A Note on an Extension of PDL

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Abstract

Recently visibly pushdown automata have been extended to so called $k$-phase multi-stack visibly pushdown automata ($k$-MVPAs). On the occasion of introducing $k$-MVPAs, it has been asked whether the extension of Propositional Dynamic Logic with $k$-MVPAs still leads to a decidable logic. This question is answered negatively here.

Key words: Propositional Dynamic Logic, Visibly Pushdown Automata, Multi-Stack Visibly Pushdown Automata, Decidability, Satisfiability

1 Introduction

Propositional Dynamic Logic (PDL) is a modal logic introduced by Fischer and Ladner [1] which allows to reason about regular programs. In PDL, there are two syntactic entities: formulas, built from boolean and modal operators and interpreted as sets of worlds of a Kripke structure; and programs, built from the operators test, union, composition, and Kleene star and interpreted as binary relations in a Kripke structure. Hence, the occurring programs can be seen as a regular language over an alphabet that consists of tests and atomic propositions. However, the mere usage of regular programs limits the expressiveness of PDL as for example witnessed by the set of executions of well-matched
calls and returns of a recursive procedure, cf. [2]. Therefore, non-regular extensions of PDL have been studied quite extensively [2–5]. An extension of PDL by a class \( L \) of languages means that in addition to regular languages also languages in \( L \) may occur in modalities of formulas. One interesting result on PDL extensions, among many others as summarized in [2], is the extension of PDL with visibly pushdown languages [6] which are the languages recognized by visibly pushdown automata (VPA). A VPA is a pushdown automaton, where the stack operation is determined by the input in the following way; the alphabet is partitioned into letters that prompt a push, internal, or pop action, respectively. Löding, Lutz, and Serre [4] showed that satisfiability of a PDL extension with VPL is complete for deterministic doubly exponential time. Note that for this result, every visibly pushdown language occurring in a formula must be over the same partition of the alphabet.

Recently, \( k \)-phase multi-stack visibly pushdown automata (\( k \)-MVPAs), a natural extension of VPAs, have been introduced in [7]. A \( k \)-MVPA is an automaton equipped with \( n \) stacks where, again, the actions on the stacks are determined by the input, more precisely, every input symbol specifies on which stack a push or pop operation or whether an internal operation is done. Moreover, a \( k \)-MVPA is restricted to accept only words that can be obtained by concatenating at most \( k \) phases, where a phase is a sequence of input symbols that invoke pop actions from at most one stack. Note that \( k \)-MVPAs with one stack coincide with VPAs. The language class \( k \)-MVPL that is described by \( k \)-MVPAs has various effective closure properties and a decidable nonemptiness problem. Therefore it is an interesting question to ask if the corresponding extension of PDL is still decidable. This question was raised in [7] and is answered negatively in this note. We prove \( \Sigma_1^1 \)-completeness for this PDL extension. A \( \Sigma_1^1 \) lower bound already holds, if we extend PDL with two fixed languages each accepted by some 2-MVP over two stacks, namely \( \{(a_1 b_2)^n(a_2 b_1)^n \mid n \geq 1\} \) and \( \{(a_2 b_1)^n(a_1 b_2)^n \mid n \geq 1\} \). For this, we give an easy reduction of PDL with the two languages \( \{a^n b^n \mid n \geq 1\} \) and \( \{b^n a^n \mid n \geq 1\} \), proven to be \( \Sigma_1^1 \) hard in [3], to the latter PDL extension.

### 2 Propositional Dynamic Logic Extensions

Fix some countable set \( P \) of atomic propositions, some finite alphabet \( \Sigma \) and a class of languages \( L \subseteq 2^{\Sigma^*} \). The set of formulas \( \Phi \) and the set of tests Tests Tests of the logic PDL+L are the smallest sets that satisfy the following conditions:

(i) if \( p \in P \), then \( p \in \Phi \),
(ii) if \( \varphi_1, \varphi_2 \in \Phi \), then \( \varphi_1 \lor \varphi_2, \neg \varphi_1 \in \Phi \),
(iii) if \( \varphi \in \Phi \), then \( \varphi? \in \text{Tests} \), and
(iv) if \( \varphi \in \Phi \) and \( \Psi \subseteq \text{Tests} \) is finite, then \( \langle \chi \rangle \varphi \in \Phi \),

where \( \chi \) is a regular expression over \( \Sigma \cup \Psi \) or \( \chi \in L \).

A Kripke structure is a tuple \( K = (X, \{\rightarrow_a \mid a \in \Sigma\}, \{X_p \mid p \in P\}) \), where \( X \) is
We say that every set of worlds, \( \rightarrow_a \subseteq X \times X \) is a binary relation for each \( a \in \Sigma \), and \( X_p \subseteq X \) is a unary relation for each \( p \in \mathbb{P} \). For each \( \varphi \in \Phi \) and for each \( w \in (\Sigma \cup \text{Tests})^* \), define the binary relation \( [w]_K \subseteq X \times X \) and the set \( [\varphi]_K \subseteq X \) via mutual induction as follows:

- \( [\varepsilon]_K = \{(x, x) \mid x \in X\} \),
- if \( \varphi' \in \text{Tests} \), then \( [\varphi']_K = \{(x, x) \mid x \in X \land x \in [\varphi]_K\} \),
- if \( a \in \Sigma \), then \( [a]_K = \rightarrow_a \),
- if \( w \in (\Sigma \cup \text{Tests})^* \) and \( \tau \in \Sigma \cup \text{Tests} \), then \( [w \tau]_K = [w]_K \circ [\tau]_K \),
- if \( p \in \mathbb{P} \), then \( [p]_K = X_p \),
- \( [\varphi_1 \lor \varphi_2]_K = [\varphi_1]_K \cup [\varphi_2]_K \),
- \( [\neg \varphi]_K = X \setminus [\varphi]_K \),
- \( [\langle x \rangle \varphi]_K = \{x \in X \mid \exists y \in X \exists w \in L(\chi) : (x, y) \in [w]_K \land y \in [\varphi]_K\} \).

We say that \( K \) is a model for \( \varphi \), if \( x \in [\varphi]_K \) for some world \( x \) of \( K \). We say that a formula \( \varphi \) is satisfiable, if there exists a model for \( \varphi \). The satisfiability problem asks, given a formula \( \varphi \), whether \( \varphi \) is satisfiable.

### 3 \( \Sigma_1^1 \)-Completeness of PDL+\( k \)-MVPL

It is not hard to see that satisfiability of PDL+\( k \)-MVPL is in \( \Sigma_1^1 \). Firstly, we can easily adapt the proof of Proposition 9.4 in [2] and show that every satisfiable PDL+\( k \)-MVPL formula \( \varphi \) has a countable tree model. Secondly, we can write down an existential second-order formula over \( \mathbb{N} \) that is valid if and only if \( \varphi \) is satisfiable.

Before we give the matching \( \Sigma_1^1 \) lower bound let us, for finite words \( u \) and \( v \), define the language \( u^Nv^N := \{u^nv^n \mid n \geq 1\} \). In the following, fix the language class \( \mathcal{L}_0 = \{(a_2b_2)^N(a_1b_1)^N, (a_2b_1)^N(a_1b_2)^N\} \). The following Proposition is obvious.

**Proposition 1** There exist 2-MVPA \( M_1, M_2 \) over a common 2-stack alphabet such that \( L(M_1) = (a_1b_2)^N(a_2b_1)^N \) and \( L(M_2) = (a_1b_1)^N(a_1b_2)^N \).

For the lower bound, we prove that already satisfiability for PDL+\( \mathcal{L}_0 \) is \( \Sigma_1^1 \)-hard. For this, we use the following result.

**Theorem 2** ([3]) Satisfiability of PDL+\{\( a^N b^N, b^N a^N \}\} is \( \Sigma_1^1 \)-hard.

Hence by Fact 1 and Theorem 2 it remains to give a satisfiability preserving translation from PDL+\{\( a^N b^N, b^N a^N \}\) to PDL+\( \mathcal{L}_0 \). The translation is straightforward. Define a homomorphism \( h : \{a, b\}^* \rightarrow \{a_1, a_2, b_1, b_2\}^* \) as follows, \( h(a) = a_1b_2 \) and \( h(b) = a_2b_1 \). For a PDL+\{\( a^N b^N, b^N a^N \}\) formula \( \varphi \), let the PDL+\( \mathcal{L}_0 \) formula \( \varphi' \) emerge from \( \varphi \) by replacing each occurrence of \( a \) by \( h(a) \).
and each occurrence of $b$ by $h(b)$. Finally, it suffices to prove the following lemma.

**Lemma 3** The formula $\varphi$ is satisfiable if and only if $\varphi'$ is satisfiable.

**PROOF.** “Only-if”: Assume $\varphi$ is satisfiable. Let $K = (X, \{\rightarrow_a, \rightarrow_b\}, \{X_p \mid p \in P\})$ be a model of $\varphi$, i.e. $x_0 \in [\varphi]_K$ for some $x_0 \in X$. Let $K'$ be the Kripke structure that is obtained from $K$ by (i) replacing each transition $x \rightarrow_a y$ by a sequence of two transitions $x \rightarrow_{a_1} z \rightarrow_{b_2} y$ for some fresh world $z$ in $K'$, (ii) replacing each transition $x \rightarrow_b y$ by a sequence of two transitions $x \rightarrow_{a_2} z \rightarrow_{b_1} y$ for some fresh world $z$ in $K'$, and (iii) keeping $X_p$ unchanged for each $p \in P$. Now for each $x, y \in X$ and for each $w \in \{a, b\}^*$ we have $(x, y) \in [w]_K$ if and only if $(x, y) \in [h(w)]_{K'}$. By an induction on the structure of $\varphi$, one can prove that $x_0 \in [\varphi']_{K'}$. Thus, $K'$ is a model for $\varphi'$.

“If”: Assume $\varphi'$ is satisfiable. Let $K' = (X, \{\rightarrow_a, \rightarrow_b\} \mid i = 1, 2, \{X_p \mid p \in P\})$ be a model of $\varphi'$, i.e. $x_0 \in [\varphi']_{K'}$ for some $x_0 \in X$. Now define the Kripke structure $K = (X, \{\rightarrow_a, \rightarrow_b\}, \{X_p \mid p \in P\})$ where $\rightarrow_a = \{(x, y) \in X \times X \mid \exists z : x \rightarrow_{a_1} z \rightarrow_{b_2} y\}$ and $\rightarrow_b = \{(x, y) \in X \times X \mid \exists z : x \rightarrow_{a_2} z \rightarrow_{b_1} y\}$. As above for each $x, y \in X$ and for each $w \in \{a, b\}^*$ we have $(x, y) \in [w]_K$ if and only if $(x, y) \in [h(w)]_{K'}$. Again, by an induction on the structure of $\varphi$, one can prove that $x_0 \in [\varphi]_K$. Thus, $K$ is a model for $\varphi$.

**References**


