CONVERGENCE WITHOUT POINTS

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Abstract. We introduce a pointfree theory of convergence on lattices and coframes. A convergence lattice is a lattice L with a monotonic map $\lim_L$ from the lattice of filters on L to L, meant to be an abstract version of the map sending every filter of subsets to its set of limits. This construction exhibits the category of convergence spaces as a coreflective subcategory of the opposite of the category of convergence lattices. We extend this construction to coreflections between limit spaces and the opposite of so-called limit lattices and limit coframes, between pretopological convergence spaces and the opposite of so-called pretopological convergence coframes, between adherence spaces and the opposite of so-called adherence coframes, between topological spaces and the opposite of so-called topological coframes. All of our pointfree categories are complete and cocomplete, and topological over the category of coframes. Our final pointfree category, that of topological coframes, shares with the category of frames the property of being in a dual adjunction with the category of topological spaces. We show that the latter arises as a retract of the former, and that this retraction restricts to a reflection between frames and so-called strong topological coframes.

1. Introduction

Locales are pointfree analogues of topological spaces. This is by now a well-known concept [7, 9]. Our aim is to propose a pointfree analogue of convergence spaces. That may seem curious at best, since convergence is a relation between sequences, or better, filters, and *points*. Nevertheless, as we shall see, there is a natural way of doing so, generalizing the classical Stone duality between topological spaces and locales to one between the category **Conv** of convergence spaces and the opposite category of a category $\mathbf{CF}^{\mathrm{conv}}$ of coframes with additional structure representing an abstract form of convergence. We extend those dualities to limit spaces, pretopological spaces, and topological spaces.

Our developments do not require the Axiom of Choice.

We introduce our basic notion of convergence **C**-object, where **C** is a category of inf-semilattices, in Section 2. When **C** is a category of lattices, we obtain a coreflective adjunction $\mathbb{P} \dashv \mathrm{pt}$ between **Conv** and the opposite of the category $\mathbf{C}^{\mathrm{conv}}$ of convergence **C**-objects. When **C** is a category of coframes, we show that $\mathbf{C}^{\mathrm{conv}}$ is complete and cocomplete in Section 3, by showing that it is topological over the category **CF** of coframes.

We then proceed to similar constructions for limit spaces, pretopological and topological spaces. We organize our main results in the following commutative diagram of adjunctions. The bottom row consists of familiar categories of convergence

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spaces, limit spaces, pretopological convergence spaces and adherence spaces, and topological spaces. The top row are the matching pointfree situations.

(1.1)
$$\begin{tikzcd}
(\mathbf{CF}^{\mathrm{conv}})^{op} \arrow[r, shift left] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathrm{pt}\ (\mathrm{Sec.}\ 4)"] \arrow[d, phantom, "\dashv"] & (\mathbf{CF}^{\mathrm{lim}})^{op} \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathrm{pt}\ (\mathrm{Sec.}\ 5)"] \arrow[d, phantom, "\dashv"] & (\mathbf{CF}^{\mathrm{pretop}})^{op} \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathrm{pt}\ (\mathrm{Sec.}\ 7)"] \arrow[d, phantom, "\dashv"] & (\mathbf{CF}^{\mathrm{adh}})^{op} \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathrm{pt}\ (\mathrm{Sec.}\ 8)"] \arrow[d, phantom, "\dashv"] & (\mathbf{CF}^{\mathrm{top}})^{op} \arrow[l, shift left] \arrow[d, shift left, "\mathrm{pt}"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Conv} \arrow[u, shift left, "\mathbb{P}"] \arrow[r, shift left, "M_{\mathrm{lim}}"] \arrow[r, phantom, "\perp"] & \mathbf{Lim} \arrow[u, shift left, "\mathbb{P}"] \arrow[l, shift left, "\supseteq"] \arrow[r, shift left, "M_{\mathrm{pretop}}"] \arrow[r, phantom, "\perp"] & \mathbf{PreTop} \arrow[u, shift left, "\mathbb{P}"] \arrow[l, shift left, "\supseteq"] \arrow[r, equal] & \mathbf{Adh} \arrow[u, shift left, "\mathbb{P}"] \arrow[r, shift left, "M_{\mathrm{top}}"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[u, shift left, "\mathbb{P}"] \arrow[l, shift left, "\supseteq"]
\end{tikzcd}$$

In addition, Section 6 introduces the notions of open and closed elements in a convergence **C**-object, and adherence operators associated with a convergence. This is notably required in the study of the connection between pretopological **C**-objects and adherence **C**-objects of Section 7.

2. Convergence Lattices

We use [3] as a modern reference on convergence spaces.

An *inf-semilattice* is a poset with all finite infima. Some authors call that a bounded inf-semilattice, and reserve the term "inf-semilattice" for posets with binary infima (a.k.a., binary meets), possibly missing the top element $\top$. Given a poset L, L^{op} is its opposite poset, whose ordering is reversed. A *sup-semilattice* L is such that L^{op} is an inf-semilattice. A *lattice* has all finite infima and all finite suprema.

Given an inf-semilattice L, a *filter* on L is a non-empty upwards-closed subset of L that is closed under binary meets. We do allow L itself as a filter, contrarily to, say, [3, Section II.2], although that is not essential. All others are called *proper*. In general, an upwards-closed subset is proper, i.e., different from L, if and only if it does not contain the least element $\bot$ of L.

We write $\mathbb{F}L$ for the set of filters on L. For $L = \mathbb{P}X$, the lattice of all subsets of a set X, a filter on L is called a *filter of subsets* of X. A maximal proper filter of subsets of X is an *ultrafilter* on X.

A *convergence space* is a pair $(X, \to)$ of a set X and a binary relation $\to$ between filters $\mathcal{F}$ of subsets of X and points x of X (in notation, $\mathcal{F} \to x$, meaning "$\mathcal{F}$ converges to x", or "x is a limit of $\mathcal{F}$") such that:

- (Point Axiom.) $\dot{x} \to x$, where $\dot{x} = \{S \subseteq X \mid x \in S\}$ is the *principal ultrafilter* at x;
- (Monotonicity Axiom.) If $\mathcal{F} \to x$ and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G} \to x$.

We shall write X for $(X, \to)$, leaving the notion of convergence $\to$ implicit.

A standard example of a convergence space is given by topological spaces. The *standard convergence* on a topological space X is given by $\mathcal{F} \to x$ if and only if every open neighborhood of x belongs to $\mathcal{F}$. There are many examples of convergence spaces that do not arise from a topological space this way, see [3, Section III.1].

Convergence spaces form a category **Conv**. A morphism $f : X \to Y$ is a *continuous map*, meaning one that preserves convergence: if $\mathcal{F} \to x$ then $f[\mathcal{F}] \to f(x)$, where the *image filter* $f[\mathcal{F}]$ is $\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{F}\}$. A map f is *initial* (in the categorical sense [1, Definition 8.6]) if and only if that implication is an equivalence. The injective initial maps are exactly the embeddings of convergence spaces.

2.1. The pointfree analogue of convergence spaces.

We obtain an analogous theory of *pointfree* convergence by abstracting away from the lattice $\mathbb{P}(X)$, and replacing it by an inf-semilattice L, possibly with extra structure. Interesting inf-semilattices with extra structure are complete Boolean algebras, such as $\mathbb{P}(X)$, and also frames and coframes. Let us recall that a *frame* is a complete lattice in which arbitrary suprema distribute over binary infima. A *coframe* is a poset L whose opposite L^{op} is a frame. In other words, a coframe is a complete lattice in which arbitrary infima distribute over binary suprema. Morphisms of coframes are required to preserve all infima, and finite suprema.

In order to justify our definitions, we shall often start with the standard definition, using points, and massage it into a definition that no longer mentions points, and recast it in an arbitrary poset, lattice, or coframe, depending on what we require.

Honor to whom honor is due, let us start with the notion of *limit*. Writing $\lim_{\mathbb{P}(X)} \mathcal{F}$ for the set of limits of a filter of subsets $\mathcal{F}$, and ordering both filters and $\mathbb{P}(X)$ by inclusion, the Monotonicity Axiom states that $\lim_{\mathbb{P}(X)}$ is monotonic. The Point Axiom is meaningless in a pointfree setting, and accordingly we ignore it for the time being. It will resurface naturally later.

Definition 2.1. A *convergence semilattice* is an inf-semilattice L together with a monotonic map $\lim_L \colon \mathbb{F}(L) \to L$.

Remark 2.2. There is another view on convergence semilattices, obtained as follows. A filter on L is an *ideal* of L^{op}. Then $\mathbb{F}L$ is equal to $\mathbf{I}(L^{op})$, where $\mathbf{I}$ denotes ideal completion (the poset of all ideals, ordered by inclusion). A convergence semilattice L is then exactly the same thing as a sup-semilattice $\Omega := L^{op}$, together with an *antitone* map $\lim \colon \mathbf{I}(\Omega) \to \Omega$.

By extension, a convergence lattice is a lattice L with a monotonic map, and similarly for convergence Boolean algebras, convergence frames, convergence coframes. By varying the underlying category of inf-semilattices, we obtain:

Definition 2.3 (Convergence $\mathbf{C}$-object)**.** Let $\mathbf{C}$ be a category of inf-semilattices. A *convergence* $\mathbf{C}$*-object* is an object L of $\mathbf{C}$ together with a map $\lim_L \colon \mathbb{F}(L) \to L$ that is monotonic.

The category $\mathbf{C}^{\mathrm{conv}}$ has as objects the convergence $\mathbf{C}$-objects, and as morphisms $\varphi \colon L \to L'$ the morphisms from L to L' in $\mathbf{C}$ that are *continuous*, in the sense that for every $\mathcal{F} \in \mathbb{F}(L')$,

$$\lim_{L'} \mathcal{F} \leq \varphi(\lim_L \varphi^{-1}(\mathcal{F})). \tag{2.1}$$

The morphism φ is *final* if and only if equality holds in (2.1).

We shall often write $\lim \mathcal{F}$ instead of $\lim_L \mathcal{F}$, when the ambient inf-semilattice L is clear from context. We shall also write L instead of the convergence $\mathbf{C}$-object $(L, \lim_L)$, when $\lim_L$ is unambiguous.

The morphism condition (2.1) is obtained in such a way that for every continuous map f between convergence spaces, the inverse image map is a morphism of convergence posets. Let us do the exercise. A map $f \colon X \to Y$ between convergence spaces is continuous if and only if, for every filter $\mathcal{F}$ of subsets of X, every point x such that $\mathcal{F} \to x$, namely every point x of $\lim_{\mathbb{P}(X)} \mathcal{F}$, is such that $f(x) \in \lim_{\mathbb{P}(Y)} f[\mathcal{F}]$: in other words, the continuity condition for f reads $\lim_{\mathbb{P}(X)} \mathcal{F} \subseteq f^{-1}(\lim_{\mathbb{P}(Y)} f[\mathcal{F}])$, and that no longer mentions points. Write φ for $f^{-1} \colon \mathbb{P}(Y) \to \mathbb{P}(X)$. Then $f[\mathcal{F}] = \{B \subseteq Y \mid \varphi(B) \in \mathcal{F}\} = \varphi^{-1}(\mathcal{F})$, and the condition $\lim_{\mathbb{P}(X)} \mathcal{F} \subseteq f^{-1}(\lim_{\mathbb{P}(Y)} f[\mathcal{F}])$ is then exactly (2.1).

Remark 2.4. Our notion of final morphism agrees with the usual categorical notion [1, Definition 8.10], and is a pointfree analogue of *initial* morphisms in **Conv**. Categorically, a morphism $\varphi\colon L \to L'$ in $\mathbf{C}^{\mathrm{conv}}$ is final if and only if for every object L'' in $\mathbf{C}$, for every morphism $\psi\colon L' \to L''$ in $\mathbf{C}$ such that $\psi \circ \varphi$ is continuous, ψ is continuous. If φ is final in this sense, then take $L'' = L'$, $\psi = \mathrm{id}_{L'}$, and $\lim_{L''} \mathcal{F} = \varphi(\lim_L \varphi^{-1}(\mathcal{F}))$: then φ is final in the sense of Definition 2.3. The converse implication is immediate. Here $\lim_{L''}$ is an example of a final convergence structure on L'. We shall generalize this argument in Corollary 3.3.

Our constructions will work on various categories of lattices, but not all. We shall use the following, non-minimal but convenient set of assumptions.

Definition 2.5. A category $\mathbf{C}$ of inf-semilattices is *admissible* if and only if:

- for every set X, $\mathbb{P}(X)$ is an object of $\mathbf{C}$;
- there are two classes of index sets $\mathcal{I}$ and $\mathcal{J}$ such that, for all objects L and L' of $\mathbf{C}$, the morphisms from L to L' are exactly the monotonic maps that preserve all I-indexed infima that exist in L, $I \in \mathcal{I}$, and all J-indexed suprema that exist in L, $J \in \mathcal{J}$.

For example, the categories of complete Boolean algebras, of frames, of coframes, are admissible. In the case of frames, $\mathcal{I}$ can be taken to be the class of finite sets and $\mathcal{J}$ can be taken to be the class of all sets.

Proposition 2.6. *Let $\mathbf{C}$ be an admissible category of inf-semilattices. There is a functor $\mathbb{P}\colon \mathbf{Conv} \to (\mathbf{C}^{\mathrm{conv}})^{op}$ defined:*

- *on objects by: $\mathbb{P}(X)$ is the powerset of X, with inclusion ordering $\subseteq$, and with $\lim_{\mathbb{P}(X)} \mathcal{F} = \{x \in X \mid \mathcal{F} \to x\}$;*
- *on morphisms by $\mathbb{P}(f)(S) = f^{-1}(S)$, for every continuous map $f\colon X \to Y$ and every $S \in \mathbb{P}(Y)$.*

Proof. The main point consists in checking that, for a continuous map $f\colon X \to Y$, $\mathbb{P}(f)$ is a morphism of convergence $\mathbf{C}$-objects from $\mathbb{P}(Y)$ to $\mathbb{P}(X)$. $\mathbb{P}(f)$ preserves all infima and all suprema, hence is certainly a morphism in $\mathbf{C}$ by admissibility. Checking (2.1) was the purpose of our preliminary exercise. $\square$

2.2. **The point functor.** We now go the other way around, and define a so-called point functor $\mathrm{pt}\colon \mathbf{C}^{\mathrm{conv}} \to \mathbf{Conv}$.

To do so, we shall need to replace $\mathbf{C}$ by an (admissible) category *of lattices*. By that we mean that all objects of $\mathbf{C}$ should be lattices, but also that all morphisms should be lattice morphisms, that is, they should preserve all finite suprema and finite infima. A similar convention will apply later, when we further restrict to categories of frames, or of coframes. We will never need to restrict to categories of complete Boolean algebras (despite the temptation provided by $\mathbb{P}(X)$), and coframes will turn out to be the right notion, in particular to define the functor $S\ell$ in the final part of this paper (Lemma 8.23).

There is a general category-theoretic notion of point of an object L in a category. A point is a morphism from a terminal object 1 to L. In **Conv**, the terminal object is the one-element set $1 = \{*\}$, with the only possible notion of convergence on it. $\mathbb{P}(1)$, where $\mathbb{P}$ is the functor given in Proposition 2.6, is the two-element coframe $\{\emptyset, 1\}$, with $\lim_{\mathbb{P}(1)}$ mapping every filter to 1. (We must have $\lim_{\mathbb{P}(1)}\{1\} = \{*\} = 1$, since $\{1\} = \dot{*}$, and by monotonicity $\lim_{\mathbb{P}(1)}\{\emptyset, 1\} = 1$. This completes the argument, since $\{1\}$ and $\{\emptyset, 1\}$ are the only two filters of $\mathbb{P}(1)$.)

Lemma 2.7. *Let* $\mathbf{C}$ *be an admissible category of lattices.* $\mathbb{P}(1)$ *is a terminal object in* $(\mathbf{C}^{\mathrm{conv}})^{op}$*.*

Proof. In other words, we claim that $\mathbb{P}(1)$ is an initial object in $\mathbf{C}^{\mathrm{conv}}$. It is an object at all, by admissibility. Let L be an arbitrary convergence $\mathbf{C}$-object. Any morphism $\varphi\colon \mathbb{P}(1) \to L$, being a lattice morphism, must map 1 to the top element $\top$ of L, and $\emptyset$ to the bottom element $\perp$ of L. (That would not work with a mere inf-semilattice morphism, hence our assumption that $\mathbf{C}$ is a category of lattices.) Now let φ be that map. For every $\mathcal{F} \in \mathbb{F}(L)$, $\varphi(\lim_{\mathbb{P}(1)} \varphi^{-1}(\mathcal{F})) = \varphi(1) = \top \geq \lim_L \mathcal{F}$, establishing (2.1). ☐

Accordingly:

Definition 2.8 (Point)**.** Let $\mathbf{C}$ be an admissible category of lattices. For every convergence $\mathbf{C}$-object L, the *points* of L are the morphisms from L to $\mathbb{P}(1)$ in $\mathbf{C}^{\mathrm{conv}}$. We write $\mathrm{pt}\, L$ for the set of points of L.

The notion of point therefore depends on the chosen category $\mathbf{C}$. We explore the notion, in a few selected cases, in the following remarks.

Remark 2.9. In all cases, it will be profitable to note that $\varphi\colon L \to \mathbb{P}(1)$ is continuous, in the sense that it satisfies (2.1), if and only if $\varphi(\lim_L \varphi^{-1}(\mathcal{G})) = 1$ for every filter $\mathcal{G}$ on $\mathbb{P}(1)$ (since $\lim_{\mathbb{P}(1)} \mathcal{G} = 1$), if and only if $\varphi(\lim_L \varphi^{-1}(\{1\})) = 1$ (since all involved maps are monotonic, and $\{1\}$ is the smallest filter on $\mathbb{P}(1)$), if and only if $\lim_L \mathcal{F} \in \mathcal{F}$, where $\mathcal{F}$ is the filter $\varphi^{-1}(\{1\})$. Hence it is fair to equate the points of L in $\mathbf{C}^{\mathrm{conv}}$ with certain filters $\mathcal{F}$ such that $\lim_L \mathcal{F} \in \mathcal{F}$.

Remark 2.10 (Points of convergence lattices). Let $\mathbf{C}$ be the category of lattices. A map $\varphi\colon L \to \mathbb{P}(1)$ is a point if and only if $\mathcal{F} = \varphi^{-1}(\{1\})$ is a *prime* filter, namely a proper filter that does not contain $\ell_1 \vee \ell_2$ unless it already contains ℓ_1 or ℓ_2. Points can then be equated with prime filters $\mathcal{F}$ such that $\lim_L \mathcal{F} \in \mathcal{F}$.

Remark 2.11 (Points of convergence frames). Let $\mathbf{C}$ be the category of frames, or more generally any admissible category of frames. A map $\varphi\colon L \to \mathbb{P}(1)$ is a frame morphism if and only if $\mathcal{F} = \varphi^{-1}(\{1\})$ is a *completely prime* filter, in the sense that $\bigvee_{i \in I} \ell_i \in \mathcal{F}$ implies $\ell_i \in \mathcal{F}$ for some $i \in I$. The space of completely prime elements of the frame L will be written $\mathrm{pt}^{\mathrm{s}}\, L$, and is one half of the famous Stone adjunction between topological spaces and frames. Hence points of convergence frames can be equated with those completely prime filters $\mathcal{F}$ such that $\lim_L \mathcal{F} \in \mathcal{F}$.

In turn, the completely prime filters $\mathcal{F}$ are exactly the families $\complement{\downarrow}\, \ell$, where ℓ is a meet-prime element, the largest element of L that is not in $\mathcal{F}$. (We write $\complement$ for complement.) An element ℓ of L is *meet-prime* if and only if $\bigwedge_{i=1}^{n} \ell_i \leq \ell$ implies $\ell_i \leq \ell$ for some $i \in I$. We can then equate points of convergence frames with meet-primes ℓ such that $\lim_L \complement{\downarrow}\, \ell \not\leq \ell$.

Remark 2.12. Following up on Remark 2.11, $\mathrm{pt}\, L$ is included in $\mathrm{pt}^{\mathrm{s}}\, L$ for every frame L, but one should stress that the inclusion is in general proper. Consider for example the *discrete* convergence defined by $\lim_L \mathcal{F} = \perp$, for every filter $\mathcal{F}$ on L. In that case, $\mathrm{pt}\, L$ is empty, although $\mathrm{pt}^{\mathrm{s}}\, L$ can be arbitrary large.

Remark 2.13 (Points of convergence coframes). Let $\mathbf{C}$ be the category of coframes, or more generally any admissible category of coframes. A map $\varphi\colon L \to \mathbb{P}(1)$ is a coframe morphism if and only if $\mathcal{F} = \varphi^{-1}(\{1\})$ is a filter that is closed under arbitrary infima. Such filters are exactly those of the form $\uparrow \ell$, where ℓ is a *join-prime* of L, namely: for any finite family of elements $\ell_1, \cdots, \ell_n$ of L such that $\ell \leq \bigvee_{i=1}^n \ell_i$, there is an index i such that $\ell \leq \ell_i$ already. Equivalently, a join-prime is an element different from $\bot$ such that $\ell \leq \ell_1 \vee \ell_2$ implies $\ell \leq \ell_1$ or $\ell \leq \ell_2$.

It is therefore natural to (re)define the points of a convergence coframe L as the join-primes ℓ of L such that $\ell \leq \lim_L \uparrow \ell$.

In general, given an admissible category $\mathbf{C}$ of lattices, it is worthwhile to see what the points of the convergence $\mathbf{C}$-object $\mathbb{P}(X)$ are, for a given convergence space X. We shall see that the Point Axiom emerges naturally here.

Remark 2.14. Let $\mathbf{C}$ be an admissible category of lattices, and X be a convergence space. For every $x \in X$, consider the constant map $\overline{x} \colon * \mapsto x$ from 1 to X. By Proposition 2.6, $\mathbb{P}(\overline{x})$ is a morphism from $\mathbb{P}(X)$ to $\mathbb{P}(1)$ in $\mathbf{C}$. By Remark 2.9, this can be equated with the filter $\mathbb{P}(\overline{x})^{-1}(\{1\}) = \{S \in \mathbb{P}(X) \mid \overline{x}^{-1}(S) = 1\} = \{S \in \mathbb{P}(X) \mid x \in S\} = \dot{x}$. That always defines a point in $\mathrm{pt}\, L$: the condition of being a point reads $\lim_{\mathbb{P}(X)} \dot{x} \in \dot{x}$, and is exactly what the Point Axiom states. Hence $\mathrm{pt}\, \mathbb{P}(X)$ always contains the points $\mathbb{P}(\overline{x}) \cong \dot{x}$, for every $x \in X$.

Remark 2.15 (Points of $\mathbb{P}(X)$ qua convergence lattice). Let $\mathbf{C}$ be the category of all lattices. The prime filters of $\mathbb{P}(X)$ are exactly the ultrafilters on X. Hence the points of $\mathbb{P}(X)$, qua convergence lattice, are the ultrafilters $\mathcal{U}$ such that $\lim_{\mathbb{P}(X)} \mathcal{U} \in \mathcal{U}$. Those are exactly the *compact* ultrafilters on X. (We let the reader check that this notion agrees with the usual notion of compactness for filters, once specialized to ultrafilters [3, Definition IX.7.1].) Among all compact ultrafilters, one finds the principal ultrafilters $\dot{x}$, if only because of Remark 2.14, but there are others in general. Look indeed at the special case where X is topological: then every compact ultrafilter is principal if and only if X is Noetherian; hence any non-Noetherian topological space will have non-principal compact ultrafilters.

Remark 2.16 (Points of $\mathbb{P}(X)$ qua convergence frame). Let $\mathbf{C}$ be the category of frames, or more generally any admissible category of frames. The situation is much simpler here. We rely on Remark 2.11. The meet-primes ℓ of $\mathbb{P}(X)$ are exactly the complements of one-element sets $\{x\}$, $x \in X$. The condition $\lim_L \complement\!\downarrow \ell \not\leq \ell$ rewrites as $x \in \lim_{\mathbb{P}(X)} \dot{x}$. The latter is just the Point Axiom $\dot{x} \to x$. Hence $\mathrm{pt}\, \mathbb{P}(X)$ can be equated with X itself.

Remark 2.17 (Points of $\mathbb{P}(X)$ qua convergence coframe). Let $\mathbf{C}$ be the category of coframes, or more generally any admissible category of coframes. The situation is equally simple. We rely on Remark 2.13. The join-primes ℓ of $\mathbb{P}(X)$ are exactly the one-element sets $\{x\}$, $x \in X$. The condition $\ell \leq \lim_L \uparrow \ell$ rewrites as $x \in \lim_{\mathbb{P}(X)} \dot{x}$. As for frames, this is the Point Axiom. Hence, again, $\mathrm{pt}\, \mathbb{P}(X)$ can be equated with X itself.

In order to define a lim operator on $\mathrm{pt}\, L$, we introduce the following. For a filter $\mathcal{F}$ of subsets of $\mathrm{pt}\, L$, $\mathcal{F}^\circ$ will be a corresponding filter on L.

Definition 2.18. Let $\mathbf{C}$ be a category of lattices, and L be a convergence $\mathbf{C}$-object. For every $\ell \in L$, let $\ell^\bullet$ be the set of points $\varphi \in \mathrm{pt}\, L$ such that $\varphi(\ell) = 1$.

For every $\mathcal{F} \in \mathbb{FP}(\mathrm{pt}\, L)$, let:

$$\mathcal{F}^\circ := \{\ell \in L \mid \ell^\bullet \in \mathcal{F}\}.$$

Lemma 2.19. *Let* $\mathbf{C}$ *be an admissible category of lattices, and* L *be a convergence* $\mathbf{C}$*-object. The following hold:*

(1) *For all* ℓ, ℓ' *in* L*, if* $\ell \leq \ell'$ *then* $\ell^\bullet \subseteq \ell'^\bullet$.
(2) *For all* $\mathcal{F}, \mathcal{G} \in \mathbb{FP}(\operatorname{pt} L)$*, if* $\mathcal{F} \subseteq \mathcal{G}$ *then* $\mathcal{F}^\circ \subseteq \mathcal{G}^\circ$.
(3) *For all* $\ell_1, \ell_2, \cdots, \ell_n \in L$ $(n \in \mathbb{N})$*,* $\bigcup_{i=1}^n \ell_i^\bullet = (\bigvee_{i=1}^n \ell_i)^\bullet$.
(4) *For all* $\ell_1, \ell_2, \cdots, \ell_n \in L$ $(n \in \mathbb{N})$*,* $\bigcap_{i=1}^n \ell_i^\bullet = (\bigwedge_{i=1}^n \ell_i)^\bullet$.
(5) *For every* $\mathcal{F} \in \mathbb{FP}(\operatorname{pt} L)$*,* $\mathcal{F}^\circ$ *is a filter on* L*, i.e.,* $\mathcal{F}^\circ \in \mathbb{F}L$.
(6) *For every* $\varphi \in \operatorname{pt} L$*,* $\dot{\varphi}^\circ = \varphi^{-1}(\{1\})$.
(7) *For every* $x \in \operatorname{pt} L$*,* x *is in* $(\lim_L \dot{x}^\circ)^\bullet$.

Proof. (1) and (2) are obvious. (3) For every point φ, φ is in $(\bigvee_{i=1}^n \ell_i)^\bullet$ if and only if $\varphi(\bigvee_{i=1}^n \ell_i) = 1$, if and only if $\bigvee_{i=1}^n \varphi(\ell_i) = 1$ (since φ is a lattice morphism), if and only if $\varphi(\ell_i) = 1$ for some i, if and only if φ is in some $\ell_i^\bullet$. (4) is proved similarly.

(5) We first check that $\mathcal{F}$ is upwards-closed. Let $\ell \in \mathcal{F}^\circ$ and $\ell \leq \ell'$. By definition, $\ell^\bullet$ is in $\mathcal{F}$. Using (1), $\ell'^\bullet$ is also in $\mathcal{F}$, so ℓ' is in $\mathcal{F}^\circ$. $\mathcal{F}^\circ$ is non-empty: since $\top^\bullet = \operatorname{pt} L$ is in $\mathcal{F}$, $\top$ is in $\mathcal{F}^\circ$. Finally, for any two elements ℓ, ℓ' of $\mathcal{F}^\circ$, $\ell^\bullet$ and $\ell'^\bullet$ are in $\mathcal{F}$, so $\ell^\bullet \cap \ell'^\bullet$ is in $\mathcal{F}$. Using (4), $(\ell \wedge \ell')^\bullet$ is in $\mathcal{F}$, so $\ell \wedge \ell'$ is in $\mathcal{F}^\circ$.

(6) The set $\dot{\varphi}^\circ$ is the set of elements $\ell \in L$ such that $\ell^\bullet \in \dot{\varphi}$, equivalently such that $\varphi \in \ell^\bullet$, equivalently such that $\varphi(\ell) = 1$.

(7) For every $\varphi \in \operatorname{pt} L$, by Remark 2.9, $\varphi(\lim_L \varphi^{-1}(\{1\})) = 1$. Using (6), this means that $\varphi(\lim_L \dot{\varphi}^\circ) = 1$, and by definition this is equivalent to $\varphi \in (\lim_L \dot{\varphi}^\circ)^\bullet$. ☐

This allows us to define:

Definition 2.20. Let $\mathbf{C}$ be a category of lattices. For every convergence $\mathbf{C}$-object L, define $\lim_{\mathbb{P}(\operatorname{pt} L)} \mathcal{F}$ as $(\lim_L \mathcal{F}^\circ)^\bullet$. In other words, $\mathcal{F} \to x$ in $\operatorname{pt} L$ if and only if $x \in (\lim_L \mathcal{F}^\circ)^\bullet$.

Lemma 2.21. *Let* $\mathbf{C}$ *be a category of lattices. For every convergence* $\mathbf{C}$*-object* L*,* $(\operatorname{pt} L, \to)$ *as defined in Definition 2.20 is a convergence space.*

Proof. By (1) and (2) of Lemma 2.19, and since $\lim_L$ is monotonic, $\lim_{\mathbb{P}(\operatorname{pt} L)}$ is monotonic as well. The Point Axiom is item (7) of the same lemma. ☐

In the following, $\operatorname{pt} L$ will always define the convergence space obtained with that notion of convergence.

2.3. **The** $\mathbb{P} \dashv \operatorname{pt}$ **adjunction.** The pt construction satisfies the following universal property.

Proposition 2.22. *Let* $\mathbf{C}$ *be an admissible category of lattices,* X *be a convergence space, and* L *be a convergence* $\mathbf{C}$*-object. For every morphism* $\varphi \colon L \to \mathbb{P}(X)$ *in* $\mathbf{C}^{\mathrm{conv}}$*, there is a unique map* $\varphi^\dagger \colon X \to \operatorname{pt} L$ *such that, for every* $\ell \in L$*,* $\mathbb{P}(\varphi^\dagger)(\ell^\bullet) = \varphi(\ell)$*, and it is continuous.*

Proof. If $\varphi^\dagger$ exists, then for every $\ell \in L$,

$$\begin{aligned} \mathbb{P}(\varphi^\dagger)(\ell^\bullet) &= {\varphi^\dagger}^{-1}(\ell^\bullet) \\ &= \{x \in X \mid \varphi^\dagger(x) \in \ell^\bullet\} \\ &= \{x \in X \mid \varphi^\dagger(x)(\ell) = 1\} \end{aligned} \tag{2.2}$$

must be equal to $\varphi(\ell)$. This forces $\varphi^\dagger(x)(\ell)$ to be equal to 1 if $x \in \varphi(\ell)$, and to $\emptyset$ otherwise, establishing uniqueness.

Now define $\varphi^\dagger$ as mapping each $x \in X$ to the map:

$$\ell \mapsto \begin{cases} 1 & \text{if } x \in \varphi(\ell) \\ \emptyset & \text{otherwise.} \end{cases} \tag{2.3}$$

We show that $\varphi^\dagger(x)$ is a point of L, that is, an element of $\operatorname{pt} L$, by the following argument. Recall from Remark 2.14 that, for every $x \in X$, $\mathbb{P}(\overline{x})$ is a point of $\mathbb{P}(X)$, i.e., a morphism from $\mathbb{P}(X)$ to $\mathbb{P}(1)$ in $\mathbf{C}^{\mathrm{conv}}$. Now we check that $\varphi^\dagger(x) = \mathbb{P}(\overline{x}) \circ \varphi$, and is therefore a morphism from L to $\mathbb{P}(1)$ in $\mathbf{C}^{\mathrm{conv}}$, that is, a point of L.

For every $\ell \in L$, in view of (2.2),

$$\mathbb{P}(\varphi^\dagger)(\ell^\bullet) = \{x \in X \mid \varphi^\dagger(x)(\ell) = 1\} = \varphi(\ell),$$

by (2.3). It remains to be shown that $\varphi^\dagger$ is continuous. For every filter $\mathcal{F}$ of subsets of X, the image filter $\varphi^\dagger[\mathcal{F}]$ is the set of subsets B of $\operatorname{pt} L$ such that ${\varphi^\dagger}^{-1}(B) \in \mathcal{F}$. In particular, $\varphi^\dagger[\mathcal{F}]^\circ$ is the set of elements $\ell \in L$ such that $\ell^\bullet \in \varphi^\dagger[\mathcal{F}]$, namely such that ${\varphi^\dagger}^{-1}(\ell^\bullet) \in \mathcal{F}$. We have just seen that ${\varphi^\dagger}^{-1}(\ell^\bullet) = \varphi(\ell)$. Therefore $\varphi^\dagger[\mathcal{F}]^\circ = \varphi^{-1}(\mathcal{F})$. The continuity condition (2.1) reads $\lim_{\mathbb{P}(X)} \mathcal{F} \subseteq \varphi(\lim_L \varphi^{-1}(\mathcal{F}))$. In particular, if $\mathcal{F}$ converges to x in X, x must belong to $\varphi(\lim_L \varphi^{-1}(\mathcal{F})) = \varphi(\lim_L \varphi^\dagger[\mathcal{F}]^\circ)$. Recall from (2.3) that $x \in \varphi(\ell)$ if and only if $\varphi^\dagger(x)(\ell) = 1$, for every $\ell \in L$. Hence $\varphi^\dagger(x)(\ell) = 1$ where $\ell = \lim_L \varphi^\dagger[\mathcal{F}]^\circ$. By definition, this means that $\varphi^\dagger(x)$ is in $\ell^\bullet$. However, $\ell^\bullet = (\lim_L \varphi^\dagger[\mathcal{F}]^\circ)^\bullet = \lim_{\mathbb{P}(\operatorname{pt} L)} \varphi^\dagger[\mathcal{F}]$ by Definition 2.20. Therefore $\varphi^\dagger(x)$ is in $\lim_{\mathbb{P}(\operatorname{pt} L)} \varphi^\dagger[\mathcal{F}]$, and this completes our continuity argument. □

Lemma 2.23. *Let* $\mathbf{C}$ *be an admissible category of lattices, and* L *be a convergence* $\mathbf{C}$*-object. The map* $\epsilon_L \colon L \to \mathbb{P}(\operatorname{pt} L)$ *defined by* $\epsilon_L(\ell) = \ell^\bullet$ *is a morphism in* $\mathbf{C}^{\mathrm{conv}}$*, and is final.*

Proof. Consider the classes $\mathcal{I}$ and $\mathcal{J}$ posited in Definition 2.5. By the same argument as for Lemma 2.19 (3) and (4), we show that ϵ_L preserves I-indexed infima, $I \in \mathcal{I}$, and J-indexed suprema, $J \in \mathcal{J}$. Explicitly, $(\bigwedge_{i\in I} \ell_i)^\bullet$ is the set of points φ such that $\varphi(\bigwedge_{i\in I} \ell_i) = \bigwedge_{i\in I} \varphi(\ell_i)$ is equal to 1 (since φ preserves I-indexed infima), namely such that $\varphi(\ell_i) = 1$ for every $i \in I$, and this is the set $\bigcap_{i\in I} \ell_i^\bullet$. As far as suprema are concerned, the argument is similar, and uses the fact that a supremum of elements in the two-element lattice $\mathbb{P}(1)$ is equal to 1 if and only if one of those elements is equal to 1. Therefore ϵ_L is a morphism in $\mathbf{C}$.

As far as continuity is concerned, let $\mathcal{F}$ be a filter of subsets of $\operatorname{pt} L$. Noticing that $\mathcal{F}^\circ = \epsilon_L^{-1}(\mathcal{F})$, we have:

$$\lim_{\mathbb{P}(\operatorname{pt} L)} \mathcal{F} = (\lim_L \mathcal{F}^\circ)^\bullet = \epsilon_L(\lim_L \epsilon_L^{-1}(\mathcal{F}))$$

which shows that ϵ_L is not just continuous but final. □

By general categorical arguments (e.g., [8, Chap. IV, Sec. 1, Theorem 2]), one can define an adjoint pair of functors $F \dashv U$ by specifying a functor U, a family of morphisms $\epsilon_A \colon A \to UF(A)$ for each object A, in such a way that for every morphism $f \colon A \to U(B)$, there is a unique morphism $f^\dagger \colon F(A) \to B$ such that $U(f^\dagger) \circ \epsilon_A = f$. Here $U = \mathbb{P}$, $F = \operatorname{pt}$, ϵ_L maps ℓ to $\ell^\bullet$ and is a morphism by Lemma 2.23, so that Proposition 2.22 defines an adjoint pair of functors $\operatorname{pt} \dashv \mathbb{P}$ between $\mathbf{Conv}^{op}$ and $\mathbf{C}^{\mathrm{conv}}$.

We take the opposite categories, and obtain a dual adjunction where pt and $\mathbb{P}$ are exchanged.

Theorem 2.24. *Let* $\mathbf{C}$ *be an admissible category of lattices. Then* $\mathbb{P} \dashv \mathrm{pt}$ *is an adjunction between* **Conv** *and* $(\mathbf{C}^{\mathrm{conv}})^{op}$*.*

Referring again to Proposition 2.22, the unit $\epsilon_L \colon L \to \mathbb{P}(\mathrm{pt}\, L)$ (defined by $\epsilon_L(\ell) = \ell^\bullet$) of the dual adjunction $\mathrm{pt} \dashv \mathbb{P}$ in the opposite categories is the *counit* of the adjunction $\mathbb{P} \dashv \mathrm{pt}$.

In our recollection of adjoint pairs $F \dashv U$, we required U to be a functor, but not F. It is a fact that F automatically gives rise to a functor in the converse direction, whose action on morphisms $\varphi \colon A \to A'$ is given by $F(\varphi) = (\epsilon_{A'} \circ \varphi)^\dagger$. As a result, and seeing points of convergence lattices as certain filters, following Remark 2.9:

Lemma 2.25. *Let* $\mathbf{C}$ *be an admissible category of lattices, and let* $\varphi \colon L \to L'$ *be a morphism in* $\mathbf{C}^{\mathrm{conv}}$*. Then*

$$\mathrm{pt}\, \varphi = \varphi^{-1}.$$

Proof. In view of the discussion above, $\mathrm{pt}\, \varphi \colon \mathrm{pt}\, L' \to \mathrm{pt}\, L$ is equal to $(\epsilon_{L'} \circ \varphi)^\dagger$. By (2.3), for every $\psi \in \mathrm{pt}\, L'$ and for every $\ell \in L$, $\mathrm{pt}\, \varphi(\psi)(\ell) = 1$ if and only if $\psi \in \epsilon_{L'}(\varphi(\ell)) = \varphi(\ell)^\bullet$, if and only if $\psi(\varphi(\ell)) = 1$, from which we deduce that $\mathrm{pt}\, \varphi(\psi) = \psi \circ \varphi$. If we equate points $\psi \in \mathrm{pt}\, L'$ with filters $\mathcal{F} := \psi^{-1}(\{1\})$ as in Remark 2.9, we obtain that $\mathrm{pt}\, \varphi(\mathcal{F}) = \varphi^{-1}(\mathcal{F})$, that is, $\mathrm{pt}\, \varphi = \varphi^{-1}$. $\square$

The *unit* $\eta_X \colon X \to \mathrm{pt}\, \mathbb{P}(X)$ of the adjunction is $\mathrm{id}^\dagger_{\mathbb{P}(X)}$. Recall that $\varphi^\dagger(x) = \mathbb{P}(\overline{x}) \circ \varphi$ for any φ, so η_X maps each $x \in X$ to $\mathbb{P}(\overline{x}) \in \mathrm{pt}\, \mathbb{P}(X)$. Equating points with certain filters as in Remark 2.14, η_X simply maps x to the principal filter $\dot{x}$.

The map η_X has additional properties.

Lemma 2.26. *Let* $\mathbf{C}$ *be an admissible category of lattices. The map* $\eta_X \colon X \to \mathrm{pt}(\mathbb{P}X)$ *is injective and initial. It is an isomorphism if* $\mathbf{C}$ *is an admissible category of frames or of coframes.*

Proof. We equate points of $\mathbb{P}(X)$ with certain filters $\mathcal{U}$, such that $\lim_{\mathbb{P}(X)} \mathcal{U} \in \mathcal{U}$, following Remark 2.9. Then $\eta_X(x) = \dot{x}$, and $\ell^\bullet = \{\mathcal{U} \in \mathrm{pt}\, \mathbb{P}(X) \mid \ell \in \mathcal{U}\}$ for every $\ell \in \mathbb{P}(X)$. That η_X is injective is clear. To establish initiality, let $\mathcal{F} \in \mathbb{F}\mathbb{P}(X)$ be arbitrary, and compute $(\eta_X[\mathcal{F}])^\circ$: this is the set of elements $\ell \in \mathbb{P}(X)$ such that $\ell^\bullet \in \eta_X[\mathcal{F}]$, i.e., such that $\eta_X^{-1}(\ell^\bullet) \in \mathcal{F}$. However,

$$\begin{aligned} \eta_X^{-1}(\ell^\bullet) &= \{x \in X \mid \dot{x} \in \ell^\bullet\} \\ &= \{x \in X \mid \ell \in \dot{x}\} \\ &= \{x \in X \mid x \in \ell\} = \ell. \end{aligned} \tag{2.4}$$

Therefore $(\eta_X[\mathcal{F}])^\circ$ is just $\mathcal{F}$. It follows that

$$\lim_{\mathbb{P}(\mathrm{pt}\, \mathbb{P}(X))} \eta_X[\mathcal{F}] = (\lim_{\mathbb{P}(X)} (\eta_X[\mathcal{F}])^\circ)^\bullet = (\lim_{\mathbb{P}(X)} \mathcal{F})^\bullet.$$

In particular, for every $x \in X$, $\eta_X[\mathcal{F}]$ converges to $\eta_X(x)$ if and only if $\eta_X(x) \in (\lim_{\mathbb{P}(X)} \mathcal{F})^\bullet$, if and only if $x \in \lim_{\mathbb{P}(X)} \mathcal{F}$ (using (2.5)), if and only if $\mathcal{F}$ converges to x in X.

If $\mathbf{C}$ is an admissible category of frames, by Remark 2.11 and Remark 2.16, the only points of $\mathrm{pt}(\mathbb{P}X)$ are the principal filters $\dot{x}$, so that η_X is surjective, hence an isomorphism, in this case. In the case of coframes, we use Remark 2.13 and Remark 2.17 instead. $\square$

A functor that has a fully faithful left adjoint is called a coreflector. It is well-known that a coreflector is also a right adjoint functor whose unit is an isomorphism: see [4, Proposition 1.3], where this is stated for the dual case of reflections. We note that the proof does not make use of the Axiom of Choice.

Corollary 2.27. *Let* $\mathbf{C}$ *be an admissible category of frames, or of coframes.* $\mathbf{Conv}$ *is a coreflective subcategory of* $(\mathbf{C}^{\mathrm{conv}})^{op}$*, through the coreflector* pt.

We have just observed that $\mathbb{P}\colon \mathbf{Conv} \to (\mathbf{C}^{\mathrm{conv}})^{op}$ is fully faithful in that case, allowing us to equate convergence spaces X with the convergence (co)frames $\mathbb{P}(X)$.

2.4. **Classical convergence lattices.** We shall progressively discover the importance of complemented elements of lattices L. Recall that an element ℓ of L is *complemented* if and only if it has a *complement* $\overline{\ell}$, namely an element such that $\ell \vee \overline{\ell} = \top$ and $\ell \wedge \overline{\ell} = \bot$. If L is distributive, the complement is unique if it exists, and $\overline{\overline{\ell}} = \ell$. Let us denote by $\mathcal{C}_L$ the set of complemented elements of a lattice L.

Definition 2.28 (Classical)**.** Let $\mathbf{C}$ be a category of lattices. A convergence $\mathbf{C}$-object L is *classical* if and only if any two filters L that contain the same complemented elements have the same limit, that is, for every two filters $\mathcal{F}$ and $\mathcal{G}$ on L,

$$\mathcal{F} \cap \mathcal{C}_L = \mathcal{G} \cap \mathcal{C}_L \Longrightarrow \lim_L \mathcal{F} = \lim_L \mathcal{G}.$$

We write $\mathbf{C}^{\mathrm{conv}*}$ for the full subcategory of classical convergence $\mathbf{C}$-objects in $\mathbf{C}^{\mathrm{conv}}$.

In other words, $\lim_L$ is classical if and only if $\lim_L \mathcal{F}$ only depends on the complemented elements of $\mathcal{F}$.

For every upwards-closed subset $\mathcal{A}$ of L, there is a smallest upwards-closed set $(\mathcal{A})_c$ that contains the same complemented elements as $\mathcal{A}$, namely:

$$(\mathcal{A})_c := \uparrow(\mathcal{A} \cap \mathcal{C}_L). \tag{2.5}$$

If $\mathcal{A}$ is a filter $\mathcal{F}$, then $(\mathcal{F})_c$ is a filter again, and $(\mathcal{F})_c \subset \mathcal{F}$, hence $\lim_L(\mathcal{F})_c \leq \lim_L \mathcal{F}$ if L is a convergence $\mathbf{C}$-object. Classical convergent $\mathbf{C}$-objects are those for which the reverse inequality is true:

Lemma 2.29. *Let* $\mathbf{C}$ *be a category of lattices. A convergence* $\mathbf{C}$*-object* L *is classical if and only if, for every filter* $\mathcal{F}$ *on* L*,* $\lim_L \mathcal{F} = \lim_L(\mathcal{F})_c$ *(equivalently,* $\lim_L \mathcal{F} \leq \lim_L(\mathcal{F})_c$*).* □

If L is a Boolean algebra, that is, if every element is complemented, then every convergence structure on L is classical. This is the case in particular of the convergence structure on $\mathbb{P}(X)$, for every convergence space X. Using Theorem 2.24 and Lemma 2.26, it follows that:

Proposition 2.30. *Let* $\mathbf{C}$ *be an admissible category of lattices. The adjunction* $\mathbb{P} \dashv$ pt *restricts to an adjunction between* $\mathbf{Conv}$ *and* $(\mathbf{C}^{\mathrm{conv}*})^{op}$*. This is a coreflection if* $\mathbf{C}$ *is an admissible category of frames or of coframes.* □

3. Completeness, Cocompleteness, and More

The category $\mathbf{Frm}$ of frames is complete and cocomplete [9, Section IV.3], and therefore so is the category $\mathbf{CF}$ of coframes. In order to extend this to $\mathbf{Frm}^{\mathrm{conv}}$ or $\mathbf{CF}^{\mathrm{conv}}$, or more generally to categories of the form $\mathbf{C}^{\mathrm{conv}}$, a swift approach is to show that the forgetful functor $|\_|\colon \mathbf{C}^{\mathrm{conv}} \to \mathbf{C}$, which maps any convergence $\mathbf{C}$-object $(L, \lim_L)$ to the underlying $\mathbf{C}$-object L, is topological (see [1], Chapter VI, Section 21; we shall recall the necessary elements as we progress along). For that, we shall see that $\mathbf{C}$ had better be a category of coframes.

Given a $\mathbf{C}$-object L, its *fiber* along $|\_|$ is the collection of convergence $\mathbf{C}$-objects whose image by $|\_|$ is equal to L. Hence we can equate the fiber of L with the set of monotonic maps $\lim_L\colon L \to L$. We call those maps the *convergence structures* on L.

Note that $|\_|$ is *fiber-small*, in the sense that every fiber is a set, not a proper class. That set of convergence structures on L is ordered by the pointwise ordering: $\lim_L \leq \lim'_L$ if and only if $\lim_L \mathcal{F} \leq \lim_{L'} \mathcal{F}$ for every $\mathcal{F} \in \mathbb{F}L$. In that case, we say that $\lim_L$ is *finer than* $\lim_{L'}$, or that $\lim_{L'}$ is *coarser than* $\lim_L$ [3, Definition III.1.7]. With that ordering, the fiber of L is a complete lattice.

In the theory of topological functors, it is common usage to order the fibers by $L \sqsubseteq L'$ if and only if the identity morphism on $|L| = |L'|$ lifts to a morphism from L to L'. In view of (2.1), $\sqsubseteq$ is the opposite of $\leq$.

The essence of topologicity is the following simple observation. The collection of indices I is allowed to be any class, including a proper class.

Proposition 3.1. *Let $\mathbf{C}$ be a category of coframes, and L be a $\mathbf{C}$-object. Let also $\varphi_i\colon |L_i| \to L$ be morphisms of $\mathbf{C}$, where each L_i is a convergence $\mathbf{C}$-object, $i \in I$. There is a coarsest convergence structure $\lim_L$ on L such that φ_i is continuous for each $i \in I$.*

Proof. By definition, φ_i is continuous if and only if $\lim_L \mathcal{F} \leq \varphi_i(\lim_{L_i} \varphi_i^{-1}(\mathcal{F}))$ for every $\mathcal{F} \in \mathbb{F}L$. The required map is then defined by:

$$\lim_L \mathcal{F} := \bigwedge_{i \in I} \varphi_i(\lim_{L_i} \varphi_i^{-1}(\mathcal{F})). \tag{3.1}$$

Note that arbitrary infima exist because L is a coframe. ☐

Proposition 3.2. *Under the same assumptions as Proposition 3.1, if each L_i is classical, then $(L, \lim_L)$ is classical.*

Proof. Assume $\mathcal{F}$ and $\mathcal{G}$ contain the same complemented objects. We note that, for every complemented element ℓ of L_i, $i \in I$, we have $\varphi_i(\ell) \vee \varphi_i(\bar{\ell}) = \varphi_i(\ell \vee \bar{\ell}) = \varphi_i(\top) = \top$, and similarly $\varphi_i(\ell) \wedge \varphi_i(\bar{\ell}) = \bot$, so $\varphi_i(\ell)$ is complemented. For each $i \in I$, $\varphi_i^{-1}(\mathcal{F})$ and $\varphi_i^{-1}(\mathcal{G})$ then have the same complemented elements: if ℓ is complemented in $\varphi_i^{-1}(\mathcal{F})$, then $\varphi_i(\ell)$ is complemented and in $\mathcal{F}$, hence in $\mathcal{G}$, so ℓ is in $\varphi_i^{-1}(\mathcal{G})$. It follows that $\lim_{L_i} \varphi_i^{-1}(\mathcal{F}) = \lim_{L_i} \varphi_i^{-1}(\mathcal{G})$, since L_i is classical. From (3.1), $\lim_L \mathcal{F} = \lim_L \mathcal{G}$. ☐

Categorically, a family $(\varphi_i\colon |L_i| \to L)_{i \in I}$ of morphisms with a fixed L is a *sink*. A *lift* of that sink is an I-indexed family of morphisms whose images by $|\_|$ coincide with φ_i. In our case, this is the same thing as a convergence structure $\lim_L$ on L making every φ_i continuous. Such a lift is *final* if and only if, for every $\mathbf{C}$-morphism $\psi\colon L \to |L'|$, ψ is continuous from $(L, \lim_L)$ to L' if and only if $\psi \circ \varphi_i$ is continuous for every $i \in I$.

Corollary 3.3. *Let $\mathbf{C}$ be a category of coframes. Then $|\_|\colon \mathbf{C}^{\mathrm{conv}} \to \mathbf{C}$ is topological: every sink $(\varphi_i\colon |L_i| \to L)_{i \in I}$ has a unique final lift, and this is $\lim_L$, as given in Proposition 3.1. Similarly for the restriction of $|\_|$ to $\mathbf{C}^{\mathrm{conv}*}$.*

Proof. In view of (the dual of) [1, Proposition 10.43], if it has a final lift then it is the coarsest convergence structure $\lim_L$ of Proposition 3.1, and is therefore unique.

To show existence, we check that $(L, \lim_L)$ is a final lift of $(\varphi_i \colon |L_i| \to L)_{i \in I}$. Let $\psi \colon L \to |L'|$ be such that $\psi \circ \varphi_i$ is continuous for every $i \in I$. We aim to show that ψ is continuous from $(L, \lim_L)$ to L', and for that we consider an arbitrary filter $\mathcal{F}$ on L', and show that $\lim_{L'} \mathcal{F} \leq \psi(\lim_L(\psi^{-1}(\mathcal{F})))$. Indeed:

$$\begin{aligned}
\psi(\lim_L(\psi^{-1}(\mathcal{F}))) &= \psi(\bigwedge_{i \in I} \varphi_i(\lim_{L_i} \varphi_i^{-1}(\psi^{-1}(\mathcal{F})))) && \text{by (3.1)} \\
&= \bigwedge_{i \in I} \psi(\varphi_i(\lim_{L_i} \varphi_i^{-1}(\psi^{-1}(\mathcal{F})))) && \text{since } \psi \text{ is a morphism} \\
& && \text{of coframes} \\
&\geq \lim_{L'} \mathcal{F}
\end{aligned}$$

since each $\psi \circ \varphi_i$ is continuous. □

Remark 3.4. The usual definition of a topological functor is the dual property that every source has a unique initial lift (dual in the sense that a source is a sink in the opposite category, and that a lift is initial if and only if it is final in the opposite category). Being pedantic, what we have shown in Corollary 3.3 is that $|\_| \colon (\mathbf{C}^{\mathrm{conv}})^{op} \to \mathbf{C}^{op}$ is topological. However, that is equivalent to $|\_| \colon \mathbf{C}^{\mathrm{conv}} \to \mathbf{C}$ being topological, by the Topological Duality Theorem [1, Theorem 21.9].

Given a topological functor between two categories $\mathbf{C}$ and $\mathbf{D}$, $\mathbf{C}$ is (co)complete if and only if $\mathbf{D}$ is [1, Theorem 21.16 (1)]. Hence, writing $\mathbf{CF}$ for the category of coframes:

Fact 3.5. *The categories* $\mathbf{CF}^{\mathrm{conv}}$ *and* $\mathbf{CF}^{\mathrm{conv}*}$ *are complete and cocomplete.*

Topologicity has many other consequences. For example, the category of frames is not co-wellpowered [9, Section IV.6.6], hence neither is $\mathbf{CF}$. Given a fiber-small topological functor between two categories $\mathbf{C}$ and $\mathbf{D}$, $\mathbf{C}$ is (co-)wellpowered if and only if $\mathbf{D}$ is [1, Theorem 21.16 (2)], so:

Fact 3.6. *The categories* $\mathbf{CF}^{\mathrm{conv}}$ *and* $\mathbf{CF}^{\mathrm{conv}*}$ *are not co-wellpowered.*

4. Limit Lattices and Variants

A *limit space*, also known as a *finitely deep* convergence space [3, Section III.1], is a convergence space X where, for any two filters $\mathcal{F}$ and $\mathcal{G}$ of subsets of X that converge to the same point x, $\mathcal{F} \cap \mathcal{G}$ also converges to x. It is equivalent to require that $\lim_{\mathbb{P}(X)}(\mathcal{F} \cap \mathcal{G}) = \lim_{\mathbb{P}(X)} \mathcal{F} \cap \lim_{\mathbb{P}(X)} \mathcal{G}$. Those form a full subcategory $\mathbf{Lim}$ of $\mathbf{Conv}$.

Accordingly, call *limit* $\mathbf{C}$*-object* any convergence $\mathbf{C}$-object L such that $\lim_L(\mathcal{F} \cap \mathcal{G}) = \lim_L \mathcal{F} \wedge \lim_L \mathcal{G}$ for all $\mathcal{F}, \mathcal{G} \in \mathbb{F}L$, that is, such that $\lim_L$ preserves binary infima. Those form a full subcategory $\mathbf{C}^{\mathrm{lim}}$ of $\mathbf{C}^{\mathrm{conv}}$. The classical limit $\mathbf{C}$-objects form another full subcategory $\mathbf{C}^{\mathrm{lim}*}$.

Proposition 4.1. *Let* $\mathbf{C}$ *be an admissible category of lattices. The adjunction* $\mathbb{P} \dashv \mathrm{pt}$ *restricts to an adjunction between* $\mathbf{Lim}$ *and* $(\mathbf{C}^{\mathrm{lim}})^{op}$*, resp. between* $\mathbf{Lim}$ *and* $(\mathbf{C}^{\mathrm{lim}*})^{op}$*.*

Proof. Given a limit space X, $\mathbb{P}(X)$ is a (classical) limit $\mathbf{C}$-object by construction. Conversely, let L be a limit $\mathbf{C}$-object, and consider two filters $\mathcal{F}$ and $\mathcal{G}$ of subsets of $\mathrm{pt}\, L$. Then:

$$\begin{aligned}
\lim_{\mathbb{P}(\mathrm{pt}\, L)}(\mathcal{F} \cap \mathcal{G}) &= (\lim_L (\mathcal{F} \cap \mathcal{G})^\circ)^\bullet \\
&= (\lim_L(\mathcal{F}^\circ \cap \mathcal{G}^\circ))^\bullet \quad \text{by definition} \\
&= (\lim_L \mathcal{F}^\circ \wedge \lim_L \mathcal{G}^\circ)^\bullet \quad \text{since } \lim_L \text{ preserves binary infima} \\
&= (\lim_L \mathcal{F}^\circ)^\bullet \cap (\lim_L \mathcal{G}^\circ)^\bullet \quad \text{by Lemma 2.19 (4)} \\
&= \lim_{\mathbb{P}(\mathrm{pt}\, L)} \mathcal{F} \cap \lim_{\mathbb{P}(\mathrm{pt}\, L)} \mathcal{G}.
\end{aligned}$$

□

We recall that the traditional definitions of convergence and limit spaces only consider convergence sets of *proper* filters. We have chosen to also include the non-proper filter on L: this is the top element of $\mathbb{F}L$, namely L itself. This raises the question whether $\lim_L$ preserves top elements, namely whether $\lim_L L = \top$. A convergence $\mathbf{C}$-object L is *strict* if and only if $\lim_L L = \top$. Those form the category $\mathbf{C}^{\mathrm{sconv}}$, resp. $\mathbf{C}^{\mathrm{sconv}*}$ for the classical variant.

There is no need to define strict convergence spaces: for every convergence space X, $\lim_{\mathbb{P}(X)}$ is strict, since for every $x \in X$, $\lim_{\mathbb{P}(X)} \mathbb{P}(X) \supseteq \lim_{\mathbb{P}(X)} \dot{x} \supseteq \{x\}$, by the Point Axiom.

Fact 4.2. *Let $\mathbf{C}$ be an admissible category of lattices. The adjunction $\mathbb{P} \dashv \mathrm{pt}$ restricts to an adjunction between* **Conv** *and* $(\mathbf{C}^{\mathrm{sconv}})^{op}$*, resp. between* **Conv** *and* $(\mathbf{C}^{\mathrm{sconv}*})^{op}$. □

Finally, let $\mathbf{C}^{\mathrm{slim}}$ be the category of strict limit $\mathbf{C}$-objects, i.e., convergence $\mathbf{C}$-objects L such that $\lim_L$ preserves finite infima. Let $\mathbf{C}^{\mathrm{slim}*}$ be the full subcategory of classical strict limit $\mathbf{C}$-objects.

Fact 4.3. *Let $\mathbf{C}$ be an admissible category of lattices. The adjunction $\mathbb{P} \dashv \mathrm{pt}$ restricts to an adjunction between* **Lim** *and* $(\mathbf{C}^{\mathrm{slim}})^{op}$*, resp. between* **Lim** *and* $(\mathbf{C}^{\mathrm{slim}*})^{op}$. □

All the categories considered in this section are complete and cocomplete, assuming that $\mathbf{C}$ is a category of coframes. We show this by appealing to the following result. Let $|\_|$ be a fiber-small topological functor from a category $\mathbf{D}$ to a category $\mathbf{C}$. A *deflationary functor* on $\mathbf{D}$ is a functor $S^1 \colon \mathbf{D} \to \mathbf{D}$ such that $|S^1|$ is the identity functor and such that $S^1(C) \sqsubseteq C$ for every object C of $\mathbf{D}$ (i.e., the identity map on $|C|$ lifts to a morphism from $S^1(C)$ to C). Let S^∞ map each object C of $\mathbf{D}$ to the $\sqsubseteq$-largest fixed point of S^1 below C in the fiber of $|C|$, and let $Fix(S^1)$ denote the full subcategory of $\mathbf{D}$ whose objects are the fixed points of S^1. Then S^∞ extends to a functor from $\mathbf{D}$ to $\mathbf{C}$ such that $|S^\infty|$ is the identity functor, and $Fix(S^1)$ is a coreflective subcategory of $\mathbf{D}$, with coreflector S^∞ [6, Proposition 10]. Moreover, the restriction of $|\_|$ to $Fix(S^1)$ is topological again (Lemma 15, loc. cit.)

In our case, where $\mathbf{D} = \mathbf{C}^{\mathrm{conv}}$ or $\mathbf{D} = \mathbf{C}^{\mathrm{conv}*}$, and $|\_|$ maps every convergence $\mathbf{C}$-object to the underlying $\mathbf{C}$-object, a deflationary functor S^1 on $\mathbf{C}^{\mathrm{conv}}$ is given by maps S^1 from convergence structures to convergence structures such that $S^1(\lim) \geq \lim$, and which act functorially, in the sense that for every morphism $\varphi \colon L \to L'$ in $\mathbf{C}^{\mathrm{conv}}$, φ should again be continuous from $S^1(L) = (L, S^1(\lim_L))$ to $S^1(L') =$

$(L', S^1(\lim_{L'}))$. We may define functors S^1 on $\mathbf{C}^{\mathrm{conv}}$, resp. $\mathbf{C}^{\mathrm{conv}*}$, by one of the following formulae:

$$S^1(\lim_L) \quad : \quad \mathcal{F} \mapsto \bigvee_{\substack{\mathcal{F}_1, \mathcal{F}_2 \in \mathbb{F}L \\ \mathcal{F} \supseteq \mathcal{F}_1 \cap \mathcal{F}_2}} \lim_L \mathcal{F}_1 \wedge \lim_L \mathcal{F}_2 \tag{4.1}$$

$$S^1(\lim_L) \quad : \quad \mathcal{F} \mapsto \begin{cases} \lim_L \mathcal{F} & \text{if } \mathcal{F} \text{ is proper} \\ \top & \text{otherwise} \end{cases} \tag{4.2}$$

$$S^1(\lim_L) \quad : \quad \mathcal{F} \mapsto \bigvee_{\substack{n \in \mathbb{N} \\ \mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathbb{F}L \\ \mathcal{F} \supseteq \bigcap_{i=1}^n \mathcal{F}_i}} \bigwedge_{i=1}^{n} \lim_L \mathcal{F}_i. \tag{4.3}$$

Proposition 4.4. *Each of the formulas* (4.1), (4.2), *and* (4.3) *defines a deflationary functor, that is,* $S^1(\lim_L) \geq \lim_L$.

Proof. We check that S^1 defines a functor in the case of (4.1), the other cases being similar. Let φ be a morphism in $\mathbf{C}^{\mathrm{conv}}$. In view of showing the continuity condition $S^1(\lim_{L'})(\mathcal{F}) \leq \varphi(S^1(\lim_L)(\varphi^{-1}(\mathcal{F})))$, we pick two arbitrary filters $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $\mathcal{F} \supseteq \mathcal{F}_1 \cap \mathcal{F}_2$. Then $\varphi^{-1}(\mathcal{F}) \supseteq \varphi^{-1}(\mathcal{F}_1) \cap \varphi^{-1}(\mathcal{F}_2)$, and that implies that $\lim_L \varphi^{-1}(\mathcal{F}_1) \wedge \lim_L \varphi^{-1}(\mathcal{F}_2) \leq S^1(\lim_L)(\varphi^{-1}(\mathcal{F}))$ by definition of $S^1(\lim_L)$. We apply φ on both sides (recall that φ commutes with finite infima, and that $\lim_{L'} \mathcal{F}_i \leq \varphi(\lim_L(\varphi^{-1}(\mathcal{F}_i)))$ for each i by the continuity of φ) so as to obtain $\lim_{L'} \mathcal{F}_1 \wedge \lim_{L'} \mathcal{F}_2 \leq \varphi(S^1(\lim_L)(\varphi^{-1}(\mathcal{F})))$. We now take suprema over all $\mathcal{F}_1$, $\mathcal{F}_2$ such that $\mathcal{F} \supseteq \mathcal{F}_1 \cap \mathcal{F}_2$ and conclude.

It is also the case that $S^1(\lim_L)$ is classical whenever L is classical. We show this as follows, again in the case of (4.1) only. Given a filter $\mathcal{F}$ on L, and two filters $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $\mathcal{F} \supseteq \mathcal{F}_1 \cap \mathcal{F}_2$, we have $(\mathcal{F})_c \supseteq (\mathcal{F}_1)_c \cap (\mathcal{F}_2)_c$, so $\lim_L \mathcal{F}_1 \wedge \lim_L \mathcal{F}_2 = \lim_L (\mathcal{F}_1)_c \wedge \lim_L (\mathcal{F}_2)_c \leq S^1(\lim_L)((\mathcal{F})_c)$ since L is classical. Taking suprema over $\mathcal{F}_1$, $\mathcal{F}_2$, $S^1(\lim_L)(\mathcal{F}) \leq S^1(\lim_L)((\mathcal{F})_c)$, and we conclude by Lemma 2.29. □

The pointwise supremum of any family of classical convergence structures is again classical. It follows that $S^\infty(\lim_L)$, which is obtained as a transfinite supremum of iterates of S^1, is classical whenever L is.

The convergence $\mathbf{C}$-objects that are fixed points of S^1 are exactly the limit $\mathbf{C}$-objects, the strict convergence $\mathbf{C}$-objects, and the strict limit $\mathbf{C}$-objects, respectively (and similarly for the classical variants, as we have just argued). This allows us to conclude:

Proposition 4.5. *Let* $\mathbf{C}$ *be a category of lattices. The functor* $|\_|$ *restricts to a topological functor from* $\mathbf{C}^{\mathrm{lim}}$*, resp.* $\mathbf{C}^{\mathrm{sconv}}$*, resp.* $\mathbf{C}^{\mathrm{slim}}$ *to* $\mathbf{C}$*. Those categories are reflective subcategories of* $\mathbf{C}^{\mathrm{conv}}$.

The functor $|\_|$ *restricts to a topological functor from* $\mathbf{C}^{\mathrm{lim}*}$*, resp.* $\mathbf{C}^{\mathrm{sconv}*}$*, resp.* $\mathbf{C}^{\mathrm{slim}*}$ *to* $\mathbf{C}$*. Those categories are reflective subcategories of* $\mathbf{C}^{\mathrm{conv}*}$.

$\mathbf{CF}^{\mathrm{lim}}$, $\mathbf{CF}^{\mathrm{sconv}}$, $\mathbf{CF}^{\mathrm{slim}}$*, and their classical variants* $\mathbf{CF}^{\mathrm{lim}*}$, $\mathbf{CF}^{\mathrm{sconv}*}$ *and* $\mathbf{CF}^{\mathrm{slim}*}$ *are complete and cocomplete.* □

There is a well-known adjunction between **Conv** and **Lim**. The *finitely deep modification* $M_{\mathrm{lim}}X$ of a convergence space X is X together with the finest finitely deep convergence lim coarser than $\lim_{\mathbb{P}(X)}$ [3, Definition III.5.3]; namely $\lim = S^\infty(\lim_{\mathbb{P}(X)})$ where S^∞ is the least fixed point functor derived from the S^1 functor of (4.1). This is left-adjoint to the forgetful functor from **Lim** to **Conv**, and by definition the left-adjoints commute in the following diagram:

$$\begin{tikzcd}
(\mathbf{C}^{\mathrm{conv}})^{op} \arrow[r, shift left, "S^\infty"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathrm{pt}"] \arrow[d, phantom, "\dashv"] & (\mathbf{C}^{\mathrm{lim}})^{op} \arrow[l, shift left, "\supseteq"] \arrow[d, shift left, "\mathrm{pt}"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Conv} \arrow[u, shift left, "\mathbb{P}"] \arrow[r, shift left, "M_{\mathrm{lim}}"] \arrow[r, phantom, "\perp"] & \mathbf{Lim} \arrow[u, shift left, "\mathbb{P}"] \arrow[l, shift left, "\supseteq"]
\end{tikzcd}$$

It follows that the right-adjoints commute, too, since they are uniquely determined by the left-adjoints. A similar diagram is obtained with $\mathbf{C}^{\mathrm{slim}}$ in place of $\mathbf{C}^{\mathrm{lim}}$, or with $\mathbf{C}^{\mathrm{conv}*}$ instead of $\mathbf{C}^{\mathrm{conv}}$ and $\mathbf{C}^{\mathrm{lim}*}$ (or $\mathbf{C}^{\mathrm{slim}*}$) instead of $\mathbf{C}^{\mathrm{lim}}$.

The latter yields the leftmost square of (1.1) in the case $\mathbf{C} = \mathbf{CF}$.

5. Pretopological Coframes

A *pretopological space* is a convergence space X in which $\lim_{\mathbb{P}(X)}$ commutes with arbitrary intersections [3, Proposition V.1.4]. Those form a full subcategory **PreTop** of **Conv**.

Similarly:

Definition 5.1 (Pretopological)**.** Given a category $\mathbf{C}$ of coframes, a $\mathbf{C}$-object L is *pretopological* if and only if $\lim_L \bigcap_{i\in I} \mathcal{F}_i = \bigwedge_{i\in I} \lim_L \mathcal{F}_i$, for every family $(\mathcal{F}_i)_{i\in I}$ of filters on L. They define a full subcategory $\mathbf{C}^{\mathrm{pretop}}$ of $\mathbf{C}^{\mathrm{conv}}$, and a further full subcategory $\mathbf{C}^{\mathrm{pretop}*}$ of classical pretopological $\mathbf{C}$-objects.

Lemma 5.2. *Let $\mathbf{C}$ be a category of coframes, and L be a convergence $\mathbf{C}$-object. The following hold:*

(1) *For every family $(\ell_i)_{i\in I}$ of elements of L, $\bigcap_{i\in I} \ell_i^\bullet = (\bigwedge_{i\in I} \ell_i)^\bullet$.*

Proof. Recall that ϵ_L defined by $\epsilon_L(\ell) = \ell^\bullet$ is a $\mathbf{C}^{\mathrm{conv}}$ morphism by Lemma 2.23, hence preserves arbitrary infima. □

Proposition 5.3. *Let $\mathbf{C}$ be an admissible category of coframes. The coreflection $\mathbb{P} \dashv \mathrm{pt}$ restricts to a coreflection of* **PreTop** *into $(\mathbf{C}^{\mathrm{pretop}})^{op}$, resp. into $(\mathbf{C}^{\mathrm{pretop}*})^{op}$.*

Proof. Given a pretopological space X, $\mathbb{P}(X)$ is a (classical) pretopological $\mathbf{C}$-object by construction. Conversely, let L be a pretopological $\mathbf{C}$-object, and consider any family of filters $\mathcal{F}_i$ of subsets of $\mathrm{pt}\, L$, $i \in I$. Then:

$$\begin{aligned}
\lim_{\mathbb{P}(\mathrm{pt}\, L)}\Big(\bigcap_{i\in I} \mathcal{F}_i\Big) &= \Big(\lim_L \Big(\bigcap_{i\in I} \mathcal{F}_i\Big)^{\circ}\Big)^\bullet \\
&= \Big(\lim_L\Big(\bigcap_{i\in I} \mathcal{F}_i{}^\circ\Big)\Big)^\bullet \quad \text{by definition} \\
&= \Big(\bigwedge_{i\in I} \lim_L \mathcal{F}_i{}^\circ\Big)^\bullet \quad \text{since } \lim_L \text{ preserves arbitrary infima} \\
&= \bigcap_{i\in I} (\lim_L \mathcal{F}_i{}^\circ)^\bullet \quad \text{by Lemma 5.2} \\
&= \bigcap_{i\in I} \lim_{\mathbb{P}(\mathrm{pt}\, L)} \mathcal{F}_i.
\end{aligned}$$

□

Using the technology of topological functors developed in Section 4, define:

$$S^1(\lim_L) \quad : \quad \mathcal{F} \mapsto \bigvee_{\substack{\Phi \in \mathbb{P}(\mathbb{F}L) \\ \mathcal{F} \supseteq \bigcap \Phi}} \bigwedge_{\mathcal{G} \in \Phi} \lim_L \mathcal{G}. \tag{5.1}$$

Lemma 5.4. *S^1 given by* (5.1) *is a functor. Moreover, S^1, hence S^∞, preserves classicality.*

Proof. We proceed as we did for limit spaces and (4.1). Let φ be a morphism in $\mathbf{C}^{\mathrm{conv}}$, and assume that $\mathbf{C}$ is a category of coframes. In view of showing the continuity condition $S^1(\lim_{L'})(\mathcal{F}) \leq \varphi(S^1(\lim_L)(\varphi^{-1}(\mathcal{F})))$, we pick a family Φ of filters on L such that $\mathcal{F} \supseteq \bigcap \Phi$. Then $\varphi^{-1}(\mathcal{F}) \supseteq \bigcap_{\mathcal{G} \in \Phi} \varphi^{-1}(\mathcal{G})$, and that implies that $\bigwedge_{\mathcal{G} \in \Phi} \lim_L \varphi^{-1}(\mathcal{G}) \leq S^1(\lim_L)(\varphi^{-1}(\mathcal{F}))$ by definition of $S^1(\lim_L)$. We apply φ on both sides (recall that φ commutes with arbitrary infima, and that $\lim_{L'} \mathcal{G} \leq \varphi(\lim_L(\varphi^{-1}(\mathcal{G})))$ for each $\mathcal{G} \in \Phi$ by the continuity of φ) so as to obtain $\bigwedge_{\mathcal{G} \in \Phi} \lim_{L'} \mathcal{G} \leq \varphi(S^1(\lim_L)(\varphi^{-1}(\mathcal{F})))$. We now take suprema over all Φ such that $\mathcal{F} \supseteq \bigcap \Phi$ and conclude.

We also prove that S^1 preserves classicality similarly as for (4.1). $\square$

The pretopological convergence $\mathbf{C}$-objects are exactly the fixed points of S^1, whence:

Proposition 5.5. *Let $\mathbf{C}$ be a category of coframes. The functor $|\_|$ restricts to a topological functor from $\mathbf{C}^{\mathrm{pretop}}$ (resp., $\mathbf{C}^{\mathrm{pretop}*}$) to $\mathbf{C}$. $\mathbf{C}^{\mathrm{pretop}}$ and $\mathbf{C}^{\mathrm{pretop}*}$ are reflective subcategories of $\mathbf{C}^{\mathrm{conv}}$. $\mathbf{CF}^{\mathrm{pretop}}$ and $\mathbf{C}^{\mathrm{pretop}*}$ are complete and cocomplete.* $\square$

There is a well-known adjunction between **Lim** (or **Conv**) and **PreTop**. The *pretopological modification* $M_{\mathrm{pretop}}X$ of a convergence space X is X together with the finest pretopological convergence lim coarser than $\lim_{\mathbb{P}(X)}$ [3, Proposition V.1.6]. This is left-adjoint to the forgetful functor from **PreTop** to **Lim**, and by definition the left-adjoints commute in the following diagram:

$$\begin{tikzcd}
(\mathbf{C}^{\mathrm{lim}})^{op} \arrow[r, shift left, "S^\infty"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathrm{pt}"] \arrow[d, phantom, "\dashv"] & (\mathbf{C}^{\mathrm{pretop}})^{op} \arrow[l, shift left, "\supseteq"] \arrow[d, shift left, "\mathrm{pt}"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Lim} \arrow[u, shift left, "\mathbb{P}"] \arrow[r, shift left, "M_{\mathrm{lim}}"] \arrow[r, phantom, "\perp"] & \mathbf{PreTop} \arrow[u, shift left, "\mathbb{P}"] \arrow[l, shift left, "\supseteq"]
\end{tikzcd}$$

where $\mathbf{C}$ is an admissible category of coframes. The right-adjoints commute, too, since they are uniquely determined by the left-adjoints. A similar diagram is obtained with $\mathbf{C}^{\mathrm{slim}}$ in place of $\mathbf{C}^{\mathrm{lim}}$, or with $\mathbf{C}^{\mathrm{pretop}*}$ instead of $\mathbf{C}^{\mathrm{pretop}}$ and $\mathbf{C}^{\mathrm{lim}*}$ (or $\mathbf{C}^{\mathrm{slim}*}$) instead of $\mathbf{C}^{\mathrm{lim}}$.

The latter yields the second square of (1.1) from the left, in the case $\mathbf{C} = \mathbf{CF}$.

In the realm of convergence spaces, we know that it is equivalent to assume a convergence space to be pretopological, or to assume that it arises from a non-idempotent closure operator, i.e., an inflationary operator ν that preserves finite suprema (the *adherence* operator). The situation is more complicated in the point-free world. We shall deal with it in Section 7. Classicality will play an important role there. More importantly, we shall need to define suitable notions of closed elements and adherence operators, a task that will occupy Section 6.

6. The Sierpiński Convergence Coframe, Open and Closed Elements, and Adherence

Every topological space X gives rise to a convergence space by defining $\mathcal{F} \to x$ if and only if every open neighborhood of x belongs to $\mathcal{F}$.

The Sierpiński space $\mathbb{S}$ is the topological space with two elements 0 and 1, with open sets $\emptyset$, $\{1\}$, and the whole space. We can therefore also view $\mathbb{S}$ as a convergence space, where convergence is given by $\mathcal{F} \to x$ if and only if $x = 0$, or $x = 1$ and $\{1\} \in \mathcal{F}$. Explicitly: there are four filters of subsets of $\mathbb{S}$, and they are $\uparrow\{0,1\}$, $\uparrow\{0\}$, $\uparrow\{1\}$, and $\uparrow\emptyset$. The first two converge to 0, the last two converge to both 0 and 1.

Using the $\mathbb{P}$ functor, we obtain the *Sierpiński convergence coframe* $\mathbb{P}(\mathbb{S})$, with $\lim_{\mathbb{P}(\mathbb{S})} \uparrow\{0,1\} = \lim_{\mathbb{P}(\mathbb{S})} \uparrow\{0\} = \{0\}$, and $\lim_{\mathbb{P}(\mathbb{S})} \uparrow\{1\} = \lim_{\mathbb{P}(\mathbb{S})} \uparrow\emptyset = \{0,1\}$. Said more briefly, $\lim_{\mathbb{P}(\mathbb{S})} \mathcal{F}$ equals $\{0,1\}$ if $\mathcal{F}$ contains $\{1\}$, and $\{0\}$ otherwise.

One way of defining an open subset U of a topological space X is by giving its indicator function $\chi_U \colon X \to \mathbb{S}$. That is a continuous function, and all continuous functions from X to $\mathbb{S}$ are of this form. This observation naturally leads us to study the morphisms $\varphi\colon L \to \mathbb{P}(\mathbb{S})$ in $(\mathbf{C}^{\mathrm{conv}})^{op}$, that is, the morphisms of convergence $\mathbf{C}$-objects $\varphi\colon \mathbb{P}(\mathbb{S}) \to L$, and to attempt to retrieve a suitable notion of open element of L.

Lemma 6.1. *Let $\mathbf{C}$ be a category of lattices, and L be a $\mathbf{C}$-object.*

(1) *For every morphism $\varphi\colon \mathbb{P}(\mathbb{S}) \to L$ in $\mathbf{C}^{\mathrm{conv}}$, the elements $u := \varphi(\{1\})$ and $c := \varphi(\{0\})$ satisfy:*
  - (*i*) *$u \wedge c = \bot$,*
  - (*ii*) *$u \vee c = \top$, and*
  - (*iii*) *for every filter $\mathcal{F}$ on L, $u \in \mathcal{F}$ or $\lim_L \mathcal{F} \leq c$.*

(2) *Conversely, if $\mathbf{C}$ is admissible, then for any pair of elements u, c of L satisfying (i)–(iii), the map $\varphi\colon \mathbb{P}(\mathbb{S}) \to L$ that sends $\emptyset$ to $\bot$, $\{0,1\}$ to $\top$, $\{0\}$ to c and $\{1\}$ to u, is a morphism in $\mathbf{C}^{\mathrm{conv}}$.*

Proof. (1) We have $u \wedge c = \varphi(\{1\}) \wedge \varphi(\{0\}) = \varphi(\{1\} \cap \{0\}) = \varphi(\emptyset) = \bot$, proving (*i*). Claim (*ii*) is proved similarly. By the continuity condition (2.1), for every filter $\mathcal{F}$ on L, $\lim_L \mathcal{F} \leq \varphi(\lim_{\mathbb{P}(\mathbb{S})} \varphi^{-1}(\mathcal{F}))$. If $\varphi^{-1}(\mathcal{F})$ contains $\{1\}$, then $u := \varphi(\{1\})$ is in $\mathcal{F}$. Otherwise, $\lim_{\mathbb{P}(\mathbb{S})} \varphi^{-1}(\mathcal{F}) = \{0\}$, and then $\lim_L \mathcal{F} \leq \varphi(\{0\}) = c$.

(2) φ preserves top and bottom by definition, and binary infima and binary suprema by (*i*) and (*ii*). Due to the low cardinality of $\mathbb{P}(\mathbb{S})$, this is enough to guarantee that φ preserves all infima and all suprema, hence is a morphism of $\mathbf{C}$ by admissibility.

We need to check continuity. Let $\mathcal{F}$ be a filter on L. If $\varphi^{-1}(\mathcal{F})$ contains $\{1\}$, then $\lim_L \mathcal{F} \leq \varphi(\lim_{\mathbb{P}(\mathbb{S})} \varphi^{-1}(\mathcal{F}))$ holds trivially, since the right-hand side is $\varphi(\{0,1\}) = \top$. Otherwise, $\varphi(\lim_{\mathbb{P}(\mathbb{S})} \varphi^{-1}(\mathcal{F})) = \varphi(\{0\}) = c$. By (*iii*) we must have $\lim_L \mathcal{F} \leq c$, that is, $\lim_L \mathcal{F} \leq \varphi(\lim_{\mathbb{P}(\mathbb{S})} \varphi^{-1}(\mathcal{F}))$. ☐

In a pair (u, c) satisfying (*i*)–(*iii*), u is *open* and c is *closed*. Conditions (*i*) and (*ii*) express that u and c are complements. In a distributive lattice, complements are determined uniquely, so one may try to define open elements, and closed elements, without referring to their complement. This can be done by using the notion of *grill*, imitated from the eponymous notion introduced by Choquet on convergence spaces [2] and widely used since then (e.g., [3]).

Definition 6.2. Let L be a lattice and $\mathcal{A}$, $\mathcal{B}$ two subsets of L.

We say that $\mathcal{A}$ and $\mathcal{B}$ *mesh*, or that $\mathcal{A}$ *meshes with* $\mathcal{B}$, in notation $\mathcal{A} \# \mathcal{B}$, if and only if for every $a \in \mathcal{A}$, for every $b \in \mathcal{B}$, $a \wedge b \neq \bot$.

The *grill* of $\mathcal{A}$, $\mathcal{A}^{\#}$ is the set of elements $\ell \in L$ such that, for every $a \in A$, $a \wedge \ell \neq \bot$.

Lemma 6.3. *Let L be a lattice.*

1. *For every subset $\mathcal{A}$, $\mathcal{A}^{\#}$ is upwards-closed.*
2. *If L is distributive, then for every filter $\mathcal{F}$, $\mathcal{F}^{\#}$ is prime, in the sense that for every finite family of elements $\ell_1, \ldots, \ell_n$ of L such that $\bigvee_{i=1}^n \ell_i \in \mathcal{F}^{\#}$, some ℓ_i is already in $\mathcal{F}^{\#}$.*
3. *For any two subsets $\mathcal{A}$ and $\mathcal{B}$ of L, if $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{B}^{\#} \subseteq \mathcal{A}^{\#}$.*
4. *For any two subsets $\mathcal{A}$ and $\mathcal{B}$ of L, $\mathcal{A} \# \mathcal{B}$ if and only if $\mathcal{A} \subseteq \mathcal{B}^{\#}$.*
5. *For every filter $\mathcal{G}$ on L, $\mathcal{G}$ is proper if and only if $\mathcal{G} \subseteq \mathcal{G}^{\#}$.*

Proof. (1), (3) and (4) are obvious. (2) Let $\bigvee_{i=1}^n \ell_i \in \mathcal{F}^{\#}$, and assume for the sake of contradiction that no ℓ_i is in $\mathcal{F}^{\#}$. For every i, there is an element m_i in $\mathcal{F}$ such that $\ell_i \wedge m_i = \bot$. Let $m = \bigwedge_{i=1}^n m_i$. Then m is in $\mathcal{F}$, since $\mathcal{F}$ is a filter, and $\ell_i \wedge m = \bot$ for every i since $\ell_i \wedge m \leq \ell_i \wedge m_i = \bot$. Taking suprema over all i, and using distributivity, $\bigvee_{i=1}^n \ell_i \wedge m = \bot$, contradicting $\bigvee_{i=1}^n \ell_i \in \mathcal{F}^{\#}$.

(5) If $\mathcal{G}$ is proper, then $\mathcal{G}$ meshes with itself: for all $\ell_1 \in \mathcal{G}$ and $\ell_2 \in \mathcal{G}$, $\ell_1 \wedge \ell_2$ is in $\mathcal{G}$, since $\mathcal{G}$ is a filter, and is different from $\bot$ since $\mathcal{G}$ is proper. It follows that $\mathcal{G} \subseteq \mathcal{G}^{\#}$, using (4). Conversely, if $\mathcal{G} \subseteq \mathcal{G}^{\#}$, then $\mathcal{G}$ meshes with itself by (4). In particular, for every $\ell \in \mathcal{G}$, $\ell = \ell \wedge \ell$ is different from $\bot$, so $\mathcal{G}$ is proper. $\square$

The $\#$ relation is better understood on filters. The ideal completion of a sup-semilattice is a complete lattice. It follows that, given an inf-semilattice L, $\mathbb{F}L = \mathbf{I}(L^{op})$ (see Remark 2.2) is a complete lattice as well. Infima are given by intersections. For suprema, we have the following. The supremum of the empty family is the smallest filter, $\{\top\}$. Directed suprema are unions. For binary suprema, $\mathcal{F} \vee \mathcal{G}$ is the upward closure of the set of elements $m \wedge \ell$, $m \in \mathcal{F}$, $\ell \in \mathcal{G}$. It follows that:

Fact 6.4. *Two filters $\mathcal{F}$ and $\mathcal{G}$ on a lattice L mesh if and only if $\mathcal{F} \vee \mathcal{G}$ is proper.*

A *pseudocomplement* of $\ell \in L$ is an element ℓ^* such that $m \leq \ell^*$ if and only if $m \wedge \ell = \bot$. If ℓ has a complement $\overline{\ell}$ in a distributive lattice L, then $\overline{\ell}$ is a pseudocomplement of ℓ. In a frame, every element has a unique pseudocomplement. In a coframe L, every element has a unique pseudocomplement in L^{op}; if it is also a pseudocomplement (in L), then it is a complement.

Lemma 6.5. *Let L be a lattice, and $\mathcal{F}$ be an upward-closed subset of L. For every element ℓ with a pseudocomplement ℓ^*, the following are equivalent:*

1. *$\ell \in \mathcal{F}^{\#}$;*
2. *$\ell^* \notin \mathcal{F}$.*

Proof. We show instead that $\ell \notin \mathcal{F}^{\#}$ is equivalent to $\ell^* \in \mathcal{F}$.

If $\ell^* \in \mathcal{F}$, then $\ell \wedge \ell^* = \bot$ prevents ℓ from being in $\mathcal{F}^{\#}$. Conversely, if ℓ is not in $\mathcal{F}^{\#}$, then there is an $m \in \mathcal{F}$ such that $\ell \wedge m = \bot$. Equivalently, $m \leq \ell^*$, which implies that ℓ^* is in $\mathcal{F}$ because $\mathcal{F}$ is upwards-closed. $\square$

Corollary 6.6. *Let $\varphi : L \to L'$ be a lattice morphism where L' is a distributive lattice, and let $a \in L$ be a complemented element. Then $\varphi(a)$ is complemented and*

$\overline{\varphi(a)} = \varphi(\overline{a})$. *Moreover, if $\mathcal{F}$ is an upwards-closed subset of L' then*

$$a \in \varphi^{-1}(\mathcal{F})^{\#} \iff \varphi(a) \in \mathcal{F}^{\#}.$$

Proof. Since $\varphi(a \wedge \overline{a}) = \varphi(a) \wedge \varphi(\overline{a})$ and $\varphi(a \wedge \overline{a}) = \varphi(\bot_L) = \bot_{L'}$, we have $\varphi(a) \wedge \varphi(\overline{a}) = \bot_{L'}$. Similarly, $\varphi(a \vee \overline{a}) = \varphi(a) \vee \varphi(\overline{a})$ and $\varphi(a \vee \overline{a}) = \varphi(\top_L) = \top_{L'}$. Thus $\varphi(\overline{a})$ is the complement of $\varphi(a)$ in L' (which is unique because L' is distributive).

In view of Lemma 6.5, $\varphi(a) \in \mathcal{F}^{\#}$ if and only if $\overline{\varphi(a)} \notin \mathcal{F}$, that is, $\varphi(\overline{a}) \notin \mathcal{F}$, equivalently, $\overline{a} \notin \varphi^{-1}(\mathcal{F})$. By Lemma 6.5, this is equivalent to $a \in \varphi^{-1}(\mathcal{F})^{\#}$ because $\varphi^{-1}(\mathcal{F})$ is an upwards-closed subset of L if $\mathcal{F}$ is an upwards-closed subset of L'. □

Corollary 6.7. *Let L be a lattice, and $\mathcal{F}$, $\mathcal{G}$ be two filters on L.*

$$\mathcal{F} \cap \mathcal{C}_L \subset \mathcal{G} \cap \mathcal{C}_L \Longrightarrow \mathcal{G}^{\#} \cap \mathcal{C}_L \subset \mathcal{F}^{\#} \cap \mathcal{C}_L.$$

In particular, if $\mathcal{F} \cap \mathcal{C}_L = \mathcal{G} \cap \mathcal{C}_L$ then $\mathcal{F}^{\#} \cap \mathcal{C}_L = \mathcal{G}^{\#} \cap \mathcal{C}_L$.

Proof. For every complemented element a of $\mathcal{G}^{\#}$, $\overline{a}$ is not in $\mathcal{G}$ by Lemma 6.5. Then $\overline{a}$ cannot be in $\mathcal{F}$, since every complemented element of $\mathcal{F}$ is in $\mathcal{G}$. By Lemma 6.5 again, it follows that a is in $\mathcal{F}^{\#}$. □

Following Lemma 6.1, one is tempted to define closed elements as those complemented elements c such that every filter $\mathcal{F}$ on L that does not contain the complement of c satisfies $\lim_L \mathcal{F} \leq c$. In light of Lemma 6.5, we define:

Definition 6.8 (Closed)**.** Let $\mathbf{C}$ be a category of lattices, and L be a convergence $\mathbf{C}$-object. An element c of L is *quasi-closed* if and only if:

for every filter $\mathcal{F}$ on L such that $c \in \mathcal{F}^{\#}$, $\lim_L \mathcal{F} \leq c$.

A *closed element* of L is a complemented quasi-closed element of L.

Summing up:

Fact 6.9. *Let $\mathbf{C}$ be an admissible category of distributive lattices. The morphisms $\varphi\colon \mathbb{P}(\mathbb{S}) \to L$ in $\mathbf{C}^{\mathrm{conv}}$ are in one-to-one correspondence with the closed elements of L.*

Remark 6.10. Since every element is complemented in $\mathbb{P}(X)$, its quasi-closed elements and its closed elements coincide. Explicitly, the closed elements are the subsets C that contain all the limits of filters $\mathcal{F}$ such that $C \in \mathcal{F}^{\#}$. This is the standard definition of a closed subset of a convergence space.

Closed sets can instead be obtained as (pre-)fixed points of a so-called adherence operator. The following definition is the obvious choice, but a better one will be given in Definition 6.21, which takes complemented elements into account more finely.

Definition 6.11 (Raw adherence)**.** Let $\mathbf{C}$ be a category of lattices, and L be a convergence $\mathbf{C}$-object. For every $\ell \in L$, the *raw adherence* of ℓ is defined by:

$$\mathrm{adh}^0_L\, \ell := \bigvee_{\substack{\mathcal{F} \in \mathbb{F}L \\ \ell \in \mathcal{F}^{\#}}} \lim_L \mathcal{F}. \tag{6.1}$$

Fact 6.12. *The quasi-closed elements of L are the pre-fixpoints of $\mathrm{adh}^0_L\colon L \to L$, that is, the elements $\ell \in L$ such that $\mathrm{adh}^0_L\, \ell \leq \ell$.*

Proposition 6.13. *Let* $\mathbf{C}$ *be a category of distributive lattices, and* L *be a convergence* $\mathbf{C}$*-object. The operator* $\mathrm{adh}^0_L \colon L \to L$ *preserves finite suprema, and is in particular monotonic.*

Proof. Let $\ell_1, \ldots, \ell_n$ be finitely many elements of L. For every filter $\mathcal{F}$ on L, by Lemma 6.3 (1) and (2), $\bigvee_{i=1}^n \ell_i \in \mathcal{F}^\#$ if and only if $\ell_i \in \mathcal{F}^\#$ for some i, $1 \leq i \leq n$. Hence:

$$\begin{aligned}
\mathrm{adh}^0_L(\bigvee_{i=1}^n \ell_i) &= \bigvee_{\mathcal{F}\in\mathbb{F}L, \bigvee_{i=1}^n \ell_i \in \mathcal{F}^\#} \lim_L \mathcal{F} \\
&= \bigvee_{\mathcal{F}\in\mathbb{F}L, \ell_i \in \mathcal{F}^\# \text{ for some } i} \lim_L \mathcal{F} \\
&= \bigvee_{i=1}^n \bigvee_{\mathcal{F}\in\mathbb{F}L, \ell_i \in \mathcal{F}^\#} \lim_L \mathcal{F} \\
&= \bigvee_{i=1}^n \mathrm{adh}^0_L \ell_i.
\end{aligned}$$

□

Remark 6.14. The adherence operator adh^0_L is not idempotent in general, even on convergence $\mathbf{C}$-objects of the form $\mathbb{P}(X)$, see [3, Example V.4.6]. More surprising, perhaps, is the failure of the law $\mathrm{adh}^0_L \ell \geq \ell$, since Lemma 6.17 below, for one, will state that this law holds on every convergence $\mathbf{C}$-object of the form $\mathbb{P}(X)$. However, it certainly fails on the *bottom* convergence $\mathbf{C}$-objects L, namely those such that $\lim_L \mathcal{F} = \bot$ for every $\mathcal{F} \in \mathbb{F}L$.

Definition 6.15 (Centered)**.** Let $\mathbf{C}$ be a category of lattices. The convergence $\mathbf{C}$-object L is *centered* if and only if, for every $\ell \in L$, $\ell \leq \mathrm{adh}^0_L \ell$.

Remark 6.16. A *preconvergence space* is defined like a convergence space, but without requiring the Point Axiom [3]. It is easily seen that a preconvergence space X is a convergence space if and only if its adherence operator $\mathrm{adh} : \mathbb{P}X \to \mathbb{P}X$ defined by $\mathrm{adh}\, A = \bigcup_{A\in\mathcal{F}\in\mathbb{F}\mathbb{P}X} \lim \mathcal{F}$ satisfies $A \subset \mathrm{adh}\, A$ for all $A \in \mathbb{P}X$. Hence the condition of being centered is a pointfree analog of the Point Axiom!

Lemma 6.17. *Every convergence* $\mathbf{C}$*-object of the form* $\mathbb{P}(X)$*, where* X *is a convergence space, is centered.*

Proof. Let $\ell \in \mathbb{P}(X)$. For every $x \in \ell$, x is in $\lim_{\mathbb{P}(X)} \mathcal{F}$ where $\mathcal{F} = \dot{x}$. Clearly $\ell \in \mathcal{F}^\#$, so x is in $\mathrm{adh}^0_{\mathbb{P}(X)} \ell$. □

Fact 6.18. *The quasi-closed elements of a centered convergence* $\mathbf{C}$*-object* L *are the fixed points of* $\mathrm{adh}^0_L \colon L \to L$. □

Every morphism of coframes $\varphi \colon L \to L'$ is an (order-theoretic) right adjoint, since it preserves all infima. Write its left adjoint as $\varphi_!$. When φ is the inverse image map f^{-1}, where $f \colon X \to Y$, $\varphi_!$ is the corresponding direct image map.

Lemma 6.19. *If* $\varphi : L \to L'$ *is a morphism of coframes,* $\varphi_! : L' \to L$ *denotes its left adjoint, and* $\mathcal{F} \in \mathbb{F}L'$*, then*

$$\varphi_!\left(\mathcal{F}^\#\right) \subset \left(\varphi^{-1}(\mathcal{F})\right)^\# .$$

Proof. Let $\ell' \in \mathcal{F}^{\#}$. Since $\varphi_!$ is left adjoint to φ, $\ell' \leq \varphi(\varphi_!(\ell'))$. Then, for every $\ell \in \varphi^{-1}(\mathcal{F})$, $\varphi(\varphi_!(\ell') \wedge \ell) = \varphi(\varphi_!(\ell')) \wedge \varphi(\ell) \geq \ell' \wedge \varphi(\ell)$. Since ℓ' is in $\mathcal{F}^{\#}$ and $\varphi(\ell)$ is in $\mathcal{F}$, $\ell' \wedge \varphi(\ell)$ is different from $\perp$, which implies $\varphi(\varphi_!(\ell') \wedge \ell) \neq \perp$ (for otherwise $\perp = \varphi(\varphi_!(\ell') \wedge \ell) = \varphi(\varphi_!(\ell')) \wedge \varphi(\ell) \geq \ell' \wedge \varphi(\ell)$ because φ is a coframe morphism). Thus $\varphi_!(\ell') \wedge \ell \neq \perp$ and we conclude that $\varphi_!(\ell')$ is in $\left(\varphi^{-1}(\mathcal{F})\right)^{\#}$. □

The following is then the pointfree analogue of the standard fact that the images of adherences are contained in the adherences of the images, in convergence spaces.

Proposition 6.20. *Let* $\mathbf{C}$ *be a category of coframes, and let* $\varphi \colon L \to L'$ *be a morphism in* $\mathbf{C}^{\mathrm{conv}}$*. Then, for every* $\ell' \in L'$*,*

$$\varphi_!(\mathrm{adh}^0_{L'}(\ell')) \leq \mathrm{adh}^0_L(\varphi_!(\ell')), \tag{6.2}$$

$$\mathrm{adh}^0_{L'}(\ell') \leq \varphi(\mathrm{adh}^0_L(\varphi_!(\ell'))). \tag{6.3}$$

Proof. It follows from Lemma 6.19 that, if $\ell' \in \mathcal{F}'^{\#}$, then $\lim_L \varphi^{-1}(\mathcal{F}') \leq \mathrm{adh}^0_L(\varphi_!(\ell'))$. Indeed, the left-hand side appears as one of the disjuncts in the definition of the right-hand side. Since φ is continuous, $\lim_{L'} \mathcal{F}' \leq \varphi(\lim_L \varphi^{-1}(\mathcal{F}'))$, so $\lim_{L'} \mathcal{F}' \leq \varphi(\mathrm{adh}^0_L(\varphi_!(\ell')))$. Taking suprema over all filters $\mathcal{F}'$ such that $\ell' \in \mathcal{F}'^{\#}$, we obtain $\mathrm{adh}^0_{L'}(\ell') \leq \varphi(\mathrm{adh}^0_L(\varphi_!(\ell')))$. That is (6.3). The inequality (6.2) is equivalent, by the definition of left adjoints. □

There is another natural notion of adherence on convergence coframes. While it is less natural at first sight, this is the one we shall need in the sequel.

Definition 6.21 (Adherence)**.** Let $\mathbf{C}$ be a category of coframes, and L be a convergence $\mathbf{C}$-object. For every $\ell \in L$, the *adherence* of ℓ is:

$$\mathrm{adh}_{\lim_L} \ell := \mathrm{adh}_L \ell := \bigwedge_{a \in \mathcal{C}_L, a \geq \ell} \mathrm{adh}^0_L a. \tag{6.4}$$

Proposition 6.22. *Let* $\mathbf{C}$ *be a category of coframes, and* L *be a convergence* $\mathbf{C}$*-object. The operator* $\mathrm{adh}_L \colon L \to L$*:*

1. *satisfies* $\mathrm{adh}_L \ell \geq \mathrm{adh}^0_L \ell$ *for every* $\ell \in L$*,*
2. *coincides with* adh^0_L *on complemented elements,*
3. *is monotonic,*
4. *preserves finite suprema of complemented elements,*
5. *and satisfies* $\mathrm{adh}_L \ell = \bigwedge_{a \in \mathcal{C}_L, a \geq \ell} \mathrm{adh}_L a$ *for every* $\ell \in L$*.*

Proof. (1) For every complemented $a \geq \ell$, $\mathrm{adh}^0_L a \geq \mathrm{adh}^0_L \ell$ by monotonicity. Taking infima over a yields the result. (2) If ℓ is complemented, then one can take $a = \ell$ in (6.4), so that $\mathrm{adh}_L \ell \leq \mathrm{adh}^0_L \ell$. Equality follows from (1). (3) is obvious. (4) On complemented elements, adh_L and adh^0_L coincide by (2), so the conclusion follows from Proposition 6.13. (5) By (6.4) and (2). □

Proposition 6.22 together with Fact 6.12 together imply the following.

Fact 6.23. *The closed elements of* L *are the complemented pre-fixpoints of* $\mathrm{adh}_L \colon L \to L$*, that is, the complemented elements* $\ell \in L$ *such that* $\mathrm{adh}_L \ell \leq \ell$*.*

Proposition 6.24. *Let* $\mathbf{C}$ *be a category of coframes, and let* $\varphi \colon L \to L'$ *be a morphism in* $\mathbf{C}^{\mathrm{conv}}$*. Then, for every* $\ell' \in L'$*,*

$$\varphi_!(\mathrm{adh}_{L'}(\ell')) \leq \mathrm{adh}_L(\varphi_!(\ell')), \tag{6.5}$$

$$\mathrm{adh}_{L'}(\ell') \leq \varphi(\mathrm{adh}_L(\varphi_!(\ell'))). \tag{6.6}$$

Proof. Let $\ell' \in L'$. By Proposition 6.2, in particular by (6.3), $\mathrm{adh}^0_{L'}(a') \leq \varphi(\mathrm{adh}^0_L(\varphi_!(a')))$ for every complemented $a' \geq \ell'$. Hence:

$$\begin{aligned} \mathrm{adh}_{L'}(\ell') &= \bigwedge_{a' \in \mathcal{C}_{L'}, a' \geq \ell'} \mathrm{adh}^0_{L'}(a') \\ &\leq \bigwedge_{a' \in \mathcal{C}_{L'}, a' \geq \ell'} \varphi(\mathrm{adh}^0_L(\varphi_!(a'))) \\ &= \varphi(\bigwedge_{a' \in \mathcal{C}_{L'}, a' \geq \ell'} \mathrm{adh}^0_L(\varphi_!(a'))) \end{aligned}$$

since φ preserves all infima. Among the complemented elements a', we find those of the form $\varphi(a)$ with a complemented in L, so:

$$\begin{aligned} \mathrm{adh}_{L'}(\ell') &\leq \varphi(\bigwedge_{\substack{a \in \mathcal{C}_L \\ \ell' \leq \varphi(a)}} \mathrm{adh}^0_L(\varphi_!(\varphi(a)))) \\ &\leq \varphi(\bigwedge_{\substack{a \in \mathcal{C}_L \\ \ell' \leq \varphi(a)}} \mathrm{adh}^0_L(a)) \quad \text{since } \varphi_!(\varphi(a)) \leq a \\ &= \varphi(\bigwedge_{\substack{a \in \mathcal{C}_L \\ \varphi_!(\ell') \leq a}} \mathrm{adh}^0_L(a)) = \varphi(\mathrm{adh}_L(\varphi_!(\ell'))). \end{aligned}$$

This shows (6.6). (6.5) is an equivalent formulation. □

Corollary 6.25. *Let* $\mathbf{C}$ *be a category of coframes, and let* $\varphi\colon L \to L'$ *be a morphism in* $\mathbf{C}^{\mathrm{conv}}$*. If* $c \in L$ *is closed then* $\varphi(c)$ *is closed in* L'*.*

Proof. We use Fact 6.23. In view of (6.6) with $\ell' = \varphi(c)$,

$$\mathrm{adh}_{L'}(\varphi(c)) \leq \varphi(\mathrm{adh}_L(\varphi_!(\varphi(c)))) \leq \varphi(\mathrm{adh}_L\, c) \leq \varphi(c),$$

where the second inequality follows from $\varphi_!(\varphi(c)) \leq c$ because $\varphi_!$ is left-adjoint to φ and the last inequality follows from Fact 6.23 because c is closed in L. We conclude that $\varphi(c)$ is closed in L'. □

7. Adherence coframes

We have already mentioned the fact that it is equivalent to assume a convergence space X to be pretopological, or to assume that it arises from a non-idempotent closure operator, i.e., an inflationary operator ν on $\mathbb{P}(X)$ that preserves finite suprema. This is also known as a *Čech closure operator*; inflationary means that $\ell \leq \nu(\ell)$ for every ℓ.

To make this formal, call *adherence space* any space X with a Čech closure operator $\nu_{\mathbb{P}(X)}$. Adherence spaces form a category **Adh**, whose morphisms $f\colon X \to X'$ are the *continuous* maps, namely those maps such that $f(\nu_{\mathbb{P}(X)}(A)) \subseteq \nu_{\mathbb{P}(Y)}(f(A))$ for every $A \in \mathbb{P}(X)$. The fact that pretopological convergence spaces can be described equivalently as adherence spaces translates to the fact that there is an equivalence between the categories **PreTop** and **Adh**.

7.1. **Pretopological coframes and adherence coframes.** That equivalence is lost in the pointfree case, in general. What will remain is an adjunction between $\mathbf{C}^{\text{pretop}}$ (resp., $\mathbf{C}^{\text{pretop}*}$) with a new category $\mathbf{C}^{\text{adh}}$ of $\mathbf{C}$-objects L with an adherence-like operator $\nu \colon L \to L$. We shall drop the inflationary requirement, since $\nu = \text{adh}_L$ need not be inflationary, in view of Remark 6.14. This is analogous to our dropping the Point Axiom in the definition of convergence lattices.

The axioms that ν should satisfy are obtained by looking at the properties of adh_L, summarized in Proposition 6.22 and Proposition 6.24.

Definition 7.1 (Adherence coframe)**.** Let $\mathbf{C}$ be a category of coframes. An *adherence* $\mathbf{C}$*-object* is an object L of $\mathbf{C}$ together with an operator $\nu_L \colon L \to L$ that is monotonic, preserves finite suprema of complemented elements, and satisfies $\nu_L(\ell) = \bigwedge_{a \in \mathcal{C}_L, a \geq \ell} \nu_L(a)$ for every $\ell \in L$.

The adherence $\mathbf{C}$-objects form a category $\mathbf{C}^{\text{adh}}$, whose morphisms $\varphi \colon L \to L'$ are the $\mathbf{C}$-morphisms that are *continuous* in that they satisfy:

$$\nu_{L'}(\ell') \leq \varphi(\nu_L(\varphi_!(\ell'))) \tag{7.1}$$

for every $\ell' \in L'$.

We shall call *adherence structure* on L any operator $\nu \colon L \to L$ that is monotonic, preserves finite suprema of complemented elements, and satisfies $\nu(\ell) = \bigwedge_{a \in \mathcal{C}_L, a \geq \ell} \nu(a)$ for every $\ell \in L$. The adherence $\text{adh}_{\lim}$ of a convergence structure lim on L is always an adherence structure, by Proposition 6.22.

The following shows that adherence structures on coframes L are, equivalently, those finite supremum preserving self-maps $\nu \colon L \to L$ such that $\nu(\ell) = \bigwedge_{a \in \mathcal{C}_L, a \geq \ell} \nu(a)$ for every $\ell \in L$.

Lemma 7.2. *Let L be a coframe. Every adherence structure ν on L preserves all finite suprema.*

Proof. Let $\ell_1, \ldots, \ell_n$ be finitely many elements. The inequality $\nu(\ell_1 \vee \cdots \vee \ell_n) \geq \nu(\ell_1) \vee \cdots \vee \nu(\ell_n)$ follows from monotonicity. In the other direction, and letting a, $a_1, \ldots, a_n$ range over complemented elements:

$$\begin{aligned}
\nu(\ell_1 \vee \cdots \vee \ell_n) &= \bigwedge_{a \geq \ell_1 \vee \cdots \vee \ell_n} \nu(a) \\
&\leq \bigwedge_{a_1 \geq \ell_1, \ldots, a_n \geq \ell_n} \nu(a_1 \vee \cdots \vee a_n) \\
&\quad \text{since } a_1 \geq \ell_1, \ldots, a_n \geq \ell_n \text{ imply } a := a_1 \vee \cdots \vee a_n \geq \ell_1 \vee \cdots \vee \ell_n \\
&= \bigwedge_{a_1 \geq \ell_1, \ldots, a_n \geq \ell_n} (\nu(a_1) \vee \cdots \vee \nu(a_n)) \\
&= \left(\bigwedge_{a_1 \geq \ell_1} \nu(a_1) \right) \vee \cdots \vee \left(\bigwedge_{a_n \geq \ell_n} \nu(a_n) \right) \\
&\quad \text{by the coframe distributivity law} \\
&= \nu(\ell_1) \vee \cdots \vee \nu(\ell_n).
\end{aligned}$$

$\square$

Lemma 7.3. *Let* $\mathbf{C}$ *be a category of coframes, and* L *be a* $\mathbf{C}$*-object. Every adherence structure* ν *on* L *defines a classical pretopological convergence structure* $\lim_\nu$ *by:*

$$\lim_\nu \mathcal{F} := \bigwedge_{a \in \mathcal{C}_L, a \in \mathcal{F}^\#} \nu(a). \tag{7.2}$$

Moreover:

(1) *The mapping* $\nu \mapsto \lim_\nu$ *is monotonic: if* $\nu \leq \nu'$ *then* $\lim_\nu \leq \lim_{\nu'}$*.*
(2) *The mapping* $\lim \mapsto \mathrm{adh}_{\lim}$ *that sends every convergence structure to its adherence is monotonic: if* $\lim_1 \leq \lim_2$ *then* $\mathrm{adh}_{\lim_1} \leq \mathrm{adh}_{\lim_2}$*.*
(3) *For every convergence structure* $\lim$ *on* L*, for every filter* $\mathcal{F}$ *on* L*,* $\lim \mathcal{F} \leq \lim_{\mathrm{adh}_{\lim}} \mathcal{F}$*.*
(4) *For every adherence structure* ν *on* L*, for every* $\ell \in L$*,* $\mathrm{adh}_{\lim_\nu} \ell \leq \nu(\ell)$*.*

In other words, the $\lim \mapsto \mathrm{adh}_{\lim}$ construction is left adjoint to the $\nu \mapsto \lim_\nu$ construction.

Proof. Clearly $\lim_\nu$ is monotonic: if $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G}^\# \subseteq \mathcal{F}^\#$ by Lemma 6.3 (3), so $\lim_\nu \mathcal{F} \leq \lim_\nu \mathcal{G}$. Pretopologicity is the reason why we only quantify over complemented elements a in (7.2). Indeed, let $(\mathcal{F}_i)_{i \in I}$ be a family of filters on L, and let $\mathcal{F} := \bigcap_{i \in I} \mathcal{F}_i$. If a is complemented then a is in $\mathcal{F}^\#$ if and only if its complement $\overline{a}$ is not in $\mathcal{F}$, by Lemma 6.5, if and only if there is an $i \in I$ such that $\overline{a}$ is not in $\mathcal{F}_i$, if and only if there is an $i \in I$ such that a is in $\mathcal{F}_i^\#$. It follows that

$$\begin{aligned} \lim_\nu \mathcal{F} &= \bigwedge_{a \in \mathcal{C}_L, a \in \bigcup_{i \in I} \mathcal{F}_i^\#} \nu(a) \\ &= \bigwedge_{i \in I} \bigwedge_{a \in \mathcal{C}_L, a \in \mathcal{F}_i^\#} \nu(a) = \bigwedge_{i \in I} \lim_\nu \mathcal{F}_i. \end{aligned}$$

The fact that $\lim_\nu$ is classical follows from the definition of $\lim_\nu$ and the second part of Corollary 6.7.

Claims (1) and (2) are immediate. For (3),

$$\begin{aligned} \lim_{\mathrm{adh}_{\lim}} \mathcal{F} &= \bigwedge_{a \in \mathcal{C}_L, a \in \mathcal{F}^\#} \bigwedge_{b \in \mathcal{C}_L, b \geq a} \bigvee_{\substack{\mathcal{G} \in \mathbb{F}L \\ b \in \mathcal{G}^\#}} \lim \mathcal{G} \\ &= \bigwedge_{a \in \mathcal{C}_L, a \in \mathcal{F}^\#} \bigvee_{\substack{\mathcal{G} \in \mathbb{F}L \\ a \in \mathcal{G}^\#}} \lim \mathcal{G} \\ &\qquad \text{since } \mathcal{F}^\# \text{ is upwards-closed by Lemma 6.3 (1)} \\ &\geq \lim \mathcal{F} \quad \text{since one can take } \mathcal{G} = \mathcal{F} \text{ in the supremum.} \end{aligned}$$

For (4),

$$\begin{aligned}\mathrm{adh}_{\lim_\nu}\ell &= \bigwedge_{a\in\mathcal{C}_L, a\geq\ell}\ \bigvee_{\substack{\mathcal{F}\in\mathbb{F}L\\ a\in\mathcal{F}^\#}}\ \bigwedge_{b\in\mathcal{C}_L, b\in\mathcal{F}^\#}\nu(b)\\ &\leq \bigwedge_{a\in\mathcal{C}_L, a\geq\ell}\ \bigvee_{\substack{\mathcal{F}\in\mathbb{F}L\\ a\in\mathcal{F}^\#}}\nu(a)\\ &= \bigwedge_{a\in\mathcal{C}_L, a\geq\ell}\nu(a)=\nu(\ell)\end{aligned}$$

where the last equality is part of the definition of an adherence structure. Note that we have used the equality:

$$\bigvee_{\substack{\mathcal{F}\in\mathbb{F}L\\ a\in\mathcal{F}^\#}}\nu(a)=\nu(a),$$

which looks obvious but actually requires some care. That equality holds unless there is no filter $\mathcal{F}$ such that $a\in\mathcal{F}^\#$ (in which case the left-hand side is $\bot$) and $\nu(a)\neq\bot$. If $a\neq\bot$, there there is a filter $\mathcal{F}$ such that $a\in\mathcal{F}^\#$, for example $\{\top\}$. If $a=\bot$, then $\nu(a)=\bot$ since ν preserves finite suprema (hence empty suprema) of complemented elements, so the equality holds in all cases. ☐

Proposition 7.4. *Let* $\mathbf{C}$ *be an admissible category of coframes. There is an identity-on-morphisms functor* $(L,\lim_L)\mapsto(L,\mathrm{adh}_L)$ *from* $\mathbf{C}^{\mathrm{pretop}}$ *to* $\mathbf{C}^{\mathrm{adh}}$*, which is right adjoint to the identity-on-morphisms functor* $(L,\nu)\mapsto(L,\lim_\nu)$ *from* $\mathbf{C}^{\mathrm{adh}}$ *to* $\mathbf{C}^{\mathrm{pretop}}$*. Similarly with* $\mathbf{C}^{\mathrm{pretop}*}$ *in lieu of* $\mathbf{C}^{\mathrm{pretop}}$*.*

Proof. Write U for the first functor, and F for the second. By "identity-on-morphisms", we mean that $U(\varphi)=\varphi$ and $F(\varphi)=\varphi$ for every morphism φ. If φ is a morphism from $(L,\lim_L)$ to $(L',\lim_{L'})$, $U(\varphi)$ is a morphism from $U(L,\lim_L)=(L,\mathrm{adh}_L)$ to $U(L',\lim_{L'})=(L',\mathrm{adh}_{L'})$ by Proposition 6.24.

As far as F is concerned, consider a morphism $\varphi\colon(L,\nu_L)\to(L',\nu_{L'})$ of adherence $\mathbf{C}$-objects. We must show that φ is continuous from $(L,\lim_{\nu_L})$ to $(L',\lim_{\nu_{L'}})$. Fix a filter $\mathcal{F}$ on L'.

$$\begin{aligned}\lim_{\nu_L}\varphi^{-1}(\mathcal{F}) &= \bigwedge_{a\in\mathcal{C}_L\cap\varphi^{-1}(\mathcal{F})^\#}\nu_L(a)\\ &\geq \bigwedge_{a\in\mathcal{C}_L\cap\varphi^{-1}(\mathcal{F})^\#}\nu_L(\varphi_!(\varphi(a)))\quad\text{since }\varphi_!\text{ is left adjoint to }\varphi\\ &\geq \bigwedge_{a'\in\mathcal{C}_{L'}\cap\mathcal{F}^\#}\nu_L(\varphi_!(a')),\end{aligned}$$

since by Corollary 6.6, for every complement a in $\varphi^{-1}(\mathcal{F})^\#$, one can find a complemented element $a':=\varphi(a)$ in $\mathcal{F}^\#$.

Applying φ and using the fact that φ preserves infima,

$$\begin{aligned}\varphi(\lim_{\nu_L}\varphi^{-1}(\mathcal{F})) &\geq \bigwedge_{a'\in\mathcal{C}_{L'}\cap\mathcal{F}^\#}\varphi(\nu_L(\varphi_!(a')))\\ &\geq \bigwedge_{a'\in\mathcal{C}_{L'}\cap\mathcal{F}^\#}\nu_{L'}(a')=\lim_{\nu_{L'}}\mathcal{F}.\end{aligned}$$

The second inequality stems from the continuity of φ as a morphism of adherence **C**-objects, namely condition (7.1). We recognize the formula for continuity as a morphism of convergence **C**-objects, namely (2.1).

Showing $F \dashv U$ is now easy. The unit $\eta_L : (L, \nu_L) \to UF(L, \nu_L) = (L, \mathrm{adh}_{\lim_{\nu_L}})$ is the identity map. Continuity (7.1) boils down to checking that for every $\ell \in L$, $\mathrm{adh}_{\lim_{\nu_L}}(\ell) \leq \nu_L(\ell)$, and that is Lemma 7.3 (4). The counit $\epsilon_L : FU(L, \lim_L) = (L, \lim_{\mathrm{adh}_L}) \to (L, \lim_L)$ is also the identity map. Continuity (2.1) boils down to checking that for every filter $\mathcal{F}$ on L, $\lim_L \mathcal{F} \leq \lim_{\mathrm{adh}_L} \mathcal{F}$, and that is Lemma 7.3 (3). The fact that η_L and ϵ_L are natural in L, and that $\epsilon_{F(L)} \circ F(\eta_L)$ and $U(\epsilon_L) \circ \eta_{U(L)}$ are identities are obvious since all concerned maps are identities. ☐

The counit of that adjunction is the identity map from $(L, \lim_{\mathrm{adh}_L})$ to $(L, \lim_L)$, and the unit is the identity map from (L, ν_L) to $(L, \mathrm{adh}_{\lim_{\nu_L}})$.

7.2. **Complete distributivity and the equivalence between pretopological and adherence structures.** We now investigate sufficient conditions for the inequalities in Lemma 7.3 (3), (4) to be equalities.

A frame Ω is *spatial* if and only if every element is an infimum of meet-prime elements. Under the Axiom of Choice, this is equivalent to requiring that Ω be order-isomorphic to the open-set lattice of some topological space, but we will not use that.

Definition 7.5. We say that a coframe L is *spatial* if and only if its dual frame L^{op} is spatial. In other words, L is spatial if and only if every element is a supremum of join-primes.

On a complete lattice L, and following [5], we say that ℓ is *way-way-below* ℓ', in notation $\ell \lll \ell'$, if and only if for every subset S of L such that $\ell' \leq \bigvee S$, some element of S is larger than or equal to ℓ. L itself is *prime-continuous* if and only if every element ℓ of L is the supremum of elements way-way-below ℓ. The prime-continuous complete lattices are exactly the completely distributive complete lattices, as first observed by Raney [10], see [5, Exercice IV-3.31], but that observation requires the Axiom of Choice.

Again using the Axiom of Choice, prime-continuity implies spatiality. In fact, prime-continuity implies continuity and distributivity, and the continuous distributive spatial lattices are exactly the completely distributive lattices [5, Proposition VII-2.10 (7)]. The following is choice-free, as is everything else in this paper.

Proposition 7.6. *Let* **C** *be a category of coframes, and L be a* **C***-object.*

(1) *If L is spatial, then $\nu = \mathrm{adh}_{\lim_\nu}$ for every adherence structure ν on L.*
(2) *If L is prime-continuous, then* $\lim = \lim_{\mathrm{adh}_{\lim}}$ *for every classical pretopological convergence structure* $\lim$ *on L.*
(3) *If L is both, then the maps $\nu \mapsto \lim_\nu$ and $\lim \mapsto \mathrm{adh}_{\lim}$ define an order isomorphism between adherence structures and classical pretopological structures on L.*

Proof. (1) Let ν be an adherence structure on L. In order to show that $\nu = \mathrm{adh}_{\lim_\nu}$, and considering Lemma 7.3 (4), it is enough to show that $\mathrm{adh}_{\lim_\nu} \ell \geq \nu(\ell)$ for every $\ell \in L$. Since $\nu_L(\ell) = \bigwedge_{a \in \mathcal{C}_L, a \geq \ell} \nu_L(a)$ for every $\ell \in L$, and by the definition of $\mathrm{adh}_{\lim_\nu}$, it is enough to show $\mathrm{adh}^0_{\lim_\nu} a \geq \nu(a)$ for every complemented element a of L.

For that, we consider an arbitrary join-prime $\ell \leq \nu(a)$, and we claim that there is a filter $\mathcal{F}$ on L such that $a \in \mathcal{F}^{\#}$ and, for every complemented element b of $\mathcal{F}^{\#}$, $\ell \leq \nu(b)$. We simply define $\mathcal{F}$ as the upward closure of the set of complements $\bar{b}$ of complemented elements b such that $\ell \not\leq \nu(b)$. Since ℓ is join-prime, $\ell \not\leq \bot$. Since ν preserves finite suprema of complemented elements, in particular the supremum $\bot$ of the empty family, $\ell \not\leq \nu(b)$ for $b = \bot$. Hence $\mathcal{F}$ contains the complement $\top$ of $\bot$, hence is not empty. In order to show that $\mathcal{F}$ is a filter, we only have to show that for any two complements $\bar{b}_1$ and $\bar{b}_2$ such that $\ell \not\leq \nu(b_1), \nu(b_2)$, their infimum $\bar{b}_1 \wedge \bar{b}_2 = \overline{b_1 \vee b_2}$ is such that $\ell \not\leq \nu(b_1 \vee b_2)$. This is obtained by contraposition: if $\ell \leq \nu(b_1 \vee b_2)$, then $\ell \leq \nu(b_1)$ or $\ell \leq \nu(b_2)$ because ν preserves finite suprema of complemented elements and because ℓ is join-prime. Therefore $\mathcal{F}$ is a filter. Since $\ell \leq \nu(a)$, the complement $\bar{a}$ of a is not in $\mathcal{F}$, and by Lemma 6.5, a is in $\mathcal{F}^{\#}$.

Having proved this, we obtain that for every join-prime $\ell \leq \nu(a)$,

$$\ell \leq \bigvee_{\mathcal{F} \in \mathbb{F}L, a \in \mathcal{F}^{\#}} \bigwedge_{b \in \mathcal{C}_L \cap \mathcal{F}^{\#}} \nu(b) = \mathrm{adh}^0_{\lim_\nu}(a).$$

Because every element is a supremum of join-primes, $\nu(a) \leq \mathrm{adh}^0_{\lim_\nu}(a)$, and this concludes our argument.

(2) Let lim be a classical pretopological convergence structure on L. In order to show that $\lim = \lim_{\mathrm{adh}_{\lim}}$, and considering Lemma 7.3 (3), it is enough to show that $\lim \mathcal{F} \geq \lim_{\mathrm{adh}_{\lim}} \mathcal{F}$ for every filter $\mathcal{F}$ on L. Since L is prime-continuous, it is enough to show that, for every $\ell \lll \lim_{\mathrm{adh}_{\lim}} \mathcal{F}$, $\ell \leq \lim \mathcal{F}$. Recall that

$$\lim_{\mathrm{adh}_{\lim}} \mathcal{F} = \bigwedge_{a \in \mathcal{C}_L \cap \mathcal{F}^{\#}} \mathrm{adh}_{\lim}(a) = \bigwedge_{a \in \mathcal{C}_L \cap \mathcal{F}^{\#}} \mathrm{adh}^0_{\lim}(a)$$

by Proposition 6.22 (2), so that

$$\lim_{\mathrm{adh}_{\lim}} \mathcal{F} = \bigwedge_{a \in \mathcal{C}_L \cap \mathcal{F}^{\#}} \bigvee_{\mathcal{G} \in \mathbb{F}L, a \in \mathcal{G}^{\#}} \lim \mathcal{G}.$$

Since $\ell \lll \lim_{\mathrm{adh}_{\lim}} \mathcal{F}$, for every complemented element a in $\mathcal{F}^{\#}$, there is a filter $\mathcal{G}$ such that $a \in \mathcal{G}^{\#}$ and $\ell \leq \lim \mathcal{G}$. Consider the family $\mathcal{A}$ of all filters $\mathcal{G}$ such that $\ell \leq \lim \mathcal{G}$ and such that $\mathcal{G}^{\#}$ contains a complemented element of $\mathcal{F}^{\#}$, and look at the filter $\bigcap \mathcal{A}$. Since lim is pretopological, $\lim \bigcap \mathcal{A} = \bigwedge_{\mathcal{G} \in \mathcal{A}} \lim \mathcal{G} \geq \ell$.

For every complemented element a of $\bigcap \mathcal{A}$, we claim that a is in $\mathcal{F}$. Otherwise, by Lemma 6.5 its complement $\bar{a}$ is in $\mathcal{F}^{\#}$, and in that case we have seen that there is a filter $\mathcal{G}$ such that $\bar{a} \in \mathcal{G}^{\#}$ and $\ell \leq \lim \mathcal{G}$, namely, $\mathcal{G}$ is in $\mathcal{A}$. By Lemma 6.5 again and since $\bar{a} \in \mathcal{G}^{\#}$, a is not in $\mathcal{G}$, hence not in $\bigcap \mathcal{A}$.

Therefore $(\bigcap \mathcal{A})_c \subseteq \mathcal{F}$ (see (2.5)). Since lim is classical, $\lim \bigcap \mathcal{A} = \lim (\bigcap \mathcal{A})_c \leq \lim \mathcal{F}$, and since $\lim \bigcap \mathcal{A} \geq \ell$, we conclude that $\ell \leq \lim \mathcal{F}$.

(3) is immediate from (1) and (2). □

Corollary 7.7. *Let* $\mathbf{C}$ *be an admissible category of coframes. The adjunction of Proposition 7.4 restricts to:*

(1) *a coreflection of the full subcategory of spatial objects of* $\mathbf{C}^{\mathrm{pretop}}$ *(resp., of* $\mathbf{C}^{\mathrm{pretop}*}$*) into that of spatial objects of* $\mathbf{C}^{\mathrm{adh}}$*;*
(2) *a reflection of the full subcategory of prime-continuous objects of* $\mathbf{C}^{\mathrm{pretop}*}$ *into that of prime-continuous objects of* $\mathbf{C}^{\mathrm{adh}}$*;*
(3) *an adjoint equivalence between the full subcategory of spatial prime-continuous objects of* $\mathbf{C}^{\mathrm{pretop}*}$ *and that of spatial prime-continuous objects of* $\mathbf{C}^{\mathrm{adh}}$*.*

Proof. The unit is an isomorphism in the first case, the counit is an isomorphism in the second, and both are in the final case. □

Remark 7.8. Even without using the Axiom of Choice, one can check that $\mathbb{P}(X)$ is spatial and prime-continuous, for every set X. Its join-primes are the one-element sets $\{x\}$, and spatiality boils down to the fact that every subset A of X is the union of its one-element subsets. The way-way-below relation is given by $A \lll B$ if and only if $A \subseteq \{x\}$ for some $x \in B$. Then, each subset B is the union of its one-element subsets $\{x\}$, and for each $x \in B$, $\{x\} \lll B$; this shows prime-continuity.

7.3. **A square of adjunctions.** In a world with points, Remark 7.8 and Proposition 7.6, together with the fact that $\mathbb{P}(X)$ is a Boolean algebra, allow us to recover the well-known fact that $\lim \mapsto \mathrm{adh}_{\lim}$ and $\nu \mapsto \lim_\nu$ are inverse bijections between pretopological convergence structures on $\mathbb{P}(X)$ and Čech closure operators on $\mathbb{P}(X)$. We refine this as follows.

Lemma 7.9. *The functors* $(X, \lim) \mapsto (X, \mathrm{adh}_{\lim})$ *and* $(X, \nu) \mapsto (X, \lim_\nu)$ *define an adjoint equivalence of categories between* **PreTop** *and* **Adh**.

Proof. What remains to be proved is that those are actual functors. Let f be a continuous map from $(X, \lim_{\mathbb{P}(X)})$ to $(Y, \lim_{\mathbb{P}(Y)})$ in **Conv**. Then $\varphi := f^{-1}$ is continuous from $(\mathbb{P}(Y), \lim_{\mathbb{P}(Y)})$ to $(\mathbb{P}(X), \lim_{\mathbb{P}(X)})$ in $\mathbf{CF}^{\mathrm{conv}}$ (Proposition 2.6). By Proposition 6.24, φ is also continuous from $(\mathbb{P}(Y), \mathrm{adh}_{\lim_{\mathbb{P}(Y)}})$ to $(\mathbb{P}(X), \mathrm{adh}_{\lim_{\mathbb{P}(X)}})$. Now read (6.5), and recall that $\varphi_!$ is the direct f-image operator: this shows that $f(\mathrm{adh}_{\lim_{\mathbb{P}(X)}}(A)) \subseteq \mathrm{adh}_{\lim_{\mathbb{P}(Y)}}(f(A))$ for every subset A of X, namely that f is continuous in the sense of adherence spaces.

Conversely, let $f \colon (X, \nu_{\mathbb{P}(X)}) \to (X, \nu_{\mathbb{P}(Y)})$ be a morphism of adherence spaces. Similarly, $\varphi := f^{-1}$ is a morphism from $(\mathbb{P}(Y), \nu_{\mathbb{P}(Y)})$ to $(\mathbb{P}(X), \nu_{\mathbb{P}(X)})$ in $\mathbf{C}^{\mathrm{adh}}$, hence one from $(\mathbb{P}(Y), \lim_{\nu_{\mathbb{P}(Y)}})$ to $(\mathbb{P}(X), \lim_{\nu_{\mathbb{P}(X)}})$ in $\mathbf{C}^{\mathrm{pretop}*}$ by Proposition 7.4. Therefore, for every filter $\mathcal{F}$ of subsets of X, $\lim_{\nu_{\mathbb{P}(X)}} \mathcal{F} \subseteq \varphi(\lim_{\nu_{\mathbb{P}(Y)}} \varphi^{-1}(\mathcal{F}))$. In particular, for every point $x \in X$ that $\mathcal{F}$ converges to, $f(x)$ is in $\lim_{\nu_{\mathbb{P}(Y)}} \varphi^{-1}(\mathcal{F})$. Since $\varphi^{-1}(\mathcal{F}) = f[\mathcal{F}]$, $f[\mathcal{F}]$ converges to $f(x)$ with respect to $\lim_{\nu_{\mathbb{P}(Y)}}$. □

Turning to the opposite categories, we obtain a functor $(L, \lim_L) \mapsto (L, \mathrm{adh}_L)$ from $(\mathbf{C}^{\mathrm{pretop}})^{op}$ (or $(\mathbf{C}^{\mathrm{pretop}*})^{op}$) to $(\mathbf{C}^{\mathrm{adh}})^{op}$, left adjoint to the functor $(L, \nu) \mapsto (L, \lim_\nu)$ from $(\mathbf{C}^{\mathrm{adh}})^{op}$ to $(\mathbf{C}^{\mathrm{pretop}})^{op}$. Composing that with the coreflection $\mathbb{P} \dashv \mathrm{pt}$ between **PreTop** and $(\mathbf{C}^{\mathrm{pretop}})^{op}$, we obtain an adjunction between **PreTop** and $\mathbf{C}^{\mathrm{adh}}$, hence also between **Adh** and $\mathbf{C}^{\mathrm{adh}}$. This is the rightmost adjunction in the following square, which is the third square from the left in (1.1) if we define $\mathbf{C}$ as **CF**.

(7.3)
$$\begin{tikzcd}
(\mathbf{C}^{\mathrm{pretop}*})^{op} \arrow[r, shift left] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathrm{pt}"] & (\mathbf{C}^{\mathrm{adh}})^{op} \arrow[l, shift left] \arrow[d, shift left, "\mathrm{pt}"] \\
\mathbf{PreTop} \arrow[u, shift left, "\mathbb{P}"] \arrow[u, phantom, "\dashv"] \arrow[r, equal] & \mathbf{Adh} \arrow[u, shift left, "\mathbb{P}"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

We again write $\mathbb{P} \dashv \mathrm{pt}$ for that adjunction between **Adh** and $(\mathbf{C}^{\mathrm{adh}})^{op}$, and our purpose is now to make it explicit.

The left adjoint $\mathbb{P}$ maps every pretopological convergence space X to $(\mathbb{P}(X), \mathrm{adh}_{\mathbb{P}(X)})$ and every continuous map $f \colon X \to Y$ to $\mathbb{P}(f) = f^{-1}$. Note that $\mathrm{adh}_{\mathbb{P}(X)}$ coincides with the raw adherence $\mathrm{adh}^0_{\mathbb{P}(X)}$ because every element of $\mathbb{P}(X)$ is complemented, and that the raw adherence is the usual notion of adherence on (pretopological) convergence spaces.

Writing $U \dashv F$ for the adjunction between $(\mathbf{C}^{\text{pretop}*})^{op}$ and $(\mathbf{C}^{\text{adh}})^{op}$ on the top row of (7.3), at least temporarily, the unit of the rightmost $\mathbb{P} \dashv \text{pt}$ adjunction is:

$$X \xrightarrow{\eta_X} \text{pt}\,\mathbb{P}(X) \xrightarrow{\text{pt}\,u_{\mathbb{P}(X)}} \text{pt}\,FU\mathbb{P}(X)$$

where $\eta_X : x \mapsto \dot{x}$ is the unit of the adjunction $\mathbb{P} \dashv \text{pt}$ between **Conv** (or **PreTop**) and $(\mathbf{C}^{\text{conv}})^{op}$ (or $(\mathbf{C}^{\text{pretop}*})^{op}$), and $u_L \colon FUL \to L$ is the unit of the adjunction $U \dashv F$, written as a morphism in the opposite category. By Proposition 7.6 and Remark 7.8, $u_{\mathbb{P}(X)}$ is an isomorphism, and by Lemma 2.26 η_X is an isomorphism. It follows that the unit of the rightmost adjunction of (7.3) is an isomorphism, whence:

Proposition 7.10. *Let* $\mathbf{C}$ *be an admissible category of coframes.* **Adh** *is a coreflective subcategory of* $(\mathbf{C}^{\text{adh}})^{op}$.

The right adjoint, which we again write as pt, maps every adherence $\mathbf{C}$-object (L, ν_L) to $\text{pt}(L, \lim_{\nu_L})$. Let us spell out what this means.

Lemma 7.11. *Let* $\mathbf{C}$ *be an admissible category of coframes, and* L *be an adherence* $\mathbf{C}$*-object. The points of* $(L, \lim_{\nu_L})$ *are exactly the join-prime elements* x *of* L *such that* $x \leq \nu_L(x)$.

Proof. Following Remark 2.13, a point of $(L, \lim_{\nu_L})$ is a join-prime x of L such that $x \leq \lim_{\nu_L}(\uparrow x)$. Now $\lim_{\nu_L}(\uparrow x)$ is equal to

$$\bigwedge_{a \in (\uparrow x)^{\#}} \nu_L(a) = \bigwedge_{a \in \mathcal{C}_L : \overline{a} \not\geq x} \nu_L(a),$$

where $\overline{a}$ is the complement of a. Since x is join-prime, and $\top = a \vee \overline{a} \geq x$, a or $\overline{a}$ is larger than or equal to x. Since x is join-prime again, x is different from $\bot$, so a and $\overline{a}$ cannot be both larger than or equal to x. It follows that the condition $\overline{a} \not\geq x$ is equivalent to $a \geq x$. We obtain that $\lim_{\nu_L}(\uparrow x)$ is equal to $\bigwedge_{a \geq x} \nu_L(a)$, where a ranges over the complemented elements of L. That is equal to $\nu_L(x)$, whence the conclusion. ☐

We make that into a definition.

Definition 7.12 (Point)**.** Let $\mathbf{C}$ be an admissible category of coframes. For every adherence $\mathbf{C}$-object L, the *points* of L are the join-prime elements x such that $x \leq \nu_L(x)$, namely the points of $(L, \lim_{\nu_L})$. We write again $\text{pt}\,L$ for the set of points of L.

Define again $\ell^\bullet$ as $\{x \in \text{pt}\,L \mid x \leq \ell\}$. The only change compared to Definition 2.18 is that $\text{pt}\,L$ now has to be understood per Definition 7.12. The following is an analogue of Lemma 2.19 (1), (3), (4) for adherence coframes; item (4) is new, and could have been observed back then, too.

Lemma 7.13. *Let* $\mathbf{C}$ *be an admissible category of coframes, and* L *be an adherence* $\mathbf{C}$*-object. The following hold:*

(1) *For all* ℓ, ℓ' *in* L*, if* $\ell \leq \ell'$ *then* $\ell^\bullet \subseteq \ell'^\bullet$.
(2) *For all* $\ell_1, \ell_2, \cdots, \ell_n \in L$ *(*$n \in \mathbb{N}$*),* $\bigcup_{i=1}^n \ell_i^\bullet = (\bigvee_{i=1}^n \ell_i)^\bullet$.
(3) *For every family* $(\ell_i)_{i \in I}$ *in* L*,* $\bigcap_{i \in I} \ell_i^\bullet = (\bigwedge_{i \in I} \ell_i)^\bullet$.

(4) *For every complemented element a of L, the complement of $a^\bullet$ in* $\mathrm{pt}\, L$ *is* $(\overline{a})^\bullet$*, where $\overline{a}$ is the complement of a in L.*

Proof. (1), (3) are obvious, and (2) follows from the fact that the elements of $(\bigvee_{i=1}^n \ell_i)^\bullet$ are join-prime. In order to show (4), we use (2) to obtain $a^\bullet \cup (\overline{a})^\bullet = (a \vee \overline{a})^\bullet = \top^\bullet = \mathrm{pt}\, L$, and (3) to obtain $a^\bullet \cap (\overline{a})^\bullet = (a \wedge \overline{a})^\bullet = \bot^\bullet = \emptyset$. ☐

The convergence space $\mathrm{pt}\, L$ is then pretopological. Equivalently, $\mathrm{pt}\, L$ is a space with a Čech closure operator $\nu_{\mathbb{P}\,\mathrm{pt}\, L} = \mathrm{adh}_{\lim_{\mathbb{P}\,\mathrm{pt}\, L}}$, which we now describe explicitly.

Lemma 7.14. *Let* $\mathbf{C}$ *be an admissible category of coframes. For every adherence* $\mathbf{C}$*-object L, for every $S \in \mathbb{P}\,\mathrm{pt}\, L$, $\nu_{\mathbb{P}\,\mathrm{pt}\, L}(S) = (\nu_L(\bigvee S))^\bullet$.*

Here S is a set of points, hence in particular a subset of L: $\bigvee S$ is the supremum of that set in L.

Proof. Define $\nu(S)$ as $(\nu_L(\bigvee S))^\bullet$. The outline of the proof is as follows. We check that ν is a Čech-closure operator, that its associated pretopological convergence $\lim_\nu$ is exactly $\lim_{\mathbb{P}\,\mathrm{pt}\, L}$, and we shall conclude that ν is equal to $\mathrm{adh}_{\lim_{\mathbb{P}\,\mathrm{pt}\, L}}$ by Proposition 7.6.

Let us start by checking that ν is a Čech-closure operator, namely that ν commutes with finite suprema. Since ν_L is monotonic and by Lemma 7.13 (1), ν is monotonic. Let $S = S_1 \cup \cdots \cup S_n$. Since ν is monotonic, $\bigcup_{i=1}^n \nu(S_i) \subseteq \nu(S)$. To show the converse inequality, let $a_1, \ldots, a_n$ be arbitrary complemented elements such that $a_i \geq \bigvee S_i$, $1 \leq i \leq n$. Then $a := a_1 \vee \cdots \vee a_n \geq \bigvee S$, and $\nu_L(a) = \bigvee_{i=1}^n \nu_L(a_i)$. Taking infima over $a_1, \ldots, a_n$, we obtain

$$\nu(\bigvee S) \leq \bigwedge_{\substack{\forall i=1,\cdots,n.\\ a_i \geq \bigvee S_i}} \bigvee_{i=1}^n \nu_L(a_i) = \bigvee_{i=1}^n \bigwedge_{a_i \geq \bigvee S_i} \nu_L(a_i)$$

by the coframe distributivity law, and $\bigvee_{i=1}^n \bigwedge_{a_i \geq \bigvee S_i} \nu_L(a_i) = \bigvee_{i=1}^n \nu_L(\bigvee S_i)$. We now use Lemma 7.13 (1) and (2) to obtain

$$\nu(S) = (\nu_L(\bigvee S))^\bullet \subseteq (\bigvee_{i=1}^n \nu_L(\bigvee S_i))^\bullet = \bigcup_{i=1}^n \nu_L(\bigvee S_i)^\bullet = \bigcup_{i=1}^n \nu(S_i).$$

Since $\mathbb{P}\,\mathrm{pt}\, L$ is Boolean, ν is therefore a (centered) adherence structure on $\mathbb{P}\,\mathrm{pt}\, L$.

As a preparation to proving that $\lim_\nu = \lim_{\mathbb{P}\,\mathrm{pt}\, L}$, we establish the following facts, where $\mathcal{F}$ is a filter of subsets of $\mathrm{pt}\, L$, and S is a subset of $\mathrm{pt}\, L$:

(A) For every complemented element a in $(\mathcal{F}^\circ)^\#$, $a^\bullet$ is in $\mathcal{F}^\#$. If $a^\bullet$ were not in $\mathcal{F}^\#$, then by Lemma 6.5 the complement of $a^\bullet$ would be in $\mathcal{F}$, that is, $(\overline{a})^\bullet$ would be in $\mathcal{F}$, by Lemma 7.13 (4). By definition, $\overline{a}$ would then be in $\mathcal{F}^\circ$, contradicting the fact that $a \in (\mathcal{F}^\circ)^\#$, in view of Lemma 6.5.

(B) If $S \in \mathcal{F}^\#$, then for every complemented element $a \geq \bigvee S$, a is in $(\mathcal{F}^\circ)^\#$. Otherwise, $\overline{a}$ would be in $\mathcal{F}^\circ$, namely $(\overline{a})^\bullet$ would be in $\mathcal{F}$. By Lemma 7.13 (4) again, the complement of $a^\bullet$ would be in $\mathcal{F}$, so $a^\bullet$ would not be in $\mathcal{F}^\#$. We notice that S is included in $(\bigvee S)^\bullet$, since every point of S is by definition smaller than or equal to $\bigvee S$. Using Lemma 7.13 (1), $(\bigvee S)^\bullet \subseteq a^\bullet$, so $S \subseteq a^\bullet$. Since $\mathcal{F}^\#$ is upwards-closed (Lemma 6.3 (1)), it follows that S cannot be in $\mathcal{F}^\#$, a contradiction.

It follows that:

(C) For every filter $\mathcal{F}$ of subsets of $\operatorname{pt} L$, $\bigcap_{S \in \mathcal{F}^{\#}} \nu(S) = (\bigwedge_{a \in (\mathcal{F}^\circ)^{\#}} \nu_L(a))^{\bullet}$. Indeed, in one direction, for every $S \in \mathcal{F}^{\#}$,

$$\nu_L(\bigvee S) = \bigwedge_{a \in \mathcal{C}_L, a \geq \bigvee S} \nu_L(a) \geq \bigwedge_{a \in (\mathcal{F}^\circ)^{\#}} \nu_L(a)$$

by (B). Using Lemma 7.13 (1), $\nu(S) = \nu_L(\bigvee S)^{\bullet} \supseteq (\bigwedge_{a \in (\mathcal{F}^\circ)^{\#}} \nu_L(a))^{\bullet}$. In the other direction, for every complemented element $a \in (\mathcal{F}^\circ)^{\bullet}$, $S := a^{\bullet}$ is in $\mathcal{F}^{\#}$ by (A). Moreover, $\bigvee S \leq a$ since every point in S is by definition less than or equal to a, so $\nu_L(\bigvee S) \leq \nu_L(a)$, and hence $\nu(S) = \nu_L(\bigvee S)^{\bullet} \subseteq \nu_L(a)^{\bullet}$ by Lemma 7.13 (1). It follows that $\nu_L(a)^{\bullet} \supseteq \bigcap_{S \in \mathcal{F}^{\#}} \nu(S)$. Taking infima over all complemented a, we obtain that:

$$\bigcap_{S \in \mathcal{F}^{\#}} \nu(S) \subseteq \bigcap_{a \in (\mathcal{F}^\circ)^{\#}} \nu_L(a)^{\bullet} = \left(\bigwedge_{a \in (\mathcal{F}^\circ)^{\#}} \nu_L(a) \right)^{\bullet},$$

where the last equality is by Lemma 7.13 (3).

For every filter $\mathcal{F}$ of subsets of $\operatorname{pt} L$,

$$\lim_{\mathbb{P} \operatorname{pt} L} \mathcal{F} = (\lim_L \mathcal{F}^\circ)^{\bullet} = \left(\bigwedge_{a \in \mathcal{C}_L \cap (\mathcal{F}^\circ)^{\#}} \nu_L(a) \right)^{\bullet}.$$

By (C), this is equal to $\bigcap_{S \in \mathcal{F}^{\#}} \nu(S)$. In other words, $\lim_{\mathbb{P} \operatorname{pt} L}$ is equal to $\lim_{\nu}$.

Since $\mathbb{P} \operatorname{pt} L$ is spatial and prime-continuous, by Remark 7.8, we can apply Proposition 7.6: ν is equal to $\operatorname{adh}_{\lim_{\mathbb{P} \operatorname{pt} L}}$, which is by definition equal to $\nu_{\mathbb{P} \operatorname{pt} L}$. □

7.4. **Completeness, cocompleteness.** It is natural to compare adherence structures $\nu, \nu' \colon L \to L$ on a $\mathbf{C}$-object L by saying that ν is *finer* than ν', in notation $\nu \leq \nu'$, or that ν' is *coarser* than ν, if and only if $\nu(\ell) \leq \nu'(\ell)$ for every $\ell \in L$.

The identity map from (L, ν) to (L, ν') is continuous (per (7.1)) if and only if $\nu' \leq \nu$, if and only if ν is coarser than ν'. Let us write $|\_|$ for the forgetful functor from $\mathbf{C}^{\mathrm{adh}}$ to $\mathbf{C}$ mapping (L, ν) to L.

Proposition 7.15. *Let $\mathbf{C}$ be a category of coframes, and L be a $\mathbf{C}$-object. Let also $\varphi_i \colon |L_i| \to L$ be morphisms of $\mathbf{C}$, where each L_i is an adherence $\mathbf{C}$-object, $i \in I$. There is a coarsest adherence structure ν_L on L such that φ_i is continuous for each $i \in I$.*

Proof. By definition φ_i is continuous if and only if $\nu_L(\ell) \leq \varphi_i(\nu_{L_i}(\varphi_{i!}(\ell)))$ for every $\ell \in L$. In particular, if φ_i is continuous for every $i \in I$, then for every $\ell \in L$, and for every finite family $a_1, \ldots, a_n$ of complemented elements such that $\ell \leq \bigvee_{j=1}^n a_j$, we have $\nu_L(\ell) \leq \bigvee_{j=1}^n \nu_L(a_j) \leq \bigvee_{j=1}^n \bigwedge_{i \in I} \varphi_i(\nu_{L_i}(\varphi_{i!}(a_j)))$. That means that $\nu_L(\ell)$ is less than or equal to:

$$\bigwedge_{\substack{a_1, \ldots, a_n \in \mathcal{C}_L \\ \ell \leq \bigvee_{j=1}^n a_j}} \bigvee_{j=1}^n \bigwedge_{i \in I} \varphi_i(\nu_{L_i}(\varphi_{i!}(a_j))). \tag{7.4}$$

In particular, if (7.4) defines an adherence structure on L that makes every φ_i continuous, then this is the desired coarsest adherence structure.

Define $\nu_L(\ell)$ as (7.4). Then ν_L is monotonic. We check that ν_L preserves finite suprema of complemented elements. It is enough to check this for the 0-ary supremum, $\bot$, and for binary suprema. By taking $n = 1$ and $a_1 = \bot$ in (7.4), we obtain that $\nu_L(\bot) \leq \bigwedge_{i\in I} \varphi_i(\nu_{L_i}(\varphi_{i!}(\bot)))$. Since $\varphi_{i!}$ is a left-adjoint, it preserves all suprema, in particular the least element $\bot$. We know that ν_{L_i} preserves finite suprema of complemented elements, hence $\bot$, and φ_i is a coframe morphism, so $\varphi_i(\nu_{L_i}(\varphi_{i!}(\bot))) = \bot$ for every $i \in I$. It follows that $\nu_L(\bot) \leq \bot$. In the case of binary suprema of complemented elements a and a', it suffices to show that $\nu_L(a) \vee \nu_L(a') \geq \nu_L(a \vee a')$, since the converse inequality is implied by monotonicity. We have the following, where $a_1, \ldots, a_m$ and $a'_1, \ldots, a'_n$ range over finite families of complemented elements:

$$\begin{aligned}
\nu_L(a) \vee \nu_L(a') &= \bigwedge_{\substack{a_1,\ldots,a_m \\ a\leq\bigvee_{j=1}^m a_j}} \bigvee_{j=1}^{m} \bigwedge_{i\in I} \varphi_i(\nu_{L_i}(\varphi_{i!}(a_j))) \vee \bigwedge_{\substack{a'_1,\ldots,a'_n \\ a\leq\bigvee_{k=1}^n a'_k}} \bigvee_{k=1}^{n} \bigwedge_{i\in I} \varphi_i(\nu_{L_i}(\varphi_{i!}(a'_k))) \\
&= \bigwedge_{\substack{a_1,\ldots,a_m \\ a'_1,\ldots,a'_n \\ a\leq\bigvee_{j=1}^m a_j \\ a'\leq\bigvee_{k=1}^n a'_k}} \left(\bigvee_{j=1}^{m} \bigwedge_{i\in I} \varphi_i(\nu_{L_i}(\varphi_{i!}(a_j))) \vee \bigvee_{k=1}^{n} \bigwedge_{i\in I} \varphi_i(\nu_{L_i}(\varphi_{i!}(a'_k))) \right)
\end{aligned}$$

by the coframe distributivity law. The conditions $a \leq \bigvee_{j=1}^m a_j$ and $a' \leq \bigvee_{k=1}^n a'_k$ imply $a \vee a' \leq \bigvee_{j=1}^m a_j \vee \bigvee_{k=1}^n a'_k$, so:

$$\begin{aligned}
\nu_L(a) \vee \nu_L(a') &\geq \bigwedge_{\substack{a_1,\ldots,a_m \\ a'_1,\ldots,a'_n \\ a\vee a'\leq\bigvee_{j=1}^m a_j \vee\bigvee_{k=1}^n a'_k}} \left(\bigvee_{j=1}^{m} \bigwedge_{i\in I} \varphi_i(\nu_{L_i}(\varphi_{i!}(a_j))) \vee \bigvee_{k=1}^{n} \bigwedge_{i\in I} \varphi_i(\nu_{L_i}(\varphi_{i!}(a'_k))) \right) \\
&= \nu_L(a \vee a').
\end{aligned}$$

Next, we verify that $\nu_L(\ell) \geq \bigwedge_{a\geq\ell} \nu_L(a)$, where a ranges over complemented elements. Equality will follow by monotonicity. We merely observe that for every finite family of complemented elements $a_1, \ldots, a_n$ such that $\ell \leq \bigvee_{j=1}^n a_j$, there is a complemented element a such that $\ell \leq a$ and $a \leq \bigvee_{j=1}^n a_j$, namely $a := \bigvee_{j=1}^n a_j$. It follows:

$$\begin{aligned}
\nu_L(\ell) &= \bigwedge_{\substack{a_1,\ldots,a_n \\ \ell\leq\bigvee_{j=1}^n a_j}} \bigvee_{j=1}^{n} \bigwedge_{i\in I} \varphi_i(\nu_{L_i}(\varphi_{i!}(a_j))) \\
&\geq \bigwedge_{\substack{a\geq\ell \\ a_1,\ldots,a_n \\ a\leq\bigvee_{j=1}^n a_j}} \bigvee_{j=1}^{n} \bigwedge_{i\in I} \varphi_i(\nu_{L_i}(\varphi_{i!}(a_j))) = \bigwedge_{a\geq\ell} \nu_L(a)
\end{aligned}$$

where again $a, a_1, \ldots, a_n$ range over complemented elements.

Therefore ν_L is an adherence structure on L. In order to show that each φ_i is continuous, namely that $\nu_L(\ell)$ is less than or equal to $\varphi_i(\nu_{L_i}(\varphi_{i!}(\ell))$ for every $\ell \in L$ and every $i \in I$, we proceed as follows. Fix $\ell \in L$ and $i \in I$. For every complemented $b \geq \varphi_{i!}(\ell)$, $a := \varphi_i(b)$ is complemented in L, since φ_i preserves binary suprema and binary infima. Moreover, $a \geq \ell$ since $\varphi_{i!}$ is left-adjoint to φ_i.

Therefore, still understanding a and b as complemented elements,

$$\begin{aligned}\bigwedge_{a\geq \ell}\nu_{L_i}(\varphi_{i!}(a)) &\leq \bigwedge_{b\geq\varphi_{i!}(\ell)}\nu_{L_i}(\varphi_{i!}(\varphi_i(b)))\\ &\leq \bigwedge_{b\geq\varphi_{i!}(\ell)}\nu_{L_i}(b) \quad \text{since } \varphi_{i!}(\varphi_i(b))\leq b\\ &= \nu_{L_i}(\varphi_{i!}(\ell)).\end{aligned}$$

Applying φ_i on both sides, and remembering that φ_i preserves arbitrary infima,

$$\bigwedge_{a\geq\ell}\varphi_i(\nu_{L_i}(\varphi_{i!}(a))) \leq \varphi_i(\nu_{L_i}(\varphi_{i!}(\ell))).$$

Taking $n=1$ (and $a_1=a$) in (7.4) shows that $\nu_L(\ell)$ is smaller than or equal to the left-hand side of the latter inequality, hence also to the right-hand side. □

Corollary 7.16. *Let* $\mathbf{C}$ *be a category of coframes. Then* $|\_|\colon \mathbf{C}^{\mathrm{adh}}\to\mathbf{C}$ *is topological: every sink* $(\varphi_i\colon |L_i|\to L)_{i\in I}$ *has a unique final lift, and this is* ν_L*, as given in Proposition 7.15.*

Proof. As noticed in the proof of Corollary 3.3, uniqueness follows from the dual of [1, Proposition 10.43]. To show existence, we check that (L,ν_L) of Proposition 3.1 is a final lift of $(\varphi_i\colon |L_i|\to L)_{i\in I}$. Let $\psi\colon L\to|L'|$ be such that $\psi\circ\varphi_i$ is continuous for every $i\in I$. We aim to show that ψ is continuous from (L,ν_L) to L', and for that we consider an arbitrary element ℓ' of L', and show that $\nu_{L'}(\ell')\leq\psi(\nu_L(\psi_!(\ell')))$. Indeed, writing again $a_1,\ldots,a_n$ for complemented elements:

$$\begin{aligned}\psi(\nu_L(\psi_!(\ell'))) &= \psi\left(\bigwedge_{\substack{a_1,\ldots,a_n\\ \psi_!(\ell')\leq\bigvee_{j=1}^n a_j}}\bigvee_{j=1}^n\bigwedge_{i\in I}\varphi_i(\nu_{L_i}(\varphi_{i!}(a_j)))\right) \quad \text{by (7.4)}\\ &= \bigwedge_{\substack{a_1,\ldots,a_n\\ \psi_!(\ell')\leq\bigvee_{j=1}^n a_j}}\bigvee_{j=1}^n\bigwedge_{i\in I}\psi(\varphi_i(\nu_{L_i}(\varphi_{i!}(a_j))))\\ &\qquad \text{since } \psi \text{ is a morphism of coframes}\\ &\geq \bigwedge_{\substack{a_1,\ldots,a_n\\ \psi_!(\ell')\leq\bigvee_{j=1}^n a_j}}\bigvee_{j=1}^n\bigwedge_{i\in I}\psi(\varphi_i(\nu_{L_i}(\varphi_{i!}(\psi_!(\psi(a_j))))))\\ &\qquad \text{since } \psi_!(\psi(a_j))\leq a_j \text{ by definition of left-adjoints}\\ &\geq \bigwedge_{\substack{a_1,\ldots,a_n\\ \psi_!(\ell')\leq\bigvee_{j=1}^n a_j}}\bigvee_{j=1}^n\nu_{L'}(\psi(a_j))\\ &\qquad \text{since } \psi\circ\varphi_i \text{ is continuous, and } \varphi_{i!}\circ\psi_!=(\psi\circ\varphi_i)_!\\ &= \bigwedge_{\substack{a_1,\ldots,a_n\\ \ell'\leq\bigvee_{j=1}^n\psi(a_j)}}\bigvee_{j=1}^n\nu_{L'}(\psi(a_j))\end{aligned}$$

since $\psi_!(\ell') \leq \bigvee_{j=1}^n a_j$ is equivalent to $\ell' \leq \bigvee_{j=1}^n \psi(a_j)$, by the definition of a left-adjoint, and since ψ preserves finite suprema. The elements $b_j := \psi(a_j)$ are all complemented, so the latter infimum is larger than or equal to $\bigwedge_{\substack{b_1,\ldots,b_n \\ \ell' \leq \bigvee_{j=1}^n b_j}} \bigvee_{j=1}^n \nu_{L'}(b_j)$. That, in turn, is equal to $\bigwedge_{\substack{b_1,\ldots,b_n \\ \ell' \leq \bigvee_{j=1}^n b_j}} \nu_{L'}(\bigvee_{j=1}^n b_j)$, which is larger than or equal to $\nu_{L'}(\ell')$. Hence $\nu_{L'}(\ell') \leq \psi(\nu_L(\psi_!(\ell')))$, showing that ψ is continuous. □

The usual consequences follow.

Fact 7.17. *The category* $\mathbf{CF}^{\mathrm{adh}}$ *is complete and cocomplete.*

Fact 7.18. *The category* $\mathbf{CF}^{\mathrm{adh}}$ *is not co-wellpowered.*

7.5. **Closed elements.** Fact 6.23 shows that the notion of a closed element of a convergence $\mathbf{C}$-object only depends on the adherence adh_L. Hence:

Definition 7.19 (Closed)**.** Let $\mathbf{C}$ be a category of coframes, and L be an adherence $\mathbf{C}$-object. An element c of L is *quasi-closed* if and only if $\nu_L(c) \leq c$. A *closed* element of L is a complemented quasi-closed element of L.

Remark 7.20. Let L be a $\mathbf{C}$-object, where $\mathbf{C}$ is an admissible category of coframes. The closed elements of L can also be reconstructed as morphisms from $\mathbb{P}(\mathbb{S})$ to L in $\mathbf{C}^{\mathrm{adh}}$, as we did for convergence lattices (Lemma 6.1). For that, we equip $\mathbb{P}(\mathbb{S})$ with the adherence operator $\nu_{\mathbb{P}(\mathbb{S})}(A) := \downarrow A$, where $\downarrow$ is downward closure with respect to the ordering $0 \leq 1$. One checks easily that the corresponding convergence structure $\lim_{\nu_{\mathbb{P}(\mathbb{S})}}$ coincides with the operator $\lim_{\mathbb{P}(\mathbb{S})}$ of Section 6.

Given a morphism $\varphi\colon \mathbb{P}(\mathbb{S}) \to L$ in $\mathbf{C}^{\mathrm{adh}}$, the elements $u := \varphi(\{1\})$ and $c := \varphi(\{0\})$ satisfy $u \wedge c = \bot$, $u \vee c = \top$, and $\nu_L(c) \leq c$. The latter follows from the continuity condition (7.1): $\nu_L(c) \leq \varphi(\nu_{\mathbb{P}(\mathbb{S})}(\varphi_!(c))$. Since $\varphi_!(c) = \varphi_!(\varphi(\{0\})) \subseteq \{0\}$ by adjointness, $\nu_L(c) \leq \varphi(\downarrow\{0\}) = \varphi(\{0\}) = c$.

Conversely, if c is complemented and $\nu_L(c) \leq c$, let u be the complement of c, φ be the unique lattice (hence coframe) morphism mapping $\{0\}$ to c and $\{1\}$ to u. We check that φ is continuous, namely that for every $\ell \in L$, $\nu_L(\ell) \subseteq \varphi(\downarrow \varphi_!(\ell))$, as follows. If $\varphi_!(\ell) = \{0\}$, then $\varphi_!(\ell) \subseteq \{0\}$ hence $\ell \leq \varphi(\{0\}) = c$ by adjointness; in that case, $\nu_L(\ell) \leq \nu_L(c) \leq c = \varphi(\{0\}) = \varphi(\downarrow \varphi_!(\ell))$. Otherwise, $\varphi(\downarrow \varphi_!(\ell)) = \varphi(\{0, 1\}) = \top \geq \nu_L(\ell)$.

The inverse image of closed subsets by continuous maps are closed. Analogously:

Proposition 7.21. *Let* $\mathbf{C}$ *be a category of coframes. For every morphism* $\varphi\colon L \to L'$ *in* $\mathbf{C}^{\mathrm{adh}}$*,* φ *maps quasi-closed elements of* L *to quasi-closed elements of* L'*, and closed elements of* L *to closed elements of* L'*.*

Proof. Let c be quasi-closed in L. By continuity, $\nu_{L'}(\varphi(c)) \leq \varphi(\nu_L(\varphi_!(\varphi(c))))$. By adjointness, $\varphi_!(\varphi(c)) \leq c$, so $\nu_{L'}(\varphi(c)) \leq \varphi(\nu_L(c))$, and that is less than or equal to $\varphi(c)$ since c is quasi-closed. If additionally c is complemented, then $\varphi(c)$ is complemented, hence closed. □

Remark 7.22. That the images of closed elements are closed is obvious (assuming $\mathbf{C}$ admissible): in the light of Remark 7.20, that boils down to the fact that given a morphism $\varphi\colon L \to L'$ and a morphism $\psi\colon \mathbb{P}(\mathbb{S}) \to L$ in $\mathbf{C}^{\mathrm{adh}}$, $\varphi \circ \psi$ is a morphism from $\mathbb{P}(\mathbb{S})$ to L'.

Lemma 7.23. *Let* $\mathbf{C}$ *be a category of coframes, and* L *be an adherence* $\mathbf{C}$*-object. For every quasi-closed element* c *of* L*,* $c^\bullet = \{x \in \operatorname{pt} L \mid x \leq c\}$ *is a closed subset of* $\operatorname{pt} L$*.*

Proof. We use Lemma 7.14: $\nu_{\mathbb{P}\operatorname{pt} L}(c^\bullet) = (\nu_L(\bigvee c^\bullet))^\bullet$. Since every element of $c^\bullet$ is less than or equal to c, $\bigvee c^\bullet \leq c$, so $\nu_L(\bigvee c^\bullet) \leq \nu_L(c) \leq c$. It follows that $\nu_{\mathbb{P}\operatorname{pt} L}(c^\bullet) \subseteq c^\bullet$, so that $c^\bullet$ is quasi-closed. It is closed because every element of $\mathbb{P}(\operatorname{pt} L)$ is complemented. ☐

A *subcoframe* of a coframe L is a subset of L that is closed under finite suprema and arbitrary infima. A *sublattice* is only closed under finite suprema and finite infima.

Proposition 7.24. *Let* $\mathbf{C}$ *be a category of coframes, and* L *be an adherence* $\mathbf{C}$*-object.*

1. *The quasi-closed elements form a subcoframe of* L*.*
2. *The closed elements of* L *form a sublattice of* L*.*

Proof. (1) Let c_i be quasi-closed, $i \in I$. Then $\nu_L(\bigwedge_{i\in I} c_i) \leq \bigwedge_{i\in I} \nu_L(c_i) \leq \bigwedge_{i\in I} c_i$, where the first inequality is by the monotonicity of ν_L, and the second one is because each c_i is quasi-closed.

Let now $c_1, \ldots, c_n$ be quasi-closed. By Lemma 7.2, $\nu(c_1 \vee \cdots \vee c_n) = \nu(c_1) \vee \cdots \vee \nu(c_n)$, and this is less than or equal to $c_1 \vee \cdots \vee c_n$ because each c_i is quasi-closed.

(2) follows from (1), using the fact that finite infima and finite suprema of complemented elements in a distributive lattice are complemented. ☐

8. Topological Coframes

The standard pointfree analogue of topological spaces are locales [7, 9], i.e., the objects of the opposite category of frames. The idea is that all we need to know about its topological spaces is its frame of open subsets, not its points. We might as well study coframes (of closed subsets). Proposition 7.24 leads to another pointfree analogue of topological spaces, where the closed elements are embedded in a larger coframe, and only form a sublattice, not a subcoframe.

Definition 8.1 (Topological coframe)**.** Let $\mathbf{C}$ be a category of coframes. A *topological* $\mathbf{C}$*-object* is an object L of $\mathbf{C}$ together with a sublattice $C(L)$ of complemented elements, called its *closed elements*.

The topological $\mathbf{C}$-objects form a category $\mathbf{C}^{\mathrm{top}}$, whose morphisms $\varphi\colon L \to L'$ are the $\mathbf{C}$-morphisms that are *continuous*, namely that map closed elements to closed elements: for every $c \in C(L)$, $\varphi(c) \in C(L')$.

We shall call *topological structure* on a coframe L any sublattice C of complemented (in L) elements of L.

8.1. **The adherence of a topological coframe.** Lemma 7.24 shows that every adherence $\mathbf{C}$-object (L, ν_L) defines a topological $\mathbf{C}$-object $(L, C(L))$, where $C(L)$ is the sublattice of closed elements of (L, ν_L). Conversely:

Lemma 8.2. *Let* $\mathbf{C}$ *be a category of coframes, and* L *be a* $\mathbf{C}$*-object. Every topological structure* C *on* L *defines an adherence structure* ν_C *on* L *by:*

$$\nu_C(\ell) = \bigwedge\{c \in C \mid c \geq \ell\}. \tag{8.1}$$

Moreover:

(1) *ν_C is centered, that is, $\nu_C(\ell) \geq \ell$ for every $\ell \in L$;*
(2) *for every $\ell \in L$, for every $c \in C$, $c \geq \nu_C(\ell)$ if and only if $c \geq \ell$;*
(3) *for every $c \in C$, $\nu_C(c) = c$;*
(4) *ν_C is idempotent, that is, $\nu_C(\nu_C(\ell)) = \nu_C(\ell)$ for every $\ell \in L$.*

Proof. Clearly, ν_C is monotonic. For finitely many elements $\ell_1, \ldots, \ell_n$, $\nu_C(\ell_1 \vee \cdots \vee \ell_n) \geq \nu_C(\ell_1) \vee \cdots \vee \nu_C(\ell_n)$ by monotonicity. In the converse direction, and using the coframe distributivity law, $\nu_C(\ell_1) \vee \cdots \vee \nu_C(\ell_n)$ is the infimum of the elements of the form $c_1 \vee \cdots \vee c_n$, where $c_1, \ldots, c_n \in C$ and $c_1 \geq \ell_1, \ldots, c_n \geq \ell_n$. For each such choice of elements $c_1, \ldots, c_n$, $c := c_1 \vee \cdots \vee c_n$ is again in C and larger than or equal to $\ell_1 \vee \cdots \vee \ell_n$, so $\nu_C(\ell_1) \vee \cdots \vee \nu_C(\ell_n) \geq \bigwedge\{c \in C \mid c \geq \ell_1 \vee \cdots \vee \ell_n\} = \nu_C(\ell_1 \vee \cdots \vee \ell_n)$.

In order to show that ν_C is an adherence operator, it remains to show that $\nu_C(\ell) \geq \bigwedge_{a \geq \ell} \nu_C(a)$, where a ranges over complemented elements. This follows from the fact that for every $c \in C$ such that $c \geq \ell$, there is a complemented element a such that $c \geq a$ and $a \geq \ell$, namely c itself.

(1) The fact that ν_C is centered is obvious. (2) If $c \geq \nu_C(\ell)$, then $c \geq \ell$ by (1). Conversely, if $c \in C$ is such that $c \geq \ell$, then by definition $c \geq \bigwedge\{c \in C \mid c \geq \ell\} = \nu_C(\ell)$. (3) Taking $\ell := c$ in (2), we obtain $c \geq \nu_C(c)$, and the converse inequality is by (1). (4) $\nu_C(\nu_C(\ell)) = \bigwedge\{c \in C \mid c \geq \nu_C(\ell)\} = \bigwedge\{c \in C \mid c \geq \ell\}$ (by (2)) $= \nu_C(\ell)$. ☐

Corollary 8.3. *The set $\bigwedge C$ of infima of elements of C is the set of fixed points of ν_C.*

Proof. If $\ell = \nu_C(\ell)$ then by definition ℓ is an infimum of elements of C. Conversely, if $\ell = \bigwedge_{i \in I} c_i$ where each $c_i \in C$, then

$$\nu_C(\ell) = \nu_C\left(\bigwedge_{i \in I} c_i\right) \leq \bigwedge_{i \in I} \nu_C(c_i) = \bigwedge_{i \in I} c_i = \ell,$$

because of Lemma 8.2 (3). Since ν_C is centered (Lemma 8.2 (1)), $\nu_C(\ell) = \ell$. ☐

We compare topological structures by inclusion $\supseteq$, and we say that C is *finer* than C' if and only if $C \supseteq C'$. In that case, we also say that C' is *coarser* than C.

Lemma 8.4. *Let $\mathbf{C}$ be a category of coframes, and L be a $\mathbf{C}$-object. For an adherence structure ν on L, let C_ν denote its lattice of closed elements. Then:*

(1) *The mapping $\nu \mapsto C_\nu$ is monotonic: if $\nu \leq \nu'$ then $C_{\nu'} \subset C_\nu$.*
(2) *The mapping $C \mapsto \nu_C$ is monotonic: if $C' \subset C$ then $\nu_C \leq \nu_{C'}$.*
(3) *For every topological structure C, $C_{\nu_C} \supseteq C$.*
(4) *For every adherence structure ν, $\nu \leq \nu_{C_\nu}$.*

Proof. (1) if $\nu \leq \nu'$, then every complemented element c such that $\nu'(c) \leq c$ is such that $\nu(c) \leq c$. (2) if C and C' are two lattices of complemented elements and C contains C', then certainly $\bigwedge\{c \in C \mid c \geq \ell\} \leq \bigwedge\{c \in C' \mid c \geq \ell\}$. (3) Given a lattice C of complemented elements, C_{ν_C} is the set of complemented elements a such that $\nu_C(a) \leq a$. Every element c of C is complemented by definition, and $\nu_C(c) = c$ by Lemma 8.2 (3), hence is in C_{ν_C}. (4) For every $\ell \in L$, $\nu_{C_\nu}(\ell) = \bigwedge\{c \in C_\nu \mid c \geq \ell\} = \bigwedge\{c \text{ complemented} \mid \nu(c) \leq c, c \geq \ell\}$. For each complemented c such that $\nu(c) \leq c$ and $c \geq \ell$, $\nu(c) \geq \nu(\ell)$ since ν is monotonic, so $c \geq \nu(\ell)$. It follows that $\nu_{C_\nu}(\ell) \geq \nu(\ell)$. ☐

It follows that the $\nu \mapsto C_\nu$ construction is (order-theoretically) left-adjoint to the $C \mapsto \nu_C$ construction.

Proposition 8.5. *Let* $\mathbf{C}$ *be an admissible category of coframes. There is an identity-on-morphisms functor* $(L, \nu_L) \mapsto (L, C_{\nu_L})$ *from* $\mathbf{C}^{\mathrm{adh}}$ *to* $\mathbf{C}^{\mathrm{top}}$*, which is right adjoint to the identity-on-morphisms functor* $(L, C) \mapsto (L, \nu_C)$*.*

Proof. Write U for the first functor, and F for the second. For every morphism $\varphi\colon L \to L'$ in $\mathbf{C}^{\mathrm{adh}}$, $U(\varphi) = \varphi$ is a morphism in $\mathbf{C}^{\mathrm{top}}$ by Proposition 7.21. Conversely, for every morphism $\varphi\colon L \to L'$ in $\mathbf{C}^{\mathrm{top}}$, we check that $\nu_{C(L')}(\ell') \leq \varphi(\nu_{C(L)}(\varphi_!(\ell')))$ for every $\ell' \in L'$:

$$\begin{aligned}
\varphi(\nu_{C(L)}(\varphi_!(\ell'))) &= \varphi\left(\bigwedge\{c \in C(L) \mid c \geq \varphi_!(\ell')\}\right)\\
&= \varphi\left(\bigwedge\{c \in C(L) \mid \varphi(c) \geq \ell'\}\right)\\
&= \bigwedge\{\varphi(c) \mid c \in C(L), \varphi(c) \geq \ell'\}
\end{aligned}$$

since φ preserves all infima. Since φ maps $C(L)$ to $C(L')$, this is larger than or equal to $\bigwedge\{c' \in C(L') \mid c' \geq \ell'\} = \nu_{C(L')}(\ell')$. Therefore both U and F are functors.

By Lemma 8.4 (3), the identity map is continuous from $(L, C(L))$ to $(L, C_{\nu_{C(L)}}) = UF(L, C(L))$ in $\mathbf{C}^{\mathrm{top}}$: this is the unit. By Lemma 8.4 (4), the identity map is continuous from $(L, \nu_{C_{\nu_L}}) = FU(L, \nu_L)$ to (L, ν_L): this is the counit. The laws that they should satisfy hold because all considered maps are identities. ☐

8.2. **A square of adjunctions.** The latter is the topmost adjunction in the following square, which is the rightmost square in (1.1):

$$\begin{tikzcd}
(\mathbf{C}^{\mathrm{adh}})^{op} \arrow[r, shift left] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathrm{pt}"] \arrow[d, phantom, "\dashv"] & (\mathbf{C}^{\mathrm{top}})^{op} \arrow[l, shift left] \arrow[d, shift left, "\mathrm{pt}"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Adh} \arrow[u, shift left, "\mathbb{P}"] \arrow[r, shift left, "M_{\mathrm{top}}"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[u, shift left, "\mathbb{P}"] \arrow[l, shift left, "\supseteq"]
\end{tikzcd} \tag{8.2}$$

The bottommost adjunction is formed as follows. The inclusion functor $\mathbf{Top} \to \mathbf{Adh}$ maps every topological space X to the adherence space (X, cl), where cl maps every subset of X to its closure, namely the smallest closed set containing it. The *topological modification* functor M_{top} maps every adherence space $(X, \nu_{\mathbb{P}(X)})$ to the topological space X, whose closed subsets are defined as the fixed points of $\nu_{\mathbb{P}(X)}$.

The rightmost adjunction, which we write $\mathbb{P} \dashv \mathrm{pt}$ again, is built as follows. For every topological space X, $\mathbb{P}(X)$ is the coframe of all subsets of X, and $C(\mathbb{P}(X))$ is the sublattice of closed subsets of X. For every continuous map $f\colon X \to Y$, $\mathbb{P}(f) = f^{-1}$. In the converse direction, we define:

Definition 8.6 (Point)**.** Let $\mathbf{C}$ be an admissible category of coframes. For every topological $\mathbf{C}$-object L, the *points* of L are its join-prime elements. Let $\mathrm{pt}\, L$ be the set of points of L.

For each $\ell \in L$, let $\ell^\bullet := \{x \in \mathrm{pt}\, L \mid x \leq \ell\}$, and let $C(\mathbb{P}\,\mathrm{pt}\, L)$ be the set of elements of the form $c^\bullet$, where c is an infimum of elements of $C(L)$.

Note that c is not taken directly from $C(L)$ in the definition of the set $C(\mathbb{P}\,\mathrm{pt}\, L)$ of closed subsets of $\mathrm{pt}\, L$. Let us write $\bigwedge C(L)$ for the set of infima of elements of $C(L)$.

Lemma 8.7. *Let* $\mathbf{C}$ *be an admissible category of coframes, and* L *be a topological* $\mathbf{C}$*-object. The set* $\bigwedge C(L)$ *of infima of elements of* $C(L)$ *is a subcoframe of* L*.*

Proof. Clearly $\bigwedge C(L)$ is closed under arbitrary infima. Since $C(L) \subseteq \bigwedge C(L)$ and $\bot \in C(L)$, the 0-ary supremum $\bot$ is in $\bigwedge C(L)$. It remains to check that given two elements $c := \bigwedge_{i\in I} c_i$ and $c' := \bigwedge_{j\in J} c'_j$ where $c_i, c'_j \in C(L)$, $c \vee c'$ is in $\bigwedge C(L)$. This follows from the coframe distributivity law: $c \vee c' = \bigwedge_{i\in I, j\in J}(c_i \vee c'_j)$, noticing that each $c_i \vee c'_j$ is in $C(L)$. ☐

Lemma 8.8. *Let* $\mathbf{C}$ *be an admissible category of coframes, and* L *be a topological* $\mathbf{C}$*-object* L*. Then:*

(1) *For every family* $(\ell_i)_{i\in I}$ *in* L*,* $\bigcap_{i\in I} \ell_i^\bullet = (\bigwedge_{i\in I} \ell_i)^\bullet$*.*
(2) *For every finite family* $(\ell_i)_{i=1}^n$ *in* L*,* $\bigcup_{i\in I} \ell_i^\bullet = (\bigvee_{i\in I} \ell_i)^\bullet$*.*
(3) $C(\mathbb{P}\,\mathrm{pt}\,L)$ *is closed under finite unions and arbitrary intersections.*
(4) *The elements of* $C(\mathbb{P}\,\mathrm{pt}\,L)$ *define the closed sets of a topology on* $\mathrm{pt}\,L$*.*

Proof. (1) $\bigcap_{i\in I} \ell_i^\bullet = \{x \in \mathrm{pt}\,L \mid \forall i \in I, x \le \ell_i\} = \{x \in \mathrm{pt}\,L \mid x \le \bigwedge_{i\in I} \ell_i\} = (\bigwedge_{i\in I} \ell_i)^\bullet$. (2) The elements x of $(\bigvee_{i=1}^n \ell_i)^\bullet$ are those join-primes such that $x \le \bigvee_{i=1}^n \ell_i$, that is, such that $x \le \ell_i$ for some ℓ_i. Those are exactly the elements of $\bigcup_{i=1}^n \ell_i^\bullet$. (3) Let $(c_i)_{i\in I}$ be a family of elements of $\bigwedge C(L)$. We have $\bigcap_{i\in I} c_i^\bullet = (\bigwedge_{i\in I} c_i)^\bullet$ by (1), and clearly $\bigwedge_{i\in I} c_i$ is in $\bigwedge C(L)$. Given finitely many elements $c_1, \ldots, c_n$ in $\bigwedge C(L)$, $\bigcup_{i=1}^n c_i^\bullet = (\bigvee_{i=1}^n c_i)^\bullet$ by (2) and $\bigvee_{i=1}^n c_i$ is in $\bigwedge C(L)$ by Lemma 8.7. (4) follows directly from (3). ☐

Therefore $\mathrm{pt}\,L$ defines a topological space. It satisfies the following universal property.

Proposition 8.9. *Let* $\mathbf{C}$ *be an admissible category of coframes. For every topological space* X*, for every topological* $\mathbf{C}$*-object* L*, and every morphism* $\varphi\colon L \to \mathbb{P}(X)$ *in* $\mathbf{C}^{\mathrm{top}}$*, there is a unique map* $\varphi^\dagger\colon X \to \mathrm{pt}\,L$ *such that, for every* $\ell \in L$*,* $\mathbb{P}(\varphi^\dagger)(\ell^\bullet) = \varphi(\ell)$*, and it is continuous.*

Proof. If $\varphi^\dagger$ exists, then for every $\ell \in L$, $\varphi(\ell) = \mathbb{P}(\varphi^\dagger)(\ell^\bullet) = \{x \in X \mid \varphi^\dagger(x) \le \ell\}$. For each x, this forces $\varphi^\dagger(x)$ to be the least element $\ell \in L$ such that $x \in \varphi(\ell)$. Since $x \in \varphi(\ell)$ is equivalent to $\{x\} \subseteq \varphi(\ell)$, hence to $\varphi_!(\{x\}) \le \ell$, this forces $\varphi^\dagger(x)$ to be equal to $\varphi_!(\{x\})$.

Hence define $\varphi^\dagger(x)$ as $\varphi_!(\{x\})$. We check that this is join-prime: if $\varphi_!(\{x\}) \le \bigvee_{i=1}^n \ell_i$, then $\{x\} \subseteq \varphi(\bigvee_{i=1}^n \ell_i) = \bigvee_{i=1}^n \varphi(\ell_i)$, so $\{x\} \subseteq \varphi(\ell_i)$ for some i, from which $\varphi_!(\{x\}) \le \ell_i$.

It remains to show that $\varphi^\dagger$ is continuous. Consider an arbitrary closed subset $c^\bullet$ of $\mathrm{pt}\,L$, where c is an infimum $\bigwedge_{i\in I} c_i$ of elements of $C(L)$. Then $(\varphi^\dagger)^{-1}(c^\bullet) = \{x \in X \mid \varphi_!(\{x\}) \le c\} = \{x \in X \mid \{x\} \subseteq \varphi(c)\} = \varphi(c) = \bigwedge_{i\in I} \varphi(c_i)$. Since $\varphi(c_i) \in C(\mathbb{P}(X))$ by continuity, this is a closed subset of X. ☐

Lemma 8.10. *Let* $\mathbf{C}$ *be an admissible category of coframes, and* L *be a topological* $\mathbf{C}$*-object. The map* $\epsilon_L\colon L \mapsto \mathbb{P}(\mathrm{pt}\,L)$ *defined by* $\epsilon_L(\ell) = \ell^\bullet$ *is a morphism in* $\mathbf{C}^{\mathrm{top}}$*, which is injective if and only if* L *is spatial.*

Proof. First, ϵ_L is a coframe morphism by Lemma 8.8 (1) and (2). Second, ϵ_L maps each $c \in C(L)$ to $c^\bullet$, which is a closed element of $\mathbb{P}(\mathrm{pt}\,L)$ by definition, so ϵ_L is continuous.

Finally, let $\ell, \ell' \in L$ be such that $\ell^\bullet = \ell'^\bullet$. The join-prime elements below ℓ and below ℓ' are the same. If L is spatial (in the sense of Definition 7.5), each element is the supremum of join-prime elements below it, and that implies $\ell = \ell'$. Conversely, if L is not spatial, then for some $\ell \in L$, ℓ is not the supremum of the set $\ell^\bullet$ of join-primes below it. Let $\ell' = \bigvee \ell^\bullet$, so that $\ell' \neq \ell$. We check that $\ell' \leq \ell$, since ℓ is an upper bound of $\ell^\bullet$ and ℓ' is the least one. It follows that $\ell'^\bullet \subseteq \ell^\bullet$. Conversely, every point in $\ell^\bullet$ is below $\bigvee \ell^\bullet = \ell'$, hence in $\ell'^\bullet$, so $\ell^\bullet = \ell'^\bullet$. ☐

It follows that $\mathrm{pt} \dashv \mathbb{P}$ defines an adjunction between $\mathbf{Top}^{op}$ and $\mathbf{C}^{\mathrm{top}}$, with unit ϵ_L, hence by taking opposite categories:

Theorem 8.11. *Let $\mathbf{C}$ be an admissible category of coframes. Then $\mathbb{P} \dashv \mathrm{pt}$ is an adjunction between $\mathbf{Top}$ and $(\mathbf{C}^{\mathrm{top}})^{op}$.*

The counit is ϵ_L, and the unit $\eta_X \colon X \to \mathrm{pt}\,\mathbb{P}(X)$ is equal to $\mathrm{id}^\dagger_{\mathbb{P}(X)} \colon x \mapsto \mathrm{id}_{\mathbb{P}(X)!}(x) = \{x\}$.

Lemma 8.12. *Let $\mathbf{C}$ be an admissible category of coframes. For every topological space X, η_X is an isomorphism.*

Proof. Clearly η_X is bijective, and continuous by construction. Explicitly, consider any closed subset of $\mathrm{pt}\,\mathbb{P}(X)$. This is a subset of the form $c^\bullet$, where c is an infimum of closed elements of $\mathbb{P}(X)$, i.e., of closed subsets of X. In particular, c is itself a closed subset of X. Then $\eta_X^{-1}(c^\bullet) = \{x \in X \mid \{x\} \in c^\bullet\} = \{x \in X \mid \{x\} \subseteq c\} = c$. The inverse of c by the inverse map η_X^{-1} is then $c^\bullet$, which is closed. Therefore both η_X and η_X^{-1} are continuous. ☐

Corollary 8.13. *Let $\mathbf{C}$ be an admissible category of coframes.* **Top** *is a coreflective subcategory of $(\mathbf{C}^{\mathrm{top}})^{op}$, through the coreflection* pt.

Moreover, the second part of Lemma 8.10 shows that the objects that are isomorphic to $\mathbb{P}(X)$ for some topological space X in $(\mathbf{C}^{\mathrm{top}})^{op}$ are exactly the topological $\mathbf{C}$-objects $(L, C(L))$ such that L is spatial.

This finishes our description of the final side of (7.3). Clearly, the left-adjoints commute, hence also the right-adjoints.

Note that the $\mathbb{P} \dashv \mathrm{pt}$ coreflection embeds the whole of **Top** inside $(\mathbf{C}^{\mathrm{top}})^{op}$, not just the subcategory of sober spaces, as the familiar adjunction between topological spaces and locales would do. We examine the relation between topological coframes and locales in Section 8.5, and quickly examine (co)completeness questions.

8.3. **Completeness, cocompleteness.** We proceed along familiar lines. Let now $|\_| \colon \mathbf{C}^{\mathrm{top}} \to \mathbf{C}$ be the functor that maps every topological $\mathbf{C}$-object $(L, C(L))$ to the underlying $\mathbf{C}$-object L.

Proposition 8.14. *Let $\mathbf{C}$ be a category of coframes, and L be a $\mathbf{C}$-object. Let also $\varphi_i \colon |L_i| \to L$ be morphisms of $\mathbf{C}$, where each L_i is a topological $\mathbf{C}$-object, $i \in I$. There is a coarsest topological structure $C(L)$ on L such that φ_i is continuous for each $i \in I$.*

Proof. This must be the smallest sublattice of L that contains all the elements $\varphi_i(c)$, $i \in I$, $c \in C(L_i)$. Those are exactly the finite suprema of finite infima of such elements, and they are all complemented because φ_i, being a morphism of lattices, maps complemented elements to complemented elements. ☐

Corollary 8.15. *Let* $\mathbf{C}$ *be a category of coframes. Then* $|\_|\colon \mathbf{C}^{\mathrm{top}} \to \mathbf{C}$ *is topological: every sink* $(\varphi_i\colon |L_i| \to L)_{i\in I}$ *has a unique final lift, and this is the coarsest topological structure* $C(L)$ *given in Proposition 8.14.*

Proof. As for Corollary 3.3 and Corollary 7.16 uniqueness is a general categorical fact.

To show existence, we check that $(L, C(L))$ is a final lift of $(\varphi_i\colon |L_i| \to L)_{i\in I}$. Let $\psi\colon L \to |L'|$ be such that $\psi \circ \varphi_i$ is continuous for every $i \in I$. We aim to show that ψ is continuous from $(L, C(L))$ to L', and for that we consider an arbitrary element $c \in C(L)$. This c can be written as $\bigvee_{j=1}^{m} \bigwedge_{k=1}^{n_i} c_{jk}$, where each c_{jk} is of the form $\varphi_i(c_i)$, for some $i \in I$ and $c_i \in C(L_i)$. Since $\psi \circ \varphi_i$ is continuous, $\psi(c_{jk}) = \psi(\varphi_i(c_i))$ is an element of $C(L')$. It follows that $\psi(c) = \bigvee_{j=1}^{m} \bigwedge_{k=1}^{n_i} \psi(c_{jk})$ is also in $C(L')$. □

The usual consequences follow.

Fact 8.16. *The category* $\mathbf{CF}^{\mathrm{top}}$ *is complete and cocomplete.*

Fact 8.17. *The category* $\mathbf{CF}^{\mathrm{top}}$ *is not co-wellpowered.*

8.4. **Topological convergence coframes and topological modification.** Every convergence coframe L defines a sublattice $C(L)$ of closed elements (via Definition 7.19), which in turn determines a topological coframe in the sense of Definition 8.1. What are the convergence coframes whose lattice of closed elements determines the convergence?

With a convergence coframe $(L, \lim_L)$, we associate another convergence structure on L defined by

$$\lim\nolimits_{\mathrm{T}(L)} \mathcal{F} = \bigwedge_{c \in \mathcal{F}^{\#} \cap C(L)} c. \tag{8.3}$$

That (8.3) defines a convergence lattice structure is clear. Moreover, the two convergence structures share the same closed elements, that is,

$$C(\mathrm{T}(L)) = C(L). \tag{8.4}$$

Proof. Indeed,

$$\lim\nolimits_L \le \lim\nolimits_{\mathrm{T}(L)} \tag{8.5}$$

because $\lim_L \mathcal{F} \le c$ whenever $c \in \mathcal{F} \cap C(L)$, so that every $\mathrm{T}(L)$-closed set is also L-closed. Conversely, if $c \in C(L)$ and $c \in \mathcal{F}^{\#}$ then $\lim_{\mathrm{T}(L)} \mathcal{F} \le c$ by definition, that is, c is quasi-closed for $\mathrm{T}(L)$. As c is also complemented, it is $\mathrm{T}(L)$-closed. □

We call a convergence coframe *topological* if $\lim_L = \lim_{\mathrm{T}(L)}$. Clearly, topological convergence structures are determined by their lattice of closed elements.

Proposition 8.18. *If* $(L, \lim_L)$ *is a convergence coframe then* $\lim_{\mathrm{T}(L)}$ *is the finest topological convergence structure on* L *that is coarser than* $\lim_L$*.*

Thus we call $\mathrm{T}(L)$ the *topological modification of* L.

Proof. In view of (8.4), $\lim_{\mathrm{T}(L)}$ is topological, and coarser than $\lim_L$ by (8.5). If lim is another topological convergence structure on L coarser than $\lim_L$, then lim-closed elements are $\lim_L$-closed. As a result,

$$\lim \mathcal{F} = \bigwedge_{c \in \mathcal{F}^\# \cap C(\lim)} \mathcal{F} \geq \bigwedge_{c \in \mathcal{F}^\# \cap C(L)} \mathcal{F} = \lim_{\mathrm{T}(L)} \mathcal{F}.$$

□

Lemma 8.19. *If L and L' are two topological convergence coframes, then $\varphi : L \to L'$ is continuous if and only if $\varphi(c)$ is closed in L' whenever c is closed in L, that is, if and only if*

$$\varphi(C(L)) \subset C(L').$$

Proof. Corollary 6.25 states that $\varphi(C(L)) \subset C(L')$ whenever φ is continuous. Assume conversely, that $\varphi(C(L)) \subset C(L')$. Because L is topological and φ is a morphism of coframes,

$$\varphi\left(\lim_L \varphi^{-1}(\mathcal{F})\right) = \varphi\left(\bigwedge_{c \in (\varphi^{-1}(\mathcal{F}))^\# \cap C(L)} c\right) = \bigwedge_{c \in (\varphi^{-1}(\mathcal{F}))^\# \cap C(L)} \varphi(c).$$

Moreover, by assumption, $\varphi(c) \in C(L')$, and $c \in (\varphi^{-1}(\mathcal{F}))^\#$ if and only if $\varphi(c) \in \mathcal{F}^\#$ by Corollary 6.6. Thus

$$\varphi\left(\lim_L \varphi^{-1}(\mathcal{F})\right) \geq \bigwedge_{d \in \mathcal{F}^\# \cap C(L')} d = \lim_{L'} \mathcal{F},$$

because L' is topological. Thus φ is continuous. □

Corollary 8.20. *Let $\mathbf{C}$ be a category of coframes. The full subcategory $\mathbf{C}^{\mathrm{conv}_{\mathrm{T}}}$ of $\mathbf{C}^{\mathrm{conv}}$ formed by topological convergence coframes is a reflective subcategory of $\mathbf{C}^{\mathrm{conv}}$. The reflector* T *acts on objects as $L \to \mathrm{T}(L)$ and acts as identity on morphisms.*

Proof. In view of Proposition 8.18, we only need to show that T is a concrete functor, that is, if $\varphi : L \to L'$ is continuous, then $\mathrm{T}\,\varphi = \varphi : \mathrm{T}(L) \to \mathrm{T}(L')$ is also continuous. In view of Lemma 8.19 and (8.4), it is enough to show that $\varphi(C(L)) \subset C(L')$, and this follows from Corollary 6.25. □

In view of Lemma 8.19,

$$C : \mathbf{C}^{\mathrm{conv}_{\mathrm{T}}} \to \mathbf{C}^{\mathrm{top}}$$

that associates with each $\mathbf{C}^{\mathrm{conv}_{\mathrm{T}}}$-object L its lattice $C(L)$ of closed elements and acts as identity on morphisms is a full and faithful functor. Moreover, it is bijective on objects, because topological convergence coframes are determined by their lattice of closed elements. Therefore,

Theorem 8.21. *The categories : $\mathbf{C}^{\mathrm{conv}_{\mathrm{T}}}$ and $\mathbf{C}^{\mathrm{top}}$ are isomorphic.*

On the other hand,

$$\mathrm{Lim} : \mathbf{C}^{\mathrm{top}} \to \mathbf{C}^{\mathrm{conv}_{\mathrm{T}}}$$

that associates with each $\mathbf{C}^{\mathrm{top}}$-object (L, C) the topological convergence $\mathbf{C}$-object $(L, \lim_C)$ defined by

$$\lim_C \mathcal{F} := \bigwedge_{c \in \mathcal{F}^\# \cap C} c,$$

and acts as identity on morphisms is also an isomorphism of categories (consider Lemma 8.19). Moreover,

$$\mathrm{Lim} \circ C = Id_{\mathbf{C}^{\mathrm{conv}_{\mathrm{T}}}} \text{ and } C \circ \mathrm{Lim} = Id_{\mathbf{C}^{\mathrm{top}}}. \tag{8.6}$$

8.5. **Topological coframes and locales.** One would expect at this point that $(\mathbf{CF}^{\mathrm{top}})^{op}$ and the category $\mathbf{Frm}^{op}$ of locales to be strongly related, since both are connected to **Top** by an adjunction. The fact that topological spaces embed faithfully in the former but not in the latter is an indication that the two pointfree categories differ. They do not seem to be related by an adjunction either, but at least, there are functors between the two.

In one direction, every topological coframe L has a subcoframe $\bigwedge C(L)$ by Lemma 8.7. This defines a frame $(\bigwedge C(L))^{op}$. For every morphism $\varphi\colon L \to L'$ in $\mathbf{CF}^{\mathrm{top}}$, the restriction of φ to $\bigwedge C(L)$ defines a coframe morphism from $\bigwedge C(L)$ to $\bigwedge C(L')$. Hence:

Fact 8.22. *Let $\mathbf{C}$ be a category of coframes. There is a functor $\bigwedge C^{op}\colon \mathbf{CF}^{\mathrm{top}} \to$ $\mathbf{Frm}$ which maps each object L to $(\bigwedge C(L))^{op}$ and acts on morphisms by restriction.* □

In the other direction, given any frame Ω, we can form the coframe of sublocales $S\ell(\Omega)$ [9, Section III.3]. For each $u \in \Omega$, the *closed sublocale* $\mathfrak{c}(u) := \uparrow u$ is a complemented element of $S\ell(\Omega)$, and its complement is the *open sublocale* $\mathfrak{o}(u) := \{u \Rightarrow v \mid v \in \Omega\}$, where $\Rightarrow$ is implication in the Heyting algebra Ω (Proposition III.6.1.3, loc.cit.). Moreover, the map $\mathfrak{c}\colon \Omega^{op} \to S\ell(\Omega)$ is an order-embedding (Proposition III.6.1.4, loc.cit.) and a coframe morphism (Proposition III.6.1.5, loc.cit.). $S\ell(\Omega)$ then defines a topological coframe, provided we define $C(S\ell(\Omega))$ as its subcoframe of elements of the form $\mathfrak{c}(u)$, $u \in \Omega$.

We will use the following universal property of $S\ell(\Omega)$ below (Proposition III.6.3.1, loc.cit., slightly reformulated): for every frame morphism $\varphi\colon \Omega \to \mathcal{C}_{\Omega'}$, there is a unique frame morphism $\varphi^*\colon (S\ell(\Omega))^{op} \to \Omega'$ such that $\varphi^*(\mathfrak{c}(u)) = \varphi(u)$ for every $u \in \Omega$.

Lemma 8.23. *There is a functor $S\ell\colon \mathbf{Frm} \to \mathbf{CF}^{\mathrm{top}}$ which maps every frame Ω to $S\ell(\Omega)$, and every frame morphism $\varphi\colon \Omega \to \Omega'$ to the unique coframe morphism $S\ell(\varphi) := (\mathfrak{c} \circ \varphi)^*\colon S\ell(\Omega) \to S\ell(\Omega')$ that maps $\mathfrak{c}(u)$ to $\mathfrak{c}(\varphi(u))$ for every $u \in \Omega$.*

Proof. We check types first. Since $\mathfrak{c}\colon \Omega'^{op} \to S\ell(\Omega')$ is a coframe morphism, it is also a frame morphism from Ω' to $S\ell(\Omega')^{op}$. Then $(\mathfrak{c} \circ \varphi)^*$ is well-defined, since $\mathfrak{c} \circ \varphi$ sends every element to a complemented element of $S\ell(\Omega')^{op}$, and is a frame morphism from $S\ell(\Omega)^{op}$ to $S\ell(\Omega')^{op}$, hence a coframe morphism from $S\ell(\Omega)$ to $S\ell(\Omega')$.

We now need to check that $(\mathfrak{c} \circ \varphi)^*$ is continuous: it maps every closed element $\mathfrak{c}(u)$ of $S\ell(\Omega)$ to $\mathfrak{c}(\varphi(u))$, which is closed by definition. □

Lemma 8.24. *For every frame Ω, there is an isomorphism between Ω and $\bigwedge C^{op}(S\ell(\Omega))$, which maps each $u \in \Omega$ to $\mathfrak{c}(u)$, and it is natural in Ω.*

Proof. The closed elements of $S\ell(\Omega)$ are exactly the elements $\mathfrak{c}(u)$, $u \in \Omega$. Those elements are closed under arbitrary infima, since $\mathfrak{c}$ is a coframe morphism, hence $\bigwedge C^{op}(S\ell(\Omega)) = C(S\ell(\Omega))^{op}$. Since $\mathfrak{c}$ is an order-embedding, this shows the isomorphism part.

For every frame morphism $\varphi\colon \Omega \to \Omega'$, naturality is the fact that $\mathfrak{c}(\varphi(u)) = S\ell(\varphi)(\mathfrak{c}(u))$ for every $u \in \Omega$, and that is the property we stated of $S\ell(\varphi)$ in Lemma 8.23. □

In other words, the category $\mathbf{Frm}^{op}$ of locales arises as a retract (up to equivalence of categories) of $(\mathbf{CF}^{\mathrm{top}})^{op}$.

One can characterize a natural full subcategory of $(\mathbf{CF}^{\mathrm{top}})^{op}$ in which the category $\mathbf{Frm}^{op}$ of locales will embed reflectively, as follows.

Definition 8.25. A *strong* topological coframe is a topological coframe L such that $C(L)$ is closed under infima taken in L.

Fact 8.26. *For every frame Ω, $S\ell(\Omega)$ is a strong topological coframe.* □

Proposition 8.27. *There is a coreflection $S\ell \dashv \bigwedge C^{op}$ between* $\mathbf{Frm}$ *and the category of strong topological coframes.*

Proof. The unit is the isomorphism $\mathfrak{c}$ of Lemma 8.24. We claim that the counit is $\mathrm{id}^*_{C(L)}\colon S\ell(\bigwedge C^{op}(L)) \to L$. We reason as follows. Since L is strongly topological, the identity map $\mathrm{id}_{C(L)}\colon \bigwedge C(L)^{op} \to C(L)^{op}$ makes sense. Then $\mathrm{id}^*_{C(L)}$ is a coframe morphism, and $\mathrm{id}^*_{C(L)}(\mathfrak{c}(u)) = u$ for every $u \in L$.

Let us check that this is natural in L. For every morphism $\varphi\colon L \to L'$ in $\mathbf{CF}^{\mathrm{top}}$, $\mathrm{id}^*_{C(L)} \circ (S\ell(\bigwedge C^{op}(\varphi)))$ is the unique map that maps $\mathfrak{c}(u)$ to $\mathrm{id}^*_{C(L)}(\mathfrak{c}(\varphi(u))) = \varphi(u)$. Hence it coincides with $\varphi \circ \mathrm{id}^*_{C(L)}$.

It remains to check that the following compositions:

$$S\ell(\Omega) \xrightarrow{S\ell(\mathfrak{c})} S\ell(\bigwedge C^{op}(S\ell(\Omega))) \xrightarrow{\mathrm{id}^*_{C(S\ell(\Omega))}} S\ell(\Omega)$$

$$\bigwedge C^{op}(L) \xrightarrow{\mathfrak{c}} \bigwedge C^{op}(S\ell(\bigwedge C^{op}(L))) \xrightarrow{\bigwedge C^{op}(\mathrm{id}^*_{C(L)})} \bigwedge C^{op}(L)$$

are identities. The first one is the unique morphism that maps $\mathfrak{c}(u)$, for each $u \in \Omega$, to $\mathrm{id}^*_{C(S\ell(\Omega))}(\mathfrak{c}(\mathfrak{c}(u))) = \mathfrak{c}(u)$, hence indeed coincides with the identity morphism. The second one maps every closed element c of L to $\bigwedge C^{op}(\mathrm{id}^*_{C(L)})(\mathfrak{c}(c))$, which is equal to $\mathrm{id}^*_{C(L)}(\mathfrak{c}(c))$ since $\bigwedge C^{op}$ acts by restriction, and the latter is equal to c. □

Corollary 8.28. *The category of locales is a reflective subcategory of the opposite of the category of strong topological coframes.*

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