom (A11) $\boxminus(\bigvee_{x \in \mathcal{K}(\rho)}(x = \lceil d' \rceil)) \in \gamma(i[\mathcal{K}(\rho) \mapsto \lceil d' \rceil - d'])$.

Together with $\mathbb{W}(\bigwedge_{x \in \mathcal{K}(\rho)}(x = \lceil d' \rceil) \to \rho + {}^{f_-}/_{f_+}) \in \gamma(i[\mathcal{K}(\rho) \mapsto \lceil d' \rceil])$, we get $\boxminus(\rho + {}^{f_-}/_{f_+}) \in \gamma(i[\mathcal{K}(\rho) \mapsto \lceil d' \rceil])$ by axiom (A10).

24

So $\exists \rho \in \gamma(i)$ - contradict that $\mathbb{W} \neg \rho \in \gamma(i)$. Hence $\lambda_i \cup \Delta_{i+d'}$ is consistent, which implies $\Theta_{i+d'}$ is consistent.

(2) Extend to maximal consistent set Λ_i for each $i \in I$.

(i) If there exists $x \in \mathcal{K}$ s.t. $i(x) \geq d'$ and $i(x) \in \mathbb{Q}^+$, Θ_i is maximal, i.e. for any $\phi \in \mathcal{L}$, $\phi \notin \Theta_i$ implies $\neg \phi \in \Theta_i$.

By definition of Δ_i, $x = i(x) \in \Delta_i \subseteq \Theta_i$. Together with $\phi \notin \Theta_i$, $x = i(x) \rightarrow \phi \notin \Theta_i$. By definition of $\lambda_{i-d'}$, $\mathbb{W}(x = i(x) \rightarrow \phi) \in \gamma(i - d')$ implies $x = i(x) \rightarrow \phi \in \lambda_{i-d'} \subseteq \Theta_i$. So $\mathbb{W}(x = i(x) \rightarrow \phi) \notin \gamma(i - d')$. Since $\gamma(i - d')$ is maximal and $\neg(\exists(x = i(x) \wedge \neg \phi)) = \mathbb{W}(x = i(x) \rightarrow \phi)$, we have $\mathbb{W}(x = i(x) \rightarrow \phi) \notin \gamma(i - d')$ implies $\exists(x = i(x) \wedge \neg \phi) \in \gamma(i - d')$. By (A10) $\mathbb{W}(x = i(x) \rightarrow \neg \phi) \in \gamma(i - d')$. So $x = i(x) \rightarrow \neg \phi \in \lambda_{i-d'} \subseteq \Theta_i$ by the definition of $\lambda_{i-d'}$. Then we have $\neg \phi \in \Theta_i$.

So $\Lambda_i = \Theta_i$, if there exists $x \in \mathcal{K}$ s.t. $i(x) \geq d'$ and $i(x) \in \mathbb{Q}^+$.

(ii) If for all $x \in \mathcal{K}$, $i(x) \geq d'$ and $i(x) \in \mathbb{R}_{\geq 0}$ but $\notin \mathbb{Q}^+$, define the following function, which given an interpretation i and finite set of clocks $\mathcal{K}_f$, maps to another interpretation:

$$i^{\mathcal{K}_f} = \begin{cases} i(x) & x \in \mathcal{K}_f \\ \lceil d' \rceil & \text{otherwise} \end{cases}$$

Then we define $\Lambda_i = \bigcup_{\mathcal{K}_f \subset \mathcal{K}} \{\phi \mid \mathcal{K}(\phi) = \mathcal{K}_f, \ \phi \in \Pi_{i^{\mathcal{K}_f}}\}$.

Now we prove that Λ_i above is maximal consistent.

Obviously for any $x \in \mathcal{K}_f$ and any finite $\mathcal{K}_f \subset \mathcal{K}$, $i^{\mathcal{K}_f}(x) - d' \geq 0$. And according to the first case, $\Lambda_{i^{\mathcal{K}_f}}$ is maximal consistent. So Λ_i is maximal.

Suppose Λ_i is not consistent, i.e. $\Lambda_i \vdash \perp$. Suppose $\Psi \subseteq \Lambda_i$ s.t. $\Psi \vdash \perp$, and $\mathcal{K}(\Psi)$ is the set of clocks of Ψ. Since even in the infinitary rules, the number of the clocks is finite, $\mathcal{K}(\Psi)$ is finite. For any $\psi \in \Psi$, $\psi \in \Lambda_{i^{\mathcal{K}(\psi)}}$ by the definition of Λ_i.

By the first case, $\Lambda_{i^{\mathcal{K}(\psi)}} = \Theta_{i^{\mathcal{V}(\psi)}}$, so $\lambda_{i^{\mathcal{K}(\psi)}-d'} \cup \Delta_{i^{\mathcal{K}(\psi)}} \vdash \psi$. Suppose $\Psi' \subseteq \lambda_{i^{\mathcal{K}(\psi)}-d'} \cup \Delta_{i^{\mathcal{K}(\psi)}}$ s.t. $\Psi' \vdash \psi$. $\Psi'\{y'/y\} \vdash \psi$ for any $y \in \mathcal{K}(\Psi)\backslash\mathcal{K}(\psi), y' \notin \mathcal{K}(\Psi)$. Since $\Psi'\{y'/y\}$ has no clocks in $\mathcal{K}(\Psi)\backslash\mathcal{K}(\psi)$, and $i^{\mathcal{K}(\Psi)}$ and $i^{\mathcal{K}(\psi)}$ only differ for the clocks in $\mathcal{K}(\Psi)\backslash\mathcal{K}(\psi)$, it is easy to have $\Psi'\{y'/y\} \in \lambda_{i^{\mathcal{K}(\Psi)}-d'} \cup \Delta_{i^{\mathcal{K}(\Psi)}}$. Then $\psi \in \Theta_{i^{\mathcal{K}(\Psi)}}$. Hence $\Psi \subseteq \Theta_{i^{\mathcal{K}(\Psi)}}$, which implies $\Theta_{i^{\mathcal{K}(\Psi)}} \vdash \perp$ - contradiction.

(iii) If for all $x \in \mathcal{K}$, $i(x) < d'$.

$\Lambda_i = \{\phi \mid \phi + \lceil d' \rceil \in \Lambda_{i+\lceil d' \rceil}\}$.

Obviously Λ_i is a consistent set since $\Lambda_{i+\lceil d' \rceil}$ is consistent. Now we prove Λ_i is also maximal. Suppose $\phi \notin \Lambda_i$, so $\phi + \lceil d' \rceil \notin \Lambda_{i+\lceil d' \rceil}$, which implies $\neg(\phi + \lceil d' \rceil) \in \Lambda_{i+\lceil d' \rceil}$ since $\Lambda_{i+\lceil d' \rceil}$ is maximal. Then we have $\neg \phi \in \Lambda_i$. So Λ_i is maximal.

(3) Let $\gamma_1(i) = \Lambda_i$.

We prove that γ_1 is a coherent function, i.e., γ_1 satisfies the two conditions in Definition 6.4.

The first condition holds obviously.

25

Before we prove γ_1 satisfies the second condition, we first prove the following:

(a) For any bijection σ on $\mathcal{K}$, any $\delta : \mathcal{K} \to \mathbb{R}$ s.t. $\forall x \in \mathcal{K}$, $i(x) \geq d'$, $i(x) + \delta(x) \geq d'$, and any $f_-, f_+ : \mathcal{K} \to \mathbb{Q}$ s.t. $f_- \leq \delta \leq f_+$, we have: $\phi \in \Theta_i \Rightarrow \phi +_\sigma {}^{f_-}/_{f_+} \in \Theta_{(i+\delta)\circ\sigma^{-1}}$.

$\phi \in \Theta_i$ implies $\lambda_{i-d'} \cup \Delta_i \vdash \phi$. Because γ is interpretation coherent, so $\lambda_{i-d'} +_\sigma {}^{f_-}/_{f_+} \subseteq \lambda_{(i+\delta)\circ\sigma^{-1}-d'}$. $\Delta_i +_\sigma {}^{f_-}/_{f_+} \subseteq \Delta_{(i+\delta)\circ\sigma^{-1}}$ by the definition of Δ_i. So we have $\lambda_{(i+\delta)\circ\sigma^{-1}-d'} \cup \Delta_{(i+\delta)\circ\sigma^{-1}} \vdash \phi +_\sigma {}^{f_-}/_{f_+}$. Hence $\phi \in \Theta_i \Rightarrow \phi +_\sigma {}^{f_-}/_{f_+} \in \Theta_{(i+\delta)\circ\sigma^{-1}}$.

(b) For any bijection σ on $\mathcal{K}$, any $\delta : \mathcal{K} \to \mathbb{R}$ s.t. $\forall x \in \mathcal{K}$, $i(x) \geq d'$, $i(x) + \delta(x) \geq d'$, and any $f_-, f_+ : \mathcal{K} \to \mathbb{Q}$ s.t. $f_- \leq \delta \leq f_+$, we have: $\phi \in \Lambda_i \Rightarrow \phi +_\sigma {}^{f_-}/_{f_+} \in \Lambda_{(i+\delta)\circ\sigma^{-1}}$.

If $\Lambda_i = \Theta_i$, it's the (a) case.

If not, $\phi \in \Lambda_{i^{\mathcal{K}(\phi)}} = \Theta_{i^{\mathcal{V}(\phi)}}$. $i(x) + \delta(x) \geq 0$ implies $i^{\mathcal{K}(\phi)}(x) + \delta^{\mathcal{K}(\phi)}(x) \geq 0$, so by the first case, $\phi +_\sigma {}^{f_-}/_{f_+} \in \Theta_{(i^{\mathcal{K}(\phi)}+\delta^{\mathcal{K}(\phi)})\circ\sigma^{-1}}$.

$(i^{\mathcal{K}(\phi)} + \delta^{\mathcal{K}(\phi)}) \circ \sigma^{-1}(x) = i^{\mathcal{K}(\phi)}(\sigma^{-1}(x)) + \delta^{\mathcal{K}(\phi)}(\sigma^{-1}(x)) = (i \circ \sigma^{-1})^{\mathcal{K}(\phi+_\sigma {}^{f_-}/_{f_+})}(x) + (\delta \circ \sigma^{-1})^{\mathcal{K}(\phi+_\sigma {}^{f_-}/_{f_+})}(x)$ (because $\sigma^{-1}(x) \in \mathcal{K}(\phi)$ iff $x \in \mathcal{K}(\phi +_\sigma {}^{f_-}/_{f_+})) = ((i+\delta) \circ \sigma^{-1})^{\mathcal{K}(\phi+_\sigma {}^{f_-}/_{f_+})}(x)$.

So we have $\phi +_\sigma {}^{f_-}/_{f_+} \in \Theta_{((i+\delta)\circ\sigma^{-1})^{\mathcal{K}(\phi+_\sigma {}^{f_-}/_{f_+})}}$.

Hence $\phi \in \Lambda_i \Rightarrow \phi +_\sigma {}^{f_-}/_{f_+} \in \Lambda_{(i+\delta)\circ\sigma^{-1}}$.

Now we are ready to prove the second condition: for any bijection σ on $\mathcal{K}$, any $\delta : \mathcal{K} \to \mathbb{R}$ s.t. $\forall x \in \mathcal{K}$, $i(x) + \delta(x) \geq 0$, and any $f_-, f_+ : \mathcal{K} \to \mathbb{Q}$ s.t. $f_- \leq \delta \leq f_+$, we have: $\phi \in \gamma(i) \Rightarrow \phi +_\sigma {}^{f_-}/_{f_+} \in \gamma((i + \delta) \circ \sigma^{-1})$.

- If $i \geq d'$ and $i + \delta \geq d'$, it's the (b) case above.

- If $i \not\geq d'$ and $i + \delta \geq d'$:

  $\phi + \lceil d' \rceil \in \gamma_1(i + \lceil d' \rceil)$ by the definition of γ_1. Let $i' = i + \lceil d' \rceil$, $\delta' = \delta - \lceil d' \rceil$, $f'_- = f_- - \lceil d' \rceil$ and $f'_+ = f_+ - \lceil d' \rceil$. Then we have $i' \geq d', i' + \delta' = i + \delta \geq d'$. So we have $(\phi + \lceil d' \rceil) +_\sigma {}^{f'_-}/_{f'_+} \in \gamma_1((i' + \delta') \circ \sigma^{-1})$, which implies $\phi +_\sigma {}^{f_-}/_{f_+} \in \gamma((i + \delta) \circ \sigma^{-1})$.

- If $i \geq d'$ and $i + \delta \not\geq d'$:

  Then $\phi +_\sigma {}^{f_- + \lceil d' \rceil}/_{f_+ + \lceil d' \rceil} \in \gamma_1((i + \delta + \lceil d' \rceil) \circ \sigma^{-1})$.

  Induction on ϕ:

  [The case $x \geq r$]:

  $(x \geq r) +_\sigma {}^{f_-}/_{f_+} + \lceil d' \rceil =$
  $$\begin{cases} \sigma(x) \geq r + f_-(x) + \lceil d' \rceil, & \text{if } r + f_-(x) \geq 0 \\ \sigma(x) \geq \lceil d' \rceil, & \text{otherwise} \end{cases}$$

  So $(x \geq r) +_\sigma {}^{f_-}/_{f_+} + \lceil d' \rceil \in \gamma_1((i + \delta + d') \circ \sigma^{-1})$.

  [The case $x > r$]: similar as above.

  [The case $x \leq r$]:

$(x \leq r) +_\sigma {}^{f_-}/_{f_+} + \lceil d' \rceil =$
$$\begin{cases} \sigma(x) \leq r + f_+(x) + \lceil d' \rceil, & \text{if } r + f_+(x) \geq 0 \\ \sigma(x) \leq \lceil d' \rceil, & \text{otherwise} \end{cases}$$

So $(x \leq r) +_\sigma {}^{f_-}/_{f_+} + \lceil d' \rceil \in \gamma_1((i + \delta + d') \circ \sigma^{-1})$.

[The case $x < r$]: similar as above.

Other cases hold obviously.

So we have $\phi +_\sigma {}^{f_-}/_{f_+} \in \gamma_1((i + \delta) \circ \sigma^{-1})$.

- If $i \not\geq d'$ and $i + \delta \not\geq d'$:
  $\phi \in \gamma_1(i)$ implies $\phi + \lceil d' \rceil \in \gamma_1(i + \lceil d' \rceil)$. Then $(\phi + \lceil d' \rceil) +_\sigma {}^{f_-}/_{f_+} \in \gamma_1((i + \lceil d' \rceil + \delta) \circ \sigma^{-1})$. So $\phi +_\sigma {}^{f_- + \lceil d' \rceil}/_{f_+ + \lceil d' \rceil} \in \gamma_1((i + \lceil d' \rceil + \delta) \circ \sigma^{-1})$. Then we can do the same deduction as the above case.

So we proved that there exists γ_1 constructed as above, s.t. $\gamma_1 = d' \oplus \gamma$.

- Now we prove that: $\exists \gamma_2$ s.t. $\gamma_2 = d \oplus \gamma_1$.

In the following we prove that $\gamma_3 = d \oplus \gamma_1$, i.e. for any $i \in \mathfrak{I}(\mathcal{K})$, $\mathbb{W}\phi \in \gamma_1(i)$ implies $\phi \in \gamma_3(i + d)$.

Secondly we need to prove that exists γ_2 s.t. $\gamma_2 = d \oplus \gamma_1$. We prove that $\gamma_3 = d \oplus \gamma_1$, i.e. for any $i \in \mathfrak{I}(\mathcal{K})$, for any $\mathbb{W}\phi \in \gamma_1(i)$ implies $\phi \in \gamma_3(i + d)$.

For any $i \in \mathfrak{I}(\mathcal{K})$, $\mathbb{W}\phi \in \gamma_1(i)$ implies $x \leq r \wedge \mathbb{W}\phi \in \gamma_1(i)$ for $r \in \mathbb{Q}^+$ s.t. $i(x) \leq r \leq i(x) + d$. Then $x \leq r + \lceil d' \rceil \wedge \mathbb{W}(\phi + \lceil d' \rceil) \in \gamma_1(i + \lceil d' \rceil)$. Since $\gamma_1 = d' \oplus \gamma$, we get $\exists (x \leq r + \lceil d' \rceil \wedge \mathbb{W}(\phi + \lceil d' \rceil)) \in \gamma(i + \lceil d' \rceil - d')$. By (A9), $\mathbb{W}(x \geq r + \lceil d' \rceil \rightarrow \mathbb{W}(\phi + \lceil d' \rceil)) \in \gamma(i + \lceil d' \rceil - d')$. So $x \geq r + \lceil d' \rceil \rightarrow \mathbb{W}(\phi + \lceil d' \rceil) \in \gamma_3(i + \lceil d' \rceil + d)$ since $\gamma_3 = (d + d') \oplus \gamma$. And $r \leq i(x) + d$, which implies $x \geq r + \lceil d' \rceil \in \gamma_3(i + \lceil d' \rceil + d)$. Then $\mathbb{W}(\phi + \lceil d' \rceil) \in \gamma_3(i + \lceil d' \rceil + d)$. Hence $\mathbb{W}\phi \in \gamma_3(i + d)$.

So there exists $\gamma_2 = \gamma_3$ s.t. $\gamma_2 = d \oplus \gamma_1$.

Hence $\mathcal{W} = (\Gamma, \theta, \oplus)$ defined above is a timed labeled transition system. ∎