

Comparison of Algorithms for Checking Emptiness on Büchi Automata

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Abstract. We re-investigate the problem of LTL model-checking for finite-state systems. Typical solutions, like in Spin, work on the fly, reducing the problem to Büchi emptiness. This can be done in linear time, and a variety of algorithms with this property exist. Nonetheless, subtle design decisions can make a great difference to their actual performance in practice, especially when used on-the-fly. We compare a number of algorithms experimentally on a large benchmark suite, measure their actual run-time performance, and propose improvements. Compared with the algorithm implemented in Spin, our best algorithm is faster by about 33 % on average. We therefore recommend that, for on-the-fly explicit-state model checking, nested DFS should be replaced by better solutions. An abridged version of this paper has appeared in [7].

1 Introduction

Model checking is the problem of determining whether a given hardware or software system meets its specification. In the automata-theoretic approach, the system may have finitely many states, and the specification is an LTL formula, which is translated into a Büchi automaton, intersected with the system, and checked for emptiness. Thus, model checking becomes a graph-theoretic problem.

Because of its importance, the problem has been intensively investigated. For instance, *symbolic* algorithms use efficient data structures such as BDDs to work on sets of states; a survey of them can be found in [5]. Moreover, *parallel* model-checking algorithms have been developed [1]. The best known symbolic or parallel solutions have suboptimal asymptotic complexity ($\mathcal{O}(n \log n)$, where n is the number of states), but are often faster than that in practice.

Büchi emptiness can also be solved in $\mathcal{O}(n)$ time. All known linear algorithms are *explicit*, i.e. they construct and explore states one by one, by depth-first search (DFS). Typically, they compute some data about each state: its unique *state descriptor* and some *auxiliary data* needed for the emptiness check. Since the state descriptor is usually much larger than the auxiliary data, approximative techniques such as bitstate hashing have been developed that avoid them, storing just the auxiliary information in a hash table [13]. This entails the risk

* The author was supported by the DFG Graduiertenkolleg 1480 (PUMA).

of undetectable hash collisions; however the probability of a wrong result can be reduced below a chosen threshold by repeating the emptiness test with different hash functions. Thus they represent a trade-off between time and memory requirements. Henceforth, we shall refer to non-approximative methods that do use state descriptors as *exact* methods.

We further identify two subgroups of explicit algorithms: *Nested-DFS* methods directly look for accepting cycle in a Büchi automaton; they need very little auxiliary memory and work well with bitstate hashing. *SCC-based* algorithms identify strongly connected components containing accepting cycles; they require more auxiliary memory but can find counterexamples more quickly.

All explicit algorithms can work “on-the-fly”, i.e. the (intersected) Büchi automaton is not known at the outset. Rather, one begins with a Büchi automaton for the formula (typically small) and a compact system description and extracts the initial state from these. Successor states are computed during exploration as needed. If non-emptiness is detected, the algorithms terminate before constructing the entire intersection. Moreover, in this approach the transition relation need not be stored in memory. As we shall see, the on-the-fly nature of explicit algorithms is very significant when evaluating their performance properly.

In this paper, we investigate performance aspects of explicit, exact, on-the-fly algorithms for Büchi emptiness. The best-known example for such a tool is Spin [12], which uses the nested-DFS algorithm proposed by Holzmann et al [13], henceforth called HPY. The reasons for this choice are partly historic; the faster detection capabilities of SCC-based algorithm were not known when Spin was designed, having first been pointed out by Couvreur in 1999 [3]. Thus, the status of HPY as the best choice is questionable, all the more so since the memory advantages of nested DFS are comparatively scant in our setting. Moreover, improved nested DFS algorithms have been proposed in the meantime.

We therefore evaluate several algorithms based on their actual running time and memory usage on a large suite of benchmarks. Previous papers, especially those on SCC-based algorithms [10, 15, 4, 11], provided similar experimental results, however, experiments were few or random and unsatisfying in one important aspect: they worked from pre-computed Büchi automata, rather than truly on-the-fly. This aspect will play a significant role in our evaluation.

To summarize, this paper contains the following contributions and findings:

- We provide improvements in both subgroups, nested DFS and SCC-based. These concern the algorithms of Couvreur [3] and Schwon/Esparza [15].
- One of the algorithms we study can be extended to generalized Büchi automata, and we investigate this aspect.
- We implemented existing and new algorithms and compare them on a large benchmark suite. We analyze the structural properties of Büchi automata that cause performance differences.

We make the following observations: The overall memory consumption of all algorithms is dominated by the state descriptors, the differences in auxiliary memory play virtually no role. The running times depend practically exclusively on the number of successor computations. When experimenting with

pre-computed automata – as done in some other papers – this operation becomes cheap, which causes misleading results. Our results allow to derive detailed recommendations which algorithms to use in which circumstances. These recommendations revise those from [15]; Couvreur’s algorithm which was recommended there, is shown to have weak performance; however, the modification mentioned above amends it. Moreover, our modification of Schwoon/Esparza improves the previous best nested-DFS algorithm.

In addition, this paper provides new, self-contained proofs of both improved algorithms. Since the original algorithms are already known to be correct, one could easily give non-self-contained proofs by showing that the modifications do not affect correctness. However, we feel that there are still good reasons to provide completely new proofs.

First, the nested-DFS algorithm was derived through a succession of modifications, from [2] via [13], [8], and [15], during which the mechanics of the algorithm have changed sufficiently to merit a new proof.

Secondly, self-contained proofs are a necessity if improved Büchi emptiness algorithms are ever to be taught in verification classes. In the authors’ experience, DFS algorithms are notoriously difficult to explain, yet the proofs we give are still reasonably simple. For instance, the proof of the new SCC-based algorithm is based on eight simple facts that are easy to understand and prove. In our experience, these proofs can be used in a classroom setting even if the students are previously unfamiliar with the concepts of DFS and SCCs.

We proceed as follows: Section 2 establishes preliminaries, Sections 3 and 4 present nested-DFS and SCC-based algorithms, including our modifications. Section 5 details our experimental results and concludes.

2 Preliminaries

A *Büchi automaton* (BA) is a tuple $\mathcal{B} = (S, s_I, \text{post}, A)$, where S is a finite set of *states*, $s_I \in S$ is the *initial state*, $\text{post}: S \rightarrow 2^S$ is the *successor function*, and $A \subseteq S$ are the *accepting states*. A *path* of \mathcal{B} is a sequence of states $s_1 \cdots s_m$ for some $m \geq 1$ such that $s_{i+1} \in \text{post}(s_i)$ for all $1 \leq i < m$. If a path from s to t exists, we write $s \rightarrow^* t$. When $m > 1$, we write $s \rightarrow^+ t$, and if additionally $s = t$, we call the path a *loop*. A *run* of \mathcal{B} is an infinite sequence $(s_i)_{i \geq 0}$ such that $s_0 = s_I$ and $s_{i+1} \in \text{post}(s_i)$ for all $i \geq 0$. A run is called *accepting* if $s_i \in A$ for infinitely many different i . The *emptiness problem* is to determine whether no accepting run exists. If an accepting run exists, it is also called a *counterexample*. From now on, we assume a fixed Büchi automaton \mathcal{B} .

Note that we omit the usual input alphabet because we are just interested in emptiness checks. Moreover, the transition relation is given as a mapping from each state to its successors, which is suitable for on-the-fly algorithms.

A *strongly connected component* (SCC) of \mathcal{B} is a subset $C \subset S$ such that for each pair $s, t \in C$, we have $s \rightarrow^* t$, and moreover, no other state can be added to C without violating this property. An SCC C is called *trivial* if $|C| = 1$ and for the singleton $s \in C$, $s \notin \text{post}(s)$. The following two facts are well-known:

- (1) A counterexample exists iff there exists some $s \in A$ such that $s_I \rightarrow^* s$ and $s \rightarrow^+ s$. This fact is exploited by nested-DFS algorithms.
- (2) A counterexample exists iff there exists a non-trivial SCC C reachable from s_I such that $C \cap A \neq \emptyset$. This fact is exploited by SCC-based algorithms.

A Büchi automaton is called *weak* if each of its SCCs is either contained in A or in $S \setminus A$. This implies the following fact:

- (3) Each loop in a weak BA is entirely contained in A or in $S \setminus A$.

A *generalized Büchi automaton* (GBA) is a tuple $\mathcal{G} = (S, s_I, \text{post}, \mathcal{A})$, where S , s_I , and post are as before, and $\mathcal{A} = (A_1, \dots, A_k)$ is a *set* of acceptance conditions, i.e. $A_j \subseteq S$ for all $j = 1, \dots, k$. Paths and runs are defined as for normal Büchi automata; a run $(s_i)_{i \geq 0}$ of \mathcal{G} is called *accepting* iff for each $j = 1, \dots, k$ there exist infinitely many different i such that $s_i \in A_j$.

GBA are generally more concise than BA: a GBA with k acceptance conditions and n states can be transformed into a BA with nk states. There is no known nested-DFS algorithm that avoids this k -fold blowup for checking emptiness of a GBA, although Tauriainen’s algorithm mitigates it [17]. Some SCC-based algorithms, however, can exploit the following fact:

- (4) A counterexample exists in \mathcal{G} iff there exists a non-trivial SCC C reachable from s_I such that $C \cap A_j \neq \emptyset$ for all $j = 1, \dots, k$.

3 Nested depth-first search

Nested DFS was first proposed by Courcoubetis et al [2], and all other algorithms in this subgroup still follow the same pattern. There are two DFS iterations: the “blue” DFS is the main loop and marks every newly discovered state as blue. Upon backtracing from an accepting state s , it initiates a “red” DFS that tries to find a loop back to s , marking every encountered state as red. If a loop is found, a counterexample is reported, otherwise the blue DFS continues, but the established red markings remain. Thus, both blue and red DFS visit each state at most once each. Only two bits of auxiliary data are required per state.

This pattern of searching for accepting loops in post-order ensures that multiple red searches do not interfere; states in “deep” SCCs are coloured red first, and when a red DFS terminates, red states are guaranteed not to be part of any counterexample. While being memory-efficient and simple, this has two disadvantages. First, nested DFS prefers long counterexamples over shorter ones; secondly, the blue DFS never notices that a complete counterexample has already been explored and continues exploring potentially many more states than necessary before eventually noticing the counterexample during backtracking. Also, nested DFS computes the successors of many states twice.

Several improvements have been suggested in the past, e.g. the HPY algorithm [13], implemented in Spin, and the SE algorithm [15]. We present an improvement of SE, shown in Figure 1. We first describe the differences w.r.t. SE; a detailed proof is given below.

```

1  procedure new_dfs ()
2    call dfs_blue( $s_I$ )

3  procedure dfs_blue ( $s$ )
4    allred := true;
5    s.colour := cyan;
6    for all  $t \in \text{post}(s)$  do
7      if  $t.colour = cyan$ 
8         $\wedge (s \in A \vee t \in A)$  then
9        report cycle
10     else if  $t.colour = white$  then
11       call dfs_blue( $t$ );
12     if  $t.colour \neq red$  then
13       allred := false;

14     if allred then
15       s.colour := red
16     else if  $s \in A$  then
17       call dfs_red( $s$ );
18       s.colour := red
19     else
20       s.colour := blue
21     procedure dfs_red ( $s$ )
22       for all  $t \in \text{post}(s)$  do
23         if  $t.colour = cyan$  then
24           report cycle
25         else if  $t.colour = blue$  then
26           t.colour := red;
27         call dfs_red( $t$ )

```

Fig. 1. New Nested-DFS algorithm.

The additions to SE are in lines 4 and from 12 to 15. These exploit the fact that red states cannot be part of any counterexample; therefore a state that has only red successors cannot be either. This avoids certain initiations of the red search. The improvement is similar in spirit to [8], but avoids some unnecessary invocations of `post`. Like in [2], only two bits per state are used. Our experiments shall show that it performs best among the known nested DFS algorithms.

Finally, we remark that for weak automata a much simpler algorithm suffices, as observed by Černá and Pelánek [18]. Exploiting Fact (3), one can simply omit the red search because all counterexamples are bound to be reported by line 9 in Figure 1. In that case, `post` is only invoked once per state.

3.1 Proof of the new algorithm

Colour changes We assume that all newly discovered states are initialized to white. There are four colours, meaning that the auxiliary data can be encoded with two bits. There are five statements that change the colour of states, in lines 5, 15, 18, 20, and 26

The procedure *dfs_blue* is only invoked on white states in lines 3 and 11. Thus, the statement in line 5 changes only white states into cyan. There is no statement that changes states back to white, therefore *dfs_blue* is only invoked once per state. The statement in line 26 changes only blue states to red. Therefore, when *dfs_blue*(s) reaches line 14, s must still be cyan, and its colour is changed by of the statements in lines 15, 18, or 20 to either red or blue.

Meaning of colours From the above, we can deduce the following:

- A state is white if and only if it has never been touched by *dfs_blue*.
- A state is cyan if and only if its invocation of *dfs_blue* is still running, (i.e., it is on the “search stack” of *dfs_blue*), and every cyan state can reach s , if *dfs_blue*(s) is the currently active instance of *dfs_blue*.

- A state is blue if and only if it is non-accepting and its invocation of *dfs_blue* has terminated.
- If a state is red, its invocation of *dfs_blue* has terminated, and it is not part of any counterexample.

The last part of this statement is proved in the next paragraph.

Red states We prove that red states are never part of any counterexample. More precisely, whenever an invocation of *dfs_blue* terminates, all states that have been coloured red by that time are not part of any counterexample. We proceed by induction on the states in the post-order implied by *dfs_blue*, or, put differently, we show that this property is an invariant of the program.

Obviously, the statement holds initially because there are no red states. Now, suppose that some state s is made red by line 15. Then, all its successor states are red, so by induction hypothesis none of them are part of any counterexample. Since any counterexample including s also has to include one of its successors, s cannot be part of a counterexample.

It remains to show that lines 17 and 18 preserve the invariant. Assume therefore that the call to *dfs_red* in line 17 terminates. We now show that in this case, no state s' visited by *dfs_red* is part of any counterexample. Assume by contradiction that s' is part of a counterexample. Then there must be some accepting state t reachable from s' (and therefore from s), and there must be a path from s via s' to t in which all states were non-red before line 17 was reached (by induction hypothesis, because these states are part of a counterexample). However, such a state t cannot exist:

- t cannot be white because it is reachable from s , and therefore it must have been visited by *dfs_blue* before *dfs_blue*(s) could have reached line 14.
- t cannot be cyan because it is reachable from s by non-red states, and therefore *dfs_red* would terminate when reaching t .
- t cannot be blue because it is accepting.
- t cannot be red because this means that its invocation of *dfs_blue* has already finished, in which case, by induction hypothesis, it is not part of any counterexample.

Correctness, part 1 We now show that whenever the algorithm reports a cycle, a counterexample indeed exists. Cycles are reported in lines 9 and 24.

In line 9, there is a transition from s to t . Since t is cyan, there is also a path from t to s , and either s or t are accepting. Therefore, a counterexample exists.

In line 24, there is a transition from s to t . Assume that s' is the “seed” of the current red DFS, i.e. s' was the state that most recently reached line 17. Then, s' is accepting and can reach s . Moreover, since t is cyan, it can reach s' , completing the counterexample.

Correctness, part 2 We now show that whenever a counterexample exists, the algorithm reports one. Let s be an accepting state within the loop of such a

counterexample. Then, either the algorithm reaches line 17 in the *dfs_blue* invocation on s , or it will terminate even earlier with a counterexample. We show that in the first case the red DFS on s will still find a counterexample.

Consider the states forming the loop of the counterexample at the time when *dfs_red*(s) is called. None of them can be red, and none of them can be white because they are all reachable from s and therefore have been considered by *dfs_blue* earlier. This, all of them are either blue or cyan. In particular, at least one state in the loop, i.e., s itself, is still cyan. Therefore, the red search is guaranteed to find a cyan state and report a counterexample.

4 SCC-based algorithms

An efficient algorithm for determining SCCs that works on-the-fly was first proposed by Tarjan [16]. However, for model-checking purposes Tarjan’s algorithm was deemed unsuitable because it used more memory than nested DFS while offering no advantages. More recent innovations by Geldenhuys/Valmari [10] and Couvreur [3] change the picture, however: their modifications allow SCC-based analysis to report a counterexample as soon as all its states and transitions were discovered, no matter in which order. In other words, if the order in which successors are explored by the DFS is fixed, both can find a counterexample in optimal time (w.r.t. to the exploration order).

Space constraints prevent us from presenting the algorithms in detail. However, we mention a few salient points. Tarjan places all newly discovered states onto a stack (henceforth called *Tarjan stack*) and numbers them in pre-order. Certain properties of the DFS ensure that at any time during the algorithm, states belonging to the same SCC are stored consecutively on the stack and therefore also numbered consecutively. The *root* of an SCC is the state explored first during DFS, having the lowest number and being deepest on the Tarjan stack. For each state s , Tarjan computes a so-called “lowlink” number, which is identical to the number of s iff s is a root, and less than that otherwise. An SCC is completely explored when backtracking from its root, and at that point it can be identified as a complete SCC and removed from the Tarjan stack.

Geldenhuys/Valmari (GV) exploit properties of lowlinks; they remember the number of the deepest accepting state on the current search path, say k , and when a state with lowlink $\leq k$ is found, a counterexample is reported. They also propose some memory savings that are of minor importance in our context.

Couvreur (C99) omits both Tarjan stack and lowlinks but introduces a *roots stack* that stores the roots of all partially explored SCCs on the current search path. When one finds a transition to a state with number k , properties of the numbering imply that no state with number larger than k can be a root, prompting their removal from the roots stack. This effectively merges some SCCs, and one checks whether the merger creates an SCC with the conditions from Fact (2).

Both algorithms report a counterexample after seeing the same states and transitions, provided they work with the same exploration order. However, it turns out that the removal of the Tarjan stack in C99, while more memory

```

1  procedure couv ()
2    count := 0;
3    Roots :=  $\emptyset$ ; Active :=  $\emptyset$ ;
4    call couv_dfs(sI)
5  procedure couv_dfs(s):
6    count := count + 1;
7    s.dfsnum := count;
8    s.current := true;
9    push(Roots, (s, A(s)));
10   push(Active, s);
11   for all t  $\in$  post(s) do
12     if t.dfsnum = 0 then
13       call couv_dfs(t)
14   else if t.current then
15     B :=  $\emptyset$ ;
16     repeat
17       (u, C) := pop(Roots);
18       B := B  $\cup$  C;
19       if B = K then report cycle
20       until u.dfsnum  $\leq$  t.dfsnum;
21       push(Roots, (u, B));
22   if top(Roots) = (s, ?) then
23     pop(Roots);
24     repeat
25       u := pop(Active);
26       u.current := false
27     until u = s

```

Fig. 2. Amendment of Couvreur’s algorithm.

efficient, was a crucial oversight: when backtracking from a root, another DFS is necessary to mark these states as “removed”. These extra post computations severely impede its performance. This makes GV superior to C99 in practice.

We propose to amend C99 by re-inserting the Tarjan stack.³ This amendment makes it competitive with GV while using slightly less memory; more crucially, C99 can deal directly with GBAs, which GV cannot. Since GBAs tend to be smaller than BAs for the same LTL formula, the amended algorithm can hope to explore fewer states and be faster.

The amended algorithm, working with GBAs, is shown in Figure 2, and a proof of correctness is given below. Note that in C99 acceptance conditions are annotated on the transitions, whereas here we place them on the states, which is only a minor difference. Figure 2 assumes k acceptance sets, denoting $\mathcal{A}(s) := \{j \mid s \in A_j\}$ and $K := \{1, \dots, k\}$. Note that if k is “small”, the union operation in line 18 can be implemented with bit parallelism.

4.1 Proof of the new algorithm

We provide a detailed proof of correctness. The proof works from scratch and assumes only very basic knowledge of graph theory plus the concept of SCCs.

Basic Definitions A *DFS numbering* of \mathcal{B} is a pre-order numbering starting at the initial state s_I . In general, depending on the order in which successors are explored, an automaton has many possible DFS numberings; here we assume one externally fixed order and therefore one fixed DFS numbering. The number assigned to state s is denoted $num(s)$. Note that states are added to the Tarjan stack (called *Active* in Figure 2) in the order of their numbering.

³ The problem with C99 was first hinted at in [15]. After creating this improvement independently, we learned that similar changes were already proposed in [4] and [11].

The *root* of an SCC within \mathcal{B} is the state visited first by *couv_dfs* during the algorithm. (Precisely which state within an SCC is a root may also depend on the exploration order.)

At any time during the algorithm, we mean by *search path* the sequence of currently unfinished calls to *couv_dfs*.

Subgraphs of \mathcal{B} A state s is called *explored* when *couv_dfs*(s) has been called. A transition from s to t is called *explored* when t appears in the for-loop during execution of *couv_dfs*(s). At any time during the algorithm, we mean by *explored graph* the subgraph \mathcal{E} consisting of all explored states and transitions.

We call an SCC of \mathcal{E} *active* if the search path contains at least one of its states. Note that the SCCs of \mathcal{E} may be different from those of \mathcal{B} ! In particular, due to unexplored transitions, two SCCs of \mathcal{E} may be part of the same SCC of \mathcal{B} .

A state is called *active* if it is part of an active SCC. The state itself need not be on the search path.

At any time during the algorithm, we mean by *active graph* the subgraph \mathcal{A} induced by the active states.

Facts

1. Let $s_0 \cdots s_n$ be the search path at any time. Then $num(s_i) \leq num(s_j)$ iff $i \leq j$. Moreover, $s_i \rightarrow^* s_j$ if $i \leq j$.
Proof: immediate from the logic of the program.
2. A root has the least number and lies lowest on *Active* within its SCC.
Proof: obvious
3. Within each SCC, the root is the last state from which *couv_dfs* backtracks, and at that point, the SCC has been completely explored (i.e., all states and edges have been considered).
Proof: Suppose *couv_dfs* reaches root r of SCC C . At that point, no other state of C has been visited so far, and all are reachable from r . Therefore, the DFS will visit all those states (and possibly others) and backtrack from them before it can backtrack from r .
4. An SCC becomes inactive when we backtrack from its root.
Proof: follows from Fact 3.
5. An inactive SCC of \mathcal{E} is also an SCC of \mathcal{B} .
Proof: follows from Facts 3 and 4.
6. The roots of \mathcal{A} are a subsequence of the search path.
Proof: follows from Fact 4 because the root of an active SCC must be on the search path.
7. Let s be an active state and t the root of its SCC in \mathcal{A} . Then $num(t) \leq num(s)$ and there is no active root u with $num(t) < num(u) < num(s)$.
Proof: The first part is just a consequence from Fact 2. For the second part, assume that such an active root u exists. Since u is active, it is on the search stack, just like t , which follows from Fact 6. From Fact 1, we have $t \rightarrow^* u$. As *couv_dfs*(u) has not yet terminated and $num(u) < num(s)$, s must have been reached from u , i.e. $u \rightarrow^* s$. Because s and t are in the same SCC, $s \rightarrow^* t$. But then, t and u are in the same SCC and cannot both be its root.

8. Let s and t be two active states with $num(s) \leq num(t)$. Then $s \rightarrow^* t$.
 Proof: Let s', t' be the (active) roots for s and t , resp. From Fact 7 we have $num(s') \leq num(t')$, thus from Fact 1 we have $s' \rightarrow^* t'$, and therefore $s \rightarrow^* t$.

Conclusions From the facts that we have just shown, we can conclude that the active graph \mathcal{A} always has the kind of shape visualized in Figure 3, with the following properties:

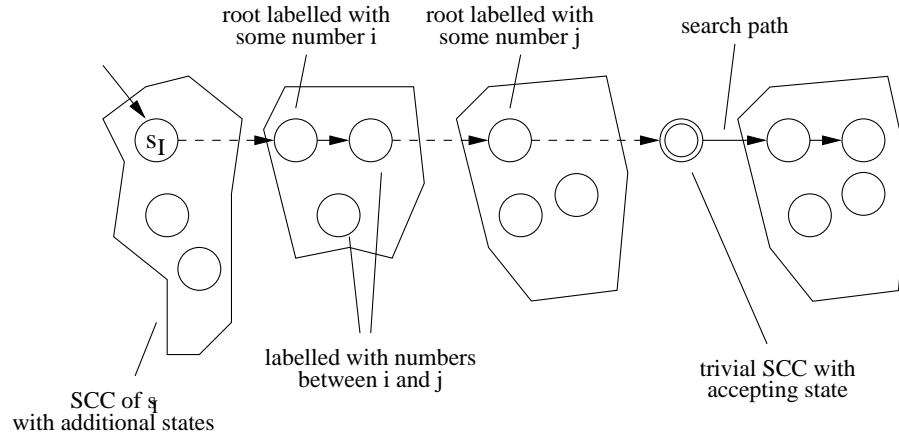


Fig. 3. Shape of the active graph

- The search path (indicated by the connected line of states at the top of the figure) is completely contained in the active graph, and its roots from a subsequence of the search path.
- The SCCs are “linearly ordered”, i.e. if one defines $C_1 < C_2$ iff C_2 can be reached from C_1 , then $<$ is a total order.
- The DFS numbering is consecutive in the sense that if i and j are the numbers of two subsequent roots on the search path, then the active states with numbers n such that $i \leq n < j$ form an SCC. From this it follows that these states are also consecutive on the Tarjan stack.

Correctness of the algorithm The correctness of the algorithm is now easy to show. We assume that all newly discovered states are initialized with a number 0 and a false *current* bit. It then suffices to prove that the algorithm maintains the following invariants after each exploration of a state or transition:

- The *Roots* stack contains the roots of the active graph (in the order implied by the search path) together with the union of all acceptance indices occurring within the corresponding SCC of \mathcal{A} .
- The *Active* stack contains exactly the active states, and exactly the active states have the *current* bit set to true.

In the beginning of the algorithm, this invariant holds because the active graph contains just s_I and no transitions. Thus, the single element of *Roots* is $(s_I, A(s_I))$, and s_I is active. This is ensured by the first part up to line 10.

The invariant is then upheld whenever a transition from some s to some t is discovered. There are five cases:

- t is a newly discovered state. In this case, \mathcal{A} is extended by t and the transition $s \rightarrow t$, and t forms a new trivial SCC within \mathcal{A} . No counterexample is generated in this way. The recursive call in line 13 and lines 6 through 10 perform the necessary actions.
- t has been visited before and is inactive. Then, its SCC has been completely explored, so s and t belong to different SCCs, so $t \not\rightarrow^* s$. The edge $s \rightarrow t$ cannot be part of a loop, the active graph does not change, so no action is necessary.
- t is active and $num(t) > num(s)$. From Fact 8 we already know that $s \rightarrow^* t$ holds, therefore this discovery does not change the SCCs and no new counterexample can be generated in this way. Thus, no action is necessary.
- t is active and $num(t) = num(s)$. Then $s = t$, and a counterexample has been discovered iff s contains all acceptance conditions. Otherwise, the SCCs of the active graph remain unchanged.
- t is active and $num(t) < num(s)$. Then from Fact 8 we know $t \rightarrow^* s$, so s and t belong to the same SCC. Let u , with $num(u) \leq num(t)$ be the root of the SCC to which t belongs. Since s is the last element on the search path, it follows from Fact 1 that all SCCs on the *Roots* stack from u downwards must be merged into one SCC. Moreover, u is the unique topmost root on *Roots* whose number is no larger than $num(t)$ according to Fact 7. Finally, the merger yields a non-trivial SCC, and a new counterexample is generated iff the SCC contains all acceptance conditions.

The last three cases are dealt with uniformly in lines 14 through 21 of Figure 2.

Finally, when backtracking from a state s , two cases can happen:

- s is a root. Then necessarily the *Roots* stack has a topmost entry with s because s is currently last on the search path, and said entry must be removed. Moreover, the entire SCC becomes inactive according to Fact 4. This is dealt with from line 22 downwards.
- s is not a root. Then the topmost *Roots* entry does not show s , no node becomes inactive, and no further action is necessary.

Thus, the invariant is upheld. A counterexample is reported as soon it is contained in the explored graph \mathcal{E} . As a consequence, if the algorithm terminates normally, no counterexample exists.

5 Experiments

We implemented a framework for testing and comparing the actual performance of all the known Büchi emptiness algorithms. For practical relevance, the best

framework for such an implementation would have been Spin. However, Spin turned out too difficult to modify for this purpose. Instead, we based our testbed on NIPS [19], a reverse-engineered Promela engine. Essentially, NIPS allows to process a Promela model, provides the initial state descriptor and a function for computing its successors. It is thus ideally suited for testing on-the-fly algorithms, and we believe that the conditions are as close to Spin as possible.

We used 266 test cases from the BEEM database [14], including many different algorithms, e.g., the Sliding Window protocol, Lamport’s Bakery algorithm, Leader Election, and many others, together with various LTL properties.

Among the algorithms tested and implemented were HPY [13], GV [10], C99 [3], SE [15], and the amended algorithms presented in Sections 3 and 4, henceforth called AND and ASCC. For weak automata, we report on simple DFS (SD, see Section 3). We also implemented and tested other algorithms, notably those from [2] and [8]. However, these were always dominated by others, and we omit them in the following. Naturally, our concrete running times and memory consumptions are subject to certain implementation-specific issues. Nonetheless, we believe that the tendencies exhibited by our experiments are transferrable.

In the following, we give a summary of our results. A more detailed description of our framework, the benchmarks, and the experimental results is given in [6]; here, we just summarize the most important findings.

We first found that, in the context of exact model checking, the differences in auxiliary memory usage was basically irrelevant. Certainly, the auxiliary memory used by the various algorithms ranged from 2 bits to 12 bytes, a comparatively large difference. However, this was dwarfed by the memory consumption of state descriptors, which ranged from 20 to 380 bytes, averaging at 130.

The only practical difference therefore was in the running time. Here, we found that, no matter which auxiliary data structures were employed, the running time was practically proportional to the number of post invocations (more precisely: the number of individual successor states generated by post), by far the most costly operation. In retrospect, these two observations may seem obvious; however, we find that they were consistently under-represented in previous papers, therefore it is worth re-emphasizing their relevance. The two main factors contributing to the running time were fast counterexample detection and whether an algorithm had to compute each transition at most once or twice.

Discussing individual test cases would not be very meaningful: for instance, the early-detection properties of some algorithms can cause arbitrarily large differences. Instead, we exhibit certain structural properties that occurred in many test cases and caused those differences. We first discuss algorithms working on “normal” Büchi automata, followed by a discussion of ASCC with GBAs.

First, we observe that most test cases constitute weak Büchi automata. Note that the intersection BA is weak if the BA arising from the formula is weak. Černá and Pelánek [18] estimate the proportion of weak formulae in practice to 90–95 %; indeed, we found that only 8 % of our test cases were non-weak. For weak test cases, five out of six tested algorithms (GV, C99, SE, AND, SD)

detect counterexamples with minimal exploration. The three main structural effects causing performance differences (which may overlap) were as follows:

- In 86 test cases, we observed many trivial SCCs consisting of one accepting state. A typical example is the LTL property GFp , which (when negated) yields a weak automaton with a looping accepting state. Then, any non-looping part of the system necessarily yields such trivial SCCs. In these cases, GV and SD dominate, sometimes with a factor of two, whereas C99, SE, and HPY fall behind because they explore the accepting trivial SCCs twice. In our test cases, the AND algorithm had the same performance as GV and SD, although this is not guaranteed in general.
- In 98 cases, we observed non-accepting SCCs not preceded by accepting SCCs. In this case, C99 falls behind all the others.
- HPY reports counterexamples only during the red DFS, whereas SE and AND discovers some during the blue DFS. This accounts for 101 test cases in which HPY fared worst, whereas all others showed the same performance.

Non-weak automata also had these effects, affecting 18, 17, and 7 out of 21 test cases. In 7 cases, GV and C99 found counterexamples more quickly than the others, being faster by a factor of up to six. Since we used the same exploration order in all algorithms, these results are directly comparable.

We then tested the ASCC algorithm with GBA, generated by the LTL2BA tool [9]. Most formulae yielded GBA with only one acceptance condition, meaning that the GBA had the same size as the corresponding BA. Notice that the running times of GBA with multiple conditions are not directly comparable with those of the corresponding BA. This is because using a different automaton changes the order of exploration, therefore in some “lucky” cases the BA-based algorithms may still find a counterexample more quickly.

The running times summed up over all 266 test cases are given in Figure 4, expressed as percentages of each other. Additionally, SD had the same performance as GV for the weak cases. Note that every set of benchmarks would lead to the same order among the algorithms because it reflects their different qualitative properties (e.g., quick counterexample detection or number of post calls). The concrete numbers in Figure 4 tell their quantitative effect in what we believe to be a representative set of benchmarks. We draw the following conclusions:

- Because of the dominance of weak test cases and GBAs with only one acceptance condition, the sum of running times yields small differences; only SE, HPY, and C99 clearly fall behind. The performance differences in the comparatively few other cases is very pronounced however.
- Overall, ASCC is the best algorithm if GBAs can be used. Due to the technical reasons explained above, it did not perform best in all examples.

algorithm	run-time
ASCC	67.0 %
GV	69.2 %
AND	69.7 %
SE	96.3 %
HPY	100.0 %
C99	128.3 %

Fig. 4. Performances

- Among the BA-based algorithms, GV is the best for general formulae; it is never outperformed on any test case by any other BA-based algorithm. ASCC performs equally well when used with simple BAs.
- For weak formulae, SD is the best algorithm for bitstate hashing.
- For general formulae, AND is the best algorithm for bitstate hashing, improving the previous best algorithm for this setting (SE) by 28 %.
- There remains no reason to use either SE, HPY, or C99.

Acknowledgements: The authors would like to thank Michael Weber for creating and helping us use the NIPS framework.

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