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QRB-DOMAINS AND THE PROBABILISTIC POWERDOMAIN *

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Abstract. Is there any Cartesian-closed category of continuous domains that would be closed under Jones and Plotkin's probabilistic powerdomain construction? This is a major open problem in the area of denotational semantics of probabilistic higher-order languages. We relax the question, and look for quasi-continuous dcpos instead. We introduce a natural class of such quasi-continuous dcpos, the omega-QRB-domains. We show that they form a category omega-QRB with pleasing properties: omega-QRB is closed under the probabilistic powerdomain functor, under finite products, under taking bilimits of expanding sequences, under retracts, and even under so-called quasi-retracts. But... omega-QRB is not Cartesian closed. We conclude by showing that the QRB domains are just one half of an FS-domain, merely lacking control.

1. Introduction

1.1. **The Jung-Tix Problem.** A famous open problem in denotational semantics is whether the probabilistic powerdomain $\mathbf{V}_1(X)$ of an **FS**-domain X is again an **FS**-domain [JT98], and similarly with **RB**-domains in lieu of **FS**-domains. $\mathbf{V}_1(X)$ (resp. $\mathbf{V}_{\leq 1}(X)$) is the dcpo of all continuous probability (resp., subprobability) valuations over X: this construction was introduced by Jones and Plotkin to give a denotational semantics to higher-order probabilistic languages [JP89].

More generally, is there a category of nice enough dcpos that would be Cartesian-closed and closed under $\mathbf{V}_1$? We call this the *Jung-Tix problem*. By "nice enough", we mean nice enough to do any serious mathematics with, e.g., to establish definability or full abstraction results in extensional models of higher-order, probabilistic languages. It is traditional to equate "nice enough" with "continuous", and this is justified by the rich theory of continuous domains [GHK+03].

However, *quasi-continuous* dcpos (see [GLS83], or [GHK+03, III-3]) generalize continuous dcpos and are almost as well-behaved. We propose to widen the scope of the problem, and ask for a category of quasi-continuous dcpos that would be closed under $\mathbf{V}_1$. We show that, by mimicking the construction of **RB**-domains [AJ94], with some flavor of "quasi",

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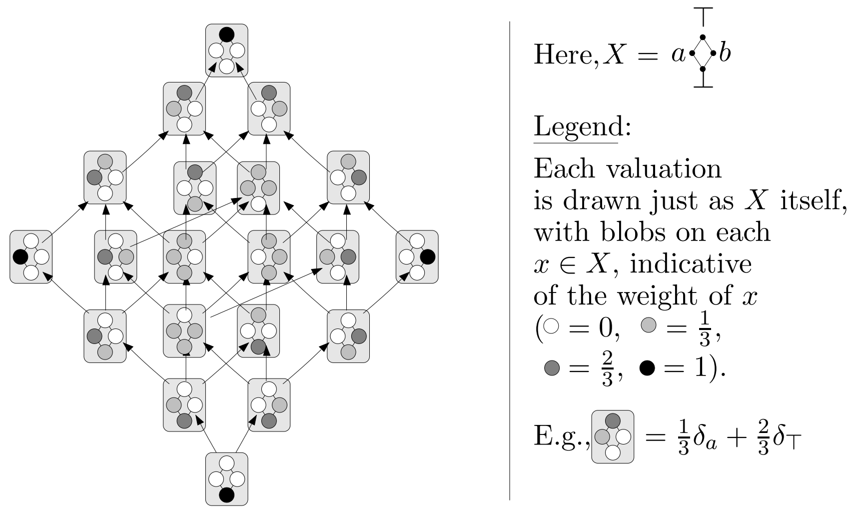


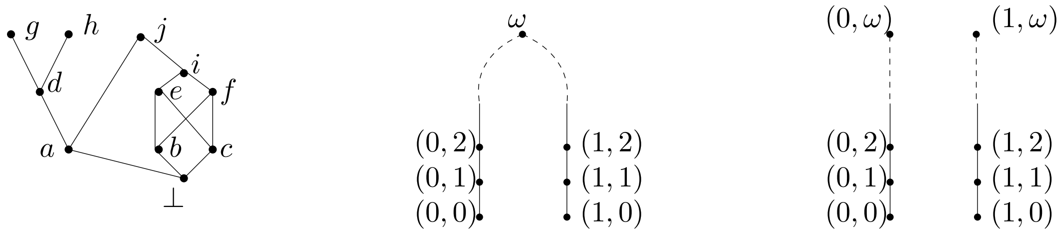
Figure 1: Part of the Hasse Diagram of $\mathbf{V}_1(X)$

we obtain a category $\omega\mathbf{QRB}$ of so-called $\omega\mathbf{QRB}$-domains that not only has many desired, nice mathematical properties (e.g., it is closed under taking bilimits of expanding sequences, and every $\omega\mathbf{QRB}$-domain is stably compact), but is also closed under $\mathbf{V}_1$.

We failed to solve the Jung-Tix problem: $\omega\mathbf{QRB}$ is indeed not Cartesian-closed. In spite of this, we believe our contribution to bring some progress towards settling the question, and at least to understand the structure of $\mathbf{V}_1(X)$ better. To appreciate this, recall what is currently known about $\mathbf{V}_1$. There are two landmark results: $\mathbf{V}_1(X)$ is a continuous dcpo as soon as X is ([Eda95], building on Jones [JP89]), and $\mathbf{V}_1(X)$ is stably compact (with its weak topology) whenever X is [JT98, AMJK04]. Since then, no significant progress has been made. When it comes to solving the Jung-Tix problem, we must realize that there is *little choice*: the only known Cartesian-closed categories of (pointed) continuous dcpos that may suit our needs are **RB** and **FS** [JT98]. I.e., all other known Cartesian-closed categories of continuous dcpos, e.g., bc-domains or L-domains, are *not* closed under $\mathbf{V}_1$. Next, we must recognize that *little is known* about the (sub)probabilistic powerdomain of an **RB** or **FS**-domain. In trying to show that either **RB** or **FS** was closed under $\mathbf{V}_1$, Jung and Tix [JT98] only managed to show that the subprobabilistic powerdomain $\mathbf{V}_{\leq 1}(X)$ of a *finite tree* X was an **RB**-domain, and that the subprobabilistic powerdomain of a *reversed finite tree* was an **FS**-domain. This is still far from the goal.

In fact, we do not know whether $\mathbf{V}_1(X)$ is an **RB**-domain when X is even the simple poset $\{\bot, a, b, \top\}$ (a and b incomparable, $\bot \leq a, b \leq \top$, see Figure 1, right)—but it is an **FS**-domain. For a more complex (arbitrarily chosen) example, take X to be the finite pointed poset of Figure 2 (i): then $\mathbf{V}_1(X)$ and $\mathbf{V}_{\leq 1}(X)$ are continuous and stably compact, but not known to be **RB**-domains or **FS**-domains (and they are much harder to visualize, too).

No progress seems to have been made on the question since Jung and Tix' 1998 attempt. As part of our results, we show that for every finite pointed poset X, e.g. Figure 2 (i), $\mathbf{V}_1(X)$ is a continuous $\omega\mathbf{QRB}$-domain. This is also one of the basic results that we then leverage to show that $\mathbf{V}_1(X)$ is an $\omega\mathbf{QRB}$-domain for *any* $\omega\mathbf{QRB}$-domain, in particular every **RB**-domain, not just every finite pointed poset, X.



(i) A finite poset (ii) The non-continuous dcpo $\mathcal{N}_2$ (iii) $\mathbb{N}_\omega + \mathbb{N}_\omega$

Figure 2: Poset Examples

One may obtain some intuition as to why this should be so, and at the same time give an idea of what (ω)**QRB**-domains are. Let X be a finite pointed poset. In attempting to show that $\mathbf{V}_1(X)$ is an **RB**-domain, we are led to study the so-called *deflations* $f : \mathbf{V}_1(X) \to \mathbf{V}_1(X)$, i.e., the continuous maps f with *finite* range such that $f(\nu) \leq \nu$ for every continuous probability valuation ν on X, and we must try to find deflations f such that $f(\nu)$ is as close as one desires to ν. All natural definitions of f fail to be continuous, and in fact to be monotonic. (E.g., Graham's construction [Gra88] is not monotonic, see Jung and Tix.) Looking for maps f such that $f(\nu)$ is instead a finite, non-empty *set* of valuations below ν shows more promise—the monotonicity requirements are slightly more relaxed. Such a set-valued function is what we call a *quasi-deflation* below. For example, one may think of fixing $N \geq 1$ ($N = 3$ in Figure 1), and mapping ν to the collection of all valuations ν' below ν such that the measure of any subset is a multiple of $1/N$, keeping only those ν' that are maximal. (Pick them from the left of Figure 1, in our example.) This still does not provide anything monotonic, but we managed to show that one can indeed approximate every element ν of $\mathbf{V}_1(X)$, continuously in ν, using quasi-deflations. The proof is non-trivial, and rests on deep properties relating **QRB**-domains and *quasi-retractions*, all notions that we define and study.

1.2. **Outline.** We introduce most of the required notions in Section 2. Since we shall only start studying the probabilistic powerdomain in Section 6, we shall refrain from defining valuations, probabilities, and related concepts until then.

We introduce **QRB**-domains in Section 3. They are defined just as **RB**-domains are, only with a flavor of "quasi", i.e., replacing approximating elements by approximating *sets* of elements. We establish their main properties there, in particular that they are quasi-continuous, stably compact, and Lawson-compact. Much as **RB**-domains are also characterized as the retracts of bifinite domains, we show that, up to a few details, the **QRB**-domains are the quasi-retracts of bifinite domains in Section 4. This allows us to parenthesize **QRB** as quasi-(retract of bifinite domain) or as (quasi-retract) of bifinite domain. Quasi-retractions are an essential concept in the study of **QRB**-domains, and we introduce them here, as well as the related notion of quasi-projections—images by proper maps.

We also show that the category of countably-based **QRB**-domains is closed under finite products (easy) and taking bilimits of expanding sequences (hard, but similar to the case of **RB**-domains) in Section 5.

The core of the paper is Section 6, where we show that the category ω**QRB** of countably-based **QRB**-domains is closed under the probabilistic powerdomain construction. This capitalizes on all previous sections, and will follow from a variant of Jung and Tix' result that $\mathbf{V}_1(X)$ is an **RB**-domain whenever X is a finite tree, and applying suitable quasi-projections and bilimits. The key result will then be Theorem 6.5, which shows that for any quasi-projection Y of a stably compact space X, $\mathbf{V}_1(Y)$ is again a quasi-projection of $\mathbf{V}_1(X)$, again up to a few details.

We conclude in Section 7.

1.3. **Other Related Work.** Instead of solving the Jung-Tix problem, one may try to circumvent it. One of the most successful such attempts led to the discovery of *qcb-spaces* [BSS07] and to compactly generated countably-based monotone convergence spaces [BSS06], as Cartesian-closed categories of topological spaces where a reasonable amount of semantics can be done. This provides exciting new perspectives. The category of qcb-spaces accommodates two probabilistic powerdomains [BS09]. The observationally induced one is essentially $\mathbf{V}_1(X)$ (with the weak topology), but differs from the one obtained as a free algebra.

2. Preliminaries

We refer to [AJ94, GHK+03, Mis98] for background material. A *poset* X is a set with a partial ordering $\leq$. Let $\downarrow A$ be the downward closure $\{x \in X \mid \exists y \in A \cdot x \leq y\}$; we write $\downarrow x$ for $\downarrow\{x\}$, when $x \in X$. The upward closures $\uparrow A$, $\uparrow x$ are defined similarly. When $x \leq y$, x is *below* y and y is *above* x. X is *pointed* iff it has a least element $\perp$. A *dcpo* is a poset X where every directed family $(x_i)_{i \in I}$ has a least upper bound $\sup_{i \in I} x_i$; directedness means that $I \neq \emptyset$ and for every $i, i' \in I$, there is an $i'' \in I$ such that $x_i, x_{i'} \leq x_{i''}$.

Every poset, and more generally each preordered set X comes with a topology, whose opens U are the upward closed subsets such that, for every directed family $(x_i)_{i \in I}$ that has a least upper bound in U, $x_i \in U$ for some $i \in I$. This is the *Scott topology*. When we see a poset or dcpo X as a topological space, we will implicitly assume the latter, unless marked otherwise.

There is a deep connection between order and topology. Given any topological space X, its *specialization preorder* $\leq$ is defined by $x \leq y$ iff every open containing x also contains y. X is T_0 iff $\leq$ is an ordering, i.e., $x \leq y$ and $y \leq x$ imply $x = y$. The specialization preorder of a dcpo X (with ordering $\leq$, and equipped with its Scott topology), is the original ordering $\leq$.

A subset A of a topological space X is *saturated* iff it is the intersection of all opens U containing A. Equivalently, A is upward closed in the specialization preorder [Mis98, Remark after Definition 4.34]. So we can, and shall often prove inclusions $A \subseteq B$ where B is upward closed by showing that every open U containing B also contains A.

A map $f : X \to Y$ between topological spaces is *continuous* iff $f^{-1}(V)$ is open for every open subset V of Y. Every continuous map is monotonic with respect to the underlying specialization preorders. When X and Y are preordered sets, it is equivalent to require f to be *Scott-continuous*, i.e., to be monotonic and to preserve existing directed least upper bounds. A *homeomorphism* is a bijective continuous map whose inverse is also continuous.

Given a set X, and a family $\mathcal{B}$ of subsets of X, there is a smallest topology containing $\mathcal{B}$: then $\mathcal{B}$ is a *subbase* of the topology, and its elements are the *subbasic opens*. To show that $f : X \to Y$ is continuous, it is enough to show that the inverse image of every subbasic open of Y is open in X. A subbase $\mathcal{B}$ is a *base* if and only if every open is a union of elements of $\mathcal{B}$. This is the case, for example, if $\mathcal{B}$ is closed under finite intersections.

The *interior* $int(A)$ of a subset A of a topological space X is the largest open contained in A. A is a *neighborhood* of x if and only if $x \in int(A)$, and a neighborhood of a subset B if and only if $B \subseteq int(A)$. A subset Q of a topological space X is *compact* iff one can extract a finite subcover from every open cover of Q. The important ones are the *saturated* compacts. X is *locally compact* iff for each open U and each $x \in U$, there is a compact saturated subset Q such that $x \in int(Q)$ and $Q \subseteq U$. In any locally compact space, we have the following interpolation property: whenever Q is a compact subset of some open U, then there is a compact saturated subset Q_1 such that $Q \subseteq int(Q_1) \subseteq Q_1 \subseteq U$.

X is *sober* iff every irreducible closed subset is the closure of a unique point; in the presence of local compactness (and when X is T_0), it is equivalent to require that X be *well-filtered* [GHK⁺03, Theorem II-1.21], i.e., to require that, for every open U, for every filtered family $(Q_i)_{i\in I}$ of saturated compacts such that $\bigcap^{\downarrow}_{i\in I} Q_i \subseteq U$, $Q_i \subseteq U$ for some $i \in I$ already. We say that the family is *filtered* iff it is directed in the $\supseteq$ ordering, and make it explicit by using $\downarrow$ as superscript. (Symmetrically, we write $\bigcup^{\uparrow}$ for directed unions.)

Given a topological space X, let $\mathcal{Q}(X)$ be the collection of all non-empty compact saturated subsets Q of X. There are two prominent topologies one can put on $\mathcal{Q}(X)$. The *upper Vietoris* topology has a subbase of opens of the form $\Box U$, U open in X, where we write $\Box U$ for the collection of compact saturated subsets Q' included in U. We shall write $\mathcal{Q}_{\mathcal{V}}(X)$ for the space $\mathcal{Q}(X)$ with the upper Vietoris topology, and call it the *Smyth powerspace*. The specialization ordering of $\mathcal{Q}_{\mathcal{V}}(X)$ is reverse inclusion $\supseteq$. On the other hand, we shall reserve the notation $\mathcal{Q}_{\sigma}(X)$ for the *Smyth powerdomain* of X, which is equipped with the Scott topology of $\supseteq$ instead. When X is well-filtered, $\mathcal{Q}(X)$ is a dcpo, with least upper bounds of directed families computed as filtered intersections, and $\Box U$ is Scott-open for every open subset U of X, i.e., the Scott topology is finer than the upper Vietoris topology. When X is locally compact and sober (in particular, well-filtered), the two topologies coincide, and $\mathcal{Q}_{\sigma}(X)$ is then a continuous dcpo (see below), where $Q \ll Q'$ iff $Q' \subseteq int(Q)$ [GHK⁺03, Proposition I-1.24.2]. Schalk [Sch93, Chapter 7] provides a deep study of these spaces.

For every finite subset E of a topological space X, E is compact and $\uparrow E$ is saturated compact in X. We call *finitary compact* those subsets of the form $\uparrow E$ with E finite, and let $\mathrm{Fin}(X)$ be the subset of $\mathcal{Q}(X)$ consisting of the non-empty finitary compacts. $\mathrm{Fin}(X)$ can be topologized with the subspace topology from $\mathcal{Q}_{\mathcal{V}}(X)$, in which case we obtain a space we write $\mathrm{Fin}_{\mathcal{V}}(X)$, or with the Scott topology of reverse inclusion $\supseteq$, yielding a space that we write $\mathrm{Fin}_{\sigma}(X)$.

Given any poset X, any finite subset E of X, and any element x of X, we write $E \leq x$ iff $x \in \uparrow E$, i.e., iff there is a $y \in E$ such that $y \leq x$. Given any upward closed subset U of X, we shall write $U \lll x$ iff for every directed family $(x_i)_{i\in I}$ that has a least upper bound above x, then x_i is in U for some $i \in I$. Then a finite set E *approximates* x iff $\uparrow E \lll x$. This is usually written $E \ll x$ in the literature. We shall also write $y \ll x$, when $y \in X$, as shorthand for $\uparrow y \lll x$. This is the more familiar way-below relation, and a poset is *continuous* if and only if the set $\twoheaddownarrow x$ of all elements y such that $y \ll x$ is directed and has x as least upper bound. One should be aware that $\uparrow E \lll x$ means that the elements of E approximate x *collectively*, while none in particular may approximate x individually. E.g., in the poset $\mathcal{N}_2$ (Figure 2 (*ii*)), the sets $\{(0,m),(1,n)\}$ approximate ω, for all $m,n \in \mathbb{N}$; but $(0,m) \not\ll \omega$, $(1,n) \not\ll \omega$.

It may be helpful to realize that $\mathrm{Fin}(X)$ can also be presented in the following equivalent way. Given two finitary compacts $\uparrow E$ and $\uparrow E'$, $\uparrow E \supseteq \uparrow E'$ if and only if for every $x' \in E'$, there is an $x \in E$ such that $x \leq x'$, and then we write $E \leq^\sharp E'$: this is the so-called *Smyth preorder*. Then we can equate the finitary compacts $\uparrow E$ with the equivalence classes of finite subsets E, up to the equivalence $\equiv$ defined by $E \equiv E'$ iff $\uparrow E = \uparrow E'$ iff $E \leq^\sharp E'$ and $E' \leq^\sharp E$, declare that $\mathrm{Fin}(X)$ is the set of equivalence classes of non-empty finite sets, ordered by $\leq^\sharp$. But the approach based on finitary compacts is mathematically smoother.

Among the Cartesian-closed categories of continuous dcpos, one finds in particular the **B**-domains (a.k.a., the bifinite domains), the **RB**-domains, i.e., the retracts of bifinite domains [AJ94, Section 4.2.1], and the **FS**-domains [AJ94, Section 4.2.2][GHK$^+$03, Section II.2]. There are several equivalent definitions of the first two.

For our purposes, an **RB**-*domain* is a pointed dcpo X with a directed family $(f_i)_{i \in I}$ of deflations such that $\sup_{i \in I} f_i = \mathrm{id}_X$ [AJ94, Exercise 4.3.11(9)]. A *deflation* f on X is a continuous map from X to X such that $f(x) \leq x$ for every $x \in X$, and that has *finite image*. We order deflations, as well as all maps with codomain a poset, pointwise: i.e., $f \leq g$ iff $f(x) \leq g(x)$ for every $x \in X$; knowing this, directed families and least upper bounds of deflations make sense. Every **RB**-domain is a continuous dcpo, and $f_i(x) \ll x$ for every $i \in I$ and every $x \in X$.

A **B**-*domain* is defined similarly, except the deflations f_i are now required to be *idempotent*, i.e., $f_i \circ f_i = f_i$ [AJ94, Theorem 4.2.6]. This implies that $f_i(x) \ll f_i(x)$, i.e., that all the elements $f_i(x)$ are finite; hence all bifinite domains are also algebraic. Every bifinite domain is an **RB**-domain. Conversely, the **RB**-domains are exactly the retracts of bifinite domains: we shall define what this means and extend this in Section 4.

An **FS**-*domain* is defined similarly again, except the functions f_i are no longer deflations, but continuous functions that are *finitely separated from* id_X. That is, we now require that there is a finite set M_i such that for every $x \in X$, there is an $m \in M_i$ such that $f_i(x) \leq m \leq x$. We say that M_i is *finitely separating* for f_i on X.

Every deflation is finitely separated from id_X: take M_i to be the image of f_i. The converse fails. E.g., for every $\epsilon > 0$, the function $x \mapsto \max(x - \epsilon, 0)$ is finitely separated from the identity on $[0, 1]$, but is not a deflation [JT98, Section 3.2]. Every **RB**-domain is an **FS**-domain. The converse is not known.

A *quasi-continuous dcpo* X (see [GLS83] or [GHK$^+$03, Definition III-3.2]) is a dcpo such that, for every $x \in X$, the collection of all $\uparrow E \in \mathrm{Fin}(X)$ that approximate x ($\uparrow E \lll x$) is directed (w.r.t. $\supseteq$) and their least upper bound in $\mathcal{Q}(X)$ is $\uparrow x$, i.e., $\bigcap_{\substack{\uparrow E \in \mathrm{Fin}(X) \\ \uparrow E \lll x}} \uparrow E = \uparrow x$. The theory of quasi-continuous dcpos is less well explored than that of *continuous dcpos*, but quasi-continuous dcpos retain many of the properties of the latter. (Every continuous dcpo is quasi-continuous, but not conversely. A counterexample is given by $\mathcal{N}_2$, see Figure 2 (*ii*).) Every quasi-continuous dcpo X is locally compact and sober in its Scott topology [GHK$^+$03, III-3.7]. In a quasi-continuous dcpo X, for every $\uparrow E \in \mathrm{Fin}(X)$, the set $\twoheaduparrow E$ defined as $\{x \in X \mid \uparrow E \lll x\}$, is open, and equals the interior $int(\uparrow E)$ [GHK$^+$03, III-3.6(ii)]; every open U is the union of all the subsets $\twoheaduparrow E$ with $\uparrow E \in \mathrm{Fin}(X)$ contained in U [GHK$^+$03, III-5.6]; and for every compact saturated subset Q and every open subset U containing Q, there is a finitary compact subset $\uparrow E$ of X such that $Q \subseteq \twoheaduparrow E$ and $\uparrow E \subseteq U$ [GHK$^+$03, III-5.7]. In particular, $Q = \bigcap^{\downarrow}_{\uparrow E \in \mathrm{Fin}(X),\, Q \subseteq \twoheaduparrow E} \uparrow E$. Another consequence is *interpolation*: writing $\uparrow E \lll \uparrow E'$ for $\uparrow E \lll y$ for every y in E' (equivalently, $\uparrow E' \subseteq \twoheaduparrow E$), if $\uparrow E \lll x$ in a quasi-continuous dcpo X, for some $\uparrow E \in \mathrm{Fin}(X)$, and $x \in X$, then $\uparrow E \ll \uparrow E' \ll x$ for some $\uparrow E' \in \mathrm{Fin}(X)$.

If X is a quasi-continuous dcpo, the formula $Q = \bigcap^{\downarrow}_{\uparrow E \in \mathrm{Fin}(X),\, Q \subseteq \Uparrow E} \uparrow E$, valid for every $Q \in \mathcal{Q}(X)$, shows that Q is the filtered intersection of its finitary compact neighborhoods, equivalently the directed least upper bound of those non-empty finitary compacts $\uparrow E$ ($E \in \mathrm{Fin}(X)$) that are way-below Q. In other words, the finitary compacts form a *basis* of $\mathcal{Q}(X)$.

3. **QRB**-Domains

We model **QRB**-domains after **RB**-domains, replacing single approximating elements $f_i(x)$, where f_i is a deflation, by finite subsets, as in quasi-continuous dcpos.

Definition 3.1 (**QRB**-Domain)**.** A *quasi-deflation* on a poset X is a continuous map $\varphi : X \to \mathrm{Fin}_\sigma(X)$ such that $x \in \varphi(x)$ for every $x \in X$, and $\mathrm{im}\,\varphi = \{\varphi(x) \mid x \in X\}$ is finite.

A **QRB**-*domain* is a pointed dcpo X with a *generating family of quasi-deflations*, i.e., a directed family of quasi-deflations $(\varphi_i)_{i \in I}$ with $\uparrow x = \bigcap^{\downarrow}_{i \in I} \varphi_i(x)$ for each $x \in X$.

We order quasi-deflations pointwise, i.e., $\varphi \leq \psi$ iff $\varphi(x) \supseteq \psi(x)$ for every $x \in X$. Above, we write $\bigcap^{\downarrow}$ instead of $\bigcap$ to stress the fact that the family $(\varphi_i(x))_{i \in I}$ of which we are taking the intersection is *filtered*, i.e., for any two $i, i' \in I$, there is an $i'' \in I$ such that $\varphi_{i''}(x)$ is contained in both $\varphi_i(x)$ and $\varphi_{i'}(x)$. It is equivalent to say that $(\varphi_i(x))_{i \in I}$ is directed in the $\supseteq$ ordering of $\mathrm{Fin}(X)$.

One can see the finitary compacts $\varphi_i(x)$ as being smaller and smaller upward closed sets containing x. The intersection $\bigcap^{\downarrow}_{i \in I} \varphi_i(x)$ is then just the least upper bound of $(\varphi_i(x))_{i \in I}$ in the Smyth powerdomain $\mathcal{Q}(X)$. On the other hand, X embeds into $\mathcal{Q}_{\mathcal{V}}(X)$ by equating $x \in X$ with $\uparrow x \in \mathcal{Q}(X)$. Modulo this identification, the condition $\uparrow x = \bigcap^{\downarrow}_{i \in I} \varphi_i(x)$ requires that x is the least upper bound of $(\varphi_i(x))_{i \in I}$ in $\mathcal{Q}(X)$.

That φ is continuous means that φ is monotonic ($x \leq y$ implies $\varphi(x) \supseteq \varphi(y)$), and that for every directed family $(x_j)_{j \in J}$ of elements of X, $\varphi(\sup_{j \in J} x_j)$ is equal to $\bigcap^{\downarrow}_{i \in I} \varphi(x_j)$—this implies that the latter is finitary compact, in particular.

Proposition 3.2. *Every* **RB**-*domain is a* **QRB**-*domain.*

Proof. Given a directed family of deflations $(f_i(x))_{i \in I}$, define $\varphi_i(x)$ as $\uparrow f_i(x)$. If $f_i \leq f_j$, then $\varphi_i(x) \supseteq \varphi_j(x)$ for every $x \in X$, so $(\varphi_i)_{i \in I}$ is directed. Also, $\bigcap^{\downarrow}_{i \in I} \varphi_i(x)$ is the set of upper bounds of $(f_i(x))_{i \in I}$, of which the least is x. So this set is exactly $\uparrow x$. ☐

We shall improve on this in Theorem 7.3, which implies that not only the **RB**-domains, but all **FS**-domains, are **QRB**-domains.

For any deflation f, and more generally whenever f is finitely separated from the identity, $f(x)$ is way-below x [GHK+03, Lemma II-2.16]. Similarly:

Lemma 3.3. *Let X be a poset, and φ be a quasi-deflation on X. For every $x \in X$, $\varphi(x) \ll x$.*

Proof. Let $(x_j)_{j \in J}$ be a directed family having a least upper bound above x. Since φ is continuous, $\bigcap^{\downarrow}_{j \in J} \varphi(x_j) \subseteq \varphi(x)$. But since $\mathrm{im}\,\varphi$ is finite, there are only finitely many sets $\varphi(x_j)$, $j \in J$. So $\varphi(x_j) \subseteq \varphi(x)$ for some $j \in J$. Since $x_j \in \varphi(x_j)$, $x_j \in \varphi(x)$. ☐

Corollary 3.4. *Every* **QRB**-*domain is quasi-continuous.*

In general, **QRB**-domains are not continuous. E.g., $\mathcal{N}_2$ (Figure 2 (*ii*)) is not continuous. However, $\mathcal{N}_2$ is a **QRB**-domain: for all $i, j \in \mathbb{N}$, take $\varphi_{ij}(\omega) = \uparrow\{(0,i),(1,j)\}$, $\varphi_{ij}(0,m) = \uparrow\{(0,\min(m,i)),(1,j)\}$, $\varphi_{ij}(1,m) = \uparrow\{(0,i),(1,\min(m,j))\}$. Then $(\varphi_{ij})_{i,j\in\mathbb{N}}$ is the desired directed family of quasi-deflations.

RB-domains, and more generally **FS**-domains, are not just continuous domains, they are *stably compact*, i.e., locally compact, sober, compact and coherent (see, e.g., [AJ94, Theorem 4.2.18]). We say that a topological space is *coherent* iff the intersection of any two compact saturated subsets is compact (and saturated). In a stably compact space, the intersection of any family of compact saturated subsets is compact. We show that **QRB**-domains are stably compact as well.

Since every quasi-continuous dcpo is locally compact and sober [GHK$^+$03, Proposition III-3.7], and also compact since pointed, only coherence remains to be shown. For this, we need the following consequence of Rudin's Lemma, a finitary form of well-filteredness:

Proposition 3.5 ([GHK$^+$03, Corollary III-3.4])**.** *Let X be a dcpo, $(\uparrow E_i)_{i\in I}$ be a directed family in Fin(X). For every open subset U of X, if $\bigcap^{\downarrow}_{i\in I} \uparrow E_i \subseteq U$, then $\uparrow E_i \subseteq U$ for some $i \in I$.*

It follows that, if X is a dcpo, then the Scott topology on $\mathrm{Fin}(X)$ is finer than the upper Vietoris topology. Indeed, this reduces to showing that $\mathrm{Fin}(X) \cap \Box U$ is Scott-open in $\mathrm{Fin}(X)$, for every open subset U of X. And this is Proposition 3.5, plus the easily checked fact that $\Box U$ is upward closed in $\supseteq$.

Corollary 3.6. *Let X be a dcpo. The Scott topology is finer than the upper Vietoris topology on Fin(X), and coincides with it whenever X is quasi-continuous.*

Proof. It remains to show that, if X is a quasi-continuous dcpo, then every Scott-open $\mathcal{U}$ of $\mathrm{Fin}(X)$ is open in the upper Vietoris topology. Let $\uparrow E \in \mathrm{Fin}(X)$ be in $\mathcal{U}$. It suffices to show that there is an open subset U of X such that $\uparrow E \in \Box U \subseteq \mathcal{U}$. Write $E = \{x_1, \ldots, x_n\}$. For each i, $1 \leq i \leq n$, $\uparrow x_i$ is the filtered intersection of all finitary compacts $\uparrow E_i \lll x_i$. The unions $\uparrow E_1 \cup \ldots \cup \uparrow E_n = \uparrow(E_1 \cup \ldots \cup E_n)$, with $\uparrow E_1 \lll x_1, \ldots, \uparrow E_n \lll x_n$, also form a directed family in $\mathrm{Fin}(X)$, and their intersection is $\uparrow E$. So there are finitary compacts $\uparrow E_1 \lll x_1, \ldots, \uparrow E_n \lll x_n$ whose union is in $\mathcal{U}$. Since $\uparrow E_i \lll x_i$ for each i, each x_i is in the Scott-open $\Uparrow E_i$, so $\uparrow E \in \Box U$ with $U = \Uparrow E_1 \cup \ldots \cup \Uparrow E_n$. Moreover, $\Box U \subseteq \mathcal{U}$: for each $\uparrow E' \in \Box U$, $\uparrow E'$ is included in $U \subseteq \uparrow E_1 \cup \ldots \uparrow E_n$; since $\uparrow E_1 \cup \ldots \uparrow E_n$ is in $\mathcal{U}$ and $\mathcal{U}$ is upward-closed in $\supseteq$, $\uparrow E'$ is in $\mathcal{U}$. ☐

Schalk [Sch93, Chapter 7] proved that $\mathcal{Q}_{\mathcal{V}}$ defines a monad on the category of topology spaces (see [Mog91] for an introduction to monads and their importance in programming language semantics). This means first that there is a *unit map* η_X—here, η_X maps $x \in X$ to $\uparrow x \in \mathcal{Q}_{\mathcal{V}}(X)$, and this is continuous because $\eta_X^{-1}(\Box U) = U$. That $\mathcal{Q}_{\mathcal{V}}$ is a monad also means that every continuous map $h : X \to \mathcal{Q}_{\mathcal{V}}(Y)$ has an *extension* $h^\dagger : \mathcal{Q}_{\mathcal{V}}(X) \to \mathcal{Q}_{\mathcal{V}}(Y)$, i.e., $h^\dagger$ is continuous and $h^\dagger \circ \eta_X = h$. This is defined by $h^\dagger(Q) = \bigcup_{x\in Q} h(x)$ in our case. Again, $h^\dagger$ is continuous, because ${h^\dagger}^{-1}(\Box U) = \Box h^{-1}(\Box U)$. And the *monad laws* are satisfied: $\eta_X^\dagger = \mathrm{id}_{\mathcal{Q}_{\mathcal{V}}(X)}$, $h^\dagger \circ \eta_X = h$, and $(g^\dagger \circ h)^\dagger = g^\dagger \circ h^\dagger$. One should be careful here: $\mathcal{Q}_{\mathcal{V}}$ is a monad, but $\mathcal{Q}_\sigma$ is not a monad, except on specific subcategories, e.g., sober locally compact spaces X, where $\mathcal{Q}_\sigma(X) = \mathcal{Q}_{\mathcal{V}}(X)$ anyway.

The continuity claims in the following lemma are then obvious.

Lemma 3.7. *Let X, Y be topological spaces. Given any continuous map $\psi : X \to Fin_{\mathcal{V}}(Y)$, its extension $\psi^\dagger$ restricts to a continuous map $\psi^\dagger : Fin_{\mathcal{V}}(X) \to Fin_{\mathcal{V}}(Y)$. If $\operatorname{im}\psi$ is finite, then $\psi^\dagger$ maps $\mathcal{Q}_{\mathcal{V}}(X)$ continuously into $Fin_{\mathcal{V}}(Y)$.*

Proof. In each case, one only needs to show that $\psi^\dagger$ maps relevant compacts to finitary compacts. In the first case, for every finitary compact $\uparrow E \in \mathrm{Fin}(X)$, $\psi^\dagger(\uparrow E) = \bigcup_{x \in \uparrow E} \psi(x) = \bigcup_{x \in E} \psi(x)$ (because ψ is monotonic), and this is finitary compact. In the second case, $\psi^\dagger(Q) = \bigcup_{x \in Q} \psi(x)$ is a finite union of finitary compacts since $\operatorname{im}\psi$ is finite. □

One would also like $\psi^\dagger$ to be continuous from $\mathcal{Q}_\sigma(X)$ to $\mathrm{Fin}_\sigma(Y)$, in the face of the importance of the Scott topology. This is a consequence of the above when X is sober and locally compact, and Y is a quasi-continuous dcpo, since $\mathcal{Q}_\sigma(X) = \mathcal{Q}_{\mathcal{V}}(X)$ and $\mathrm{Fin}_\sigma(Y) = \mathrm{Fin}_{\mathcal{V}}(Y)$ in this case. However, one can also prove this in a more general setting, using the following observation. For each topological space Z, write Z_σ for Z with the Scott topology of its specialization preorder. For short, we shall call *quasi monotone convergence space* any space Z such that the (Scott) topology on Z_σ is finer than that of Z, i.e., such that every open subset of Z is open is Scott-open. This is a slight relaxation of the notion of *monotone convergence space*, i.e., of a quasi monotone convergence space that is a dcpo in its specialization preorder [GHK+03, Definition II-3.12]. E.g., every sober space is a monotone convergence space, and in particular a quasi monotone convergence space.

Lemma 3.8. *Let Z be a quasi monotone convergence space and Z' be a topological space. Every continuous map $f : Z \to Z'$ is Scott-continuous, i.e., continuous from Z_σ to Z'_σ.*

Proof. Since f is continuous, it is monotonic with respect to the underlying specialization preorders. Let $(z_i)_{i \in I}$ be any directed family of elements of Z, with least upper bound z. Certainly $f(z)$ is an upper bound of $(f(z_i))_{i \in I}$. Let us show that, for any other upper bound z', $f(z) \leq z'$. It is enough to show that every open neighborhood V of $f(z)$ contains z'. Since $f(z) \in V$, z is in the open subset $f^{-1}(V)$, which is Scott-open by assumption, so $z_i \in f^{-1}(V)$ for some $i \in I$. It follows that $f(z_i)$ is in V, hence also z' since V is upward closed. □

When X is sober and locally compact, the topology of $\mathcal{Q}_\sigma(X)$ coincides with that of $\mathcal{Q}_{\mathcal{V}}(X)$. In particular, $Z = \mathcal{Q}_{\mathcal{V}}(X)$ is a quasi-monotone convergence space. Taking $Z' = \mathcal{Q}_{\mathcal{V}}(Y)$ in Lemma 3.8, one obtains the following corollary.

Corollary 3.9. *Let X be a sober, locally compact space, and Y be a topological space. Every continuous map from $\mathcal{Q}_{\mathcal{V}}(X)$ to $\mathcal{Q}_{\mathcal{V}}(Y)$ is also Scott-continuous from $\mathcal{Q}(X)$ to $\mathcal{Q}(Y)$.*

Similarly, with $Z' = \mathrm{Fin}_{\mathcal{V}}(Y)$:

Corollary 3.10. *Let Y be a topological space, Z be a quasi monotone convergence space. Every continuous map from Z to $Fin_{\mathcal{V}}(Y)$ is Scott-continuous, i.e., continuous from Z_σ to $Fin_\sigma(Y)$.*

Lemma 3.11. *Let X be a* **QRB***-domain, and $(\varphi_i)_{i \in I}$ a generating family of quasi-deflations. For every open subset U of X, $\bigcup^{\uparrow}_{i \in I} \varphi_i^{-1}(\Box U) = U$.*

Proof. The union is directed, since $\varphi_i^{-1}(\Box U) \subseteq \varphi_{i'}^{-1}(\Box U)$ whenever φ_i is pointwise below $\varphi_{i'}$, i.e., when $\varphi_i(x) \supseteq \varphi_{i'}(x)$ for all $x \in X$. For every $i \in I$, $\varphi_i^{-1}(\Box U) \subseteq U$: every element x of $\varphi_i^{-1}(\Box U)$ is indeed such that $x \in \varphi_i(x) \subseteq U$. Conversely, we claim that every element x of

U is in $\varphi_i^{-1}(\Box U)$ for some $i \in I$. Indeed, $\uparrow x \subseteq U$, so $\bigcap_{i\in I} \uparrow \varphi_i(x) \subseteq U$. By Proposition 3.5, $\varphi_i(x) \subseteq U$ for some $i \in I$, i.e., $\varphi_i(x) \in \Box U$. ☐

Lemma 3.12. *Let X be a* **QRB***-domain, and $(\varphi_i)_{i\in I}$ a generating family of quasi-deflations. For every compact saturated subset Q of X, $Q = \bigcap^{\downarrow}_{i\in I} \varphi_i^{\dagger}(Q)$.*

Proof. Since $x \in \varphi_i(x)$ for every x, $\varphi_i^{\dagger}(Q)$ contains Q for every $i \in I$. So $Q \subseteq \bigcap^{\downarrow}_{i\in I} \varphi_i^{\dagger}(Q)$. Conversely, since Q is saturated, it is enough to show that every open U containing Q also contains $\bigcap^{\downarrow}_{i\in I} \varphi_i^{\dagger}(Q)$. Since $Q \subseteq U$, by Lemma 3.11, $Q \subseteq \bigcup^{\uparrow}_{i\in I} \varphi_i^{-1}(\Box U)$. By compactness, $Q \subseteq \varphi_i^{-1}(\Box U)$ for some $i \in I$, i.e., for every $x \in Q$, $\varphi_i(x) \subseteq U$. So $\varphi_i^{\dagger}(Q) \subseteq U$. ☐

Proposition 3.13. *For every* **QRB***-domain X, $\mathcal{Q}(X)$ is an* **RB***-domain.*

Proof. Assume X is a **QRB**-domain, with generating family of quasi-deflations $(\varphi_i)_{i\in I}$. The family $(\varphi_i^{\dagger})_{i\in I}$ is directed, since if φ_i is below φ_j, i.e., if $\varphi_i(x) \supseteq \varphi_j(x)$ for every $x \in X$, then $\varphi_i^{\dagger}(Q) = \bigcup_{x\in Q} \varphi_i(x) \supseteq \bigcup_{x\in Q} \varphi_j(x) = \varphi_j^{\dagger}(Q)$. Since X is quasi-continuous (Corollary 3.4), it is sober and locally compact. So Corollary 3.9 applies, showing that $\varphi_i^{\dagger}$ is Scott-continuous from $\mathcal{Q}(X)$ to $\mathcal{Q}(X)$. Lemma 3.12 states that the least upper bound of $(\varphi_i^{\dagger})_{i\in I}$ is the identity on $\mathcal{Q}(X)$. Clearly, $\varphi_i^{\dagger}$ has finite image. So $\mathcal{Q}(X)$ is an **RB**-domain. ☐

Theorem 3.14. *Every* **QRB***-domain is stably compact.*

Proof. Let X be a **QRB**-domain, with generating family of quasi-deflations $(\varphi_i)_{i\in I}$. We claim that, given any two compact saturated subsets Q and Q' of X, $Q \cap Q'$ is again compact saturated. This is obvious if $Q \cap Q'$ is empty. Otherwise, writing $\uparrow_Y y$ for the upward closure of an element y of a poset Y, $\uparrow_{\mathcal{Q}(X)} Q \cap \uparrow_{\mathcal{Q}(X)} Q'$ is an intersection of two finitary compacts in $\mathcal{Q}_{\mathcal{V}}(X)$. Since X is a quasi-continuous dcpo by Corollary 3.4, X is sober and locally compact, so $\mathcal{Q}_{\mathcal{V}}(X) = \mathcal{Q}_{\sigma}(X)$. Moreover, $\mathcal{Q}(X)$ is an **RB**-domain (Proposition 3.13), so $\mathcal{Q}_{\mathcal{V}}(X)$ is coherent. Therefore $\uparrow_{\mathcal{Q}(X)} Q \cap \uparrow_{\mathcal{Q}(X)} Q'$ is compact saturated in $\mathcal{Q}_{\mathcal{V}}(X)$. It is also non-empty: pick $x \in Q \cap Q'$, then $\uparrow_X x$ is in $\uparrow_{\mathcal{Q}(X)} Q \cap \uparrow_{\mathcal{Q}(X)} Q'$. So $\uparrow_{\mathcal{Q}(X)} Q \cap \uparrow_{\mathcal{Q}(X)} Q'$ is in $\mathcal{Q}(\mathcal{Q}_{\mathcal{V}}(X))$. Now there is a (continuous) map $\mu_X : \mathcal{Q}_{\mathcal{V}}(\mathcal{Q}_{\mathcal{V}}(X)) \to \mathcal{Q}_{\mathcal{V}}(X)$ defined as $\mathrm{id}^{\dagger}_{\mathcal{Q}_{\mathcal{V}}(X)}$—this is the so-called *multiplication* of the monad—and $\mu_X(\uparrow_{\mathcal{Q}(X)} Q \cap \uparrow_{\mathcal{Q}(X)} Q')$ is then an element of $\mathcal{Q}(X)$, i.e., a compact subset of X. We now observe that $\mu_X(\uparrow_{\mathcal{Q}(X)} Q \cap \uparrow_{\mathcal{Q}(X)} Q') = \bigcup_{\substack{Q''\in\mathcal{Q}(X)\\ Q''\subseteq Q,Q'}} Q''$ is equal to $Q \cap Q'$: the left to right inclusion is obvious, and conversely every $x \in Q \cap Q'$ defines an element $Q'' = \uparrow_X x$ of $\mathcal{Q}(X)$ that is included in Q and Q'. So $Q \cap Q'$ is compact saturated. We conclude that X is coherent.

X is compact since pointed, and also locally compact and sober, as a quasi-continuous dcpo, hence stably compact. ☐

The *Lawson topology* is the smallest topology containing both the Scott-opens and the complements of all finitary compacts $\uparrow E \in \mathrm{Fin}(X)$. When X is a quasi-continuous dcpo, since $\uparrow E$ is compact saturated and every non-empty compact saturated subset is a filtered intersection of such sets $\uparrow E$, the Lawson topology coincides with the *patch topology*, i.e., the smallest topology containing the original Scott topology and all complements of compact saturated subsets. Every stably compact space is patch-compact, i.e., compact in its patch topology [GHK+03, Section VI-6]. So:

Corollary 3.15. *Every* **QRB***-domain is Lawson-compact.*

In the sequel, we shall need some form of countability:

Definition 3.16. An ω**QRB**-*domain* is a **QRB**-domain with a *countable* generating family of quasi-deflations.

Proposition 3.17. *A pointed dcpo X is an ω**QRB**-domain iff there is a generating* sequence *of quasi-deflations $(\varphi_i)_{i\in\mathbb{N}}$, i.e., for every $i, i' \in \mathbb{N}$, $i \leq i'$, $\varphi_i(x) \supseteq \varphi_{i'}(x)$ for every $x \in X$, and $\uparrow x = \bigcap^{\downarrow}_{i\in\mathbb{N}} \varphi_i(x)$ for every $x \in X$.*

Proof. Let X be an ω**QRB**-domain, and $(\psi_j)_{j\in\mathbb{N}}$ be a countable generating family of quasi-deflations. Build a sequence $(j_i)_{i\in\mathbb{N}}$ by letting $j_0 = 0$, and j_{i+1} be any $j \in \mathbb{N}$ such that ψ_j is above ψ_i and ψ_{j_i}, by directedness. Then let $\varphi_i = \psi_{j_i}$ for every $i \in \mathbb{N}$. By construction, whenever $i \leq i'$, φ_i is below φ_{i+1}. And for every $i \in \mathbb{N}$, ψ_i is below $\varphi_i = \psi_{j_i}$, so $\uparrow x = \bigcap^{\downarrow}_{i\in\mathbb{N}} \varphi_i(x)$ for every $x \in X$. So $(\varphi_i)_{i\in\mathbb{N}}$ is the desired sequence. □

Recall that a topological space is *countably-based* if and only if it has a countable subbase, or equivalently, a countable base.

Proposition 3.18. *A* **QRB**-*domain X is an ω*QRB*-domain iff it is countably-based.*

Proof. Only if: let $(\varphi_i)_{i\in\mathbb{N}}$ be a generating sequence of quasi-deflations on X. For each $i \in \mathbb{N}$, enumerate $\operatorname{im}\varphi_i$ as $\{\uparrow E_{i1}, \ldots, \uparrow E_{in_i}\} \subseteq \mathrm{Fin}(X)$, and let E_i be the finite set $\bigcup_{j=1}^{n_i} E_{ij}$. We claim that the countably many subsets $int(\varphi_i(y))$, $y \in E_j$, $i, j \in \mathbb{N}$, form a base of the topology.

It is enough to show that, for every open U and every element $x \in U$, $x \in int(\varphi_i(y))$ for some $y \in E_j$, $i, j \in \mathbb{N}$, such that $\varphi_i(y) \subseteq U$: since $\uparrow x = \bigcap^{\downarrow}_{j\in\mathbb{N}} \varphi_j(x) \subseteq U$, use Proposition 3.5 to find $j \in \mathbb{N}$ such that $\varphi_j(x) \subseteq U$. Since $x \in \varphi_j(x)$ and $\varphi_j(x) = \uparrow E_{jk}$ for some k, there is a $y \in E_{jk} \subseteq E_j$ such that $y \leq x$, and $y \in U$. Repeating the argument on y, we find $i \in \mathbb{N}$ such that $\varphi_i(y) \subseteq U$. By Lemma 3.3, $\varphi_i(y) \lll y$, i.e., y is in $int(\varphi_i(y))$ since X is quasi-continuous. Since $y \leq x$, x is in $int(\varphi_i(y))$.

If: let $(\varphi_i)_{i\in I}$ be a generating family of quasi-deflations on X, and assume that the topology of X has a countable base $\{U_k \mid k \in \mathbb{N}\}$. Assume without loss of generality that $U_k \neq \emptyset$ for every $k \in \mathbb{N}$. For every pair $\ell, k \in \mathbb{N}$ such that $U_\ell \subseteq \uparrow E \subseteq U_k$ for some finite set E, pick one such finite set and call it $E_{\ell k}$. One can enumerate all such pairs as ℓ_m, k_m, $m \in \mathbb{N}$. By Lemma 3.12, $\bigcap^{\downarrow}_{i\in I} \varphi_i^\dagger(\uparrow E_{\ell_m k_m}) = \uparrow E_{\ell_m k_m}$. By Proposition 3.5, $\varphi_i^\dagger(\uparrow E_{\ell_m k_m}) \subseteq U_{k_m}$ for some $i \in I$: pick such an i and call it i_m. By directedness, we may also assume that φ_{i_m} is also above φ_{i_n}, $0 \leq n < m$. Define ψ_m as φ_{i_m}. This yields a non-decreasing sequence of quasi-deflations $(\psi_m)_{m\in\mathbb{N}}$.

We claim that it is generating. On one hand, $\uparrow x \subseteq \bigcap^{\downarrow}_{k\in\mathbb{N}} \psi_k(x)$ since each ψ_k is a quasi-deflation. Conversely, every open neighborhood U of x contains some U_k, $k \in \mathbb{N}$, with $x \in U_k$. Then $\uparrow x = \bigcap^{\downarrow}_{i\in I} \varphi_i(x) \subseteq U_k$, so $\varphi_i(x) \subseteq U_k$ for some $i \in I$. Write $\varphi_i(x)$ as $\uparrow E$, where E is finite. By Lemma 3.3, $\varphi_i(x) \lll x$, so $x \in \Uparrow E \subseteq \uparrow E \subseteq U_k$. As $\Uparrow E$ is open, $x \in U_\ell \subseteq \Uparrow E$ for some $\ell \in \mathbb{N}$. In particular, $U_\ell \subseteq \uparrow E \subseteq U_k$. So ℓ, k is a pair of the form ℓ_m, k_m. By definition $\psi_m^\dagger(\uparrow E_{\ell k}) \subseteq U_k$. Since $x \in U_\ell \subseteq \uparrow E_{\ell k}$, $\psi_m(x) = \psi_m^\dagger(\uparrow x) \subseteq \psi_m^\dagger(\uparrow E_{\ell k}) \subseteq U_k \subseteq U$. So every open neighborhood U of x contains $\psi_m(x)$ for some $m \in \mathbb{N}$, hence $\bigcap^{\downarrow}_{m\in\mathbb{N}} \psi_m(x)$. So $\bigcap^{\downarrow}_{m\in\mathbb{N}} \psi_m(x) \subseteq \uparrow x$, whence the equality. □

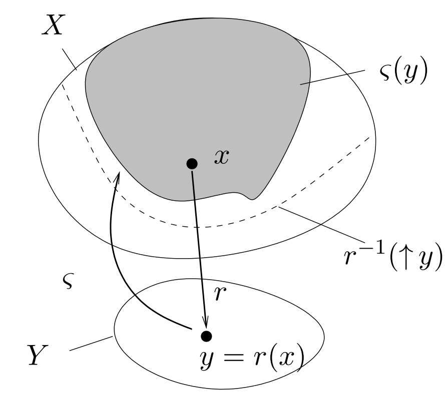


Figure 3: A quasi-retraction

4. Quasi-Retracts of Bifinite Domains

The **RB**-domains can be characterized as the retracts of bifinite domains. Recall that a *retraction* of X onto Y is a continuous map $r : X \to Y$ such that there is continuous map $s : Y \to X$ (the *section*) with $r(s(y)) = y$ for every $y \in Y$.

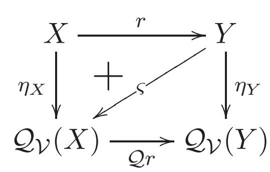
We shall show that (ω)**QRB**-domains are not just closed under retractions, but under a more relaxed notion that we shall call *quasi-retractions*. More precisely, our aim in this section is to show that the ω**QRB**-domains are exactly the quasi-retracts of bifinite domains, up to some details.

For each continuous $r : X \to Y$, define $\mathcal{Q}r : \mathcal{Q}_\mathcal{V}(X) \to \mathcal{Q}_\mathcal{V}(Y)$ by $\mathcal{Q}r(Q) = \uparrow\{r(x) \mid x \in Q\}$. $\mathcal{Q}r$ is continuous, since $\mathcal{Q}r^{-1}(\Box V) = \Box r^{-1}(V)$ for every open V. This is the action of the $\mathcal{Q}_\mathcal{V}$ functor of the Smyth powerspace monad [Sch93, Chapter 7], equivalently $\mathcal{Q}r = (\eta_Y \circ r)^\dagger$.

Definition 4.1 (Quasi-retract)**.** A *quasi-retraction* $r : X \to Y$ of X onto Y is a continuous map such that there is a continuous map $\varsigma : Y \to \mathcal{Q}_\mathcal{V}(X)$ (the *quasi-section*) such that $\mathcal{Q}r(\varsigma(y)) = \uparrow y$ for every $y \in Y$.

A topological space Y is a *quasi-retract* of X iff there is a quasi-retraction of X onto Y.

In diagram notation, we require the bottom right triangle to commute, but *not* the top left triangle—what the puncture $+$ indicates; the outer square always commutes:



(4.1)

While a section $s : Y \to X$ picks an element $s(y)$ in the inverse image $r^{-1}(y)$, continuously, a quasi-section is only required to pick a non-empty compact saturated collection of elements from $r^{-1}(\uparrow y)$ meeting $r^{-1}(y)$ (see Figure 3), continuously again.

Every retraction r (with section s) defines a canonical quasi-retraction: let $\varsigma(y) = \uparrow s(y)$, then $\mathcal{Q}r(\varsigma(y)) = \uparrow\{r(z) \mid s(y) \le z\} = \uparrow r(s(y)) = \uparrow y$.

The converse fails. For example, $\mathcal{N}_2$ is a quasi-retract of $\mathbb{N}_\omega + \mathbb{N}_\omega$ (see Figure 2 (*iii*)): r maps both $(0, \omega)$ and $(1, \omega)$ to $\omega \in \mathcal{N}_2$, and $\varsigma(y) = r^{-1}(\uparrow y)$ for every y. But Y is not a retract of X: X is a continuous dcpo, and every retract of a continuous dcpo is again one; recall that $\mathcal{N}_2$ is not continuous.

Every quasi-retraction $r : X \to Y$ induces a continuous map $\eta_Y \circ r : X \to \mathcal{Q}_{\mathcal{V}}(Y)$, which is then a retraction in the Kleisli category $\boldsymbol{C}_{\mathcal{Q}}$. A retraction in a category is a morphism $r : X \to Y$ such that there is a section morphism $s : Y \to X$, i.e., one with $r \circ s = \mathrm{id}_Y$. It is easy to see that the quasi-retractions are exactly those continuous maps $r : X \to Y$ such that $\eta_Y \circ r$ is a retraction in $\boldsymbol{C}_{\mathcal{Q}}$.

Lemma 4.2. *Every quasi-retraction $r : X \to Y$ onto a T_0 space Y is surjective. More precisely, if ς is a matching quasi-section, then every element $y \in Y$ is of the form $r(x)$ for some $x \in \varsigma(y)$.*

Proof. For every $y \in Y$, $\uparrow y = \mathcal{Q}r(\varsigma(y))$. Since $y \in \mathcal{Q}r(\varsigma(y))$, $r(x) \leq y$ for some $x \in \varsigma(y)$. But $r(x)$ is then in $\mathcal{Q}r(\varsigma(y)) = \uparrow y$, so $y \leq r(x)$. Therefore $y = r(x)$. □

The following is reminiscent of the fact that every retract of a stably compact space is again stably compact [Law87, Proposition, bottom of p.153, and subsequent discussion]: we shall show that any T_0 quasi-retract of a stably compact space is stably compact. We start with compactness.

Lemma 4.3. *Every T_0 quasi-retract Y of a compact space Y is compact.*

Proof. The image of a compact set by a continuous map is compact. Now apply Lemma 4.2. □

Lemma 4.4. *Any quasi-retract Y of a well-filtered space X is well-filtered.*

Proof. Let $r : X \to Y$ be the quasi-retraction, with quasi-section $\varsigma : Y \to \mathcal{Q}_{\mathcal{V}}(X)$.

Let $(Q_i)_{i \in I}$ be a filtered family of compact saturated subsets of Y, and assume that $\bigcap^{\downarrow}_{i \in I} Q_i \subseteq V$, where V is open in Y. Let $Q'_i = \varsigma^{\dagger}(Q_i)$. This is compact saturated, and forms a directed family, since $\varsigma^{\dagger}$ is monotonic. We claim that $\bigcap_{i \in I} Q'_i \subseteq r^{-1}(V)$. Indeed, every $x \in \bigcap_{i \in I} Q'_i$ is such that, for every $i \in I$, there is a $y_i \in Q_i$ such that $x \in \varsigma(y_i)$; then $r(x) \in \mathcal{Q}r(\varsigma(y_i)) = \uparrow y_i$, so $r(x) \in Q_i$, for every $i \in I$. Since $\bigcap^{\downarrow}_{i \in I} Q_i \subseteq V$, $r(x)$ is in V, whence the claim.

Since X is well-filtered, $Q'_i \subseteq r^{-1}(V)$ for some $i \in I$. Then, for every $y \in Q_i$, $\varsigma(y) \subseteq \varsigma^{\dagger}(Q_i) = Q'_i \subseteq r^{-1}(V)$, so $y \in \mathcal{Q}r(\varsigma(y)) \subseteq \mathcal{Q}r(r^{-1}(V)) \subseteq V$. So $Q_i \subseteq V$. □

Lemma 4.5. *Any T_0 quasi-retract Y of a coherent space X is coherent.*

Proof. Let $r : X \to Y$ be the quasi-retraction, with quasi-section $\varsigma : Y \to \mathcal{Q}_{\mathcal{V}}(X)$.

We use the fact that $\mathcal{Q}r \circ \varsigma^{\dagger}$ is the identity on $\mathcal{Q}_{\mathcal{V}}(Y)$. This is a well-known identity on monads: by the monad law $(g^{\dagger} \circ h)^{\dagger} = g^{\dagger} \circ h^{\dagger}$, and since $\mathcal{Q}r = (\eta_Y \circ r)^{\dagger}$, $\mathcal{Q}r \circ \varsigma^{\dagger} = (\mathcal{Q}r \circ \varsigma)^{\dagger}$, and this is $\eta_Y^{\dagger} = \mathrm{id}_{\mathcal{Q}_{\mathcal{V}}(Y)}$ by the first monad law.

Let Q_1, Q_2 be two compact saturated subsets of Y. Then $\varsigma^{\dagger}(Q_1) \cap \varsigma^{\dagger}(Q_2)$ is compact saturated in X, using the fact that X is coherent. So $\mathcal{Q}r(\varsigma^{\dagger}(Q_1) \cap \varsigma^{\dagger}(Q_2))$ is compact saturated in Y. We claim that $\mathcal{Q}r(\varsigma^{\dagger}(Q_1) \cap \varsigma^{\dagger}(Q_2)) = Q_1 \cap Q_2$, which will finish the proof. In one direction, every element y of $Q_1 \cap Q_2$ is in $\mathcal{Q}r(\varsigma^{\dagger}(Q_1) \cap \varsigma^{\dagger}(Q_2))$: by Lemma 4.2, pick $x \in \varsigma(y)$ such that $y = r(x)$, and observe that $x \in \varsigma^{\dagger}(Q_1)$ (indeed $x \in \varsigma(y)$, where $y \in Q_1$) and $x \in \varsigma^{\dagger}(Q_2)$. In the other direction, $\mathcal{Q}r(\varsigma^{\dagger}(Q_1) \cap \varsigma^{\dagger}(Q_2)) \subseteq \mathcal{Q}r(\varsigma^{\dagger}(Q_1)) \cap \mathcal{Q}r(\varsigma^{\dagger}(Q_2)) = Q_1 \cap Q_2$, since $\mathcal{Q}r \circ \varsigma^{\dagger}$ is the identity on $\mathcal{Q}(Y)$. □

Lemma 4.6. *Any quasi-retract Y of a locally compact space X is locally compact.*

Proof. Let $r : X \to Y$ be the quasi-retraction, with quasi-section $\varsigma : Y \to \mathcal{Q}_{\mathcal{V}}(X)$. Let y be any point of Y, and V be an open neighborhood of y. Since $y \in V$, $\mathcal{Q}r(\varsigma(y)) = \uparrow y \subseteq V$, so $\varsigma(y) \subseteq r^{-1}(V)$. Observe that $\varsigma(y)$ is compact saturated and $r^{-1}(V)$ is open in X. Use interpolation in the locally compact space X: there is a compact saturated subset Q_1 such that $\varsigma(y) \subseteq int(Q_1) \subseteq Q_1 \subseteq r^{-1}(V)$.

In particular, $\varsigma(y) \in \Box int(Q_1)$, so y is in the open subset $\varsigma^{-1}(\Box int(Q_1))$. The latter is included in the compact subset $\mathcal{Q}r(Q_1)$, since every element y' of it is such that $\varsigma(y') \subseteq int(Q_1) \subseteq Q_1$, hence $\uparrow y' = \mathcal{Q}r(\varsigma(y')) \subseteq \mathcal{Q}r(Q_1)$. In particular, y is in the interior of $\mathcal{Q}r(Q_1)$. Finally, since $Q_1 \subseteq r^{-1}(V)$, $\mathcal{Q}r(Q_1) \subseteq V$. ☐

Proposition 4.7. *Every T_0 quasi-retract Y of a stably compact space X is stably compact.*

Proof. Y is T_0 by assumption, and locally compact, well-filtered, compact, and coherent by Lemma 4.3, Lemma 4.4, Lemma 4.5, and Lemma 4.6. In the presence of local compactness, it is equivalent to require sobriety or to require the space to be T_0 and well-filtered [GHK$^+$03, Theorem II-1.21]. ☐

Call a space X *locally finitary* if and only if for every $x \in X$ and every open neighborhood U of x, there is a finitary compact $\uparrow E$ such that $x \in int(\uparrow E)$ and $\uparrow E \subseteq U$. This is the same definition as for local compactness, replacing compact saturated subsets by finitary compacts. The interpolation property of locally compact spaces refines to the following: In a locally finitary space X, if Q is compact saturated and included in some open subset U, then there is a finitary compact $\uparrow E$ such that $Q \subseteq int(\uparrow E)$ and $\uparrow E \subseteq U$. The proof is as for interpolation in locally compact spaces: for each $x \in Q$, pick a finitary compact $\uparrow E_x$ such that $x \in int(\uparrow E_x)$ and $\uparrow E_x \subseteq U$. $(int(\uparrow E_x))_{x \in Q}$ is an open cover of Q. Since Q is compact, it has a finite subcover $\uparrow E_1, \ldots, \uparrow E_n$. Then take $E = E_1 \cup \ldots \cup E_n$.

We observe right away the following analog of Lemma 4.6.

Lemma 4.8. *Any quasi-retract Y of a locally finitary space X is locally finitary.*

Proof. As in the proof of Lemma 4.6, let $y \in Y$ and V be an open neighborhood of y. By interpolation between $Q = \varsigma(y)$ and $U = r^{-1}(V)$ in the locally finitary space X, we find a finitary compact subset $Q_1 = \uparrow E_1$ of X such that $\varsigma(y) \subseteq int(Q_1) \subseteq Q_1 \subseteq r^{-1}(V)$. The rest of the proof is as for Lemma 4.6, only noticing that $\mathcal{Q}r(Q_1) = \uparrow r(E_1)$ is finitary compact. ☐

The importance of locally finitary spaces lies in the following result: see Banaschewski [Ban77], or the equivalence between Items (6) and (11) in Lawson [Law85, Theorem 2]. See also Isbell [Isb75] for the notion of locally finitary space, up to change of names.

Proposition 4.9. *The locally finitary sober spaces are exactly the quasi-continuous dcpos in their Scott topology.*

We use this, in particular, in the following proposition.

Proposition 4.10. *Every T_0 quasi-retract of an (ω)***QRB***-domain is an (ω)***QRB***-domain.*

Proof. Let X be a **QRB**-domain, Y be a T_0 space, $r : X \to Y$ be a quasi-retraction, and $\varsigma : Y \to \mathcal{Q}_{\mathcal{V}}(X)$ be a matching quasi-section. We first note that Y is stably compact, by Proposition 4.7, using the fact that X is itself stably compact (Theorem 3.14). So Y is sober. By Proposition 4.9, X is locally finitary, so Y is, too, by Lemma 4.8. By Proposition 4.9 again, Y is a quasi-continuous dcpo, and its topology is the Scott topology.

Note that Y is pointed. Letting $\perp$ be the least element of X, $r(\perp)$ is the least element of Y: for every $y \in Y$, pick some $x \in X$ such that $r(x) = y$ by Lemma 4.2, then $r(\perp) \leq r(x) = y$.

For each quasi-deflation φ on X, φ is continuous from X to $\mathrm{Fin}_{\mathcal{V}}(X)$: indeed it is continuous from X to $\mathrm{Fin}_\sigma(X)$ and $\mathrm{Fin}_\sigma(X) = \mathrm{Fin}_{\mathcal{V}}(X)$ by Corollary 3.6, since X is quasi-continuous (Corollary 3.4). So $\varphi^\dagger$ makes sense. Let $\widehat{\varphi} : Y \to \mathrm{Fin}_{\mathcal{V}}(Y)$ map y to $\mathcal{Q}r(\varphi^\dagger(\varsigma(y)))$; $\widehat{\varphi}(y)$ is in $\mathrm{Fin}(Y)$ because $\varphi^\dagger(\varsigma(y)) \in \mathrm{Fin}(X)$ (Lemma 3.7, second part), and $\mathcal{Q}r(\uparrow E) = \uparrow\{r(z) \mid z \in E\}$ is finitary compact for every finite set E.

Explicitly, $\widehat{\varphi}(y) = \uparrow\{r(z) \mid \exists x \in \varsigma(y) \cdot z \in \varphi(x)\}$.

For every open subset V of Y, $\widehat{\varphi}^{-1}(\Box V)$ is the set of all $y \in Y$ such that for every $x \in \varsigma(y)$, for every $z \in \varphi(x)$, $r(z) \in V$. I.e., for every $x \in \varsigma(y)$, $\varphi(x) \subseteq r^{-1}(V)$, that is, $\varsigma(y) \subseteq \varphi^{-1}(\Box r^{-1}(V))$. So $\widehat{\varphi}^{-1}(\Box V) = \varsigma^{-1}(\Box \varphi^{-1}(\Box r^{-1}(V)))$. Since the latter is open, and the sets $\Box V$ form a subbase of the topology of $\mathcal{Q}_{\mathcal{V}}(Y)$, $\widehat{\varphi}$ is continuous from Y to $\mathrm{Fin}_{\mathcal{V}}(Y)$. Since Y is a quasi-continuous dcpo and its topology is Scott, by Corollary 3.6 $\mathrm{Fin}_\sigma(Y) = \mathrm{Fin}_{\mathcal{V}}(Y)$, so $\widehat{\varphi}$ is also Scott-continuous from Y to $\mathrm{Fin}(Y)$. (Alternatively, apply Corollary 3.10.)

We claim that $y \in \widehat{\varphi}(y)$ for every $y \in Y$. Since $\mathcal{Q}r(\varsigma(y)) = \uparrow y$, $y \in \mathcal{Q}r(\varsigma(y))$, so there is an $x \in \varsigma(y)$ such that $r(x) \leq y$. Now $x \in \varphi(x)$, so taking $z = x$ in the definition of $\widehat{\varphi}(y)$, y is in $\widehat{\varphi}(y)$.

Let now $(\varphi_i)_{i \in I}$ be a generating family of quasi-deflations on X. Clearly, if φ_i is below φ_j, then $\widehat{\varphi}_i$ is below $\widehat{\varphi}_j$, so $(\widehat{\varphi}_i)_{i \in I}$ is directed.

It remains to show that $\bigcap^{\downarrow}_{i \in I} \widehat{\varphi}_i(y) = \uparrow y$ for every $y \in Y$. Since $y \in \widehat{\varphi}_i(y)$, it remains to show $\bigcap^{\downarrow}_{i \in I} \widehat{\varphi}_i(y) \subseteq \uparrow y$: we show that every open V containing y contains $\bigcap^{\downarrow}_{i \in I} \widehat{\varphi}_i(y)$. Since $y \in V$ and $\mathcal{Q}r(\varsigma(y)) = \uparrow y$, $\mathcal{Q}r(\varsigma(y)) \subseteq V$, so $\varsigma(y) \in \mathcal{Q}r^{-1}(\Box V) = \Box r^{-1}(V)$, i.e., $\varsigma(y) \subseteq r^{-1}(V)$. By Lemma 3.11, $\bigcup^{\uparrow}_{i \in I} \varphi_i^{-1}(\Box r^{-1}(V)) = r^{-1}(V)$. Since $\varsigma(y)$ is compact, $\varsigma(y) \subseteq \varphi_i^{-1}(\Box r^{-1}(V))$ for some $i \in I$. So y is in $\varsigma^{-1}(\Box \varphi_i^{-1}(\Box r^{-1}(V)))$, which is equal to $\widehat{\varphi}_i^{-1}(\Box V)$ (see above). It follows that V contains $\widehat{\varphi}_i(y)$, hence $\bigcap^{\downarrow}_{i \in I} \widehat{\varphi}_i(y)$. So Y is a **QRB**-domain.

The case of ω**QRB**-domains is similar, where now $(\varphi_i)_{i \in \mathbb{N}}$ is a generating *sequence* of quasi-deflations. □

Later, we shall need a refinement of the notion of quasi-retraction, which is to the latter as projections are to retractions. Recall that a *projection* is a retraction $r : X \to Y$, with section s, such that additionally $s \circ r \leq \mathrm{id}_X$. Similarly, it is tempting to define a *quasi-projection* as a quasi-retraction (with quasi-section ς) such that $x \in \varsigma(r(x))$ for every $x \in X$. If r is a retraction, with section s, and we see r as a quasi-retraction in the canonical way, defining $\varsigma(y)$ as $\uparrow s(y)$, then the quasi-projection condition $x \in \varsigma(r(x))$ is equivalent to the projection condition $(s \circ r)(x) \leq x$.

The point x shown in Figure 3 satisfies the condition $x \in \varsigma(r(x))$: x is in the gray area $\varsigma(y)$, where $y = r(x)$. However, Lemma 4.11 below shows that r is not a quasi-projection: for this to be the case, the gray area $\varsigma(y)$ should fill the whole of $r^{-1}(\uparrow y)$.

There is no need to invent a new term, though: Lemma 4.11 shows that quasi-projections are nothing else than proper surjective maps. A map $r : X \to Y$ is *proper* if and only if it is continuous, $\downarrow r(F)$ is closed in Y for every closed subset F of X, and $r^{-1}(\uparrow y)$ is compact in X for every element y of Y [GHK[+]03, Lemma VI-6.21 (i)].

Lemma 4.11. *Let X be a topological space, and Y be a T_0 topological space. For a map $r : X \to Y$, the following two conditions are equivalent:*

(1) *r is a quasi-retraction, with matching quasi-section $\varsigma : Y \to \mathcal{Q}_{\mathcal{V}}(X)$, such that additionally $x \in \varsigma(r(x))$ for every $x \in X$;*
(2) *r is proper and surjective.*

Then the quasi-section ς in (1) is unique, and it is defined by $\varsigma(y) = r^{-1}(\uparrow y)$.

Proof. We first prove the following fact, which will serve in both directions of proof: $(*)$ assume $\varsigma(y) = r^{-1}(\uparrow y)$ for every $y \in Y$, then for every open subset U of X, the complement of $\varsigma^{-1}(\Box U)$ in Y is $\downarrow r(F)$, where F is the complement of U in X. Indeed, the complement of $\varsigma^{-1}(\Box U)$ is the set of elements $y \in Y$ such that $\varsigma(y)$ is not included in U, i.e., such that there is an $x \in \varsigma(y)$ that is not in U, i.e., in F. Since $\varsigma(y) = r^{-1}(\uparrow y)$, this is the set of elements y such that there is an $x \in F$ such that $y \leq r(x)$, namely, $\downarrow r(F)$.

Assume r is a quasi-retraction, and ς is a matching quasi-section such that $x \in \varsigma(r(x))$ for every $x \in X$. We have seen that r is surjective (Lemma 4.2).

Since $\mathcal{Q}r(\varsigma(y)) = \uparrow y$, every element x of $\varsigma(y)$ is such that $r(x)$ is in $\uparrow y$, so $\varsigma(y) \subseteq r^{-1}(\uparrow y)$. Conversely, for every $x \in r^{-1}(\uparrow y)$, i.e., if $y \leq r(x)$, then $\varsigma(y) \supseteq \varsigma(r(x))$ since ς is monotonic. We have assumed that x was in $\varsigma(r(x))$, so $x \in \varsigma(y)$. It follows that $\varsigma(y) = r^{-1}(\uparrow y)$, which proves the last claim in the Lemma.

It also follows that $r^{-1}(\uparrow y)$ is compact in X. And, using $(*)$, for every closed subset F of X, with complement U, $\downarrow r(F)$ is the complement of $\varsigma^{-1}(\Box(U))$, which is open since ς is continuous, so $\downarrow r(F)$ is closed. Therefore r is proper.

Conversely, assume that r is proper and surjective. Define $\varsigma(y)$ as $r^{-1}(\uparrow y)$. Since r is surjective, $\varsigma(y)$ is non-empty. It is saturated, i.e., upward closed, because r is monotonic. Since $r^{-1}(\uparrow y)$ is compact, $\varsigma(y)$ is an element of $\mathcal{Q}(Y)$. For every open subset U of X, with complement F, $\varsigma^{-1}(\Box U)$ is the complement of $\downarrow r(F)$ by $(*)$, hence is open since r is proper. So ς is continuous.

The equation $\mathcal{Q}r(\varsigma(y)) = \uparrow y$ follows from $\mathcal{Q}r(\varsigma(y)) = \uparrow\{r(x) \mid x \in r^{-1}(\uparrow y)\}$ and the fact that r is surjective. It is clear that x is in $\varsigma(r(x)) = r^{-1}(\uparrow r(x))$ for every $x \in X$. ☐

Let us turn to bifinite domains, or rather to their countably-based variant. Countability will be needed in a few crucial places.

A pointed dcpo X is an *ω**B**-domain* (a.k.a. an SFP-domain) iff there is a non-decreasing sequence of idempotent deflations $(f_i)_{i\in\mathbb{N}}$ such that, for every $x \in X$, $x = \sup_{i\in\mathbb{N}} f_i(x)$. I.e., an ω**B**-domain is just like a **B**-domain, except that we take a non-decreasing sequence, not a general directed family of idempotent deflations.

The key lemma to prove Theorem 4.13 below is the following refinement of Rudin's Lemma [GHK$^+$03, III-3.3]. Note that Rudin's Lemma would only secure the existence of a directed family Z whose least upper bound is y, and which intersects each E_i^0; but Z may intersect each E_i^0 in more than one element y_i. We pick exactly one element y_i in each E_i^0, and for this countability seems to be needed.

Lemma 4.12. *Let Y be a dcpo, $y \in Y$, and $(\uparrow E_i^0)_{i\in\mathbb{N}}$ a non-decreasing sequence in Fin(Y) (w.r.t. $\supseteq$) such that $\uparrow y = \bigcap^{\downarrow}_{i\in\mathbb{N}} \uparrow E_i^0$. There is a non-decreasing sequence $(y_i)_{i\in\mathbb{N}}$ in Y such that $y_i \in E_i^0$ for every $i \in \mathbb{N}$, and $\sup_{i\in\mathbb{N}} y_i = y$.*

Proof. Let $E_i = E_i^0 \cap \downarrow y$ for every $i \in \mathbb{N}$. $(E_i)_{i\in\mathbb{N}}$ is a non-decreasing sequence in $\mathrm{Fin}(Y)$ such that $y \in \bigcap^{\downarrow}_{i\in\mathbb{N}} \uparrow E_i$, and $E_i \subseteq \downarrow y$.

Build a tree as follows. Informally, there is a root node, all (non-root) nodes at distance $i \geq 1$ from the root node are labeled by some element of E_{i-1}, and each such node N, labeled y_{i-1}, say, has as many successors as there are elements y_i in E_i such that $y_{i-1} \leq y_i$. Formally, one can define the nodes as being the sequences $y_0, y_1, \ldots, y_{i-1}$, $i \in \mathbb{N}$, where $y_0 \in E_0$, $y_1 \in E_1$, …, $y_{i-1} \in E_{i-1}$, and $y_0 \leq y_1 \leq \ldots \leq y_{i-1}$. Such a node is labeled y_{i-1} (if $i \geq 1$), and its successors are all the sequences $y_0, y_1, \ldots, y_{i-1}, y_i$ with y_i chosen in E_i, and above y_{i-1} if $i \geq 1$.

This tree has arbitrarily long branches (paths from the root). Indeed, for every $i \in \mathbb{N}$, pick an element $y_i \in E_i$—this is possible since $y \in \uparrow E_i$, hence E_i is non-empty—, then an element $y_{i-1} \in E_{i-1}$ below y_i—since $\uparrow E_{i-1} \supseteq \uparrow E_i$—, then an element $y_{i-2} \in E_{i-2}$ below y_{i-1}, …, and finally an element $y_0 \in E_0$ below y_1. This is a node at distance $i+1$ from the root.

It follows that the tree is infinite. It is finitely-branching, meaning that every node has only finitely many successors—because E_i is finite. Kőnig's Lemma then states that this tree must have an infinite branch. Reading the labels on non-root nodes in this branch, we obtain an infinite sequence $y_0 \leq y_1 \leq \ldots \leq y_i \leq \ldots$ of elements $y_i \in E_i$, $i \in \mathbb{N}$. Clearly, $y_i \in E_i^0$ for each $i \in \mathbb{N}$. In particular, $\sup_{i\in\mathbb{N}} y_i \in \bigcap^{\downarrow}_{i\in\mathbb{N}} \uparrow E_i^0 = \uparrow y$, so $y \leq \sup_{i\in\mathbb{N}} y_i$. Since $E_i \subseteq \downarrow y$ for every $i \in \mathbb{N}$, the converse inequality holds. So $\sup_{i\in\mathbb{N}} y_i = y$. □

Theorem 4.13. *The following are equivalent for a dcpo Y:*

- (i)**:** *Y is an ω**QRB**-domain;*
- (ii)**:** *Y is a quasi-retract of an ω**B**-domain;*
- (iii)**:** *Y is the image of an ω**B**-domain under a proper map.*

Proof. $(iii) \Rightarrow (ii)$. Because any proper surjective map is a quasi-retraction (Lemma 4.11).

$(ii) \Rightarrow (i)$. Write Y as a quasi-retract of an ω**B**-domain X. X is trivially an ω**QRB**-domain. Since Y, as a dcpo, is T_0, Proposition 4.10 applies, so Y is an ω**QRB**-domain.

$(i) \Rightarrow (iii)$. Let Y be an ω**QRB**-domain, with generating sequence of quasi-deflations $(\varphi_i)_{i\in\mathbb{N}}$. Let $\operatorname{im}\varphi_i = \{\uparrow E_{i1}, \ldots, \uparrow E_{in_i}\}$, and define E_i as the finite set $\bigcup_{j=1}^{n_i} E_{ij}$, for each $i \in \mathbb{N}$. Let X be the set of all non-decreasing sequences $\vec{y} = (y_i)_{i\in\mathbb{N}}$ in Y such that $y_i \in \bigcup_{j\leq i} E_j$, and $y_i \in \varphi_i(\sup_{k\in\mathbb{N}} y_k)$. Order X componentwise. As in [Jun88, Theorem 4.9, Theorem 4.1], X is an ω**B**-domain: for each $i_0 \in \mathbb{N}$, consider the idempotent deflation f_{i_0} defined by $f_{i_0}(\vec{y}) = (y_{\min(i,i_0)})_{i\in\mathbb{N}}$. To show that this is well-defined, we must show that $y_{\min(i,i_0)} \in \varphi_i(\sup_{k\in\mathbb{N}} y_{\min(k,i_0)})$, i.e., that $y_{\min(i,i_0)} \in \varphi_i(y_{i_0})$. If $i \leq i_0$, then $y_{\min(i,i_0)} = y_i \in \varphi_i(\sup_{k\in\mathbb{N}} y_k) \subseteq \varphi_i(y_{i_0})$ since $\vec{y} \in X$ and φ_i is monotonic, else $y_{\min(i,i_0)} = y_{i_0} \in \varphi_i(y_0)$ since φ_i is a quasi-deflation. It is easy to see that f_{i_0} is Scott-continuous.

Let now $r : X \to Y$ map $\vec{y}$ to $\sup_{i\in\mathbb{N}} y_i$. This is evidently Scott-continuous. For any fixed $y \in Y$, apply Lemma 4.12 with $\uparrow E_i^0 = \varphi_i(y)$ to obtain a non-decreasing sequence $\vec{y} = (y_i)_{i\in\mathbb{N}}$ such that $y_i \in \varphi_i(y)$ for every $i \in \mathbb{N}$ and $\sup_{i\in\mathbb{N}} y_i = y$: in particular, $\vec{y}$ is in Y, and $r(\vec{y}) = y$. So r is surjective. Let us show that it is proper.

To this end, we first remark that $r^{-1}(\uparrow y) = \{\vec{y} \in X \mid \forall i \in \mathbb{N} \cdot y_i \in \varphi_i(y)\}$. Indeed, if $\vec{y} = (y_i)_{i\in\mathbb{N}}$ is in $r^{-1}(\uparrow y)$, then $y \leq r(\vec{y}) = \sup_{k\in\mathbb{N}} y_k$, and since $\vec{y} \in X$, $y_i \in \varphi_i(\sup_{k\in\mathbb{N}} y_k) \subseteq \varphi_i(y)$, using the fact that φ_i is monotonic. Conversely, if $y_i \in \varphi_i(y)$ for every $i \in \mathbb{N}$, then $r(\vec{y}) = \sup_{i\in\mathbb{N}} y_i \in \bigcap_{i\in\mathbb{N}} \varphi_i(y) = \uparrow y$.

This remark makes it easier for us to show that $r^{-1}(\uparrow y)$ is compact for every $y \in Y$. For each $i_0 \in \mathbb{N}$, let $Q_{i_0} = \{\vec{y} \in X \mid \forall i \leq i_0 \cdot y_i \in \varphi_i(y)\}$. Let K_{i_0} be the set of all elements $\vec{y}$ of Q_{i_0} such that $y_i = y_{i_0}$ for every $i \geq i_0$. Note that K_{i_0} is finite, (recall that each y_i with $i \leq i_0$ is taken from the finite set $\bigcup_{j\leq i} E_j$), and that $Q_{i_0} = \uparrow K_{i_0}$. Indeed, for every $\vec{y} \in Q_{i_0}$, its image $f_{i_0}(\vec{y})$ by the idempotent deflation f_{i_0} is in K_{i_0}, and is below $\vec{y}$. So Q_{i_0} is (finitary) compact. Every $\omega\mathbf{B}$-domain is stably compact [AJ94, Theorem 4.2.18], and any intersection of saturated compacts in a stably compact space is compact, so $r^{-1}(\uparrow y) = \bigcap_{i_0 \in \mathbb{N}} Q_{i_0}$ is compact.

Let us now show that $\downarrow r(F)$ is closed for every closed subset F of X. Consider a directed family $(z_j)_{j \in J}$ of elements of $\downarrow r(F)$, and let $z = \sup_{j \in J} z_j$. Since $z_j \in \downarrow r(F)$, F intersects $r^{-1}(\uparrow z_j)$. The family $(r^{-1}(\uparrow z_j))_{j \in J}$ is a filtered family of compact saturated subsets of X, each of which intersects the closed set F. Since X is an $\omega\mathbf{B}$-domain, it is stably compact, hence well-filtered: so $\bigcap^{\downarrow}_{j \in J} r^{-1}(\uparrow z_j)$ intersects F. (Explicitly: if it did not, it would be included in the open complement U of F, hence some $r^{-1}(\uparrow z_j)$ would be included in U, contradicting the fact that it intersects F.) Let $\vec{y}$ be any element of $\bigcap^{\downarrow}_{j \in J} r^{-1}(\uparrow z_j) \cap F$. Then $z_j \leq r(\vec{y})$ for every $j \in J$, so $z = \sup_{j \in J} z_j \leq r(\vec{y})$, hence $z \in \downarrow r(F)$. □

5. Products, Bilimits

We first show that finite products of **QRB**-domains are again **QRB**-domains.

Lemma 5.1. *If $(\varphi_i)_{i \in I}$ (resp. $(\psi_j)_{j \in J}$) is a generating family of quasi-deflations on X (resp. Y), then $(\chi_{ij})_{i \in I, j \in J}$ is one on $X \times Y$, where $\chi_{ij}(x, y) = \varphi_i(x) \times \psi_j(y)$.*

Proof. Clearly, $(x, y) \in \chi_{ij}(x, y)$, $\chi_{ij}(x, y)$ is finitary compact, and $\operatorname{im} \chi_{ij}$ is finite. For all i, j, χ_{ij} is easily seen to be Scott-continuous, and $\bigcap^{\downarrow}_{i \in I,\, j \in J} \chi_{ij}(x, y) = \bigcap^{\downarrow}_{i \in I,\, j \in J} (\varphi_i(x) \times \psi_j(y)) = \bigcap^{\downarrow}_{i \in I} \varphi_i(x) \times \bigcap^{\downarrow}_{j \in J} \psi_j(y) = \uparrow x \times \uparrow y = \uparrow(x, y)$. □

So:

Lemma 5.2. *For any two (ω)**QRB**-domains X, Y, $X \times Y$, with the product ordering, is an (ω)**QRB**-domain.*

Recall that a retraction $p : X \to Y$, with section $e : Y \to X$, is a projection iff, additionally, $e(p(x)) \leq x$ for every $x \in X$; then e is usually called an *embedding*, and is determined uniquely from p. An *expanding system* of dcpos is a family $(X_i)_{i \in I}$, where I is a directed poset (with ordering $\leq$), with projection maps $(p_{ij})_{i,j \in I, i \leq j}$ where $p_{ij} : X_j \to X_i$, $p_{ii} = \mathrm{id}_{X_i}$, and $p_{ik} = p_{ij} \circ p_{jk}$ whenever $i \leq j \leq k$ [AJ94, Section 3.3.2]. This is nothing else than a projective system of dcpos, where the connecting maps p_{ij} must be projections. If $e_{ij} : X_i \to X_j$ is the associated embedding, then one checks that $e_{ii} = \mathrm{id}_{X_i}$ and $e_{ik} = e_{jk} \circ e_{ij}$ whenever $i \leq j \leq k$, so that $(X_i)_{i \in I}$ together with $(e_{ij})_{i,j \in I, i \leq j}$ forms an inductive system of dcpos as well. In the category of dcpos, the projective limit of the former coincides with the inductive limit of the latter (up to natural isomorphism), and is called the *bilimit* of the expanding system of dcpos. We write this bilimit as $\lim_{i \in I} X_i$, leaving the dependence on $\leq$, p_{ij}, e_{ij}, implicit. This can be built as the dcpo of all those elements $\vec{x} = (x_i)_{i \in I} \in \prod_{i \in I} X_i$ such that $p_{ij}(x_j) = x_i$ for all $i, j \in I$ with $i \leq j$, with the componentwise ordering.

General bilimits of countably-based dcpos will fail to be countably-based in general, so we shall restrict to bilimits of *expanding sequences* of dcpos [AJ94, Definition 3.3.6]: these are expanding systems of dcpos where the index poset I is $\mathbb{N}$, with its usual ordering. To make it clear what we are referring to, we shall call *ω-bilimit* of spaces any bilimit of an expanding sequence (not system) of spaces.

One can appreciate bilimits by realizing that the **B**-domains are (up to isomorphism) the bilimits of expanding systems of finite, pointed posets [AJ94, Theorem 4.2.7]. Similarly, the ω**B**-domains are the ω-bilimits of expanding sequences of finite, pointed posets.

Bilimits are harder to deal with than products. But the difficulty was solved by Jung [Jun88, Section 4.1] in the case of **RB**-domains and deflations, and we proceed in a very similar way. We first recapitulate the notion of bilimit.

Consider any set G of functions ψ from X to $\mathrm{Fin}(X)$ such that $\psi(x) \supseteq \uparrow x$, i.e., $x \in \psi(x)$, for every $x \in X$. We say that G is *qfs* (for *quasi-finitely separating*) iff given any finitely many pairs $(\uparrow E_k, x_k) \in \mathrm{Fin}(X) \times X$ with $\uparrow E_k \lll x_k$, $1 \leq k \leq n$, there is a $\psi \in G$ that *separates* the pairs, i.e., such that $\uparrow E_k \supseteq \psi(x_k) \supseteq \uparrow x_k$ (equivalently, $x_k \in \psi(x_k) \subseteq \uparrow E_k$) for every k, $1 \leq k \leq n$.

Proposition 5.3. *Let X be a poset. Then X is a* **QRB**-*domain iff X is a quasi-continuous dcpo and the set G of quasi-deflations on X is qfs.*

Proof. If X is a **QRB**-domain, then let $(\uparrow E_k, x_k) \in \mathrm{Fin}(X) \times X$ be such that $\uparrow E_k \lll x_k$ for every k, $1 \leq k \leq n$, and $(\varphi_i)_{i \in I}$ be a generating family of quasi-deflations. For each k, $1 \leq k \leq n$, $\uparrow x_k = \bigcap^{\downarrow}_{i \in I} \varphi_i(x_k) \subseteq \Uparrow E_k$, so by Proposition 3.5 there is an $i \in I$ such that $\varphi_i(x_k) \subseteq \Uparrow E_k \subseteq \uparrow E_k$. And we may pick the same i for every k, by directedness. So φ_i is the desired $\psi \in G$.

Also, X is a quasi-continuous dcpo by Corollary 3.4.

Conversely, assume that X is a quasi-continuous dcpo and G is qfs. We show that $H = \{\varphi^\dagger \circ \varphi \mid \varphi \in G\}$ is a generating family of quasi-deflations. Using Corollary 3.6, $\mathrm{Fin}_{\mathcal{V}}(X) = \mathrm{Fin}_\sigma(X)$. Write it $\mathrm{Fin}(X)$, for short. For each $\varphi \in G$, φ is continuous from X to $\mathrm{Fin}(X)$, and $\varphi^\dagger$ is continuous from $\mathrm{Fin}(X)$ to $\mathrm{Fin}(X)$ by Lemma 3.7, so $\varphi^\dagger \circ \varphi$ is continuous from X to $\mathrm{Fin}(X)$. Since $x \in \varphi(x)$, x is also in $\bigcup_{x' \in \varphi(x)} \varphi(x') = (\varphi^\dagger \circ \varphi)(x)$. Also, $\mathrm{im}(\varphi^\dagger \circ \varphi)$ is finite, since all its elements are unions of elements of the finite set $\mathrm{im}\,\varphi$. So $\varphi^\dagger \circ \varphi$ is a quasi-deflation.

Let us show that H is directed. Pick φ and φ' from G. Let $\mathrm{im}\,\varphi = \{\uparrow E_1, \dots, \uparrow E_m\}$, and $E = \bigcup_{i=1}^m E_i$. Similarly, let $\mathrm{im}\,\varphi' = \{\uparrow E'_1, \dots, \uparrow E'_n\}$ and $E' = \bigcup_{j=1}^n E'_j$. For each $y \in E$, $\varphi(y) \lll y$ by Lemma 3.3. Since X is quasi-continuous, use interpolation, and pick a finitary compact $\uparrow E_y$ such that $\varphi(y) \lll \uparrow E_y \lll y$. Similarly, let $\uparrow E'_{y'}$ be a finitary compact such that $\uparrow E'_{y'} \lll y'$ and $\varphi'(y') \lll \uparrow E'_{y'}$ for each $y' \in E'$.

Consider the finite collection of all pairs $(\uparrow E_y, y)$, $(\varphi(y), z)$, $(\uparrow E'_{y'}, y')$, and $(\varphi'(y'), z')$, where $y \in E$, $z \in E_y$, $y' \in E'$, $z' \in E_{y'}$. Since G is qfs, there is a $\psi \in G$ such that $\uparrow E'' \supseteq \psi(x) \supseteq \uparrow x$ for all the above pairs (E'', x). In particular, looking at the pair $(\uparrow E_y, y)$, we get: (a) $\uparrow E_y \supseteq \psi(y)$ for every $y \in E$. And looking at the pair $(\varphi(y), z)$, $\varphi(y) \supseteq \psi(z)$ for all $y \in E$, $z \in E_y$. So $\varphi(y) \supseteq \bigcup_{z \in E_y} \psi(z) = \bigcup_{z \in \uparrow E_y} \psi(z) = \psi^\dagger(\uparrow E_y)$. We have proved: (b) $\varphi(y) \supseteq \psi^\dagger(\uparrow E_y)$ for every $y \in E$. Then, for every $x \in X$, $(\varphi^\dagger \circ \varphi)(x) = \bigcup_{y \in \varphi(x)} \varphi(y) \supseteq \bigcup_{y \in \varphi(x)} \psi^\dagger(\uparrow E_y)$ (by (b)) $\supseteq \bigcup_{y \in \varphi(x)} (\psi^\dagger \circ \psi)(y)$ (by (a)) $= (\psi^\dagger \circ \psi)^\dagger(\varphi(x)) \supseteq (\psi^\dagger \circ \psi)^\dagger(\uparrow x)$ (since $\varphi(x) \supseteq \uparrow x$) $= (\psi^\dagger \circ \psi)^\dagger(\eta_X(x)) = (\psi^\dagger \circ \psi)(x)$ (by one of the monad laws). So $\varphi^\dagger \circ \varphi$ is below $\psi^\dagger \circ \psi$. Similarly, ${\varphi'}^\dagger \circ \varphi'$ is below $\psi^\dagger \circ \psi$, so H is directed.

Finally, we claim that $\bigcap_{\varphi \in G} (\varphi^\dagger \circ \varphi)(x) = \uparrow x$. In the $\supseteq$ direction, this is because $\varphi^\dagger \circ \varphi$ is a quasi-retraction. Conversely, let $\uparrow E \in \mathrm{Fin}(X)$ be such that $\uparrow E \lll x$. By interpolation, find $\uparrow E' \in \mathrm{Fin}(X)$ such that $\uparrow E \lll \uparrow E' \lll x$. Since G is qfs, applied to the pairs $(\uparrow E', x)$ and $(\uparrow E, y)$ for each $y \in E'$, there is an element $\varphi \in G$ such that

$\uparrow E' \supseteq \varphi(x)$ and $\uparrow E \supseteq \varphi(y)$ for every $y \in E'$. So $\uparrow E \supseteq \varphi^\dagger(\uparrow E') \supseteq (\varphi^\dagger \circ \varphi)(x)$. So $\bigcap_{\varphi \in G}(\varphi^\dagger \circ \varphi)(x) \subseteq \bigcap^{\downarrow}_{\uparrow E \in \mathrm{Fin}(X), \uparrow E \lll x} \uparrow E = \uparrow x$, as X is quasi-continuous. ☐

Theorem 5.4. *Any (ω-)bilimit of (ω)*$\mathbf{QRB}$*-domains is an (ω)*$\mathbf{QRB}$*-domain.*

Proof. Let $(X_i)_{i \in I}$ be an expanding system of $\mathbf{QRB}$-domains, with projections $p_{ij} : X_j \to X_i$ and embeddings $e_{ij} : X_i \to X_j$, $i \leq j$. Let $X = \lim_{i \in I} X_i$. There is a projection $p_i : X \to X_i$, given by $p_i(\vec{x}) = x_i$ (where $\vec{x} = (x_i)_{i \in I}$), and an embedding $e_i : X_i \to X$ for every $i \in I$.

We observe that: (a) if $\uparrow E \lll p_i(\vec{x})$ in X_i, then $\mathcal{Q}e_{ij}(\uparrow E) \lll p_j(\vec{x})$ for every $j \geq i$. Indeed, consider any directed family $(y_k)_{k \in K}$ such that $p_j(\vec{x}) \leq \sup_{k \in K} y_k$. Then $p_i(\vec{x}) = p_{ij}(p_j(\vec{x})) \leq \sup_{k \in K} p_{ij}(y_k)$, so for some $k \in K$, there is a $z \in E$ with $z \leq p_{ij}(y_k)$. Then $e_{ij}(z) \leq e_{ij}(p_{ij}(y_k)) \leq y_k$. We conclude since $e_{ij}(z) \in \mathcal{Q}e_{ij}(\uparrow E)$.

We now claim that the family $\mathcal{D}_{\vec{x}}$ of all finitary compacts of the form $\mathcal{Q}e_i(\uparrow E)$, where $\uparrow E \in \mathrm{Fin}(X_i)$ and $\uparrow E \lll p_i(\vec{x})$, $i \in I$, is directed. Given $\mathcal{Q}e_i(\uparrow E)$ and $\mathcal{Q}e_j(\uparrow E')$ in $\mathcal{D}_{\vec{x}}$, find some $k \in I$ such that $i, j \leq k$, by directedness. Then $\mathcal{Q}e_i(\uparrow E) = \mathcal{Q}e_k(\mathcal{Q}e_{ik}(\uparrow E))$, and by (a) $\mathcal{Q}e_{ik}(\uparrow E) \lll p_k(\vec{x})$, and similarly $\mathcal{Q}e_j(\uparrow E') = \mathcal{Q}e_k(\mathcal{Q}e_{jk}(\uparrow E'))$, with $\mathcal{Q}e_{jk}(\uparrow E') \lll p_k(\vec{x})$. Replacing i by k, $\uparrow E$ by the finitary compact $\mathcal{Q}e_{ik}(\uparrow E)$, j by k, and $\uparrow E'$ by $\mathcal{Q}e_{jk}(\uparrow E')$ if necessary, we can therefore simply assume that $i = j$. Since X_i is quasi-continuous, there is an $E'' \in \mathrm{Fin}(X_i)$ such that $\uparrow E, \uparrow E' \lll \uparrow E'' \lll p_i(\vec{x})$, and then $\mathcal{Q}e_i(\uparrow E'')$ is an element of $\mathcal{D}_x$ above both $\mathcal{Q}e_i(\uparrow E)$ and $\mathcal{Q}e_i(\uparrow E')$.

Moreover, we claim that $\bigcap_{\mathcal{Q}e_i(\uparrow E) \in \mathcal{D}_{\vec{x}}} \mathcal{Q}e_i(\uparrow E)$ equals $\uparrow \vec{x}$. That it contains $\vec{x}$ is obvious: whenever $\uparrow E \lll p_i(\vec{x})$, pick $z \in E$ with $z \leq p_i(\vec{x})$, so that $e_i(z) \leq e_i(p_i(\vec{x})) \leq \vec{x}$, hence $\vec{x} \in \mathcal{Q}e_i(\uparrow E)$. Conversely, every $\vec{z} \in \bigcap_{\mathcal{Q}e_i(\uparrow E) \in \mathcal{D}_{\vec{x}}} \mathcal{Q}e_i(\uparrow E)$ must be such that $z_i = p_i(\vec{z}) \in \mathcal{Q}p_i(\bigcap_{\uparrow E \lll p_i(\vec{x})} \mathcal{Q}e_i(\uparrow E)) \subseteq \bigcap_{\uparrow E \lll p_i(\vec{x})} \mathcal{Q}p_i(\mathcal{Q}e_i(\uparrow E)) = \bigcap_{\uparrow E \lll p_i(\vec{x})} \uparrow E = \uparrow p_i(\vec{x}) = \uparrow x_i$, since X_i is quasi-continuous. As this holds for every i, $\vec{x} \leq \vec{z}$. So $\bigcap_{\mathcal{Q}e_i(\uparrow E) \in \mathcal{D}_{\vec{x}}} \mathcal{Q}e_i(\uparrow E) \subseteq \uparrow \vec{x}$.

In particular, X is a quasi-continuous dcpo.

We check that the set of quasi-deflations on X is qfs. Consider a finite collection of pairs $(\uparrow \vec{D}_k, \vec{x}_k) \in \mathrm{Fin}(X) \times X$ with $\uparrow \vec{D}_k \lll \vec{x}_k$, $1 \leq k \leq n$. Recall that $\uparrow \vec{D}_k \lll \vec{x}_k$ can be rephrased equivalently as: $\vec{x}_k$ is in the open subset $\Uparrow \vec{D}_k$. Since $\bigcap_{\mathcal{Q}e_i(\uparrow E) \in \mathcal{D}_{\vec{x}_k}} \mathcal{Q}e_i(\uparrow E) = \uparrow \vec{x}_k$, by Proposition 3.5, for each k, pick $\mathcal{Q}e_i(\uparrow E_k) \in \mathcal{D}_{\vec{x}_k}$ included in $\Uparrow \vec{D}_k$, in particular above $\uparrow \vec{D}_k$. I.e., pick $i \in I$ and $\uparrow E_k \in \mathrm{Fin}(X_i)$ such that $\uparrow E_k \lll p_i(\vec{x}_k)$, and such that $\uparrow \vec{D}_k \supseteq \mathcal{Q}e_i(\uparrow E_k)$. (We can pick the same i for every k, by directedness, as above.) Since X_i is a $\mathbf{QRB}$-domain, and $\uparrow E_k \lll p_i(\vec{x}_k)$, using Proposition 3.5, there is a quasi-deflation φ on X_i such that $\varphi(p_i(\vec{x}_k)) \subseteq \Uparrow E_k$. So $\varphi(p_i(\vec{x}_k)) \subseteq \uparrow E_k$, for every k, $1 \leq k \leq n$. Consider $\psi : X \to \mathrm{Fin}(X)$ defined as $\mathcal{Q}e_i \circ \varphi \circ p_i$. $\mathcal{Q}e_i$, restricted to $\mathrm{Fin}(X_i)$, takes its values in $\mathrm{Fin}(X)$, using Lemma 3.7 and the fact that $\mathcal{Q}e_i = (\eta_X \circ e_i)^\dagger$. Moreover, ψ is continuous from X to $\mathrm{Fin}_{\mathcal{V}}(X)$, hence to $\mathrm{Fin}_\sigma(X)$ since X is quasi-continuous, by Corollary 3.6. For every $\vec{x} \in X$, $p_i(\vec{x}) \in \varphi(p_i(\vec{x}))$, since φ is a quasi-deflation. Then $e_i(p_i(\vec{x}))$ is below $\vec{x}$, and is in $\psi(\vec{x})$, so $\vec{x} \in \psi(\vec{x})$. So ψ is a quasi-deflation.

Moreover, by construction, for each k, $1 \leq k \leq n$, $\varphi(p_i(\vec{x}_k)) \subseteq \uparrow E_k$, so $\psi(\vec{x}_k) \subseteq \mathcal{Q}e_i(\uparrow E_k)$, so $\psi(\vec{x}_k) \subseteq \uparrow \vec{D}_k$, since $\uparrow \vec{D}_k \supseteq \mathcal{Q}e_i(\uparrow E_k)$. So the set of quasi-deflations on X is qfs.

By Proposition 5.3, X is then a $\mathbf{QRB}$-domain.

To deal with ω-bilimits of $\omega\mathbf{QRB}$-domains, observe that any bilimit of a countable expanding system (in particular, an expanding sequence) of countably-based quasi-continuous

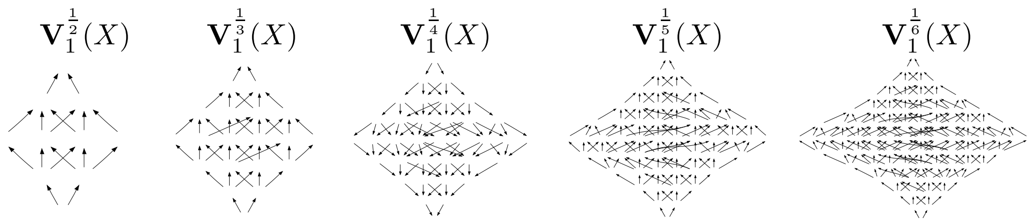


Figure 4: Discretizations of $\mathbf{V}_1(X)$, $X = \{\bot, a, b, \top\}$

dcpos is countably-based. Indeed, a countably based quasi-continuous dcpo X_i has a countable base of sets of the form $\Uparrow E_{ik}$, $\uparrow E_{ik} \in \mathrm{Fin}(X_i)$, $k \in \mathbb{N}$. The $\mathcal{D}_{\vec{x}}$ construction above, suitably modified, shows that the sets $\Uparrow \vec{E'}_{ik}$, where $\uparrow \vec{E'}_{ik} = \mathcal{Q}e_i(E_{ik})$, $i, k \in \mathbb{N}$, form a, necessarily countable, base of the topology on X. By Proposition 3.18, X is an ω**QRB**-domain. □

6. The Probabilistic Powerdomain

Let X be a fixed topological space, and let $\mathcal{O}(X)$ be the lattice of open subsets of X. A *continuous valuation* ν on X [JP89] is a map from $\mathcal{O}(X)$ to $\mathbb{R}^+$ such that $\nu(\emptyset) = 0$, which is *monotonic* ($\nu(U) \leq \nu(V)$ whenever $U \subseteq V$), *modular* ($\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$ for all opens U, V), and *continuous* ($\nu(\bigcup^{\uparrow}_{i \in I} U_i) = \sup_{i \in I} \nu(U_i)$ for every directed family $(U_i)_{i \in I}$ of opens). A *(sub)probability* valuation ν is additionally such that ν is *(sub)normalized*, i.e., that $\nu(X) = 1$ ($\nu(X) \leq 1$). Let $\mathbf{V}_1(X)$ ($\mathbf{V}_{\leq 1}(X)$) be the dcpo of all (sub)probability valuations on X, ordered pointwise, i.e., $\nu \leq \nu'$ iff $\nu(U) \leq \nu'(U)$ for every open U. $\mathbf{V}_1$ ($\mathbf{V}_{\leq 1}$) defines a endofunctor on the category of dcpos, and its action is defined on morphisms f by $\mathbf{V}_1 f(\nu)(U) = \nu(f^{-1}(U))$.

We write δ_x for the *Dirac valuation* at x, a.k.a., the point mass at x. This is the continuous valuation such that $\delta_x(U) = 1$ if $x \in U$, $\delta_x(U) = 0$ otherwise.

The probabilistic powerdomain construction $\mathbf{V}_1$ is an elusive one, and natural intuitions are often wrong. For example, one might imagine that if X has all binary least upper bounds, then so has $\mathbf{V}_1(X)$. This was dispelled by Jones and Plotkin [JP89]. Consider $X = \{\bot, a, b, \top\}$, with a and b incomparable, $\bot$ below every element and $\top$ above every element (see Figure 1, right). Then the upper bounds of $\frac{1}{2}\delta_\bot + \frac{1}{2}\delta_a$ and $\frac{1}{2}\delta_\bot + \frac{1}{2}\delta_b$ in $\mathbf{V}_1(X)$ are the probability valuations of the form $(1 - \alpha_a - \alpha_b - \alpha_\top)\delta_\bot + \alpha_a\delta_a + \alpha_b\delta_b + \alpha_\top\delta_\top$ where $\alpha_a + \alpha_\top \geq \frac{1}{2}$, $\alpha_b + \alpha_\top \geq \frac{1}{2}$, and $\alpha_a + \alpha_b + \alpha_\top \leq 1$. The minimal upper bounds are those of the form $\alpha\delta_\bot + (\frac{1}{2} - \alpha)\delta_a + (\frac{1}{2} - \alpha)\delta_b + \alpha\delta_\top$, $\alpha \in [0, 1]$. So there is no unique least upper bound; in fact, there are uncountably many of them, even on this small example.

It is unknown whether $\mathbf{V}_1(X)$, with $X = \{\bot, a, b, \top\}$ is an **RB**-domain, although it is an **FS**-domain, as a consequence of [JT98, Theorem 17]. Again, some of the most natural ideas one can have about $\mathbf{V}_1(X)$ are flawed. It seems obvious indeed that $\mathbf{V}_1(X)$ should be the bilimit of the sequence of finite posets $\mathbf{V}_1^{\frac{1}{n}}(X)$, defined as those probability valuations $(1 - \alpha_a - \alpha_b - \alpha_\top)\delta_\bot + \alpha_a\delta_a + \alpha_b\delta_b + \alpha_\top\delta_\top$ where α_a, α_b, $\alpha_\top$ are integer multiples of $\frac{1}{n}$. See Figure 4 for Hasse diagrams of a few of these posets, for n small.

That $\mathbf{V}_1(X)$ is such a bilimit is necessarily wrong, because any bilimit of finite posets is an ω**B**-domain, hence is algebraic, but $\mathbf{V}_1(X)$ is not algebraic, since no element except $\delta_\bot$ is finite.

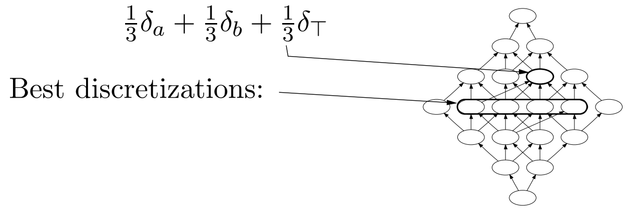


Figure 5: Largest discretizations below ν fail to be unique

However, one may imagine to define (non-idempotent) deflations f_n on $\mathbf{V}_1(X)$ directly, which would send $\nu \in \mathbf{V}_1(X)$ to some discretized probability valuation in $\mathbf{V}_1^{\frac{1}{n}}(X)$. However, all known attempts fail. A careful study of [JT98] will make this precise. Let us only note that if we decide to define $f_n(\nu)$ through its values on open sets, typically letting $f_n(\nu)(U)$ be the largest integer multiple of $\frac{1}{n}$ that is zero-or-strictly-below $\nu(U)$, we obtain a set function that is not modular. If we decide to define $f_n(\sum_{x \in X} \alpha_x \delta_x)$ as $\sum_{x \in X} \beta_x \delta_x$ where for each $x \neq \bot$ β_x is the largest integer multiple of $\frac{1}{n}$ that is zero-or-strictly-below α_x, then f_n is not monotonic. If we decide to define $f_n(\nu)$ as the largest probability valuation way-below ν in $\mathbf{V}_1^{\frac{1}{n}}(X)$, we run into the problem that there is no *unique* such largest probability valuation. For example, $\nu = \frac{1}{3}\delta_a + \frac{1}{3}\delta_b + \frac{1}{3}\delta_\top$ admits four largest probability valuations in $\mathbf{V}_1^{\frac{1}{3}}(X)$ way-below it: $\frac{1}{3}\delta_\bot + \frac{2}{3}\delta_a$, $\frac{1}{3}\delta_\bot + \frac{1}{3}\delta_a + \frac{1}{3}\delta_b$, $\frac{2}{3}\delta_\bot + \frac{1}{3}\delta_\top$, and $\frac{1}{3}\delta_\bot + \frac{2}{3}\delta_b$, see Figure 5.

Observe that the number of largest discretizations of ν in $\mathbf{V}_1^{\frac{1}{n}}(X)$ is always finite, provided X is finite. This was our original intuition that replacing deflations by quasi-deflations, hence moving from **RB**-domains to **QRB**-domains, might provide a nice enough category of domains that would be stable under the probabilistic powerdomain functor $\mathbf{V}_1$. However, defining quasi-deflations directly, as hinted above, does not work either: monotonicity fails again. This is where the characterization of **QRB**-domains as quasi-retracts of bifinite domains (up to details we have already mentioned) will be decisive.

If Y is a retract of X, then $\mathbf{V}_1(Y)$ is easily seen to be a retract of $\mathbf{V}_1(X)$, using the $\mathbf{V}_1$ endofunctor. We wish to show a similar result for quasi-retracts. We have not managed to do so. Instead we shall rely on the stronger assumptions that X is stably compact, that Y is a quasi-projection of X, not just a quasi-retract (i.e., the image of X under a proper map).

Moreover, we shall need to replace the Scott topology on $\mathbf{V}_1(X)$ by the *weak topology*, which is the smallest one containing the subbasic opens $[U > a]$, defined as $\{\nu \in \mathbf{V}_1(X) \mid \nu(U) > a\}$, for each open subset U of X and $a \in \mathbb{R}$. When X is a continuous pointed dcpo, the *Kirch-Tix Theorem* states that it coincides with the Scott topology (see [AMJK04], who attribute it to Tix [Tix95, Satz 4.10], who in turn attributes it to Kirch [Kir93, Satz 8.6]).

However, the weak topology is better behaved in the general case. For example, writing $\overline{\mathbb{R}}^+_\sigma$ for $\mathbb{R}^+ \cup \{+\infty\}$ with the Scott topology, and $[X \to \overline{\mathbb{R}}^+_\sigma]_{\mathtt{i}}$ for the space of all continuous maps from X to $\overline{\mathbb{R}}^+_\sigma$ with the Isbell topology, there is a natural homeomorphism between the space of linear continuous maps from $[X \to \overline{\mathbb{R}}^+_\sigma]_{\mathtt{i}}$ to $\overline{\mathbb{R}}^+_\sigma$ and the space of of (extended, i.e., possibly taking the value $+\infty$) continuous valuations on X, with the weak topology [Hec96, Theorem 8.1]. This is an analog of the Riesz Representation Theorem in measure theory, of which one can find variants in [Tix95, Gou07b] among others, and which we shall use silently in the proof of Theorem 6.5. Let $\mathbf{V}_{1\,wk}(X)$ be $\mathbf{V}_1(X)$ with its weak topology.

$\mathbf{V}_{1\,wk}$ defines an endofunctor on the category of topological spaces, by $\mathbf{V}_{1\,wk}(f)(\nu)(V) = \nu(f^{-1}(V))$, where $f : X \to Y$, $\nu \in \mathbf{V}_{1\,wk}(X)$, and $V \in \mathcal{O}(Y)$. That $\mathbf{V}_{1\,wk}(f)$ is continuous for every continuous f, in particular, is obvious, since for every open subset V of Y, $\mathbf{V}_{1\,wk}(f)^{-1}[V > a] = [f^{-1}(V) > a]$.

As we have said above, we shall also require X to be stably compact. If this is so, then the *cocompact topology* on X consists of all complements of compact saturated subsets. Write X^{d}, the *de Groot dual* of X, for X with its cocompact topology. Then X^{d} is again stably compact, and $X^{\mathsf{dd}} = X$ (see [AMJK04, Corollary 12] or [GHK$^+$03, Corollary VI-6.19]). The patch topology on X, mentioned earlier, is nothing else than the join of the two topologies of X and X^{d}.

Write X^{patch} for X equipped with its patch topology. If X is stably compact, then X^{patch} is not only compact Hausdorff, but the graph of the specialization preorder $\leq$ of X is closed in X^{patch}: one says that $(X^{\mathsf{patch}}, \leq)$ is a *compact pospace*. The study of compact pospaces originates in Nachbin's classic work [Nac65]. Conversely, given a compact pospace $(Z, \preceq)$, i.e., a compact space with a closed ordering $\preceq$ on it, the *upwards topology* on Z consists of those open subsets of Z that are upward closed in $\preceq$. The space $Z^{\uparrow}$, obtained as Z with the upwards topology, is then stably compact. Moreover, the two constructions are inverse of each other. (See [GHK$^+$03, Section VI-6].)

If X and Y are stably compact, then $f : X \to Y$ is proper if and only if $f : X^{\mathsf{patch}} \to Y^{\mathsf{patch}}$ is continuous, and monotonic with respect to the specialization orderings of X and Y [GHK$^+$03, Proposition VI.6.23], i.e., if and only if f is a morphism of compact pospaces.

Now, the structure of the cocompact topology on $\mathbf{V}_{1\,wk}(X)$, when X is stably compact, is as follows. For every continuous valuation ν on X, following Tix [Tix95], define $\nu^{\dagger}(Q)$ as $\inf_{U \in \mathcal{O}(X), U \supseteq Q} \nu(U)$, for every compact saturated subset Q of X. Define $\langle Q \geq a \rangle$ as the set of probability valuations ν such that $\nu^{\dagger}(Q) \geq a$. The sets $\langle Q \geq a \rangle$ are compact saturated in $\mathbf{V}_{1\,wk}(X)$, and Proposition 6.8 of [Gou10] even states that they form a subbase of compact saturated subsets. This means that the complements of the sets of the form $\langle Q \geq a \rangle$, Q compact saturated in X, $a \in \mathbb{R}$, form a base of the topology of $\mathbf{V}_{1\,wk}(X)^{\mathsf{d}}$. A similar claim was already stated in [Jun04, last lines].

Lemma 6.1. *Let X, Y be stably compact spaces, and r be a proper surjective map from X to Y. Then $\mathbf{V}_{1\,wk}(r)(\nu)^{\dagger}(Q) = \nu^{\dagger}(r^{-1}(Q))$, for every compact saturated subset Q of Y.*

Proof. We must show that $\inf_{V \supseteq Q} \nu(r^{-1}(V)) = \inf_{U \supseteq r^{-1}(Q)} \nu(U)$, where V ranges over opens in Y and U over opens in X.

For every open V containing Q, $U = r^{-1}(V)$ is an open subset of X containing the compact saturated subset $r^{-1}(Q)$, so $\inf_{V \supseteq Q} \nu(r^{-1}(V)) \geq \inf_{U \supseteq r^{-1}(Q)} \nu(U)$.

Conversely, for every open U containing $r^{-1}(Q)$, we shall build an open subset V containing Q such that $r^{-1}(V) \subseteq U$. This will establish $\inf_{V \supseteq Q} \nu(r^{-1}(V)) \leq \inf_{U \supseteq r^{-1}(Q)} \nu(U)$, hence the equality.

Recall from Lemma 4.11 that r forms a quasi-retraction, with a unique matching quasi-section $\varsigma : Y \to \mathcal{Q}_{\mathcal{V}}(X)$ such that $x \in \varsigma(r(x))$ for every $x \in X$, and such that $\varsigma(y) = r^{-1}(\uparrow y)$ for every $y \in Y$. We let $V = \varsigma^{-1}(\Box U)$. Since $r^{-1}(Q) \subseteq U$, $r^{-1}(Q)$ is in $\Box U$. For every $y \in Q$, $\varsigma(y) = r^{-1}(\uparrow y) \subseteq r^{-1}(Q)$ is then also in $\Box U$, so y is in $\varsigma^{-1}(\Box U) = V$. So $Q \subseteq V$. On the other hand, for every element x of $r^{-1}(V)$, $r(x)$ is in $V = \varsigma^{-1}(\Box U)$, so $\varsigma(r(x))$ is in $\Box U$. Then $x \in \varsigma(r(x)) \subseteq U$. So $r^{-1}(V) \subseteq U$, and we are done. $\Box$

Similarly to the formula $\mathbf{V}_{1\,wk}(f)^{-1}[V > a] = [f^{-1}(V) > a]$, which allowed us to conclude that $\mathbf{V}_{1\,wk}(f)$ was continuous for every continuous f, we obtain:

Lemma 6.2. *Let X, Y be stably compact spaces, and r be a proper surjective map from X to Y. Then $\mathbf{V}_{1\,wk}(r)^{-1}\langle Q \geq a\rangle = \langle r^{-1}(Q) \geq a\rangle$ for every compact saturated subset Q of Y, and $a \in \mathbb{R}$.*

Proof. Using Lemma 6.1, $\mathbf{V}_{1\,wk}(r)^{-1}\langle Q \geq a\rangle = \{\nu \in \mathbf{V}_{1\,wk}(X) \mid \mathbf{V}_{1\,wk}(r)(\nu)^\dagger(Q) \geq a\} = \{\nu \in \mathbf{V}_{1\,wk}(X) \mid \nu^\dagger(r^{-1}(Q)) \geq a\} = \langle r^{-1}(Q) \geq a\rangle$. □

Proposition 6.3. *Let X be a stably compact space, Y be a T_0 space, and r be a proper surjective map from X to Y. Then $\mathbf{V}_{1\,wk}(r)$ is a proper map from $\mathbf{V}_{1\,wk}(X)$ to $\mathbf{V}_{1\,wk}(X)$.*

Proof. First, since r is proper and surjective, r is a quasi-retraction (Lemma 4.11), so Y is stably compact by Proposition 4.7. $\mathbf{V}_{1\,wk}(r)$ is continuous from $\mathbf{V}_{1\,wk}(X)$ to $\mathbf{V}_{1\,wk}(Y)$. Lemma 6.2 implies that $\mathbf{V}_{1\,wk}(r)$ is also continuous from $\mathbf{V}_{1\,wk}(X)^{\mathsf{patch}}$ to $\mathbf{V}_{1\,wk}(Y)^{\mathsf{patch}}$: it suffices to check that the inverse images of subbasic patch-open subsets, of the form $[U > a]$ or whose complements are of the form $\langle Q \geq a\rangle$, are patch-open. Also, $\mathbf{V}_{1\,wk}(r)$ is monotonic with respect to the specialization orderings of $\mathbf{V}_{1\,wk}(X)$ and $\mathbf{V}_{1\,wk}(Y)$, being continuous. So $\mathbf{V}_{1\,wk}(r)$ is proper. □

Let us establish surjectivity. One possible proof goes as follows. Let $\mathcal{M}_1(Z)$ denote the space of all Radon probability measures on the space Z. If X is stably compact, then $\mathcal{M}_1(X^{\mathsf{patch}})$ is compact in the vague topology, and forms a compact pospace with the stochastic ordering, where μ is below μ' if and only if $\mu(U) \leq \mu'(U)$ for every open subset U of X [AMJK04, Theorem 31]. By [AMJK04, Theorem 36], there is an isomorphism between $\mathbf{V}_{1\,wk}(X)$ and $\mathcal{M}_1^\uparrow(X^{\mathsf{patch}})$.

Now assume a second stably compact space Y. For two measurable spaces A and B, and $f : A \to B$ measurable, let $\mathcal{M}(f)$ map the Radon measure μ to its image measure, whose value on the Borel subset E of B is $\mu(f^{-1}(E))$. A standard result [Bou69, 2.4, Lemma 1] states that for any two compact Hausdorff spaces A and B, if r is continuous surjective from A to B, then $\mathcal{M}(r)$ is surjective. The desired result follows, up to a few technical details, by taking $A = X^{\mathsf{patch}}$, $B = Y^{\mathsf{patch}}$, remembering that since r is proper from X to Y, it is continuous from X^{patch} to Y^{patch}.

Instead of working out the—technically subtle but boring—technical details, let us give a direct proof, similar to the above cited Lemma 1, 2.4 [Bou69]. Instead of using the Hahn-Banach Theorem, we rest on the following Keimel Sandwich Theorem [Kei06, Theorem 8.2]: let C be a topological cone, $q : C \to \overline{\mathbb{R}^+_\sigma}$ be a continuous superlinear map, $p : C \to \overline{\mathbb{R}^+_\sigma}$ be a sublinear map, and assume $q \leq p$; then there is a continuous linear map $\Lambda : C \to \overline{\mathbb{R}^+_\sigma}$ such that $q \leq \Lambda \leq p$. Here, a *cone* is an additive commutative monoid, with a scalar multiplication by elements of $\mathbb{R}^+$ satisfying $a(x + y) = ax + ay$, $(a + b)x = ax + bx$, $(ab)x = a(bx)$, $1x = x$, $0x = 0$ for all $a, b \in \mathbb{R}^+$, $x, y \in C$. A cone is *topological* if and only if addition and multiplication are continuous. The continuous maps $f : C \to \overline{\mathbb{R}^+_\sigma}$ are sometimes called lower semi-continuous in the literature. Such a map is *superlinear* (resp., *sublinear*, *linear*) if and only if $f(ax) = af(x)$ for all $a \in \mathbb{R}^+$, $x \in C$ and $f(x + y) \geq f(x) + f(y)$ for all $x, y \in C$ (resp., $\leq$, $=$). It is easy to see that the space $[X \to \overline{\mathbb{R}^+_\sigma}]$ of all continuous maps from X to $\overline{\mathbb{R}^+_\sigma}$, equipped with the obvious addition and scalar multiplication and with the Scott topology of the pointwise ordering, is a topological cone.

Proposition 6.4. *Let X, Y be stably compact spaces, and r be a proper surjective map from X to Y. Then $\mathbf{V}_{1\,wk}(r)$ is surjective.*

Proof. Fix some continuous probability valuation ν on Y. Let C be $[X \to \overline{\mathbb{R}^+_\sigma}]$. Since r is proper, it has an associated quasi-section ς, with $x \in \varsigma(r(x))$ for every $x \in X$, by Lemma 4.11. Define $q : C \to \overline{\mathbb{R}^+_\sigma}$ by $q(h) = \int_{y\in Y} h_*(\varsigma(y))d\nu$, where $h_*(Q) = \min_{x\in Q} h(x)$, and integration of continuous maps from Y to $\overline{\mathbb{R}^+_\sigma}$ is defined by a Choquet formula [Tix95, Gou07a], or equivalently by Heckmann's general construction [Hec96].

Note that $h_*(Q)$ is well-defined as $\min \mathcal{Q}h(Q)$, since $\mathcal{Q}h(Q)$ is compact saturated hence of the form $[a, +\infty]$ for some $a \in \overline{\mathbb{R}^+_\sigma}$—then $h_*(Q) = a$. Moreover, h_* is continuous from $\mathcal{Q}_\mathcal{V}(X)$ to $\overline{\mathbb{R}^+_\sigma}$, because $h_*^{-1}(a, +\infty] = \Box h^{-1}(a, +\infty]$. So $h_* \circ \varsigma$ is continuous, whence the integral defining q makes sense. We now claim that the map $h \mapsto h_*$ is (Scott-)continuous. First, $h \mapsto h_*$ is clearly monotonic. Now let $(h_i)_{i\in I}$ be a directed family in $[X \to \overline{\mathbb{R}^+_\sigma}]$ with a least upper bound h. By monotonicity, for every $Q \in \mathcal{Q}(X)$, $h_{i*}(Q) \leq h_*(Q)$, so $\sup_{i\in I} h_{i*}(Q)$ exists and is below $h_*(Q)$. Conversely, we must show that for every $a \in \mathbb{R}^+$ such that $a < h_*(Q)$, $a < \sup_{i\in I} h_{i*}(Q)$. The elements $Q \in \mathcal{Q}(X)$ such that $a < h_*(Q)$ are those such that for every $x \in Q$, there is an $i \in I$ such that $h_i(x) \in (a, +\infty)$, i.e., they are the elements of $\Box \bigcup_{i\in I} h_i^{-1}(a, +\infty)$. Since $\Box$ commutes with directed unions, if $a < h_*(Q)$ then $Q \in \Box h_i^{-1}(a, +\infty)$ for some $i \in I$, i.e., $h_{i*}(Q) > a$, and we are done. Since $h \mapsto h^*$ is continuous, and since the Choquet integral is Scott-continuous in the integrated function (see [Tix95, Satz 4.4], or [Hec96, Theorem 7.1 (3)]), we obtain that q is (Scott-)continuous.

For every $a \in \mathbb{R}^+$, $q(ah) = aq(h)$. Moreover, since $(h_1+h_2)_* \geq h_{1*}+h_{2*}$, and integration is linear, q is superlinear.

Define $p(h)$ as $\inf\left\{\int_{y\in Y} h'(y)d\nu \,\middle|\, h' \in [Y \to \overline{\mathbb{R}^+_\sigma}],\ h \leq h' \circ r\right\}$. Clearly, p is sublinear. Notably,

$$\begin{aligned} p(h_1) + p(h_2) &= \inf\Big\{ \int_{y\in Y} [h_1'(y) + h_2'(y)]\, d\nu \;\Big|\; h_1' \in [Y \to \overline{\mathbb{R}^+_\sigma}],\ h_1 \leq h_1' \circ r, \\ &\qquad\qquad h_2' \in [Y \to \overline{\mathbb{R}^+_\sigma}],\ h_2 \leq h_2' \circ r \Big\} \\ &\geq \inf\left\{ \int_{y\in Y} h'(y)\, d\nu \,\middle|\, h' \in [Y \to \overline{\mathbb{R}^+_\sigma}],\ h_1 + h_2 \leq h' \circ r \right\} = p(h_1 + h_2). \end{aligned}$$

Whenever $h \leq h' \circ r$, we claim that $h_*(\varsigma(y)) \leq h'(y)$. Indeed, since $y = r(x)$ for some $x \in X$ (Lemma 4.2), and since $x \in \varsigma(r(x)) = \varsigma(y)$, $h_*(\varsigma(y)) \leq h(x) \leq h'(r(x)) = h'(y)$.

It follows that $q(h) = \int_{y\in Y} h_*(\varsigma(y))d\nu \leq \int_{y\in Y} h'(y)d\nu$. By taking infs over h', $q \leq p$.

So Keimel's Sandwich Theorem applies. There is a continuous linear map $\Lambda : C \to \overline{\mathbb{R}^+_\sigma}$ such that $q \leq \Lambda \leq p$. Define $\nu_0 : \mathcal{O}(X) \to \overline{\mathbb{R}^+_\sigma}$ by $\nu_0(U) = \Lambda(\chi_U)$, where χ_U is the characteristic function of U. Then ν_0 is a continuous valuation on X; in particular, $\nu_0(U \cup V) + \nu_0(U \cap V) = \nu_0(U) + \nu_0(V)$ because $\chi_{U\cup V} + \chi_{U\cap V} = \chi_U + \chi_V$.

Now, given an open subset V of Y, take $h = \chi_{r^{-1}(V)}$. Then $h_*(Q) = 1$ iff $Q \subseteq r^{-1}(V)$, so $h_* = \chi_{\Box r^{-1}(V)}$, and therefore $h_*(\varsigma(y)) = \chi_{\Box r^{-1}(V)}(r^{-1}(\uparrow y)) = \chi_V(y)$, using the fact that r is surjective. It follows that $q(h) = \int_{y\in Y} \chi_V d\nu = \nu(V)$. On the other hand, take $h' = \chi_V$ in the definition of p, and check that $h \leq h' \circ r$. It follows that $p(h) \leq \int_{y\in Y} \chi_V(y)d\nu = \nu(V)$. Since $q(h) \leq \nu_0(r^{-1}(V)) \leq p(h)$, $\nu_0(r^{-1}(V)) = \nu(V)$. This holds for every open subset V of Y. In particular, taking $V = Y$, we obtain that ν_0 is a probability valuation: $\nu_0(X) = \nu_0(r^{-1}(Y)) = \nu(Y) = 1$. And finally, that $\nu_0(r^{-1}(V)) = \nu(V)$ holds for every open V of Y means that $\nu = \mathbf{V}_{1\,wk}(\nu_0)$. ∎

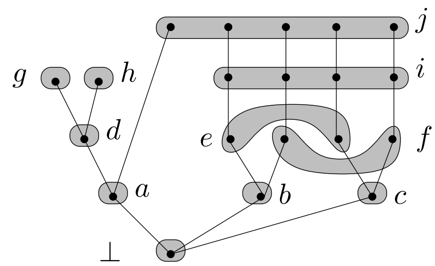


Figure 6: The path space of Figure 2 (i)

Putting together Proposition 6.3 and Proposition 6.4, we obtain:

Theorem 6.5 (Key Claim)**.** *Let X be a stably compact space, and Y be a T_0 space. If r is a proper surjective map from X to Y, then $\mathbf{V}_{1\ wk}(r)$ is a proper surjective map from $\mathbf{V}_{1\ wk}(X)$ to $\mathbf{V}_{1\ wk}(X)$.*

In particular, if Y is a quasi-projection of X, then $\mathbf{V}_{1\ wk}(Y)$ is a quasi-projection of $\mathbf{V}_{1\ wk}(X)$.

We shall apply this theorem twice, and first, to finite pointed posets. Let $<$ be the strict part of $\leq$.

Definition 6.6 (Path Space)**.** Let Y be any finite pointed poset. Write $y \to y'$ iff y is *immediately below* y', i.e., $y < y'$ and there is no $z \in Y$ such that $y < z < y'$. A *path* π in Y is any set $\{y_0, y_1, \ldots, y_n\} \subseteq Y$ with $y_0 = \bot \to y_1 \to \ldots \to y_n$. The *path space* $\Pi(Y)$ is the set of paths in Y, ordered by $\subseteq$.

Alternatively, the ordering on paths $y_0 \to y_1 \to \ldots \to y_n$ is the prefix ordering on sequences $y_0 y_1 \ldots y_n$.

Note that $\Pi(Y)$ is always a finite tree, i.e., a finite pointed poset such that the downward closure of a point is always totally ordered. Up to questions of finiteness, this is exactly how we built a tree from an ordering in the proof of Lemma 4.12, by the way.

We observe that every finite pointed poset Y is a quasi-projection of its path space $\Pi(Y)$.

Lemma 6.7. *For every finite pointed poset Y, the map $r : \Pi(Y) \to Y$ defined by $r(\pi) = \max \pi$ is proper and surjective.*

Proof. See Figure 6, which displays the path space of the space Y of Figure 2 (i). Each gray region is labeled with an element from Y, which is the image by r of every point in the region; e.g., the top right, 5-element region is mapped to j in Y.

Formally, let $X = \Pi(Y)$, and define $r : X \to Y$ by $r(\pi) = \max \pi$, i.e., $r(y_0 \to y_1 \to \ldots \to y_n) = y_n$. The map r is surjective, and monotonic. Since X and Y are finite, r is then trivially proper. □

Y is certainly not a retract of $\Pi(Y)$ in general: it is, if and only if Y is a tree. Indeed, if Y is a tree, then Y is isomorphic to $\Pi(Y)$, and conversely, every retract of a tree is a tree.

Finite trees are very special. Jung and Tix proved that $\mathbf{V}_{\leq 1}(T)$ is an **RB**-domain [JT98, Theorem 13] for every finite tree T. They noted (comment after op.cit.) that $\mathbf{V}_{\leq 1}(T)$ is even a bc-domain in this case, i.e., a pointed continuous dcpo in which every pair of elements with an upper bound has a least upper bound. It is well-known that every bc-domain is an **RB**-domain: given any finite subset A of a basis B of a bc-domain X, the map $f_A(x) = \sup(A \cap \Downarrow x)$ is a deflation, the family of these deflations is directed, and their least upper bound is the identity map.

Lemma 6.8. *For every finite tree T, $\mathbf{V}_1(T)$ is a countably-based bc-domain.*

Proof. Since T is a finite tree, it is trivially a continuous pointed dcpo, so $\mathbf{V}_1(T)$ is again continuous [Eda95, Section 3]. A basis is given by the valuations of the form $\sum_{t \in T} a_t \delta_t$ with $a_t \in [0, 1]$, $\sum_{t \in T} a_t = 1$ and each a_t rational. Since $\mathbf{V}_1(T)$ has a countable basis B, its topology has a countable base consisting of the subsets $\Uparrow b$, $b \in B$. So $\mathbf{V}_1(T)$ is countably-based.

To show that $\mathbf{V}_1(T)$ is a bc-domain, we observe that every probability valuation ν on T is entirely characterized by the values $\nu(\uparrow t)$, $t \in T$. Indeed, for every open subset U of T, let $\mathrm{Min}\, U$ be the (finite) set of minimal elements of U; the sets $\uparrow t$, $t \in \mathrm{Min}\, U$, are pairwise disjoint, so $\nu(U) = \sum_{t \in \mathrm{Min}\, U} \nu(\uparrow t)$. The map $f : T \to [0, 1]$ defined by $f(t) = \nu(\uparrow t)$ satisfies $f(\bot) = 1$ and $f(t) \geq \sum_{t' \in T, t \to t'} f(t')$ for every $t \in T$. Let us call such maps *admissible*. Given any admissible map f, there is a unique probability valuation ν such that $f(t) = \nu(\uparrow t)$ for every $t \in T$, namely $\sum_{t \in T} a_t \delta_t$ with $a_t = f(t) - \sum_{t' \in T, t \to t'} f(t')$. So $\mathbf{V}_1(T)$ is order-isomorphic to the poset of admissible maps, with the pointwise ordering. Therefore we only have to show that any two admissible maps f_1, f_2 below a third one f_0 have a least upper bound f. As a least upper bound, $f(t)$ must be above $f_1(t)$, $f_2(t)$, and $\sum_{t' \in T, t \to t'} f(t')$, so define $f(t)$ by descending induction on t by $f(t) = \max(f_1(t), f_2(t), \sum_{t' \in T, t \to t'} f(t'))$. (By descending induction, we mean induction on the largest length n of a sequence $t_0 \to t_1 \to \ldots \to t_n$ in T such that $t_0 = t$.) This is admissible if and only if $f(\bot) = 1$, and in this case will be the least upper bound of f_1, f_2. By definition $f(\bot) \geq 1$. It is easy to see that $f(t) \leq f_0(t)$ for every t, by descending induction on t: so $f(\bot) \leq f_0(\bot) = 1$, hence f is admissible. ☐

We retrieve the Jung-Tix result that $\mathbf{V}_{\leq 1}(T)$ is a bc-domain for every tree T: let $T_\bot$ be T with an extra bottom element added below all elements of T, and apply Lemma 6.8 to $\mathbf{V}_1(T_\bot) \cong \mathbf{V}_{\leq 1}(T)$.

Proposition 6.9. *For every finite pointed poset Y, $\mathbf{V}_1(Y)$ is a continuous ω**QRB**-domain.*

Proof. Y is trivially a continuous pointed dcpo. Then we know that $\mathbf{V}_1(Y)$ is again continuous [Eda95, Section 3], and that $\mathbf{V}_1(Y) = \mathbf{V}_{1\ wk}(Y)$ by the Kirch-Tix Theorem. Similarly for $\mathbf{V}_1(\Pi(Y))$. $\Pi(Y)$ is clearly stably compact, since finite. By Theorem 6.5, using Lemma 6.7, $\mathbf{V}_1(Y)$ is the image of $\mathbf{V}_1(\Pi(Y))$ under some proper surjective map. But $\Pi(Y)$ is a tree, so $\mathbf{V}_1(\Pi(Y))$ is a countably-based bc-domain by Lemma 6.8, hence a countably-based **RB**-domain, hence an ω**QRB**-domain, by Proposition 3.2 and Proposition 3.18. By Proposition 4.10, $\mathbf{V}_1(Y)$ must also be an ω**QRB**-domain. ☐

We can finally prove the main theorem of this paper.

Theorem 6.10. *The probabilistic powerdomain of any ω**QRB**-domain is an ω**QRB**-domain.*

Proof. Let Y be an ω**QRB**-domain. By Theorem 4.13, Y is the image of some ω**B**-domain $X = \lim_{i \in \mathbb{N}} X_i$ under some proper surjective map. Since $\mathbf{V}_1$ is a locally continuous functor on the category of dcpos, (as mentioned in proof of [JT98, Lemma 11]), $\mathbf{V}_1(X)$ is also a bilimit of the spaces $\mathbf{V}_1(X_i)$, $i \in I$. Each $\mathbf{V}_1(X_i)$ is a continuous ω**QRB**-domain by

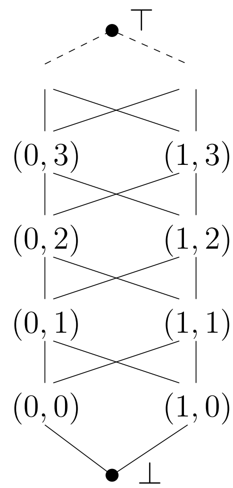


Figure 7: Plotkin's Domain T

Proposition 6.9, hence so is $\mathbf{V}_1(X)$, by Theorem 5.4 and since bilimits of continuous dcpos are continuous [AJ94, Theorem 3.3.11].

Since X is bifinite, it is stably compact, (use, e.g., Theorem 3.14), and $\mathbf{V}_1(X) = \mathbf{V}_{1\ wk}(X)$ because X is continuous and pointed, using the Kirch-Tix Theorem. So $\mathbf{V}_{1\ wk}(Y)$ is the image of $\mathbf{V}_1(X)$ under a proper surjective map, by Theorem 6.5. It is clear that $\mathbf{V}_{1\ wk}(Y)$ is T_0, so by Proposition 4.10 $\mathbf{V}_{1\ wk}(Y))$ is an ω**QRB**-domain in its specialization preorder $\preceq$, and its topology must be the Scott topology of $\preceq$.

But it is easy to see that $\preceq$ is the usual ordering on $\mathbf{V}_1(Y)$, i.e., $\nu \preceq \nu'$ iff $\nu(U) \leq \nu'(U)$ for every open U of Y: note that if $\nu \preceq \nu'$, then $\nu' \in [U > r]$ for every $r < \nu(U)$. So $\mathbf{V}_{1\ wk}(Y)) = \mathbf{V}_1(Y)$, and we conclude. ☐

Using the fact that $\mathbf{V}_1(X)$ is continuous whenever X is continuous and pointed [Eda95, Section 3], it also follows:

Corollary 6.11. *The probabilistic powerdomain of any continuous ω**QRB**-domain (in particular, every **RB**-domain) is again a continuous ω**QRB**-domain.*

7. Conclusion, Failures and Perspectives

We have shown that the category ω**QRB** of ω**QRB**-domains and continuous maps is a category of quasi-continuous, stably compact dcpos that is closed, not only under finite products, bilimits of expanding sequences, retracts (and even quasi-retracts), but also under the probabilistic powerdomain functor $\mathbf{V}_1$. It is thus reasonably well-behaved.

But ω**QRB** is *not* Cartesian-closed. Consider the space T of [AJ94, Figure 12], see Figure 7. This is an ω**QRB**-domain: define the quasi-deflations φ_i, $i \in \mathbb{N}$, as mapping $\bot$ to $\uparrow\{\bot\}$, any element (j, n) to $\uparrow\{(j, n)\}$ if $n < i$, and any other element to $\uparrow\{(0, i), (1, i)\}$.

However, $[T \to T]$ is not an ω**QRB**-domain.

Assume $(\varphi_i)_{i\in\mathbb{N}}$ were a generating sequence of quasi-deflations on $[T \to T]$. For each function $f : \mathbb{N} \to \{0, 1\}$, there is a continuous map $\hat{f} : T \to T$ that sends $\bot$ to $\bot$, $\top$ to

$\top$, $(0,n)$ to $(f(n),n)$ and $(1,n)$ to $(1-f(n),n)$ ($\hat{f}$ exchanges $(0,n)$ and $(1,n)$ if $f(n)=1$, leaves them unswapped if $f(n)=0$). Write $\varphi_i(\hat{f})$ as $\uparrow E_{i,f}$, where $E_{i,f}$ is finite.

We claim that: $(*)$ for each $f:\mathbb{N}\to\{0,1\}$, there is an index $i\in\mathbb{N}$ such that $\hat{f}\in E_{i,f}$. If there were an element g of $\varphi_i(\hat{f})$ such that $g(0,0)=\bot$, for infinitely many values of $i\in\mathbb{N}$, then this would hold for every i; but the map sending $\bot$ and $(0,0)$ to $\bot$, and all other elements to $\top$ would be in $\bigcap_{i\in\mathbb{N}}\varphi_i(\hat{f})=\uparrow\hat{f}$, which is impossible. So, for i large enough, no element g of $\varphi_i(\hat{f})$ maps $(0,0)$ to $\bot$. Similarly, for i large enough, no element g of $\varphi_i(\hat{f})$ maps $(1,0)$ to $\bot$. Since $\hat{f}\in\varphi_i(\hat{f})$, for i large enough we find $g\in E_{i,f}$ with $g(0,0)\neq\bot$, $g(1,0)\neq\bot$, and $g\leq\hat{f}$.

We check that $g=\hat{f}$. First, $g(\bot)\leq\hat{f}(\bot)=\bot$ so $g(\bot)=\hat{f}(\bot)$. Next, $g(0,0)$ is an element below $\hat{f}(0,0)=(f(0),0)$ and different from $\bot$, and the only element satisfying this is $\hat{f}(0,0)$. Similarly, $g(1,0)=\hat{f}(1,0)$. By induction on $n\in\mathbb{N}$, we show that $g(j,n)=\hat{f}(j,n)$. At rank $n+1$, $g(0,n+1)$ is an element below $\hat{f}(0,n+1)=(f(n+1),n+1)$ and above both $g(0,n)=\hat{f}(0,n)=(f(n),n)$ and $g(1,n)=\hat{f}(1,n)=(1-f(n),n)$. The only such element is $(f(n+1),n+1)=\hat{f}(0,n+1)$. Similarly, $g(1,n+1)=\hat{f}(1,n+1)$. Finally, $g(\top)$ is an element above all $g(j,n)$, hence must equal $\top=\hat{f}(\top)$.

Since $g\in E_{i,f}$, and $g=\hat{f}$, Claim $(*)$ is proved.

However, there are uncountably many functions of the form $\hat{f}$, and only countably many elements of $\bigcup_{\substack{i\in\mathbb{N}\\ f:\mathbb{N}\to\{0,1\}}} E_{i,f}$, since each set $E_{i,f}$ is finite, and for each $i\in\mathbb{N}$, there are only finitely many distinct sets $E_{i,f}$ with $f:\mathbb{N}\to\{0,1\}$. We have reached a contradiction.

Since exponentials in any full subcategory of the category of dcpos must be isomorphic to the ordinary continuous function space [Smy83], it follows:

Proposition 7.1. ω**QRB** *is not Cartesian-closed.*

The above argument also shows that, although T is both continuous (even algebraic) and an ω**QRB**-domain, T is not an **RB**-domain: so Corollary 6.11 is not enough to settle the Jung-Tix problem in the positive either.

One might hope that countability would be the problem. However, we required countability in at least two places. The first one is Lemma 4.12, which would fail in case we allowed for directed families $(E_i^0)_{i\in I}$ instead of non-decreasing sequences. The second one is in the $(i)\Rightarrow(iii)$ direction of Theorem 4.13, where we need countability to obtain Y as a quasi-projection, and not just a quasi-retract. (This is similar to an open problem in the theory of **RB**-domains, see [Jun88, Remark after Theorem 4.9].) In turn, we need quasi-projections, not just quasi-retracts, in the Key Claim, Theorem 6.5.

To get around the Jung-Tix problem using our results, one might shift the focus towards the Kleisli category $\omega\mathbf{QRB}_{\mathcal{Q}}$, for example. This is a full subcategory of Jung, Kegelmann and Moshier's pleasing category $\boldsymbol{SCS}^*$ of stably compact spaces and closed relations [JKM01].

On the other hand, we would like to point out the following deep connection between **QRB**-domains and **FS**-domains.

Definition 7.2. A *controlled quasi-deflation* on a poset X is a pair of a Scott-continuous map $f:X\to X$ and of a quasi-deflation $\varphi:X\to\mathrm{Fin}(X)$ such that $\varphi(x)\subseteq\uparrow f(x)$ for every $x\in X$. The map f is the *control*.

A *controlled* **QRB**-*domain* is a pointed dcpo X with a *generating family of controlled quasi-deflations*, i.e., a directed family of controlled quasi-deflations (in the pointwise ordering) $(f_i, \varphi_i)_{i\in I}$ such that $x = \sup_{i\in I} f_i(x)$ for every $x \in X$.

So a controlled quasi-deflation is a pair (f, φ) with the property that $\uparrow f(x) \supseteq \varphi(x) \supseteq \uparrow x$ for every $x \in X$. Every controlled **QRB**-domain is a **QRB**-domain: given a generating family of controlled quasi-deflations $(f_i, \varphi_i)_{i\in I}$, $\bigcap^{\downarrow}_{i\in I} \varphi_i(x) \subseteq \bigcap^{\downarrow}_{i\in I} \uparrow f_i(x) = \uparrow \sup_{i\in I} f_i(x) = \uparrow x$, and the converse inclusion $\uparrow x \subseteq \bigcap^{\downarrow}_{i\in I} \varphi_i(x)$ is obvious; so $(\varphi_i)_{i\in I}$ is a generating family of quasi-deflations.

Theorem 7.3. *The controlled* **QRB**-*domains are exactly the* **FS**-*domains, and hence form a Cartesian-closed category.*

Proof. If $(f_i, \varphi_i)_{i\in I}$ is a generating family of controlled quasi-deflations on a pointed dcpo X, then f_i is finitely separated from id_X: indeed, let $\mathrm{im}\,\varphi_i = \{E_{i1}, \ldots, E_{in_i}\}$ and $M_i = \bigcup_{j=1}^{n_i} E_{ij}$, then for every $x \in X$, since $x \in \varphi_i(x)$, there is a point $m \in E_{ij}$ where $\uparrow E_{ij} = \varphi_i(x)$ such that $m \leq x$, and since $\uparrow E_{ij} \subseteq \uparrow f_i(x)$, we have $f_i(x) \leq m$; so M_i is a finitely separating set for f_i on X.

Conversely, assume X is an **FS**-domain, and let $(g_i)_{i\in I}$ be a directed family of continuous maps, finitely separated from id_X, and whose pointwise least upper bound is id_X. Let $f_i = g_i \circ g_i$. By [Jun90, Lemma 2], f_i is *strongly finitely separated* from id_X, i.e., there is a finite set E_i of pairs of elements $m \ll m'$ such that for every $x \in X$, one can find a pair $(m, m') \in E_i$ such that $f_i(x) \leq m \ll m' \leq x$. Moreover, the pointwise least upper bound of $(f_i)_{i\in I}$ is again id_X.

For each pair $(m, m') \in E_i$ with $m \ll m'$, $\uparrow m' \subseteq \twoheaduparrow m \subseteq \bigcup_{j\in I} f_j^{-1}(\twoheaduparrow m)$. Indeed, for every $x \in \twoheaduparrow m$, since $x = \sup_{j\in I} f_j(x)$ and $\twoheaduparrow m$ is Scott-open, $f_j(x) \in \twoheaduparrow m$ for some $j \in I$. Since $\uparrow m'$ is compact, and the family $\left(f_j^{-1}(\twoheaduparrow m)\right)_{j\in I}$ is directed, $\uparrow m' \subseteq f_j^{-1}(\twoheaduparrow m)$ for some $j \in I$. By directedness again, we can take the same j for all pairs $m \ll m'$ in E_i. But now $\uparrow m' \subseteq f_j^{-1}(\twoheaduparrow m)$ implies that whenever $m' \leq x$, then $m \ll f_j(x)$. Using the separation property of E_i, for every $x \in X$, one can find a pair $(m, m') \in E_i$ such that $f_i(x) \leq m \ll f_j(x)$. In particular, letting M_i be the set of elements m such that $(m, m') \in E_i$ for some $m' \in X$, f_i is *finitely separated from* f_j, with separating set M_i, with the obvious meaning: for every $x \in X$, there is an $m \in M_i$ such that $f_i(x) \leq m \leq f_j(x)$. In this case, we write $f_i \prec_{M_i} f_j$.

We now define $\varphi_i(x)$ as $\uparrow(M_i \cap \uparrow f_i(x))$. Since M_i is finite, φ_i is a map from X to $\mathrm{Fin}(X)$. It is monotonic, and we claim it is Scott-continuous. Let $(x_k)_{k\in K}$ be a directed family in X. Then $\varphi_i(\sup_{k\in K} x_k) = \uparrow(M_i \cap \uparrow \sup_{k\in K} f_i(x_k))$ (since f_i is continuous) $= \uparrow(M_i \cap \bigcap_{k\in K} \uparrow f_i(x_k)) = \uparrow \bigcap_{k\in K}(M_i \cap \uparrow f_i(x_k))$. The latter intersection is an intersection of finite sets, hence there is a $k \in K$ such that $\varphi_i(\sup_{k\in K} x_k) = \varphi_i(x_k)$, from which Scott-continuity is immediate.

We must now check that $(\varphi_i)_{i\in I}$ is directed. Given $i, i' \in I$, one can find $j, j' \in I$ so that $f_i \prec_{M_i} f_j$ and $f_{i'} \prec_{M_{i'}} f_{j'}$. By directedness, there is an $\ell \in I$ such that $f_j, f_{j'} \leq f_\ell$. We claim that for every $x \in X$, $\varphi_i(x), \varphi_{i'}(x) \supseteq \varphi_\ell(x)$. For every $y \in \varphi_\ell(x)$, $f_\ell(x) \leq y$. So $f_j(x) \leq y$. Since $f_i \prec_{M_i} f_j$, there is an element $m \in M_i$ such that $f_i(x) \leq m \leq f_j(x) \leq y$. So m is in $M_i \cap \uparrow f_i(x)$, and below y, whence $y \in \varphi_i(x)$. Similarly, y is in $\varphi_{i'}(x)$.

Finally, $x \in \varphi_i(x)$ for every $x \in X$: by definition, there is a pair $(m, m') \in E_i$ such that $f_i(x) \leq m \ll m' \leq x$; so $m \in M_i$, $m \in \uparrow f_i(x)$, and m is below x. And $\bigcap^{\downarrow}_{i\in I} \varphi_i(x) \subseteq$

$\bigcap^{\downarrow}_{i \in I} \uparrow f_i(x) = \uparrow x$, while the converse inclusion is obvious. So $(\varphi_i)_{i \in I}$ is a generating family of quasi-deflations. □

Defining the *controlled* ω**QRB**-*domains* as the ω**QRB**-domains, except with sequences of controlled quasi-deflations instead of directed families, and similarly for the ω**FS**-domains (a.k.a., the countably-based **FS**-domains, again a Cartesian-closed category [Jun90, Theorem 11]), we prove similarly:

Theorem 7.4. *The controlled* ω**QRB**-*domains are exactly the* ω**FS**-*domains, and hence form a Cartesian-closed category.*

Using this last observation, Corollary 6.11 settles half of the conjecture that the probabilistic powerdomain of an ω**FS**-domain would be an ω**FS**-domain again. We are only lacking *control*.

Open Problems

(1) Is countability necessary in Theorem 4.13? Precisely, can one show that the **QRB**-domains are exactly the quasi-retracts of **B**-domains? The main difficulty seems to lie in the fact that a non-countable analog of Lemma 4.12 is missing—and Rudin's Lemma does not quite give us what we need, as discussed before the statement of the lemma.
(2) If Y is a quasi-retract of X, X is stably compact, and Y is T_0, then is $\mathbf{V}_{1\ wk}(Y)$ a quasi-retract of $\mathbf{V}_{1\ wk}(X)$? This would be the analog of Theorem 6.5, only with quasi-retractions instead of quasi-projections.
(3) Is stable compactness necessary to derive Theorem 6.5?
(4) One way of trying to prove that the probabilistic powerdomain of an ω**FS**-domain is again an ω**FS**-domain would be by inventing a new notion, say of *good maps*, and show that the ω**FS**-domains, or alternatively the controlled ω**QRB**-domains, are exactly the images under good maps of ω**B**-domains. Good maps should intuitively be intermediate between projections and proper surjective maps, in the sense that every projection should be good, and that every good map should be proper and surjective. Indeed surjective proper maps preserve the **QRB** part, but not the control, while projections preserve too much, in the sense that not all ω**QRB**-domains, only the ω**RB**-domains, are retracts of ω**B**-domains. Such a characterization of ω**FS**-domains would be of independent interest, too.

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