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Clusters, Confusion and Unfoldings

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Abstract. We study *independence* of events in the unfoldings of Petri nets, in particular indirect influences between concurrent events: *Confusion* in the sense of Smith [11] and *weak interference*. The role of structural *(conflict) clusters* is investigated, and a new unfolding semantics based on clusters motivated and introduced.

Keywords: Concurrency, occurrence nets, unfoldings, confusion, conflict clusters

1. Introduction

A central issue in the semantics of concurrent systems is the modeling of *independence* of events; it is the exploitation of independence that allows *stubborn set* or *unfolding* methods etc. to reduce the state space. In partial order semantics, the *non-ordering* or *concurrency* relation is a natural candidate for independence; yet situations described by Petri and others (see [10, 11]) as *confusion* show that events concurrent with an event e may nonetheless influence the occurrence of e. This article studies independence in the context of occurrence nets, which are the semantic domain of Petri net *unfoldings*. An unfolding- based definition of confusion, reflecting that of Smith [11], is given. We introduce and study *weak interference* and concurrency relations based on *cluster* subnets; the study of clusters also leads to a modified unfolding semantics whose events are given by the actions (steps) of clusters.
The two examples in Figure 1 illustrate contexts in which confusion arises: Transitions A and C in Net I are concurrent, yet the order of firing matters. In fact, if C fires first, B is never enabled, and C never had to face a conflict; firing A first creates (or reveals) a conflict between B and C, which C may lose.

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In Net II, Transition B is clearly in conflict with both A and C; and just as clearly, A is *concurrent* with C, i. e. if both fire, they may fire in any ordering. Yet if either of them fires first, the situation for the other is no longer the same, for B is now disabled and there is no conflict anymore.

This kind of situation, although intuitively comprehensible, is not captured by temporal logics, even those adapted to partial order semantics such as ISTL or the logics introduced in [6]; the indirect influence consists in changes of necessities and possibilities not expressible by modal operators. We will provide *relational* descriptions and investigate manifestations of these weak causalities in the framework of *occurrence nets* semantics for systems modeled by Petri nets.

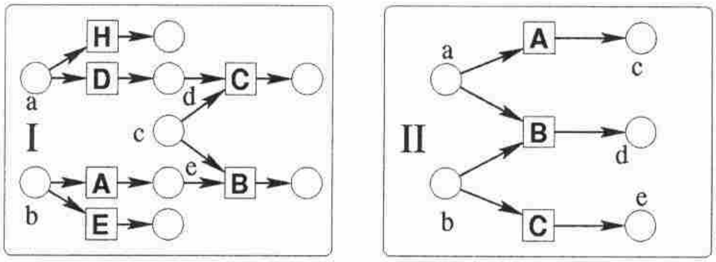

Figure 1. Confusions in Petri nets

2. Notations and Definitions

By $\mathbb{N}_0$, we denote the set of *non-negative* integers and by $\mathbb{N}$ that of the *positive* integers; $\mathbb{Z}$ is the set of (all) integers. For a set $\mathcal{E}$, we denote the power set of $\mathcal{E}$ as $\mathfrak{P}(\mathcal{E})$. For $x \in \mathcal{E}$ and a binary relation $\Re \subseteq \mathcal{E} \times \mathcal{E}$, let $\Re^{-1} := \{(y,x) \mid (x,y) \in \Re\}$ and $\Re[x] := \{y \in \mathcal{X} \mid (x,y) \in \Re\}$; by extension, if $\mu \subseteq \mathcal{E}$, write $\Re[\mu]$ for the union of all $\Re[x]$ for $x \in \mu$.

A **Petri net** (with *arc weights*) is a tuple of the form $\mathcal{N} = (\mathcal{P}, \mathcal{T}, \mathcal{W}, M_0)$. Here, $\mathcal{P} = \mathcal{P}(\mathcal{N})$ is a set of places and $\mathcal{T} = \mathcal{T}(\mathcal{N})$ a set of transitions such that $\mathcal{P} \cap \mathcal{T} = \emptyset$; we set $\mathcal{X}(\mathcal{N}) := \mathcal{P}(\mathcal{N}) \cup \mathcal{T}(\mathcal{N})$. Further,

$$\mathcal{W} : ((\mathcal{P} \times \mathcal{T}) \cup (\mathcal{T} \times \mathcal{P})) \rightarrow \mathbb{N}_0$$

is the *arc weight* function. We write $\mathcal{W}_p$ for the vector

$$(\mathcal{W}(q,p))_{q \in \mathcal{T}} \in \mathbb{N}_0^{|\mathcal{T}|}$$

of the *input* weights and $\mathcal{W}_p^\top$ for the vector

$$(\mathcal{W}(p,q))_{q \in \mathcal{T}} \in \mathbb{N}_0^{|\mathcal{T}|}$$

of the *output* weights of $p \in \mathcal{P}$, respectively. The inner product of $\mathbb{Z}^{|\mathcal{T}|}$ is $\langle \cdot, \cdot \rangle : \mathbb{Z}^{|\mathcal{T}|} \times \mathbb{Z}^{|\mathcal{T}|} \rightarrow \mathbb{Z}$. A **marking** of $\mathcal{N}$ is a multiset of places; $M_0 : \mathcal{P} \rightarrow \mathbb{N}_0$ is the **initial marking** of $\mathcal{N}$.

The set F of *arcs* of $\mathcal{N}$ is given by $\mathsf{F} := \mathcal{W}^{-1}(\mathbb{N})$. A net $\mathcal{N} = (\mathcal{P}, \mathcal{T}, \mathcal{W}, M_0)$ such that $\mathcal{W}$ takes 0 and 1 as its only values, is called **ordinary**; we will note ordinary nets as $\mathcal{N} = (\mathcal{P}, \mathcal{T}, \mathsf{F}, M_0)$. For every net $\mathcal{N} = (\mathcal{P}, \mathcal{T}, \mathcal{W}, M_0)$, one obtains an ordinary net $\mathcal{N} = (\mathcal{P}, \mathcal{T}, \mathsf{F}, M_0)$ by taking F as above and "forgetting" $\mathcal{W}$.

Transitions may fire one by one or in multi-sets; any transition multiset $\theta : \mathcal{T} \to \mathbb{N}_0$ is called a **step**. A step θ is **enabled** in a marking M, denoted $M \xrightarrow{\theta}$, iff, for all $p \in \mathcal{P}$, M has enough tokens on p to satisfy the sum of demands from θ concerning p:

$$M(p) \;\geq\; \langle \mathcal{W}_p^\top, \theta \rangle. \tag{1}$$

θ **transforms** M into M' , denoted $M \xrightarrow{\theta} M'$, iff $M \xrightarrow{\theta}$ and for all $p \in \mathcal{P}$

$$M'(p) \;=\; \left[M(p) - \langle \mathcal{W}_p^\top, \theta \rangle \right] + \langle \mathcal{W}_p, \theta \rangle. \tag{2}$$

θ is **maximal** for M if $M \xrightarrow{\theta}$ (or $M \xrightarrow{\theta} M'$) and θ satisfies **Condition CMAX**:

$$(\mathbf{CMAX}) \qquad \forall\, \overline{q} \in \mathcal{T}\, \exists\, p \in \mathcal{P}: \qquad M(p) < \langle \mathcal{W}_p^\top, \theta \rangle + \mathcal{W}(p, \overline{q});$$

denote this as $M_0 \Longrightarrow$ (or $M_0 \overset{\theta}{\Longrightarrow} M'$). In fact, Condition $(\mathbf{CMAX})$ holds iff no further firing instance of any transition $\overline{q}$ can be added to θ without rendering (1) false for some p.

A marking M is **reachable** from M_0, denoted $M_0 \xrightarrow{*} M$, iff $M = M_0$ or there exists a **firing sequence**

$$M_0 \xrightarrow{\theta_1} M_1 \xrightarrow{\theta_2} \ldots \xrightarrow{\theta_n} M_{n+1} = M \quad .$$

If, in addition, $M_{i-1} \overset{\theta_i}{\Longrightarrow} M_i$ for all $i \in \{1, \ldots, n\}$, then we say that M is reachable from M_0 under the **maximal step rule**, denoted $M_0 \overset{*}{\Longrightarrow} M$. Write $M_0 \overset{+}{\Longrightarrow} M$ ($M_0 \xrightarrow{+} M$) iff *(i)* $M_0 \neq M$, and *(ii)* $M_0 \overset{*}{\Longrightarrow} M$ (or $M_0 \xrightarrow{*} M$, respectively).

Occurrence nets are the semantic domain for *branching unfoldings* of Petri Nets. Set $< := \mathsf{F}^+$ and $\leq = \mathsf{F}^*$; the **conflict** relations ic and $\#$ are given by:

1. For $q_1, q_2 \in \mathcal{T}$, $q_1 \;\mathsf{ic}\; q_2$ iff $q_1 \neq q_2$ and $\mathsf{F}^{-1}[q_1] \cap \mathsf{F}^{-1}[q_2] \neq \emptyset$.
2. For $x, y \in \mathcal{X}$, $x \# y$ iff there exist $q_1, q_2 \in \mathcal{T}$ such that $q_1 \;\mathsf{ic}\; q_2$, $q_1 \leq x$, and $q_2 \leq y$.

For $\mathcal{U} \subseteq \mathcal{X}$, we write $\max_<$ ($\min_<$) for the set of maximal (minimal) elements of $\mathcal{U}$ w.r.t. $<$, respectively. Now, we are ready to define:

1. An ordinary net $\overline{\mathcal{N}} = (\overline{\mathcal{P}}, \overline{\mathcal{T}}, \overline{\mathsf{F}}, M_0)$ is called a **pre-occurrence net** iff:
   (a) **no backward branching**: $|\mathsf{F}^{-1}[p]| \leq 1$ for all $p \in \mathcal{P}$;
   (b) **Acyclicity**: $\neg(x < x)$ for all $x \in \mathcal{X}(\mathcal{N})$; and
   (c) **absence of auto-conflict**: $\neg(x \# x)$ for all $x \in \mathcal{X}(\mathcal{N})$;
   (d) With $\mathfrak{c}_0 = min_<$, the initial marking reduces to the set $\mathfrak{c}_0$, i.e. $M_0 = \mathbb{1}_{\mathfrak{c}_0}$.
2. A **causal net** (or CN) is a pre-occurrence net such that $|\mathsf{F}[p]| \leq 1$ for all $p \in \overline{\mathcal{P}}$.
3. An **occurrence net** (ON) is a pre-occurrence net that, in addition, is

(a) **prefix - finite** or **well-founded**: $<$ contains no infinite decreasing sequence, and

(b) **place-initialized**: $\mathfrak{c}_0 \subseteq \overline{\mathcal{P}}$.

Since we will only study *un*marked occurrence nets, we will henceforth omit the marking and describe occurrence nets as, e.g., $\overline{\mathcal{N}} = (\overline{\mathcal{P}}, \overline{\mathcal{T}}, \overline{\mathsf{F}}, \mathfrak{c}_0)$. Set $\mathsf{id} := \{(x,x) : x \in \mathcal{P} \cup \mathcal{T}\}$, $\mathsf{li} := < \cup <^{-1}$ and $\mathsf{co} := (\mathcal{P} \cup \mathcal{T})^2 - (\mathsf{li} \cup \# \cup \mathsf{id})$. For a binary relation R, denote as **kens** of R its maximal cliques, and the set of R-kens as $\mathfrak{K}(R)$. The elements of $\mathfrak{C}(\overline{\mathcal{N}}) := \mathfrak{K}(\mathsf{co})$ are called **cuts**, and those of the set $\mathfrak{C}^{\mathcal{P}}(\overline{\mathcal{N}}) := \mathfrak{C}(\overline{\mathcal{N}}) \cap \mathfrak{P}(\mathcal{P})$ are **place-cuts**. In particular, $\mathfrak{c}_0$ is a place-cut. It is easily verified that $<$ is a partial order, and that li, co, ic and $\#$ are symmetric and irreflexive. Moreover, $(\mathcal{P} \cup \mathcal{T}) \times (\mathcal{T} \cup \mathcal{P})$ is the disjoint union of id, li, co, and $\#$; finally, a pre-occurrence net $\overline{\mathcal{N}}$ is a CN iff $\# = \emptyset$.

Unfolding semantics reflects both concurrent and branching behavior of a general marked Petri net in the structure of an occurrence net. There are different rules for these *unfoldings*; Figure 6 shows two of them (cf. [3, 6, 7]) together with a third one that will be introduced below. In any case, unfoldings are occurrence nets generated inductively by a net system, reflecting the initial marking by the initial cut and representing subsequent firings of transitions by events and subsequent place markings by conditions. We will discuss below the motivation for introducing a new semantics; first, however, a closer look at the structural relations of occurrence nets and their interpretation.

3. Concurrent Runs and Weak Interference

Let $\overline{\mathcal{N}}$ be an occurrence net. With co and li as above, let $\bowtie := (\mathsf{co} \cup \mathsf{li})$ and $\Omega(\mathcal{N}) := \mathfrak{K}(\bowtie)$. Then the elements of $\Omega(\mathcal{N})$ are called the **runs** of $\overline{\mathcal{N}}$. In [6] (where runs were called *branches*), the following was shown (Lemma 2.5 there):

Lemma 3.1. If $\overline{\mathcal{N}} = (\overline{\mathcal{P}}, \overline{\mathcal{T}}, \overline{\mathsf{F}}, \mathfrak{c}_0)$ is an ON, the runs of $\overline{\mathcal{N}}$ cover $\overline{\mathcal{N}}$. Moreover, every run ω spans a causal subnet of $\overline{\mathcal{N}}$ that contains $\mathfrak{c}_0$.

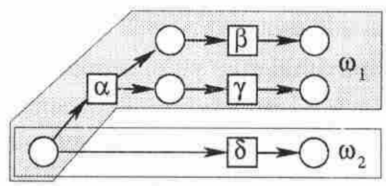

Figure 2. On occurrence sets

If $\overline{\mathcal{N}}$ is the unfolding of a net $\mathcal{N}$, then its runs represent maximal processes of $\mathcal{N}$ ([3, 4]).

We will study here *interference* between concurrent events guided by the following idea: for every node x and run ω in an occurrence net, ω is either *compatible* with x, that is $\mathsf{x} \in \omega$, or *excludes* x. For any set $\mu \subseteq \mathcal{P} \cup \mathcal{T}$, denote

$$\mathfrak{A}_\mu \;:=\; \{\omega \in \Omega(\mathcal{N}) \mid \mu \subseteq \omega\}, \tag{3}$$

where parentheses of singletons are omitted, i.e. we write $\mathfrak{A}_x$ for $\mathfrak{A}_{\{x\}}$. $\mathfrak{A}_\mu$ is called the **occurrence set** of μ. Denote the complements of node sets μ and occurrence sets $\mathfrak{A}$ as $\mu^c := (\mathcal{P} \cup \mathcal{T}) - \mu$ and $\tilde{\mathfrak{A}} := \Omega(\mathcal{N}) - \mathfrak{A}$, respectively. Applying Definition (3), on easily obtains the following fundamental properties:

Lemma 3.2. In the above setting, let $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq (\mathcal{P} \cup \mathcal{T})$ and $x, y \in (\mathcal{P} \cup \mathcal{T})$.

1. $\mathfrak{A}_{\mathfrak{c}_0} = \Omega(\mathcal{N})$ and $\Omega(\mathcal{N}) - \mathfrak{A}_\mathcal{A} = \mathfrak{A}_{\#[\mathcal{A}]}$,

2. $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathfrak{A}_{\mathcal{B}} \subseteq \mathfrak{A}_{\mathcal{A}}$,

3. $\mathfrak{A}_{\mathcal{A}\cap\mathcal{B}} = \mathfrak{A}_{\mathcal{A}} \cup \mathfrak{A}_{\mathcal{B}}$ and $\mathfrak{A}_{\mathcal{A}\cup\mathcal{B}} = \mathfrak{A}_{\mathcal{A}} \cap \mathfrak{A}_{\mathcal{B}}$,

4. $x \leq y \Rightarrow \mathfrak{A}_y \subseteq \mathfrak{A}_x$ and $x \# y \Rightarrow \mathfrak{A}_x \cap \mathfrak{A}_y = \emptyset$;

5. If $\mu \subseteq (\mathcal{P} \cup \mathcal{T})$ contains no #-pair, then $\mathfrak{A}_\mu \neq \emptyset$;

6. for all nodes x, y,

$$\mathfrak{A}_x - \mathfrak{A}_y = \bigcup_{u \,\#\, y,\ \neg z \,\#\, x} \mathfrak{A}_u.$$

If $\mathfrak{A}_y \subseteq \mathfrak{A}_x$, then the occurrence of x entails that of y, not neccessarily *after* the occurrence of x: in Figure 2, $\alpha < \beta$ and $\mathfrak{A}_\alpha = \mathfrak{A}_\beta$. Moreover, *"entails or is entailed by"* is a strictly bigger relation than causality (li) here: one has β co γ and $\mathfrak{A}_\beta = \mathfrak{A}_\gamma$. In Net I of Figure 1, $\mathfrak{A}_\beta \subseteq \mathfrak{A}_a$ and $\mathfrak{A}_\beta \neq \mathfrak{A}_a$, whereas $\mathfrak{A}_\alpha \not\subseteq \mathfrak{A}_\gamma$ and $\mathfrak{A}_\gamma \not\subseteq \mathfrak{A}_\alpha$. Let us dwell on *indirect influence* such as exerted by A on C. We express this influence by conditional relations given some $\mathcal{X} \subset (\mathcal{P} \cup \mathcal{T})$ (or, more precisely, the occurrence set $\mathfrak{A}_\mathcal{X}$). Let $\mathcal{U}, \mathcal{V}, \mathcal{X} \subset (\mathcal{P} \cup \mathcal{T})$. We write

$$(\mathcal{U} \rightsquigarrow_\mathcal{X} \mathcal{V}) \quad \text{iff} \quad \mathfrak{A}_{\mathcal{X}\cup\mathcal{U}} \neq \emptyset \text{ and } \mathfrak{A}_{\mathcal{X}\cup\mathcal{U}} \cap \overline{\mathfrak{A}}_\mathcal{V} = \emptyset,$$

read: "$\mathcal{U}$ **forces** $\mathcal{V}$ given $\mathcal{X}$". The complement $\tilde{\mathcal{U}}$ is interpreted by $\overline{\mathfrak{A}}_\mathcal{U}$, i.e.

$$\mathcal{U} \rightsquigarrow_\mathcal{X} \tilde{\mathcal{V}} \text{ iff } \mathfrak{A}_{\mathcal{X}\cup\mathcal{U}} \cap \mathfrak{A}_\mathcal{V} = \emptyset \quad \text{and} \quad \tilde{\mathcal{U}} \rightsquigarrow_\mathcal{X} \mathcal{V} \text{ iff } \mathfrak{A}_\mathcal{X} \cap \overline{\mathfrak{A}}_\mathcal{U} \cap \overline{\mathfrak{A}}_\mathcal{V} = \emptyset;$$

note that $\tilde{\mathcal{U}} \rightsquigarrow_\mathcal{X} \mathcal{V}$ is not equivalent to $\neg(\mathcal{U} \rightsquigarrow_\mathcal{X} \mathcal{V})$.

In Figure 2, $\beta \rightsquigarrow_\mathfrak{A} \gamma$ iff $\mathfrak{A}$ contains ω_1. Part I of Figure 1 shows a more subtle influence:

$$A \rightsquigarrow_H B \quad but \quad \neg\left(A \rightsquigarrow_{\{a,b\}}\right).$$

The following properties of $\rightsquigarrow_\bullet$ are easily verified:

Lemma 3.3. Let $\mathcal{U}_i, \mathcal{X} \subset (\mathcal{P} \cup \mathcal{T})$, $i \in \mathbb{N}$. Then:

1. $\rightsquigarrow_{\mathsf{c}_0} = \rightsquigarrow_\emptyset$;

2. If $\mathfrak{A}_{\mathcal{X}\cup\mathcal{U}_1} = \emptyset$, then $\rightsquigarrow_\mathcal{X}[\mathcal{U}_1] = \emptyset$;

3. $\rightsquigarrow_\mathcal{X}$ is a partial order on $\mathfrak{P}(\mathcal{P} \cup \mathcal{T})$.

We say that $\mathcal{X}$ **interferes (weakly) with** $\mathcal{U}$ iff there exists $\mathcal{A} \subset (\mathcal{P} \cup \mathcal{T})$ such that

$$\mathcal{U} \rightsquigarrow_\mathcal{X} \mathcal{A} \quad and \quad \neg(\mathcal{U} \rightsquigarrow_{\mathsf{c}_0} \mathcal{A}).$$

In the case of singletons, x trivially interferes with y if $x \# y$; interference may (but need not) occur if x and y are causally ordered. Since these relations are intended to express causal influence, there is nothing surprising in that. However, weak interference is possible in concurrency pairs: in Net I of Figure 1, E interferes weakly with D since $D \rightsquigarrow_E C$ but $\neg(D \rightsquigarrow_{\mathsf{c}_0} C)$, and yet obviously E co D. – Weak interference can thus be viewed as a generalization of causality as given by the partial ordering. As it is a property involving the space of all runs, it is not local; structural criteria for the detection of weak interference are yet to be found.

4. Confusion and Clusters

We now look at another form of weak dependency between nodes, *confusion.* A confusion is a situation in which an event e_1 is in conflict with another event e_2 such that this conflict is *influenced* by some e_3 considered independent of e_1. As the examples of Figure 1 show, such influence can be exerted either by solving the conflict via another conflict (Example *II*), or revealing the hidden conflict (Example *I*); in terms of structural relations, this leads to either $\#$- or li-connections. We propose the following general definition of confusion in terms of occurrence net structures (compare Smith's [11] definition for elementary net systems):

Definition 4.1. Let in, $\mathsf{de} \subseteq (\mathcal{P} \cup \mathcal{T}) \times (\mathcal{P} \cup \mathcal{T})$ be binary relations on $\mathcal{T}$. Then the ternary **confusion** relation $\mathsf{cf}^{\mathsf{in}}_{\mathsf{de}}$ is given by

$$\mathsf{cf}^{\mathsf{in}}_{\mathsf{de}} \;:=\; \left\{(A,B,C) \in \mathcal{T}^3 : (A \;\mathsf{de}\; B) \;\wedge\; (B \;\mathsf{ic}\; C) \;\wedge\; (A \;\mathsf{in}\; C)\right\}.$$

Here, "ic" is as above, and "in" and "de" are interpreted as *independence* and *dependence* relations, respectively; in particular, "in" can be taken as co and "de" as li. Extending the terminology of Smith [11], we will speak of **symmetric** in-**confusion** for the situations (i.e. triples) in $\mathsf{cf}^{\mathsf{in}}_{\#}$ and of **asymmetric** in-**confusion** for those in $\mathsf{cf}^{\mathsf{in}}_{\mathsf{li}}$. Setting $\mathsf{in} = \mathsf{co}$, we obtain the two cases of Figure 1 as $\mathsf{cf}^{\mathsf{co}}_{\mathsf{li}}$ for *I* and $\mathsf{cf}^{\mathsf{co}}_{\#}$ for *II*, respectively. Below, we will see that for a suitably "strong" choice of in, $\mathsf{cf}^{\mathsf{in}} := \mathsf{cf}^{\mathsf{in}}_{\#} \cup \mathsf{cf}^{\mathsf{in}}_{\mathsf{li}}$ is empty. – First, we note that asymmetric confusion is always of the "forward" type as shown in Figure 1:

Lemma 4.2. $\mathsf{cf}^{\mathsf{co}}_{\mathsf{li}} = \mathsf{cf}^{\mathsf{co}}_{<}$.

Proof: Let $B < A$ and $B \;\mathsf{ic}\; C$; then $A \# C$, which contradicts $A \;\mathsf{co}\; C$. □

So confusion is a local conflict whose resolution depends on the logical or temporal behavior of a concurrent process.

We note that there is a connection between interference and confusion: for simplicity, take three nodes x, y, z, where x interferes with y in such a way that $y \rightsquigarrow_x z$ but $\neg(y \rightsquigarrow_{\emptyset} z)$. Then this entails that the net contains either net *II* of Figure 1, i.e. *symmetric* confusion, or the subnet of net *I* of Figure 1 spanned by $\{b, c, d, e, A, B, C\}$, that is, *asymmetric* confusion, with x, y, z possibly but not necessarily in the roles of A, B, C, respectively. On the other hand, not all confusions entail interference: take the net obtained from *I* by eliminating E and G. Then one has $\mathsf{cf}^{\mathsf{co}}_{<}(A, B, C)$; still, there is no interference since $\mathfrak{A}_A = \mathfrak{A}_{c_0}$.

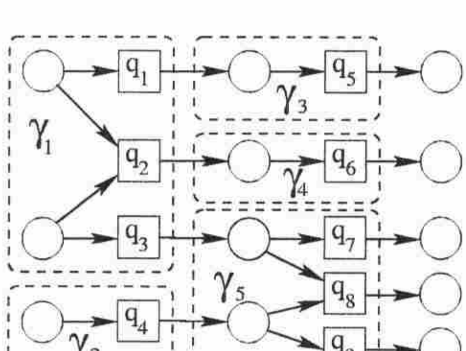

Figure 3. On strong concurrency

So, weak interference and confusion are incomparable relations, although both occur together "often". We remark here that, in a broader perspective, the notion of interference is a degenerate case of *stochastic* non-independence for some suitable probability measure; investigating this trace here would lead too far afield.

We will next see that a more restrictive *strong concurrency* relation excludes confusion.

Definition 4.3. Let $x \in (\mathcal{P} \cup \mathcal{T})$. The **(conflict) cluster**[1] $\gamma(x)$ of x is the smallest set satisfying

1. $x \in \gamma(x)$;
2. $\forall q \in \mathcal{T}$: $\mathsf{F}^{-1}[q] \cap \gamma(x) \neq \emptyset \Rightarrow q \in \gamma(x)$
3. $\forall p \in \mathcal{P}$: $\mathsf{F}[p] \cap \gamma(x) \neq \emptyset \Rightarrow p \in \gamma(x)$

A **cluster** is a set $\gamma \subseteq \mathcal{X}(\mathcal{N})$ for which there exist $x \in \mathcal{X}(\overline{\mathcal{N}})$ such that $\gamma = \gamma(x)$; we denote the set of clusters of $\mathcal{N}$ as $\Gamma(\mathcal{N})$. Note that clusters in Free Choice nets [2] contain at most one place; in event graphs, clusters contain at most one place and one transition. It is therefore *outside* these classes that clusters are most interesting, and may lead to "non-free choices".

From Definition 4.1, it is clear that different choices of independence relations yield different notions of confusion. We will now study the result of regarding clusters as *units*, i.e. assume that whatever device is used to resolve a conflict, that device must have knowledge of and control over the cluster in which the conflict arises. Then, e.g., A and C in Figure 1 are *not* independent at all, since their "controllers" are not: in I, they belong to the same cluster, and in II, A causally precedes the cluster $\gamma(C)$. We therefore introduce the following more restrictive notions of independence.

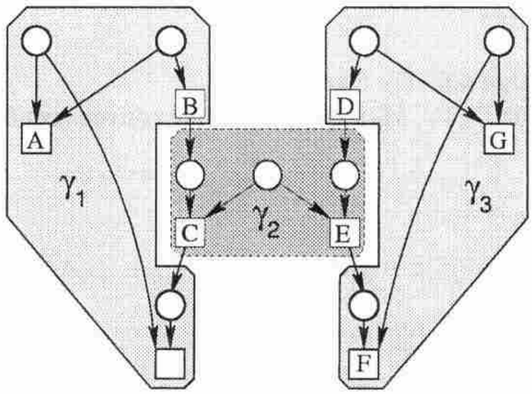

Figure 4. Clusters γ_1 and γ_3 are not convex

Definition 4.4. Let $\gamma_1, \gamma_2 \in \Gamma(\overline{\mathcal{N}})$ and $x_2 \in \mathcal{X}$; write $\mathcal{A}$ [co] $\mathcal{B}$ iff $\mathcal{B} \times \gamma_2 \subseteq \mathsf{co}$. Two nodes $x_1, x_2 \in (\mathcal{P} \cup \mathcal{T})$ are **cluster concurrent**, denoted $x_1 \ \mathsf{sc} \ x_2$, iff $\gamma(x_1)$ [co] $\gamma(x_2)$.

Note that sc is strictly stronger than co ($\mathsf{sc} \subseteq \mathsf{co}$ and $\mathsf{sc} \neq \mathsf{co}$); it eliminates both asymmetric and symmetric confusion:

Theorem 4.5. $\mathsf{cf}_{\mathsf{li}}^{\mathsf{sc}} = \mathsf{cf}_{\#}^{\mathsf{sc}} = \emptyset$.

Proof: B ic C implies $\gamma(B) = \gamma(C)$; so both $A < B$ and $A \# B$ contradict A sc C. □

This means that we were, in a way, wrong to "be confused" about the independence between A and B: those events are weakly causally connected.

Another re-enforcement of co can be obtained in the following way. Recall that a subset $\mathcal{A}$ of a poset $(\mathcal{E}, \leq)$ is **convex** iff for all $x, y \in \mathcal{A}$ and $v \in \mathcal{E}$, $x \leq v \leq y$ implies that $v \in \mathcal{A}$; equivalently, iff all v such that $\leq [v] \cap \mathcal{A} \neq \emptyset$ and $\leq^{-1} [v] \cap \mathcal{A} \neq \emptyset$ belong to $\mathcal{A}$. As Figure 4 shows, clusters need not be convex; so we define:

[1] For simplicity, we only speak of *clusters*

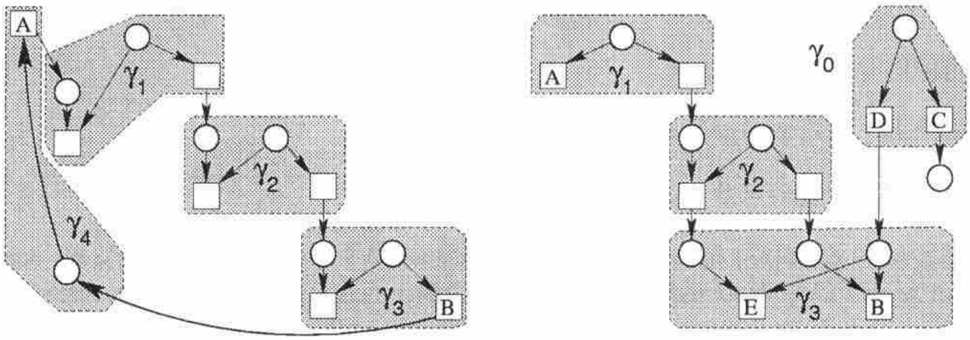

Figure 5. Ccc's not partially ordered (left); weak interference (right)

Definition 4.6. For a net
$\mathcal{N} = (\mathcal{P}, \mathcal{T}, \mathcal{W}, M_0)$ and $x \in \mathcal{X}$, the **convex (conflict) cluster (ccc)** $\gamma(x)$ of x is the smallest set satisfying

1. $x \in \gamma(x)$;
2. $\forall q \in \mathcal{T}$: $\mathsf{F}^{-1}[q] \cap \gamma(x) \neq \emptyset \Rightarrow q \in \gamma(x)$
3. $\forall p \in \mathcal{P}$: $\mathsf{F}[p] \cap \gamma(x) \neq \emptyset \Rightarrow p \in \gamma(x)$;
4. All $v \in (\mathcal{P} \cup \mathcal{T})$ such that $\leq [v] \cap \gamma(x) \neq \emptyset$ and $\leq^{-1} [v] \cap \gamma(x) \neq \emptyset$ belong to $\gamma(x)$.

A **ccc** is a set $\gamma \subseteq \mathcal{X}(\mathcal{N})$ for which there exist $x \in \mathcal{X}(\overline{\mathcal{N}})$ such that $\gamma = \gamma(x)$.

The following definition is analogous to that of sc:

Definition 4.7. Let $\gamma_1, \gamma_2 \in$ and $x_2 \in \mathcal{X}$; write $\mathcal{A}\,[\mathsf{co}]\,\mathcal{B}$ iff $(\mathcal{B} \times \gamma_2) \subseteq \mathsf{co}$. Two nodes $x_1, x_2 \in (\mathcal{P} \cup \mathcal{T})$ are **ccc concurrent**, denoted x_1 cc x_2, iff $\gamma(x_1)\,[\mathsf{co}]\,\gamma(x_2)$.

In Figure 3, q_5 sc q_8 and q_1 sc q_4, but $\neg(q_1$ sc $q_9)$ since $q_3 < q_7$ and hence $\neg\gamma_1[\mathsf{co}]\gamma_5$; also, $\neg(q_6$ sc $q_9)$ since $q_6 \# q_7$. Since all clusters here are convex, sc can be replaced by cc in the above. In general, however, sc and cc need not coincide: in Figure 4, A sc F, but A is not in cc with F; in fact, the entire structure in Figure 4 forms a single ccc. Note that γ_2 is a ccc itself; this shows that ccc's need not be disjoint, as opposed to clusters.

Now, cc is stronger than sc since every cluster γ is contained in some ccc. But even cc does not exclude weak interference: Consider the net on the right hand side of Figure 5. Obviously, A cc D (and thus A sc D), yet $D \rightsquigarrow_A B$ while $\neg(D \rightsquigarrow_\emptyset B)$.

5. Cluster Processes

So far, we have investigated independence and clusters that arise in net unfoldings; we have neither looked into clusters of the original net, nor modified the unfolding semantics; now we do both. Note first that Definition 4.3 is applicable to non-occurrence nets as well; this is not the case for Definition 4.6 since it requires an order relation, which is not given in general on an arbitrary net (quasi-orders lead to a "convexity" notion that is far too restrictive). So, denote the set of clusters of $\mathcal{N}$ as $\Gamma(\mathcal{N})$; again, clusters are pairwise disjoint. However, clusters may have forms that are not even possible in occurrence nets: consider the net in Figure 7. Its structure is a cluster, but it can never occur in an occurrence net since $d\#d$, which is excluded.

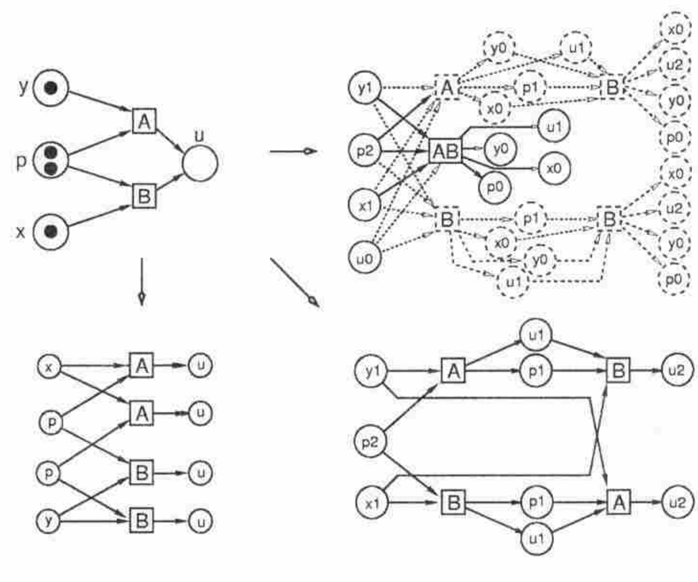

Figure 6. Three semantics

We propose to unfold a Petri net in such a way that the events of the unfolding represent instances not necessarily of single transitions – which, as we saw, may not be as independent of one another as the unfolding suggests – but of *steps*. However, using *global* steps would be unwise from a computational point of view, and also ignore the actual independence of events at a great distance (in terms of causal influence) from one another. So, rather than looking at *all* steps enabled in some global state of the net, examine the *local steps* within each *cluster*; they can be carried out independently of that in other clusters and allow to calculate the global steps as their products. So we consider the cluster viewpoint as on an intermediate scale between the local and the global one.

Consider first the existing unfolding semantics, illustrated in Figure 6. While Engelfriet's *branching processes* [3], shown on the bottom of the left hand side of Figure 6, are driven, informally speaking, by "token trajectories" and permit concurrency of events that do not compete for any individual token, *branching executions* (Vogler [13], Esparza, Römer and Vogler [5], Haar [7]), bottom right in Figure 6, regard places as variables whose values are given by the number of tokens; transitions then read from and write on these variables. As a result, *auto-concurrency* is excluded, i.e. no transition can fire more than once at a time, even if the marking would allow several concurrent firings; also, transitions accessing the same place may not act jointly.

The third point of view, top right in Figure 6, allows for an unfolding semantics with collective token view *and* auto-concurrency of transitions. It is based on *cluster steps* defined as follows:

Definition 5.1. For $\gamma \in \Gamma(\mathcal{N})$, a **$\gamma$-step** is a multi-set over $\mathcal{T}(\gamma)$ and thus a step of $\mathcal{N}$; denote as $\Theta(\gamma)$

the set of γ-steps, and set

$$\mathfrak{S}(\mathcal{N}) := \bigcup_{\gamma \in \Gamma(\mathcal{N})} \Theta(\gamma).$$

If θ satisfies $M_{|\gamma} \stackrel{\theta}{\Longrightarrow}$ (or $M_{|\gamma} \stackrel{\theta}{\Longrightarrow} M'$) for the restriction $M_{|\gamma} = M \cdot \mathbb{1}_{\mathcal{P}(\gamma)}$ of M to γ, then θ is **maximal relative γ for M**, written as $M \stackrel{\theta}{\Longrightarrow}$ (or $M \stackrel{\theta}{\Longrightarrow} M'$). Further, we will denote as

$$\begin{aligned} {}^{\circ}\gamma &:= \gamma \cap \mathcal{P} \\ \gamma^{\circ} &:= \{p \in \mathcal{P} \mid \mathsf{F}^{-1}[p] \cap \gamma \neq \emptyset\} \end{aligned}$$

the sets of *input* and *output* places of a cluster γ, respectively.

Definition 5.2. Let $\mathcal{N} = (\mathcal{P}, \mathcal{T}, \mathcal{W}, \mathcal{M}_0)$ be a net and $\overline{\mathcal{N}} = (\overline{\mathcal{P}}, \overline{\mathcal{T}}, \overline{\mathsf{F}}, \mathfrak{c}_0)$ an occurrence net. $\Pi = (\overline{\mathcal{N}}, \pi, \tau, \lambda)$ is called a **branching cluster process** or **BC process** of $\mathcal{N}$ iff the mappings $\pi : \overline{\mathcal{P}} \to \mathcal{P}$, $\lambda : \overline{\mathcal{P}} \to \mathbb{N}_0$ and $\tau : \overline{\mathcal{T}} \to \mathfrak{S}(\mathcal{N})$ satisfy

1. for all $\overline{p} \in \mathfrak{c}_0$, $\lambda(\overline{p}) = M_0(\pi(\overline{p}))$;
2. for all $\overline{q}_1, \overline{q}_2 \in \overline{\mathcal{T}}$, $(\mathsf{F}^{-1}[\overline{q}_1] = \mathsf{F}^{-1}[\overline{q}_2] \wedge \tau(\overline{q}_1) = \tau(\overline{q}_2)) \Rightarrow \overline{q}_1 = \overline{q}_2$;
3. for all $\overline{p}_1, \overline{p}_2 \in \overline{\mathcal{P}}$: $\overline{p}_1 \;\mathsf{co}\; \overline{p}_2 \Rightarrow \pi(\overline{p}_1) \neq \pi(\overline{p}_2)$;
4. for all $\overline{q} \in \overline{\mathcal{T}}$ it holds that for any $p \in ({}^{\circ}\tau(\overline{q}) \cup \tau(\overline{q})^{\circ})$, there exist $p_{in}, p_{out} \in \mathcal{P}$ such that:

$$\begin{aligned} \pi^{-1}(\{p\}) \cap \mathsf{F}^{-1}[\theta] &= \{p_{in}\}, \\ \pi^{-1}(\{p\}) \cap \mathsf{F}[\theta] &= \{p_{out}\}, \\ \lambda(p_{in}) &\geq \langle \mathcal{W}_p^{\top}, \theta \rangle, \\ \lambda(p_{out}) &= \left(\lambda(p_{in}) - \langle \theta, \mathcal{W}_p^{\top} \rangle\right) + \langle \mathcal{W}_p, \theta \rangle, \end{aligned}$$

where $\theta := \tau(\overline{q})$; compare the firability condition (1) and the firing equation (2).

Further:

1. if $\overline{\mathcal{N}}$ is a CN, Π is called a **cluster process** or **C process** of $\mathcal{N}$;
2. if for all $\overline{q}$ the step $\theta := \tau(\overline{q})$ satisfies (compare **CMAX**):

$$\forall \tilde{q} \in (\mathcal{T} \cap \gamma(\theta)) : \exists p \in \mathcal{P} : \lambda(p_{in}) < \langle \mathcal{W}_p^{\top}, \theta \rangle + \mathcal{W}(p, \tilde{q}),$$

then Π is a **max- cluster step process** of the respective type, i.e. **BCM** or **CM**.

The mappings π and τ correspond to the structural unfolding, taking conditions to places and events to steps; since conditions represent *states* of places, we also need the mapping λ to assign token numbers to conditions.

By induction, one shows as in [3] that to each cut $\mathfrak{c}$ of the cluster unfolding corresponds a unique reachable marking $M(\mathfrak{c})$ of $\mathcal{N}$, such that for two cuts $\mathfrak{c}_1, \mathfrak{c}_2$ and an event $\overline{q}$, one has

$$\mathfrak{c}_2 = \left(\mathfrak{c}_1 - \mathsf{F}^{-1}[\overline{q}]\right) \cup \mathsf{F}[\overline{q}] \quad \textit{iff} \quad M(\mathfrak{c}_1) \xrightarrow{\tau(\overline{q})} M(\mathfrak{c}_2).$$

The construction of cluster processes requires that each place (p) be represented in every instance, even with no tokens: one then has some $\overline{p}$ with $\pi(\overline{p}) = p$ and $\lambda(\overline{p}) = 0$. This creates a lot of "extra" conditions in the sense that they can be avoided under the two other semantics. Moreover, conditions representing the same place p before and after some step θ may have identical values of λ, which indicates that θ did not change the number of tokens on p, irrespective of whether p was ignored by θ or whether non-vanishing effects of θ on p cancelled out. So the event $\overline{q}$ representing an instance of the cluster step $\tau(\overline{q})$ indicates also that the system made the *"choice"* to fire exactly $\tau(\overline{q})$ and no further transition instance; and, in particular, to not use the tokens left over by $\tau(\overline{q})$ and to ignore the places that do not contribute to $\tau(\overline{q})$; and these negative actions or non-actions are exhibited in the unfoldings just as actions are. – Of course, some of the choices just described may be forced by the enabling conditions. In Figure 6, the events not admitted in the BCM process but admitted in the BC process are drawn in dashed lines.

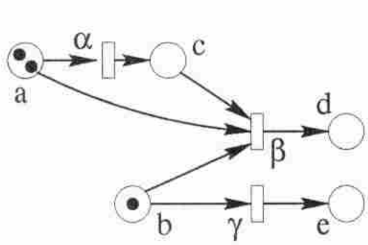

Figure 7. On step semantics

It may be desirable, for conciseness of the unfolding, to omit the extra conditions just described; we will not go into the details of the modifications needed to do this.

Note that max- cluster step semantics is halfway between the (global) maximal step firing rule and the single transition firing rule; every maximal step is expressible as a concatenation of maximal cluster steps, but not vice versa. Similarly, *not* every firing sequence can be simulated by a max- cluster step sequence: in Figure 7, Transition β will never be enabled if only max- cluster steps are allowed, since α will consume both tokens on a with two simultaneous firings in the same step. But under *single* transition firing– and *a fortiori* under general cluster step firing–, a sequence $\alpha\beta$ is possible.

BC processes allow to ignore token history and identity, i.e. they share the advantage of branching executions over branching processes; moreover, they allow auto-concurrency and simultaneous firing of transitions in structural conflict, which is important to reflect the actual time consummation of concurrent runs in the context of timed transitions; compare Winkowski's approach ([14, 15, 16]).

Let us look once more at independence in this unfolding. Suppose steps $\theta_1 \in \Theta(\gamma_1)$ and $\theta_2 \in \Theta(\gamma_2)$ are jointly enabled in M_0. They will be represented by concurrent events in the unfolding **iff**

$$^{\circ}\gamma_1 \cap \gamma_2{}^{\circ} = \emptyset \quad \textit{and} \quad {}^{\circ}\gamma_2 \cap \gamma_1{}^{\circ} = \emptyset. \tag{4}$$

In fact, suppose that $p \in {}^{\circ}\gamma_1 \cap \gamma_2{}^{\circ}$; then θ_1 and θ_2 are in conflict over which of them accesses p first, θ_1 to (possibly) take tokens from p or θ_2 to (possibly) put tokens on p. As a result, the unfolding will contain a branching between (at least) two runs, one seeing first an instance of θ_1 and then of θ_2, the other an instance of θ_2 before one of θ_2. Hence (4) defines an independence relation, based on which traces over $\mathfrak{S}(\mathcal{N})$ are generated by the system dynamics.

There are still other forms of independence that are yet to be formalized, such as **stochastic independence**, which requires an appropriate probabilistic model that we shall not develop here, and **Information independence**: This means roughly that the *decisions* leading to x and y have no effect (however indirect) on those leading to y and vice versa. For this, it has to be understood what *decision* in a complex distributed system means, a question beyond the scope of this article.

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