Submitted to
**Topology Proceedings**

# $\mathbf{\Pi}_2^0$ SUBSETS OF DOMAIN-COMPLETE SPACES AND COUNTABLY CORRELATED SPACES

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Abstract. We show every $\mathbf{\Pi}_2^0$ subset of a domain-complete space is domain-complete. This implies that Chen's countably correlated spaces are all domain-complete.

## 1. Introduction

A *domain-complete space* is a homeomorph of a $G_\delta$ subset of a continuous dcpo. Those spaces were introduced in [4], and contain all continuous dcpos, all of de Brecht's quasi-Polish spaces [3], in particular all Polish spaces, all continuous complete quasi-metric spaces in their $d$-Scott topology and in particular all completely metrizable spaces.

The following is mentioned as open problem $(v)$ in [4]: is every subspace obtained as a $\mathbf{\Pi}_2^0$ subset of a domain-complete space again domain-complete? We give a positive answer to this problem here. This also solves open problems $(vi)$ and $(vii)$ of the same paper, as we will see at the end of this paper.

## 2. Preliminaries

A *dcpo* is a poset in which every directed family $D$ has a supremum $\sup^\uparrow D$. The *way-below* relation $\ll$ on a dcpo $Y$ is defined by $x \ll y$ if and only if every directed family $D$ such that $y \leq \sup^\uparrow D$ contains an element above $x$. The following relations hold: $x \ll y$ implies $x \leq y$; $x \leq y \ll z$

2010 *Mathematics Subject Classification.* Primary 54G99;
*Key words and phrases.* Domain-complete space, $\mathbf{\Pi}_2^0$ subset, countably correlated space.

implies $x \ll z$; $x \ll y \leq z$ implies $x \ll z$. We will sometimes write $\ll_Y$ instead of $\ll$ to make it clear what dcpo we are reasoning on.

A dcpo $X$ is *continuous* if and only if, for every $x \in X$, the family $\mathord{\downarrow\!\!\downarrow} x$ of elements way-below $x$ is directed and $\sup^{\uparrow} \mathord{\downarrow\!\!\downarrow} x = x$. It is equivalent to require every $x \in X$ to be the supremum of some directed family of elements way-below it, not necessarily $\mathord{\downarrow\!\!\downarrow} x$. Continuous dcpos are commonly called *domains*.

A subset $U$ of a dcpo $X$ is *Scott-open* if and only if it is upwards-closed ($x \leq y$ and $x \in U$ imply $y \in U$) and, for every directed family $D$ such that $\sup^{\uparrow} D \in U$, some element of $D$ is already in $U$. The Scott-open subsets form a topology called the *Scott topology*. All the dcpos we will consider are equipped with the Scott topology. In a continuous dcpo $X$, the sets $\mathord{\uparrow\!\!\uparrow} x \stackrel{\text{def}}{=} \{y \in X \mid x \ll y\}$ form a base of the Scott topology.

Those are classical notions of domain theory and of topology, for which the reader is directed to [5, 1, 6].

A *domain-complete* space is any topological space that is homeomorphic to a $G_\delta$ subset of a continuous dcpo, with the subspace topology. A $G_\delta$ subset is by definition a countable intersection of open subsets, or equivalently a subset of the form $\bigcap_{n \in \mathbb{N}} V_n$, where each $V_n$ is open and $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n \supseteq \cdots$.

A *UCO* subset of $X$ is the union of a closed and an open subset, or equivalently a set of the form $U \Rightarrow V \stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ implies } x \in V\}$, where $U$ and $V$ are open. A $\mathbf{\Pi}^0_2$ *subset* of $X$ is a countable intersection of UCO subsets. Such subsets are fundamental in the study of quasi-Polish spaces [3], because the subspaces of a quasi-Polish space that are themselves quasi-Polish in the subspace topology are exactly its $\mathbf{\Pi}^0_2$ subsets [3, Corollary 23]. The fact that $G_\delta$ subsets are profitably replaced by $\mathbf{\Pi}^0_2$ subsets in the descriptive set theory of domains and further non-Hausdorff topological spaces is due to Selivanov [9].

Note that every open subset, every $G_\delta$ subset, every closed subset is $\mathbf{\Pi}^0_2$. Beware that, outside the realm of metric spaces, closed subsets need not be $G_\delta$; in a dcpo, closed subsets are downwards-closed, while $G_\delta$ subsets are upwards-closed, for instance.

## 3. Another characterization of $G_\delta$ and $\mathbf{\Pi}^0_2$ subsets

We will rely on the following characterization of $G_\delta$ and $\mathbf{\Pi}^0_2$ subsets, which is of independent interest. A real-valued map $f$ is *lower semicontinuous* if and only if $f^{-1}(]t, +\infty[)$ is open for every real number $t$.

**Lemma 3.1.** *Let $Y$ be a topological space.*

(1) *The $G_\delta$ subsets of $Y$ are exactly its subsets of the form $f^{-1}(\{1\})$, where $f$ ranges over the lower semicontinuous maps from $Y$ to $[0,1]$.*
(2) *The $\mathbf{\Pi}_2^0$ subsets of $Y$ are exactly its subsets of the form $\varphi^{-1}(\{0\})$, where $\varphi$ ranges over the differences $f-g$ of two lower semicontinuous maps from $Y$ to $[0,1]$, with $f \geq g$.*

*Proof.* 1. Let $(V_n)_{n\in\mathbb{N}}$ be any sequence of open subsets of $Y$. Define $f(x) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{N}} \frac{1}{2^{n+1}}\chi_{V_n}(x)$, for every $x \in Y$. Then $f$ is lower semicontinuous from $Y$ to $[0,1]$, and $f(x) = 1$ if and only if $x$ is in every $V_n$. Conversely, for every lower semicontinuous map $f$ from $Y$ to $[0,1]$, $f^{-1}(\{1\}) = \bigcap_{n\in\mathbb{N}} V_n$ where $V_n \stackrel{\text{def}}{=} f^{-1}(]1-1/2^n,+\infty[)$.

2. Let $X \stackrel{\text{def}}{=} \bigcap_{n\in\mathbb{N}}(U_n \Rightarrow V_n)$, where $U_n$ and $V_n$ are open in $Y$, and $V_n \subseteq U_n$. For every $x \in Y$, let $f(x) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{N}} \frac{1}{2^{n+1}}\chi_{U_n}(x)$, $g(x) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{N}} \frac{1}{2^{n+1}}\chi_{V_n}(x)$, and let $\varphi \stackrel{\text{def}}{=} f-g$. Since $V_n$ is included in $U_n$, $\chi_{U_n} - \chi_{V_n} = \chi_{U_n \smallsetminus V_n}$, so $\varphi \stackrel{\text{def}}{=} f-g$ is such that $\varphi(x) = \sum_{n\in\mathbb{N}} \frac{1}{2^{n+1}}\chi_{U_n\smallsetminus V_n}$ for every $x \in Y$. In particular, $\varphi(x) \geq 0$ for every $x \in X$, meaning that $f \geq g$. Also, $\varphi(x) > 0$ if and only if $x \in U_n \smallsetminus V_n$ for some $n \in \mathbb{N}$, if and only if $x$ is in $\bigcup_{n\in\mathbb{N}}(U_n \smallsetminus V_n)$, namely not in $X$. Hence $X = \varphi^{-1}(\{0\})$.

Conversely, let $X \stackrel{\text{def}}{=} \varphi^{-1}(\{0\})$, where $\varphi = f-g$, and $f$, $g$ are lower semicontinuous maps from $Y$ to $[0,1]$, with $f \geq g$. For every $x \in Y$, $\varphi(x) > 0$ if and only if $f(x) > g(x)$, if and only if there is a rational number $q$ such that $f(x) > q \geq g(x)$. Hence the complement of $X$ is equal to $\bigcup_{q\in\mathbb{Q}} f^{-1}(]q,+\infty[) \smallsetminus g^{-1}(]q,+\infty[)$, so $X$ is the $\mathbf{\Pi}_2^0$ subset $\bigcap_{q\in\mathbb{Q}}(f^{-1}(]q,+\infty[) \Rightarrow g^{-1}(]q,+\infty[))$. ☐

## 4. The main theorem

We will show that every $\mathbf{\Pi}_2^0$ subset $X$ of a continuous dcpo $Y$ is homeomorphic to a $G_\delta$ subset of some continuous dcpo $Z$. The plan of the proof is as follows. By Lemma 3.1, item 2, $X = \varphi^{-1}(\{0\})$ where $\varphi = f-g$ and $f$, $g$ are lower semicontinuous maps from $Y$ to $[0,1]$, $f \geq g$. We will build $Z$ as the set $\{(x,r) \in Y \times \mathbb{R}_+ \mid \varphi(x) \leq r\}$, ordered by $(x,r) \leq (y,s)$ if and only if $x \leq y$, $r \geq s$, and $\varphi(x)-r \leq \varphi(y)-s$. Despite the fact that $\varphi$ is not Scott-continuous in general, and that the ordering is somewhat strange, we will show that $Z$ is a continuous dcpo, and that the map $x \mapsto (x,0)$ defines a homeomorphism from $X$ onto a $G_\delta$ subset of $Z$.

In order to work with a more standard ordering, we will consider the function $\psi\colon Y\times]-\infty,0] \to]-\infty,1]$ defined by $\psi(y,-r) \stackrel{\text{def}}{=} \varphi(y)-r$. Then $Z$ will be isomorphic to the poset of triples $(y,r,s)$ such that $s = \psi(y,r)$ and $s \leq 0$, with the usual componentwise ordering.

4.1. **Extracting a subdcpo.** Since $\varphi$ and $\psi$ may fail to be Scott-continuous, we will need to observe that they are still continuous, but as maps from $Y_d$ to $[0,1]_\lambda$. The $d$ and $\lambda$ subscripts refer to the so-called *d-topology* [7, Section 5] and to the so-called *Lawson topology* [5, Section III.1], and will be introduced shortly.

For every subset $E$ of a dcpo $P$, let $\uparrow E$ denote its upward closure $\{y \in P \mid \exists x \in E, x \leq y\}$. The *Lawson topology* on $P$ is the coarsest topology that contains all Scott-open subsets and the complements of all the sets $\uparrow E$, $E$ finite. Let us write $P_\lambda$ for $P$ with its Lawson topology.

A subset $C$ of a dcpo $Y'$ is *d-closed* if and only if every directed family included in $C$ has a supremum in $C$. The complements of d-closed sets (the *d-open* sets) form a topology called the *d-topology* [7, Section 5]. A set $U$ is d-open if and only if every directed family whose supremum is in $U$ intersects $U$. Every Scott-open subset is d-open, but also every downwards-closed subset. We write $Y'_d$ for $Y'$ with its d-topology.

We will repeatedly use the following, easily proved fact: in a dcpo, any cofinal subfamily of a directed family $D$ is itself directed, and has the same supremum as the original family. A subfamily $E$ of $D$ is *cofinal* if and only if every element of $D$ is below some element of $E$. A first consequence, which can be used to give a simple proof that the d-topology is a topology, is as follows.

**Lemma 4.1.** *Let $U$ be a d-open subset of a dcpo $Y'$. For every directed family $(x_i)_{i\in I}$ whose supremum $x$ is in $U$, $x_i$ is in $U$ for $i$ large enough; namely, there is an $i_0 \in I$ such that, for every $i \in I$ such that $x_{i_0} \leq x_i$, $x_i$ is in $U$.*

*Proof.* We assume the contrary: for every $i_0 \in I$, there is an $i \in I$ such that $x_{i_0} \leq x_i$ but $x_i$ is not in $U$. Hence the family $E$ of points $x_i$, $i \in I$, such that $x_i \notin U$, is cofinal. $E$ is directed, included in the complement of $U$, which is d-closed by assumption, so its supremum, which must be $x$, must also be in the complement of $U$: contradiction. □

A *subdcpo* of a dcpo $Y'$ is just a d-closed subset $G$. Every subdcpo of $Y'$ is, in particular, a dcpo, and one in which suprema of directed families are computed as in the ambient dcpo $Y'$. The latter condition is important. For example, the lattice of closed subsets of a topological space $X$ is not in general a subdcpo of $\mathbb{P}(X)$ (both being ordered by inclusion), although it is a dcpo, and even a complete lattice. Indeed, directed suprema are computed as unions in the latter, and as closures of unions in the former.

**Lemma 4.2.** *Let $Y'$ and $P$ be two dcpos. Let $P'$ be the set of points of $P$, equipped with a Hausdorff topology coarser than the d-topology, and $\psi$*

*be a continuous map from $Y'_d$ to $P'$. The graph $G(\psi) \stackrel{def}{=} \{(x, \psi(x)) \mid x \in Y'\} \subseteq Y' \times P$ of $\psi$ is a subdcpo of $Y' \times P$.*

*Proof.* Let us consider a directed family $((x_i, \psi(x_i)))_{i \in I}$ in $G(\psi)$, and let $(x, t)$ be its supremum in $Y' \times P$. Note that $(x_i)_{i \in I}$ and $(\psi(x_i))_{i \in I}$ are directed, in particular—we certainly do not deduce the latter from the former, since $\psi$ is not assumed to be monotonic in any way.

If $t$ were different from $\psi(x)$, by Hausdorffness there would be disjoint $P'$-open sets $U$, $V$ containing $t$ and $\psi(x)$ respectively. Since $t = \sup^{\uparrow}_{i \in I} \psi(x_i)$ is in $U$, and $U$ is d-open, $\psi(x_i)$ is in $U$ for $i$ large enough, using Lemma 4.1, so $x_i$ is in $\psi^{-1}(U)$ for $i$ large enough. Since $x = \sup^{\uparrow}_{i \in I} x_i$ is in the d-open set $\psi^{-1}(V)$, $x_i$ is in $\psi^{-1}(V)$ for $i$ large enough. This is impossible, since $\psi^{-1}(U)$ and $\psi^{-1}(V)$ are disjoint.

Therefore $t = \psi(x)$, so $(x, t) = (x, \psi(x))$ is in $G(\psi)$. ☐

This lemma applies notably when $P' = P_d$. When $P$ is a continuous (or even a quasi-continuous dcpo), $P_\lambda$ is Hausdorff, and this allows us to weaken our assumptions on $\psi$:

**Corollary 4.3.** *Let $Y'$ be a dcpo, $P$ be a continuous dcpo, and let $\psi$ be a continuous map from $Y'_d$ to $P_\lambda$. The graph $G(\psi)$ of $\psi$ is a subdcpo of $Y' \times P$.*

**Remark 4.4.** Every Scott-continuous map $f : Y' \to P$ is continuous from $Y'_d$ to $P_\lambda$, because inverse images of Scott-open sets are Scott-open and inverse images of sets of the form $\uparrow E$ are upwards-closed, owing to the fact that $f$, being Scott-continuous, is monotonic. When $P = [0, 1]$, the topology $P_\lambda$ is the usual metric topology, for which addition and subtraction are continuous. (The latter would fail if we replaced $P_\lambda$ by $P_d$.) It follows that the map $\varphi \stackrel{\text{def}}{=} f - g$ considered in Lemma 3.1 (2) is continuous from $Y_d$ to $[0, 1]_\lambda$.

4.2. **Extracting a continuous dcpo.**

**Definition 4.5.** A *subdomain* of a continuous dcpo $Y'$ is any subdcpo $G$ which, as a dcpo, is a continuous dcpo, and whose way-below relation is the restriction of that of $Y'$.

In that case, the Scott topology on $G$ is also the subspace topology inherited from the Scott topology of $Y'$. It is not enough for the subdcpo $G$ to be a continuous dcpo in order to be a subdomain. For example, the lattice of open subsets of a locally compact space $X$ is a subdcpo of $\mathbb{P}(X)$ (ordered by inclusion), and a continuous dcpo, but $U$ is way-below $V$ in $X$ if and only if $U \subseteq Q \subseteq V$ for some compact set $Q$, while $U$ is way-below $V$ in $\mathbb{P}(X)$ if and only if $U$ is finite and included in $V$.

**Lemma 4.6.** *Let $G$ be a subdcpo of a continuous dcpo $Y'$. Then $G$ is a subdomain of $Y'$ if and only if the following cofinality condition holds: for every $g \in G$, $\twoheaddownarrow g \cap G$ is cofinal in $\twoheaddownarrow g$.*

*Proof.* We recall that $\twoheaddownarrow g$ denotes the set of points $y \in Y'$ such that $y \ll_{Y'} g$.

If $G$ is a subdomain of $Y'$, then for every $g \in G$, $g$ is the supremum of the directed family of elements $g' \in G$ such that $g' \ll_G g$, equivalently such that $g' \ll_{Y'} g$. For every $y \in \twoheaddownarrow g$, therefore, there is a $g' \in G$ such that $g' \ll_{Y'} g$ and $y \leq g'$, whence the cofinality condition holds.

Conversely, let us assume that the cofinality condition holds.

We first claim that $g \ll_{Y'} g'$ implies $g \ll_G g'$. To this end, we consider a directed family $(g_i)_{i \in I}$ in $G$, whose supremum (in $G$) lies above $g'$. Since $G$ is a subdcpo of $Y'$, namely since directed suprema are computed as in $Y'$, and since $g \ll_{Y'} g'$, $g \leq g_i$ for some $i \in I$. Hence $g \ll_G g'$.

For every $g \in G$, since $\twoheaddownarrow g \cap G$ is cofinal in $\twoheaddownarrow g$, it is directed, and has the same supremum (in $Y'$, hence also in $G$) than $\twoheaddownarrow g$, namely $g$. For every $g' \in \twoheaddownarrow g \cap G$, we have $g' \ll_{Y'} g$, hence $g' \ll_G g$. This shows that $G$ is a continuous dcpo.

In order to show that $\ll_G$ is the restriction of $\ll_{Y'}$ to $G$, we assume that $g \ll_G g'$. Since, as we have just seen, $g' = \sup^{\uparrow}(\twoheaddownarrow g' \cap G)$, there is an element $g''$ of $\twoheaddownarrow g' \cap G$ such that $g \leq g''$. Now $g \leq g'' \ll_{Y'} g'$. We have already seen that, conversely, $g' \ll_{Y'} g$ implies $g' \ll_G g$. ☐

The following corollary is well-known, see [8, Lemma 2.40] for example. One merely observes that a Scott-closed subset $G$ is the same thing as a downwards-closed subdcpo. Downward closure implies that $\twoheaddownarrow g \cap G = \twoheaddownarrow g$ for every $g \in G$, whence the cofinality requirement in Lemma 4.6 is trivial.

**Corollary 4.7.** *Every Scott-closed subset of a continuous dcpo $Y'$ is a subdomain of $Y'$.* ☐

Now we imagine that $Y'$ and $P$ are continuous dcpos. In that case, $Y' \times P$ is continuous, and its way-below relation is the product of the way-below relations. As before, we consider the subdcpo $G(\psi)$ obtained as the graph of a continuous map $\psi \colon Y'_d \to P_\lambda$. We can simplify the cofinality condition slightly in this case.

**Lemma 4.8.** *Let $Y'$ and $P$ be two continuous dcpos, and let $\psi$ be a continuous map from $Y'_d$ to $P_\lambda$. $G(\psi)$ is a subdomain of $Y' \times P$ if and only if:*

- *(*) for all points $y, x \in Y'$ such that $y \ll_{Y'} x$, there is a point $z \in Y'$ such that $y \leq z \ll_{Y'} x$ and $\psi(z) \ll_P \psi(x)$.*

*Proof.* $G(\psi)$ is a subdcpo of $Y'$ by Corollary 4.3. It suffices to show that the cofinality condition of Lemma 4.6 is equivalent to (*).

Let us show that it implies (*). Let $y \ll_{Y'} x$. Since $P$ is a continuous dcpo, there is an element $t \ll_P \psi(x)$, so $(y, t)$ is in $\twoheaddownarrow(x, \psi(x))$. The cofinality condition gives us an element $(z, \psi(z))$ of $G(\psi)$ such that $(y, t) \leq (z, \psi(z)) \ll (x, \psi(x))$, whence (*) follows.

In the converse direction, let $(x, \psi(x))$ be any point of $G(\psi)$. In order to show that $\twoheaddownarrow(x, \psi(x)) \cap G(\psi)$ is cofinal in $\twoheaddownarrow(x, \psi(x))$, we observe that for every $(y_0, s_0) \in \twoheaddownarrow(x, \psi(x))$, namely if $y_0 \ll_{Y'} x$ and $s_0 \ll_P \psi(x)$, then $x$ is in the d-open set $U \stackrel{\text{def}}{=} \psi^{-1}(\twoheaduparrow s_0) \cap \twoheaduparrow y_0$. Since $Y'$ is continuous, $x$ is the supremum of the directed family $\twoheaddownarrow x$. By Lemma 4.1, there is a point $y \in \twoheaddownarrow x$ such that, for every $z \in \twoheaddownarrow x$ such that $y \leq z$, $z$ is in $U$. Note that $y \ll_{Y'} x$, so (*) applies, giving us an element $z \in \twoheaddownarrow x$ such that $y \leq z$ and $\psi(z) \ll \psi(x)$. Hence $(z, \psi(z)) \ll (x, \psi(x))$. We have just seen that $z$ must be in $U$. Hence $y_0 \ll_{Y'} z$ and $s_0 \ll_P \psi(z)$, in particular $(y_0, s_0) \leq (z, \psi(z))$. □

We apply this to the case where $Y' \stackrel{\text{def}}{=} Y \times ]-\infty, 0]$, $P \stackrel{\text{def}}{=} ]-\infty, a]$, $a \in \mathbb{R}$. Note that $]-\infty, 0]$ and $]-\infty, a]$ are continuous dcpos, and that their way-below relation is $<$.

**Lemma 4.9.** *Let $Y$ be a continuous dcpo, $a \in \mathbb{R}$, $P \stackrel{def}{=} ]-\infty, a]$, and $\varphi$ be a continuous map from $Y_d$ to $P_\lambda$. Let $Y' \stackrel{def}{=} Y \times ]-\infty, 0]$. The function $\psi \colon Y' \to P$ defined by $\psi(y, r) \stackrel{def}{=} \varphi(y) + r$ is continuous from $Y'_d$ to $P_\lambda$, and satisfies (*).*

*Proof.* The map $\psi$ is continuous from $Y'_d$ to $P_\lambda$. Indeed, $\psi$ arises as the composition of $+\colon ]-\infty, a]_\lambda \times ]-\infty, 0]_\lambda \to ]-\infty, a]_\lambda$, of $\varphi \times \mathrm{id}\colon Y_d \times ]-\infty, 0]_\lambda \to P_\lambda \times ]-\infty, 0]_\lambda$, and of $\mathrm{id}\colon Y'_d \to Y_d \times ]-\infty, 0]_\lambda$. ($\times$ means topological, not order-theoretic product here.) The former is continuous, because the Lawson topology on real intervals is the usual metric topology, for which addition is continuous. The second map is trivially continuous. For the last one, it suffices to check that every product $C \times I$ of a d-closed subset of $Y$ with a closed interval of $]-\infty, 0]_\lambda$ is d-closed, and that follows from the fact that directed suprema are computed componentwise.

As far as condition (*) is concerned, we need to show that given $(y, r)$ and $(x, s)$ in $Y \times ]-\infty, 0]$ such that $y \ll_Y x$ and $r < s$, there is a pair $(z, t)$ in $Y \times ]-\infty, 0]$ such that: (a) $y \leq z \ll_Y x$, (b) $r \leq t < s$, and (c) $\varphi(z) + t < \varphi(x) + s$. Since $\varphi$ is continuous from $Y_d$ to $P_\lambda$, $U \stackrel{\text{def}}{=} \varphi^{-1}(P \smallsetminus \uparrow\{\varphi(x) + s - r\}) \cap \twoheaduparrow y$ is d-open in $Y$. $U$ also contains $x$, because $\varphi(x) < \varphi(x) + s - r$, and because $y \ll_Y x$. Since $Y$ is a continuous dcpo, $x = \sup^\uparrow \twoheaddownarrow x$. Using the fact that $U$ is d-open, there is an element $z \ll_Y x$

inside $U$. Since $z \in U$, in particular $y \ll_Y z$, so $y \leq z$, showing (a). We define $t$ as $r$, whence (b) follows. Since $z \in U$ again, $\varphi(z) < \varphi(x)+s-r = \varphi(x)+s-t$, so $\varphi(z)+t < \varphi(x)+s$, establishing (c). ☐

**Corollary 4.10.** *Let $Y$ be a continuous dcpo, $f$ and $g$ be two lower semicontinuous map from $Y$ to $[0,1]$ such that $f \geq g$, and let $\varphi \stackrel{def}{=} f-g$. Let also $Z \stackrel{def}{=} \{(x,r) \in Y \times \mathbb{R}_+ \mid \varphi(x) \leq r\}$, ordered by $(x,r) \leq (y,s)$ if and only if $x \leq y$, $r \geq s$, and $\varphi(x)-r \leq \varphi(y)-s$. Then:*

*(1) $Z$ is a continuous dcpo;*
*(2) the supremum $(x,r)$ of a directed family $(x_i,r_i)_{i\in I}$ in $Z$ is equal to $(\sup^\uparrow_{i\in I} x_i, \inf^\downarrow_{i\in I} r_i)$, and $\varphi(x)-r = \sup^\uparrow_{i\in I}(\varphi(x_i)-r_i)$;*
*(3) for all $(x,r)$ and $(y,s)$ in $Z$, $(x,r) \ll_Z (y,s)$ if and only if $x \ll_Y y$, $r > s$, and $\varphi(x)-r < \varphi(y)-s$.*

*Proof.* By Remark 4.4, $\varphi$ is continuous from $Y_d$ to $P_\lambda$, where $P \stackrel{\text{def}}{=} ]-\infty,1]$. Lemma 4.9 and Lemma 4.8 allow us to say that $G(\psi)$ is a subdomain of $Y' \times P$, where $Y' \stackrel{\text{def}}{=} Y \times ]-\infty,0]$. The subset $Z'$ of those points $((y,r),s)$ of $G(\psi)$ (i.e., $s=\psi(y,r)$, equivalently $s=\varphi(y)+r$) such that $s \leq 0$ is Scott-closed, as one checks easily since suprema in $G(\psi)$ are taken as in $Y' \times P$. We use Corollary 4.7, and we obtain that $Z'$ is a subdomain of $G(\psi)$, hence of $Y' \times P$. We now observe that the map $(x,r) \mapsto (x,-r,\varphi(x)-r)$ is an order isomorphism from $Z$ onto $Z'$, and the result follows. ☐

4.3. **The final argument.** We are almost done:

**Proposition 4.11.** *Every $\mathbf{\Pi}^0_2$ subset $X$ of a continuous dcpo $Y$ is homeomorphic to a $G_\delta$ subset of some continuous dcpo $Z$.*

*Proof.* Let $X$ be any $\mathbf{\Pi}^0_2$ subset of $Y$. By Lemma 3.1 (2), $X = \varphi^{-1}(\{0\})$ where $\varphi = f-g$ and $f$, $g$ are lower semicontinuous maps from $Y$ to $[0,1]$, $f \geq g$. We define $Z$ as in Corollary 4.10. It remains to show that $f\colon x \mapsto (x,0)$ defines a homeomorphism from $X$ onto a $G_\delta$ subset of $Z$.

The inverse image of the basic open set $\twoheaduparrow(y,r)$ by $f$ is the set of elements $x \in X$ such that $(y,r) \ll_Z (x,0)$, equivalently such that $y \ll_Z x$, $r > 0$, and $\varphi(y)-r < \varphi(x)-0$. The latter expression simplifies since $x \in X$ is equivalent to $\varphi(x)=0$. Hence $f^{-1}(\twoheaduparrow(y,r))$ is empty if $\varphi(y) \geq r$, otherwise is equal to $X \cap \twoheaduparrow y$. It follows that $f$ is continuous. It also follows that every basic open set $X \cap \twoheaduparrow y$ $(y \in Y)$ of $X$ is equal to $f^{-1}(\twoheaduparrow(y,r))$, for any $r > 0, \varphi(y)$. Since $f$ is clearly injective, $f$ is a topological embedding of $X$ into $Z$, hence a homeomorphism onto its image.

That image is the set of elements $(x,r)$ of $Z$ such that $r=0$, or equivalently the intersection of the countably many sets $V_n \stackrel{\text{def}}{=} \{(x,r) \in$

$Z \mid r < 1/2^n\}$. Using item (2) of Corollary 4.10, it is easy to check that $V_n$ is open in $Z$, so the image of $f$ is a $G_\delta$ subset of $Z$. □

It follows:

**Theorem 4.12.** *Every $\mathbf{\Pi}^0_2$ subset of a domain-complete space is domain-complete in the subspace topology.*

*Proof.* Let $A \stackrel{\text{def}}{=} \bigcap_{n\in\mathbb{N}}(U_n \Rightarrow V_n)$ be a $\mathbf{\Pi}^0_2$ subset of $Y$, where $Y \stackrel{\text{def}}{=} \bigcap_{m\in\mathbb{N}} W_m$, each $W_m$ is open in a continuous dcpo $X$, and $U_n$ and $V_n$ are open in $Y$. We write $U_n$ as $U'_n \cap Y$ and $V_n$ as $V'_n \cap Y$, where $U'_n$ and $V'_n$ are open in $X$. Then $U_n \Rightarrow V_n = (U'_n \Rightarrow V'_n) \cap Y$, so $A = \left(\bigcap_{n\in\mathbb{N}}(U'_n \Rightarrow V'_n)\right) \cap \bigcap_{m\in\mathbb{N}} W_m$ is a $\mathbf{\Pi}^0_2$ subset of $X$. By Proposition 4.11, $A$ is homeomorphic to a $G_\delta$ subset of some continuous dcpo, hence is domain-complete. □

An *LCS-complete* space is a homeomorph of a $G_\delta$ subset of a locally compact sober space [4]. All domain-complete spaces are LCS-complete, because every continuous dcpo is locally compact and sober, and the inclusion is strict. We should note that the analogue of Theorem 4.12 for LCS-complete spaces *fails*: by Proposition 14.5 of [4], there is a UCO subset of a compact Hausdorff space that is not LCS-complete.

## 5. Consequences

A first immediate consequence of Theorem 4.12 is the following.

**Proposition 5.1.** *A space is domain-complete if and only if it is homeomorphic to a $\mathbf{\Pi}^0_2$ subset of a continuous dcpo.*

*Proof.* Every $G_\delta$ subset is trivially $\mathbf{\Pi}^0_2$. The converse direction is by Theorem 4.12. □

A second consequence concernes Chen's *countably correlated spaces* [2]. Those are the spaces that are homeomorphic to a $\mathbf{\Pi}^0_2$ subset of $\mathbb{P}(I)$, for an arbitrary set $I$. Here $\mathbb{P}(I)$ is the space of all subsets of $I$, with the Scott topology of inclusion. This is a continuous (even algebraic) dcpo, where $A \ll B$ if and only if $A$ is a finite subset of $B$. The countably correlated spaces generalize the quasi-Polish spaces, which are exactly the homeomorphs of $\mathbf{\Pi}^0_2$ subsets of $\mathbb{P}(I)$ for $I$ *countable* [3, Corollary 23].

**Proposition 5.2.** *Every countably correlated space is domain-complete.*

*Proof.* By definition, every countably correlated space is homeomorphic to a $\mathbf{\Pi}^0_2$ subset of a continuous dcpo of the form $\mathbb{P}(I)$. □

This is a positive answer to open problem $(vi)$ of [4].

Open problem $(vii)$ asks whether every LCS-complete space is countably correlated. The answer to that problem is negative: by Remark 9.3 of [4], the space $\{0,1\}^I$ where $\{0,1\}$ is given the discrete topology and $I$ is uncountable is LCS-complete but not domain-complete, hence not countably correlated.

This leaves one new open question: is the converse of Proposition 5.2 true, namely is every domain-complete space countably correlated?

## Acknowledgments

Section 4 greatly benefited from suggestions due to the referee, whom we thank here.

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