

*Under consideration for publication in Math. Struct. in Comp. Science*

# Logical Relations for Monadic Types<sup>†</sup>

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*Received June 2005*

Logical relations and their generalizations are a fundamental tool in proving properties of lambda-calculi, e.g., yielding sound principles for observational equivalence. We propose a natural notion of logical relations able to deal with the monadic types of Moggi's computational lambda-calculus. The treatment is categorical, and is based on notions of subconing, mono factorization systems, and monad morphisms. Our approach has a number of interesting applications, including cases for lambda-calculi with non-determinism (where being in logical relation means being bisimilar), dynamic name creation, and probabilistic systems.

**Keywords:** logical relations, monads, semantics, typed lambda-calculus.

## Contents

1	Introduction	3
1.1	Motivation and context.	3
1.2	Outline.	4
2	Preliminaries	5
2.1	Related work.	5
3	Lifting of a Monad to a Scone	6

<sup>†</sup> A preliminary version of this paper was presented at the 11th Annual Conference of the European Association for Computer Science Logic (CSL'02), Edinburgh, Scotland, 22–25 September 2002 (Goubault-Larrecq et al., 2002).

<sup>‡</sup> Work partially supported by the RNTL project EVA, and the ACI jeunes chercheurs “Sécurité informatique, protocoles cryptographiques et détection d'intrusions”.

<sup>§</sup> Work partially supported by the Polish KBN grant 7 T11C 002 21.

<sup>¶</sup> Work partially supported by the ACI jeunes chercheurs “Sécurité informatique, protocoles cryptographiques et détection d'intrusions”, and the e-Society project of MEXT.

4	Lifting of a Monad to a Subcone	8
4.1	$\tilde{T}$ on objects.	10
4.2	$\tilde{T}$ on morphisms.	10
4.3	Unit $\tilde{\eta}$ .	11
4.4	Multiplication $\tilde{\mu}$ .	11
5	Lifting of a Monad to Relations	12
6	Lifting Monoidal Structures	15
6.1	Monoidal Categories	15
6.2	Unit element $\tilde{I}$ .	16
6.3	Tensor product $\tilde{\otimes}$ .	16
6.4	Associativity $\tilde{\alpha}$ .	17
6.5	Left unit $\tilde{\ell}$ .	18
6.6	Right unit $\tilde{r}$ .	18
6.7	Symmetric Monoidal Categories	19
6.8	Commutativity $\tilde{\ell}$ .	19
6.9	Lifting Cartesian Products	20
6.10	Terminal object $\tilde{1}$ .	21
6.11	Binary product $\tilde{\times}$ .	21
7	Lifting Strong, Monoidal, and Commutative Monads to a Scone and a Subcone	22
7.1	Lifting Strong Monads	22
7.2	Monoidal Monads	24
7.3	Commutative Monads	27
8	Building Monad Morphisms from Adjunctions	28
8.1	Eilenberg-Moore algebras.	28
8.2	Kleisli category.	29
8.3	Monad Morphisms from Adjunctions	29
8.4	Monoidal, and Strong Monad Morphisms from Monoidal Adjunctions	30
9	Lifting Closed Structures to the Subcone	35
9.1	Lifting Exponentials	36
9.2	Linear $\lambda$ -Calculus	41
10	Examples	42
10.1	Lift, Exceptions, State, Non-Determinism, and Continuations in <b>Set</b>	43
10.2	Lift monad $A_{\perp}$ .	43
10.3	Exception monad $A + E$ .	44
10.4	State transformer monad $(A \times \text{St})^{\text{St}}$ .	44
10.5	Finite powerset (non-determinism) monad $\mathbb{P}_{\text{fin}}(A)$ .	44
10.6	The continuation monad $\mathcal{R}^A$ .	45
10.7	Labelled transition systems and bisimulations	45
10.8	Logical relations for dynamic name creation	46
10.9	Monads of Measures and Probabilities	48
10.10	Continuous Valuations on Depos, observed from <b>Set</b>	49
10.11	Continuous Valuations on Depos, observed from <b>Cpo</b>	53
10.12	Probability Measures on Metric Spaces	57
11	Conclusions	63

Appendix A	Monoidal Monads, Commutative Monads	66
A.1	From Mediators to Strengths and Dual Strengths	66
A.2	From Strengths and Dual Strengths to Mediators	68
A.3	Monoidal Monad Morphisms	70
A.4	Commutative Monads are Monoidal	71
Appendix B	Proof of Lemma 2	72
Appendix C	Composition of Monoidal Adjunctions	76
Appendix D	Proof of Proposition 3	77

## 1. Introduction

### 1.1. Motivation and context.

Logical relations and their generalizations (Mitchell, 1996) are a fundamental tool in proving properties of lambda-calculi, e.g., characterizing lambda-definability (Plotkin, 1980; Jung and Tiuryn, 1993; Alimohamed, 1995; Fiore and Simpson, 1999), proving equational completeness (Statman, 1985; Mitchell, 1996), studying parametric polymorphism (Reynolds, 1983; Ma and Reynolds, 1992; Lazić and Nowak, 2000) notably. On the other hand, Moggi’s computational lambda-calculus (Moggi, 1991) has proved useful to define various notions of computations on top of the lambda-calculus: side-effects, input-output, continuations, non-determinism (Wadler, 1992), probabilistic computation (Ramsey and Pfeffer, 2002) in particular.

What should then be a natural notion of logical relation for Moggi’s computational lambda-calculus? Although there is no unique answer to this question, we propose one that is satisfying in practice. We shall demonstrate the relevance of our approach by illustrating our construction on monads for non-determinism, dynamic name creation, and probabilistic computation.

Moggi’s insight is based on categorical semantics: while categorical models of the  $\lambda$ -calculus are cartesian closed categories (CCCs), the computational lambda-calculus requires CCCs with a strong monad  $(\mathbf{T}, \eta, \mu, \mathbf{t})$ . The *monadic types* of the computational lambda-calculus are given by the syntax:

$$\tau ::= b \mid \tau \rightarrow \tau \mid \tau \times \tau \mid \mathbf{T}(\tau)$$

where  $b$  ranges over a set  $B$  of so-called *base types*, and  $\mathbf{T}(\tau)$  is meant to denote the type of *computations* of type  $\tau$ . Compared to the lambda-calculus, Moggi’s calculus has an additional **val** operation, of type  $\tau \rightarrow \mathbf{T}(\tau)$ , and an additional **let**  $x = u$  **in**  $v$  construct, of type  $\mathbf{T}(\tau')$  provided  $u$  has type  $\mathbf{T}(\tau)$  and  $v$  has type  $\mathbf{T}(\tau')$  under the assumption  $x : \tau$ . Every computational lambda-term has a unique interpretation as a morphism in a CCC with a strong monad. In fact the category **Comp** whose objects are types and whose morphisms are terms up to  $\beta\eta$ -conversion is the free CCC-with-a-strong-monad over the set  $B$ .

Accordingly, our study will rest on categorical principles. While there is a flurry of generalizations of logical relations (Kripke logical relations (Mitchell, 1996), lax logical

relations (Plotkin et al., 2000), pre-logical relations (Honsell and Sannella, 2002), etc.), we use *subscones* (Mitchell and Scedrov, 1993) as a unifying framework for defining logical relations. Recall that subscones over **Set** allow one to define logical relations, and subscones over the presheaf category **Set <sup>$\mathcal{I}$</sup>**  lead to  $\mathcal{I}$ -indexed Kripke logical relations (Mitchell and Scedrov, 1993). Technically, the development in (Mitchell and Scedrov, 1993) is based on unique *lifting* of the CCC structure to the subscone. Our whole endeavor then reduces to finding appropriate liftings of monads on categories **C** to the subscone category  $(\mathbf{C} \Downarrow |_{-})$  (see Section 4).

The important property of logical relations is the so-called Basic Lemma (Mitchell, 1996): meanings of a lambda-term in different models w.r.t. related environments are related. This is immediate for subscones, and stems from the fact that **Comp** is the free CCC-with-a-strong-monad on **B** (a trivial adaptation of Proposition 5.2 in (Mitchell and Scedrov, 1993)). In particular, that any two closed terms that are in logical relation are observationally equivalent is immediate.

## 1.2. Outline.

We return to preliminaries in Section 2. We then define liftings of monads to *scones* in Section 3; this is simpler than for subscones, and of independent interest. The construction is based on the use of monad morphisms. We then lift monads to *subscones* in Section 4, using a mono factorization system. The important case where the target category **C** is a product of two categories is investigated in Section 5: this is where *binary* logical relations arise, allowing us to compare two models. We terminate our lifting construction by lifting the monoidal structure and monad strength in Sections 6 and 7, respectively. Section 8 establishes a result by which adjunctions give rise to monad morphisms. In Section 9, we return to the basics of subscone theory. While the standard construction of the CCC structure over the subscone requires a functor  $|_{-}$  that commutes with finite products, we show that the use of mono factorization systems, as in Section 4, allows us to relax this requirement to  $|_{-}$  being only *monoidal*. While we do not make any use here of this observation, monoidal functors are more natural from a categorical point of view than product preserving ones, and we feel it should be interesting in future applications (we have some already, but we refrain from including them in this paper).

It remains to test the relevance of our construction (Section 10): the logical relations thus defined characterize bisimulations when **T** is the non-determinism monad (as suggested in (Lazić and Nowak, 2000)), a generalization of Larsen and Skou’s probabilistic bisimulations (Larsen and Skou, 1991) when **T** is a measure monad (Giry, 1981; Jones, 1990), and a notion close to Pitts and Stark’s logical relations for observational equivalence of programs that create names dynamically (Pitts and Stark, 1993; Stark, 1998). We conclude in Section 11.

## 2. Preliminaries

Fix two categories  $\mathbf{C}$  and  $\mathbb{C}$  and a functor  $|-| : \mathbf{C} \rightarrow \mathbb{C}$ .

Consider the comma category  $(\mathbb{C} \downarrow |-|)$ , whose objects are tuples  $\langle S, f, A \rangle$ , with  $f : S \rightarrow |A|$  in  $\mathbb{C}$  and whose morphisms are pairs  $\langle g, h \rangle : \langle S, f, A \rangle \rightarrow \langle S', f', A' \rangle$ ,  $g : S \rightarrow S'$  in  $\mathbb{C}$  and  $h : A \rightarrow A'$  in  $\mathbf{C}$ , such that the diagram on the right commutes in  $\mathbb{C}$ .

$$\begin{array}{ccc} S & \xrightarrow{f} & |A| \\ g \downarrow & & \downarrow |h| \\ S' & \xrightarrow{f'} & |A'| \end{array} \quad (1)$$

This category is the *scone of  $\mathbf{C}$  over  $\mathbb{C}$  via  $|-|$* ,  $(\mathbb{C} \downarrow |-|)$ . (We extend here terminology of (Mitchell and Scedrov, 1993), where the name *scone* was reserved to the case  $\mathbb{C} = \mathbf{Set}$ ,  $|-| = \mathbf{C}(1, -)$  only.) The projection functor  $U : (\mathbb{C} \downarrow |-|) \rightarrow \mathbf{C}$  maps  $\langle S, f, A \rangle$  to  $A$  and the morphism  $\langle g, h \rangle$  to  $h$ .

In the sequel we shall be especially interested in the case where  $\mathbb{C} = \mathbf{Set}$ , and  $|-| = \mathbf{C}(1, -)$  is the *global section* functor, where 1 is terminal in  $\mathbf{C}$ . Another interesting situation arises when  $\mathbf{C} = \mathbb{C} \times \mathbb{C}$  and  $|(A, B)| = A \times B$ , assuming that  $\mathbb{C}$  has finite products. Objects of the scone then represent binary relations between objects in  $\mathbb{C}$ . In this case, given two functors  $|-|_1 : \mathbf{C}_1 \rightarrow \mathbb{C}$  and  $|-|_2 : \mathbf{C}_2 \rightarrow \mathbb{C}$ , we may define  $|-| : \mathbf{C} \rightarrow \mathbb{C}$ , for  $\mathbf{C} = \mathbf{C}_1 \times \mathbf{C}_2$ , by  $|(A_1, A_2)| = |A_1|_1 \times |A_2|_2$ .

Further assume we are given a monad  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu})$  on  $\mathbf{C}$ . When  $\mathbf{C} = \mathbf{C}_1 \times \mathbf{C}_2$ , the monad  $\mathbf{T}$  on  $\mathbf{C}$  will be usually defined pointwise, by two monads  $\mathbf{T}_1$  and  $\mathbf{T}_2$  on  $\mathbf{C}_1$  and  $\mathbf{C}_2$ , respectively:  $\mathbf{T}(A_1, A_2) = (\mathbf{T}_1(A_1), \mathbf{T}_2(A_2))$ .

### 2.1. Related work.

One successful approach to categorical generalization of logical relations is the setting of (Hermida, 1993) based on fibrations. We do not assume in this paper the forgetful functor to be a fibration, similarly to other authors (e.g. (Plotkin et al., 2000)). Even if we lose thereby the equivalence to logic, this is justified, since we do not focus on logical rules corresponding to logical relations in this paper.

We have already said that there is no unique notion of monad lifting. One of the simplest is the lifting, proposed in (Crole and Pitts, 1992),  $\tilde{\mathbf{T}}$  of  $\mathbf{T}$ , which maps the object  $\langle S, f, A \rangle$  of the scone to  $\langle S, |\boldsymbol{\eta}_A| \circ f, \mathbf{T}A \rangle$ . This is a special case of our notion of lifting on the scone  $(\mathbb{C} \downarrow |-|)$  (Section 3), taking  $\mathbb{T}$  the identity monad and  $\sigma_A = |\boldsymbol{\eta}_A|$ . Turi (Turi, 1996) considers lifting monads to the category of coalgebras of a given endofunctor. This is a special case of our framework, when  $\mathbf{C} = \mathbb{C}$  (and  $\mathbf{T} = \mathbb{T}$ ) and moreover only objects of the form  $S \xrightarrow{f} |S|$  are taken into consideration, and only morphisms of the form  $\langle g, g \rangle$ . This defines the category of  $|-|$ -coalgebras as a proper subcategory of scones. Turi uses a simpler version of our monad morphisms (we recall monad morphisms in Section 3), namely distributivity law of a monad over an endofunctor; monad morphisms involve two monads and a functor between distinct categories. Turi's distributivity laws are similar enough to monad morphisms that we had called the latter *distributivity laws* for monads in the conference version of this paper. This also influenced (Goubault-Larrecq and Goubault, 2003), where comonad morphisms are used, but are called distributivity laws for comonads. Calling these monad morphisms distributivities turned out to be a bad choice, as distributivity laws denote a close but different concept,

due to Jon Beck (Beck, 1969); also, while distributivity laws tend to be rare, monad morphisms abound.

In (Power and Watanabe, 2002) different possible ways to combine a monad and a comonad were studied in a systematic way. In particular, the authors used a notion of a distributivity of a monad over a comonad (and dual distributivity of the comonad over the monad). This is a stronger notion than monad morphisms, as it requires commutativity of two copies of diagrams (4) below, one for the monad and another one for the comonad. On the other hand, both the monad and the comonad live in a single category, unlike in our case.

We also note that neither Pitts nor Turi deal with subscones.

(Kinoshita and Power, 1999) considers lifting of a monad on **Set** to the category of relations in **Set**; this is a particular case of our setting. Although, the main topic of this paper is a more complex situation of lax logical relations and data-refinement for the computational lambda calculus. A subsequent paper (Power and Tanaka, 2000) develops further the ideas of (Kinoshita and Power, 1999). It addresses in particular lax logical relations for the computational lambda calculus and for the linear lambda calculus.

In the same way that we lift a monad to relations, Rutten (Rutten, 1998) defines an extension of an endofunctor in **Set** to a category of relations. The latter has relations as *morphisms* between sets. An endofunctor extends to relations iff it preserves weak pullbacks (which in particular implies preserving monos), and if so, the extension is unique. (This is actually a special case of a more general fact, proved in (Carboni et al., 1990) for regular categories.) The approach taken by Rutten is different from ours, where relations are *objects* rather than morphisms. Hence, Rutten imposes a different functoriality condition: the action of a lifted endofunctor on a composition of two relations must coincide with a composition of actions of the lifted endofunctor on these two relations. This amounts to closedness under composition of relations yielded by the lifted endofunctor.

(Pitts, 1996) considers lifting of certain constructions on domains to relations on these domains. He focuses on categories on domains and does not consider (strong) monads. Instead of category of relations he assumes a relational structure, relaxing a notion of mapping (morphism) of relations.

An approach related to ours is (Goubault-Larrecq and Goubault, 2003), where a *comonad* lifting is defined. This relies on pullbacks, whereas we use mono factorization systems. Nonetheless, the so-called distributivity laws of (Goubault-Larrecq and Goubault, 2003) are comonad morphisms, dual to the monad morphisms we use.

### 3. Lifting of a Monad to a Scone

By *lifting of a monad*  $(\mathbf{T}, \eta, \mu)$  to the scone  $(\mathbb{C} \downarrow | \_ |)$  of  $\mathbf{C}$  over  $\mathbb{C}$  we mean a monad  $(\tilde{\mathbf{T}}, \tilde{\eta}, \tilde{\mu})$  on  $(\mathbb{C} \downarrow | \_ |)$  such that the diagram

$$\begin{array}{ccc}
 (\mathbb{C} \downarrow | \_ |) & \xrightarrow{\tilde{\mathbf{T}}} & (\mathbb{C} \downarrow | \_ |) \\
 \downarrow \scriptstyle U & & \downarrow \scriptstyle U \\
 \mathbf{C} & \xrightarrow{\mathbf{T}} & \mathbf{C}
 \end{array} \tag{2}$$

commutes, i.e.  $U \circ \tilde{T} = \mathbf{T} \circ U$  and moreover

$$U\tilde{\eta} = \eta_U \quad \text{and} \quad U\tilde{\mu} = \mu_U. \quad (3)$$

By  $U\tilde{\eta}$  and  $\eta_U$  we mean the two possible compositions of a natural transformation with  $U$ , similarly  $U\tilde{\mu}$  and  $\mu_U$ . The equations (3) amount to the requirement that the two diagrams on the right commute, for all objects  $X$  in  $(\mathbb{C} \downarrow | \_ |)$ :

$$\begin{array}{ccc} & & \mathbf{T}^2UX \\ & & \swarrow \mu_{UX} \\ & & \mathbf{T}UX \\ & \nearrow \eta_{UX} & \parallel (2) \\ UX & \xrightarrow{U\tilde{\eta}_X} & \mathbf{T}UX \\ & & \parallel (2) \\ & & \mathbf{T}\tilde{T}X \\ & & \swarrow \mu_{\tilde{T}X} \\ & & \mathbf{T}\tilde{T}^2X \end{array}$$

In other words, the functor  $U$  together with the identity natural transformation is a morphism of monads from  $\tilde{T}$  to  $\mathbf{T}$ . (We recall monad morphisms shortly.) Note that the equations (3) determine the  $\mathbf{C}$ -components of  $\tilde{\eta}$  and  $\tilde{\mu}$  unambiguously. Moreover, diagram (2) determines the  $\mathbf{C}$ -component of the action of  $\tilde{T}$  on objects and morphisms, i.e.  $\langle S, f, A \rangle$  is necessarily mapped to  $\langle \tilde{S}, \tilde{f}, \mathbf{T}A \rangle$ , for some  $\tilde{S}, \tilde{f}$  and a morphism  $\langle g, h \rangle$  is necessarily mapped to  $\langle \tilde{g}, \mathbf{T}h \rangle$ , for some  $\tilde{g}$ .

Our notion of lifting could be stated more generally, for an arbitrary pair of categories, a functor from the first one to the second one and a monad on the second category. In fact, in the next section we consider a lifting of  $\mathbf{T}$  to another category, namely a suitable subcategory of  $(\mathbb{C} \downarrow | \_ |)$ .

To be able to give an appropriate lifting we assume another monad  $(\mathbb{T}, \eta, \mu)$  on  $\mathbb{C}$  such that the two monads  $\mathbf{T}$  and  $\mathbb{T}$  are related by a *monad morphism* from  $\mathbf{T}$  to  $\mathbb{T}$ , i.e. a natural transformation

$$\sigma : \mathbb{T}| \_ | \Rightarrow | \mathbf{T} |$$

making the following two diagrams commute, for each object  $A$  in  $\mathbf{C}$ :

$$\begin{array}{ccc} & & \mathbb{T}^2|A| \\ & & \swarrow \mu_{|A|} \\ & & \mathbb{T}|A| \\ & \nearrow \eta_{|A|} & \downarrow \mathbb{T}\sigma_A \\ |A| & \xrightarrow{\eta_A} & \mathbb{T}|A| \\ & & \downarrow \sigma_A \\ & & |\mathbf{T}A| \\ & & \swarrow \mu_{\mathbf{T}A} \\ & & |\mathbf{T}^2A| \end{array} \quad (4)$$

To be formal, a monad morphism is a *pair*  $(| \_ |, \sigma)$  satisfying the equations above. We shall however continue to say that  $\sigma$  is a monad morphism, when  $| \_ |$  is understood.

Having  $\sigma$ , we define  $\tilde{T}$  on objects by

$$\langle S, f, A \rangle \quad \longmapsto \quad \langle \mathbb{T}S, \sigma_A \circ \mathbb{T}f, \mathbf{T}A \rangle$$

exploiting that if  $S \xrightarrow{f} |A|$  then  $\mathbb{T}S \xrightarrow{\mathbb{T}f} \mathbb{T}|A| \xrightarrow{\sigma_A} |\mathbf{T}A|$  is a morphism. On morphisms we define  $\tilde{T}$  by

$$\langle g, h \rangle \quad \longmapsto \quad \langle \mathbb{T}g, \mathbf{T}h \rangle$$

$$\text{since from } \begin{array}{ccc} S & \xrightarrow{f} & |A| \\ g \downarrow & & \downarrow |h| \\ S' & \xrightarrow{f'} & |A'| \end{array} \quad \text{we deduce that } \begin{array}{ccccc} \mathbb{T}S & \xrightarrow{\mathbb{T}f} & \mathbb{T}|A| & \xrightarrow{\sigma_A} & |\mathbb{T}A| \\ \mathbb{T}g \downarrow & & & & \downarrow |\mathbb{T}h| \\ \mathbb{T}S' & \xrightarrow{\mathbb{T}f'} & \mathbb{T}|A'| & \xrightarrow{\sigma_{A'}} & |\mathbb{T}A'| \end{array} \quad \text{commutes,}$$

by naturality of  $\sigma$ . Moreover, we put

$$\tilde{\eta}_{\langle S, f, A \rangle} = \langle \eta_S, \boldsymbol{\eta}_A \rangle \quad \text{and} \quad \tilde{\mu}_{\langle S, f, A \rangle} = \langle \mu_S, \boldsymbol{\mu}_A \rangle.$$

Checking that this defines a monad is straight-forward. First, to check that unit and multiplication are well defined it is sufficient to merge the commuting diagrams (4) and complete them with naturality squares for  $\eta$  and  $\mu$  as shown on the right.

Unit  $\tilde{\eta}$  and multiplication  $\tilde{\mu}$  are natural since they are defined pointwise and  $\boldsymbol{\eta}$ ,  $\boldsymbol{\mu}$ ,  $\eta$  and  $\mu$  are. Verifying monad laws is immediate, by the same argument.

The monad morphism (4) can be equivalently given by a lifting of  $|-|$  to the categories of algebras of the monads, i.e. by a functor  $\tilde{|-|} : \mathbf{C}_{\mathbb{T}} \rightarrow \mathbf{C}_{\mathbb{T}}$  making the diagram commute:

$$\begin{array}{ccccc} S & \xrightarrow{\eta_S} & \mathbb{T}S & \xleftarrow{\mu_S} & \mathbb{T}^2S \\ \downarrow f & & \downarrow \mathbb{T}f & & \downarrow \mathbb{T}^2f \\ & & \mathbb{T}|A| & \xleftarrow{\mu_{|A|}} & \mathbb{T}^2|A| \\ & & \downarrow \sigma_A & & \downarrow \mathbb{T}\sigma_A \\ & & |A| & \xleftarrow{\mu_A} & |\mathbb{T}A| \\ & & \downarrow \eta_{|A|} & & \downarrow \sigma_{\mathbb{T}A} \\ & & \mathbb{T}|A| & \xleftarrow{\mu_{\mathbb{T}A}} & \mathbb{T}^2|A| \end{array}$$

$$\begin{array}{ccc} \mathbf{C}_{\mathbb{T}} & \xrightarrow{\tilde{|-|}} & \mathbf{C}_{\mathbb{T}} \\ U_{\mathbb{T}} \downarrow & & \downarrow U_{\mathbb{T}} \\ \mathbf{C} & \xrightarrow{|-|} & \mathbf{C} \end{array}$$

where  $\mathbf{C}_{\mathbb{T}}$  and  $\mathbf{C}_{\mathbb{T}}$  denote categories of algebras of the monads and  $U_{\mathbb{T}}$  and  $U_{\mathbb{T}}$  denote the obvious forgetful functors. In fact, for fixed monads  $\mathbf{T}$  and  $\mathbb{T}$  and a fixed functor  $|-|$ , there is a one-to-one correspondence between the monad morphisms  $\sigma$  and liftings of  $|-|$  to algebras. The proof of this fact can be found in (Appelgate, 1965) or in (Johnstone, 1975).

#### 4. Lifting of a Monad to a Subscene

Following, and slightly extending (Mitchell and Scedrov, 1993), we may call the *subscene of  $\mathbf{C}$  over  $\mathbb{C}$*  the full subcategory  $(\mathbb{C} \Downarrow |-|)$  of  $(\mathbb{C} \downarrow |-|)$  consisting of all objects  $\langle S, f, A \rangle$  with  $f$  a mono, written  $S \xrightarrow{f} |A|$ . (We shall actually define the subscene slightly differently below.)

When  $\mathbb{C} = \mathbf{Set}$  and  $|A|$  is given by  $\mathbf{C}(1, A)$ , each object  $S \xrightarrow{f} |A|$  in the subscene represents a subset of global elements of  $A$ . In the binary case, i.e. when  $\mathbf{C} = \mathbf{C}_1 \times \mathbf{C}_2$  and  $|(A_1, A_2)| = \mathbf{C}_1(1_1, A_1) \times \mathbf{C}_2(1_2, A_2)$ ,  $S \xrightarrow{f} |(A_1, A_2)|$  corresponds to a binary relation on global elements of  $A_1$  and  $A_2$ —when  $A_1$  and  $A_2$  are the respective denotations of type  $\tau$  in two given models, this will be the logical relation at type  $\tau$ .

For technical reasons, we require that  $\mathbb{C}$  has a *mono factorization system*. This is essentially an epi-mono factorization (Adamek et al., 1990), except we relax part of the



definition: we keep the mono part but do not require the epis in the sequel. Alternately, this is a factorization system where one of the classes of morphisms is required to consist of monos only.

Formally, a mono factorization system is given by two distinguished subclasses of morphisms in  $\mathbb{C}$ , the so-called *pseudoepi*  $\twoheadrightarrow$  and the so-called *relevant monos*  $\hookrightarrow$ . The latter must be monos, while the former are not required to be epis. Both classes must be closed under composition with isomorphisms.

Each morphism  $f$  in  $\mathbb{C}$  must factor as  $f = m \circ e$  for some pseudoepi  $e$  and some relevant mono  $m$ ; and each commuting square (5) has a diagonal making both triangles commute as in (6). We call this diagonal morphism the *diagonal fill-in*. Note that the diagonal fill-in is necessarily unique and that whenever the lower-right triangle commutes, the upper-left triangle does too. Furthermore, the latter property guarantees that the factorization  $f = m \circ e$  is unique up to iso.

$$\begin{array}{ccc} \cdot & \twoheadrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \hookrightarrow & \cdot \end{array} \quad (5)$$

$$\begin{array}{ccc} \cdot & \twoheadrightarrow & \cdot \\ \downarrow & \searrow & \downarrow \\ \cdot & \hookrightarrow & \cdot \end{array} \quad (6)$$

In particular, we do not require neither pseudoepi nor relevant monos to be closed under composition, which holds true for an epi-mono factorization system, see e.g. (Adamek et al., 1990, Chapter 14). But it is easy to deduce from the diagonal fill-in property that a composition of two pseudoepi is pseudoepi indeed, and similarly a composition of two relevant monos is a relevant mono, see e.g. (Barr, 1998). It is also proved there that both classes contain all isomorphisms.

In fact, the factorization of  $f$  as  $m \circ e$  determines uniquely a so-called *relevant subobject* of the codomain, defined as follows. Two relevant monos in  $\mathbb{C}$  with the same codomain,  $S_1 \hookrightarrow S$  and  $S_2 \hookrightarrow S$  are called equivalent if and only if there exist  $g_1$  and  $g_2$  making the two triangles commute:

$$\begin{array}{ccc} S_1 & \begin{array}{c} \xrightarrow{g_2} \\ \xleftarrow{g_1} \end{array} & S_2 \\ & \begin{array}{c} \searrow f_1 \\ \swarrow f_2 \end{array} & \\ & S & \end{array}$$

i.e.,  $f_1 \circ g_1 = f_2$  and  $f_2 \circ g_2 = f_1$ . A *relevant subobject* of  $S$  is an equivalence class of relevant monos with codomain  $S$ . Equivalently, we could take as objects of the subscone all relevant subobjects of  $|A|$ , for all objects  $A$  in  $\mathbf{C}$ . We prefer to keep the simpler presentation, despite the fact that this implies that some constructions in the sequel are only determined up to isomorphism, e.g., (7) below.

We come back to the definition of the subscone:

**Definition 1.** Given two categories  $\mathbf{C}, \mathbb{C}$ , a functor  $|-| : \mathbf{C} \rightarrow \mathbb{C}$ , and a mono factorization system on  $\mathbb{C}$ , the *subcone of  $\mathbf{C}$  over  $\mathbb{C}$*  is the full subcategory  $(\mathbf{C} \downarrow \! \! \! \downarrow \! \! \! |-|)$  of  $(\mathbf{C} \downarrow \! \! \! \downarrow \! \! \! |-|)$  consisting of all objects  $\langle S, f, A \rangle$  with  $f$  a relevant mono  $S \hookrightarrow |A|$ .

It may seem that the notation  $(\mathbf{C} \downarrow \! \! \! \downarrow \! \! \! |-|)$  is too vague, as it does not mention  $\mathbf{C}$  or the

mono factorization system explicitly. It will be clear that making all parameters explicit would make the notation extremely heavy.

Additionally, we shall assume that  $\mathbb{T}e$  is pseudoepi for every morphism  $e$  in a subclass of all pseudoepis called *relevant pseudoepis*, which we shall define shortly. This will be used in Diagram (11) below. In most applications, it will suffice to check that  $\mathbb{T}$  preserves pseudoepis.

Note the following simple and important fact:

**Fact 1.** The first component  $g$  of a morphism  $\langle g, h \rangle$  (recall that  $\begin{array}{ccc} S \hookrightarrow |A| \\ g \downarrow & & \downarrow |h| \\ S' \hookrightarrow |A'| \end{array}$  commutes) in a subscone is uniquely determined by the second component  $h$ .

This is because the bottom arrow is now mono.

Let us define a lifting of the monad to the subscone by analogy with (2) and (3) for the scone. In the binary case mentioned at the beginning of this section, this corresponds to a lifting of a monad to the category of binary relations (as objects) and relation preserving functions (as morphisms).

#### 4.1. $\tilde{T}$ on objects.

The lifting  $\tilde{T}$  on objects is given by the mono part of the mono factorization of the lifting of the previous section:  $\langle S, f, A \rangle$  is taken to  $\langle \tilde{S}, m, \mathbf{T}A \rangle$  given by the diagram on the right.

$$\begin{array}{ccc} \mathbb{T}S & \xrightarrow{\mathbb{T}f} & \mathbb{T}|A| \\ e \downarrow & & \downarrow \sigma_A \\ \tilde{S} & \xrightarrow{m} & |\mathbf{T}A| \end{array} \quad (7)$$

We call pseudoepis  $e$  arising in this way  $\mathbb{T}, \sigma$ -*relevant pseudoepis*. That is, a  $\mathbb{T}, \sigma$ -relevant pseudoepi is the pseudoepi part of a factorization of a morphism of the form  $\sigma_A \circ \mathbb{T}f$ , where  $f$  is a relevant mono. For short, we shall call them *relevant pseudoepis* when  $\mathbb{T}$  and  $\sigma$  are clear from context.

Clearly  $\tilde{T}$  is defined only up to iso. Formally, the construction would be unambiguous if we worked with subobjects of  $|\mathbf{T}A|$ , which are determined uniquely.

#### 4.2. $\tilde{T}$ on morphisms.

Given a morphism  $\langle g, h \rangle$ , the diagram on the right commutes. Then the action of  $\tilde{T}$  on  $\langle g, h \rangle$  will be obtained from the unique diagonal guaranteed by (6). We construct diagram (9) below from two copies of (7).

$$\begin{array}{ccc} S \hookrightarrow |A| & & \\ g \searrow & f \longrightarrow & |A| \\ & & \downarrow |h| \\ S' \hookrightarrow |A'| & & \\ & f' \longrightarrow & |A'| \end{array} \quad (8)$$

All four given faces of the cube commute. Both front and back faces commute by definition of  $\tilde{T}$  on objects: they are copies of diagram (7). The right-hand face is a naturality square of  $\sigma$ ; the top face is by application of  $\mathbb{T}$  to diagram (8), hence commutes by definition of morphisms in the subscone.

$$\begin{array}{ccc}
 \mathbb{T}S & \xrightarrow{\mathbb{T}f} & \mathbb{T}|A| \\
 \downarrow e & \searrow \mathbb{T}g & \downarrow \mathbb{T}|h| \\
 \mathbb{T}S' & \xrightarrow{\mathbb{T}f'} & \mathbb{T}|A'| \\
 \downarrow \tilde{S}^{\mathbb{C}} & \downarrow \sigma_A & \downarrow \sigma_{A'} \\
 \tilde{S}^{\mathbb{C}} & \xrightarrow{m} & |TA| \\
 \downarrow e' & \searrow |\mathbb{T}h| & \downarrow \sigma_{A'} \\
 \tilde{S}'^{\mathbb{C}} & \xrightarrow{m'} & |TA'|
 \end{array} \quad (9)$$

Now, an instance of diagram (5) can be found in (9) by two walks from  $\mathbb{T}S$  to  $|TA'|$ : one starts with the pseudoepi  $\mathbb{T}S \xrightarrow{e} \tilde{S}$ , the other ends with the relevant mono  $\tilde{S}'^{\mathbb{C}} \xrightarrow{m'} |TA'|$ . Since all faces commute, there is an arrow  $\tilde{S} \dashrightarrow \tilde{S}'^{\mathbb{C}}$  as in diagram (6), making the two newly created faces of the cube commute. This arrow is unique by Fact 1. Now  $\tilde{T}\langle g, h \rangle$  is given by the bottom face. Functoriality follows immediately from uniqueness of the diagonal arrow in (6).

#### 4.3. Unit $\tilde{\eta}$ .

The ( $\mathbb{C}$ -component of the) unit  $\tilde{\eta}_{\langle S, f, A \rangle}$  is defined by post-composing  $\eta_S$  with the pseudoepi part of the mono factorization in (7). This is well-defined since everything in sight in the diagram on the right commutes. Indeed, the right triangle is the monad morphism diagram (4) (left), the upper square is the naturality of  $\eta$  while the lower one is a copy of (7).

$$\begin{array}{ccc}
 S^{\mathbb{C}} & \xrightarrow{f} & |A| \\
 \downarrow \eta_S & \searrow \eta_{|A|} & \downarrow |\eta_A| \\
 \mathbb{T}S & \xrightarrow{\mathbb{T}f} & \mathbb{T}|A| \\
 \downarrow e & \searrow \sigma_A & \downarrow \sigma_A \\
 \tilde{S}^{\mathbb{C}} & \xrightarrow{m} & |TA|
 \end{array} \quad (10)$$

#### 4.4. Multiplication $\tilde{\mu}$ .

The ( $\mathbb{C}$ -component of the) multiplication  $\tilde{\mu}_{\langle S, f, A \rangle}$  will be induced by a diagram similar to (9) (below).

Again, all the faces not having the dashed arrow or the required dotted arrow as edge commute. The front face and the lower half of the back face are instances of (7), defining  $\tilde{T}\langle S, f, A \rangle$  and  $\tilde{T}^2\langle S, f, A \rangle$ , respectively. The upper half of the back face is by application of  $\mathbb{T}$  to the front face. The right-hand face is the other monad morphism diagram (4) (right), which we had not used yet, while the upper one is a naturality square for  $\mu$ .

$$\begin{array}{ccc}
 \mathbb{T}^2 S & \xrightarrow{\mathbb{T}^2 f} & \mathbb{T}^2 |A| \\
 \downarrow \mathbb{T}e & \searrow \mu_S & \downarrow \mathbb{T}\sigma_A \\
 \mathbb{T}\tilde{S} & \xrightarrow{\mathbb{T}m} & \mathbb{T}|TA| \\
 \downarrow \tilde{S}^{\mathbb{C}} & \downarrow \sigma_{\mathbb{T}A} & \downarrow \mu_{|A|} \\
 \tilde{S}^{\mathbb{C}} & \xrightarrow{m} & |TA| \\
 \downarrow \tilde{e} & \downarrow \sigma_{\mathbb{T}A} & \downarrow \sigma_A \\
 \tilde{S}^{\mathbb{C}} & \xrightarrow{\tilde{m}} & |\mathbb{T}^2 A| \\
 \downarrow e & \searrow |\mu_A| & \downarrow \sigma_A \\
 \tilde{S}^{\mathbb{C}} & \xrightarrow{m} & |TA|
 \end{array} \quad (11)$$

Note that  $\mathbb{T}e$  is a pseudoepi, since  $e$  is a relevant pseudoepi by construction, and  $\mathbb{T}$  maps relevant pseudoepi to pseudoepi. The composition  $\tilde{e} \circ \mathbb{T}e$  is necessarily a pseudoepi as

well (Barr, 1998). We may use this result, or use a diagonal (6) twice. Here, and in some other cases later, we prefer to do so.

First, similarly as in diagram (9) we find an instance of diagram (5) by two walks from  $\mathbb{T}^2 S$  to  $|\mathbf{T}A|$ , one starting with  $\mathbb{T}e$  and the other ending with  $m$ . Hence, the unique dashed arrow exists and makes the two triangles commute. One of them, involving the pseudoepi  $\mathbb{T}e$ , is the upper part of the left-hand side. The other one, namely that involving the relevant mono  $m$ , allows us to apply (5) again, since the following two walks from  $\mathbb{T}\tilde{S}$  to  $|\mathbf{T}A|$  commute: one starting with the pseudoepi  $\tilde{e}$  and the other consisting of the dashed arrow followed by  $m$ . Hence, the unique dotted arrow exists and makes the bottom face as well as the triangle in the left-hand face commute. The multiplication  $\tilde{\mu}_{\langle S, f, A \rangle}$  is then defined by the bottom face of the cube.

Verification of the monad laws is a formality due to the following:

**Fact 2.** Given two parallel arrows in  $(\mathbf{C} \Downarrow |_{-}|)$ , say  $\langle g_1, h_1 \rangle$  and  $\langle g_2, h_2 \rangle$ , they are equal whenever the second components  $h_1$  and  $h_2$  are.

The proof is immediate by Fact 1. Using this fact, and knowing that second components of  $\tilde{\eta}$  and  $\tilde{\mu}$  satisfy the monad laws (as they are unit and multiplication of  $\mathbf{T}$ , respectively), we deduce immediately that  $\tilde{\eta}$  and  $\tilde{\mu}$  satisfy the monad laws too. Similarly one proves naturality of  $\tilde{\eta}$  and  $\tilde{\mu}$ . We shall use this argument extremely often in the sequel.

It is useful to summarize the ingredients we have used here. To lift a monad  $(\mathbf{T}, \eta, \mu)$  on  $\mathbf{C}$  to  $(\mathbf{C} \Downarrow |_{-}|)$ , we need:

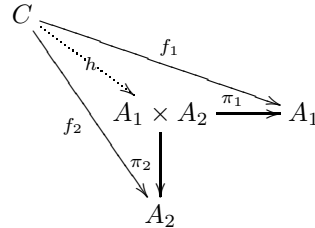
- 
- (i) a category  $\mathbb{C}$  and a functor  $|_{-}| : \mathbf{C} \rightarrow \mathbb{C}$ ,
  - (ii) a monad  $(\mathbb{T}, \eta, \mu)$  on  $\mathbb{C}$ ,  
related to  $(\mathbf{T}, \eta, \mu)$  by a monad morphism  $(|_{-}|, \sigma)$  from  $\mathbf{T}$  to  $\mathbb{T}$ ,
  - (iii) a mono factorization system on  $\mathbb{C}$ ,
  - (iv)  $\mathbb{T}$  maps relevant pseudoepis to pseudoepis.
- 

Recall that to lift the CCC structure of  $\mathbf{C}$  to the subscone, we additionally require  $\mathbb{C}$  to be a CCC with pullbacks, and  $|_{-}|$  to preserve finite products (Mitchell and Scedrov, 1993). Description of the construction can be found e.g., in (Goubault-Larrecq and Goubault, 2003), Section 5.4. We shall see in Section 9 that the existence of a mono factorization system on  $\mathbb{C}$  allows us to relax the requirements on  $|_{-}|$  somewhat.

## 5. Lifting of a Monad to Relations

Recall that we would like to lift monads to categories of binary relations as objects. Hence, assume in this section that  $\mathbf{C}$  is a product category,  $\mathbf{C} = \mathbf{C}_1 \times \mathbf{C}_2$  and that both  $\mathbf{C}_1$  and  $\mathbf{C}_2$  are equipped with monads  $\mathbf{T}_1$  and  $\mathbf{T}_2$ , and functors  $|_{-}|_1 : \mathbf{C}_1 \rightarrow \mathbb{C}$  and  $|_{-}|_2 : \mathbf{C}_2 \rightarrow \mathbb{C}$ . A monad  $\mathbf{T}$  on  $\mathbf{C}$  can be defined pairwise:  $\mathbf{T}\langle A_1, A_2 \rangle = \langle \mathbf{T}_1 A_1, \mathbf{T}_2 A_2 \rangle$  and similarly we define  $|_{-}| : \mathbf{C} \rightarrow \mathbb{C}$  by  $|(A_1, A_2)| = |A_1|_1 \times |A_2|_2$ .

To this aim we assume binary products in  $\mathbb{C}$ , i.e., for each pair of objects  $A_1, A_2$  of  $\mathbb{C}$ , an object  $A_1 \times A_2$  in  $\mathbb{C}$ , together with two morphisms  $\pi_1 : A_1 \times A_2 \rightarrow A_1$  and  $\pi_2 : A_1 \times A_2 \rightarrow A_2$  satisfying the requirement that for every morphisms  $f_1$  and  $f_2$ , there is a unique morphism  $h$  making the whole diagram commute. We write  $\langle f_1, f_2 \rangle$  for  $h$ .



In the same vein monad morphisms from  $\mathbf{T}_1$  to  $\mathbb{T}$  and from  $\mathbf{T}_2$  to  $\mathbb{T}$  induce a monad morphism from  $\mathbf{T}$  to  $\mathbb{T}$ . Indeed, given any two monad morphisms

$$\sigma^1 : \mathbb{T}|_{-}|_1 \Rightarrow |\mathbf{T}_1|_1 \quad \text{and} \quad \sigma^2 : \mathbb{T}|_{-}|_2 \Rightarrow |\mathbf{T}_2|_2,$$

we can define  $\sigma_{(A_1, A_2)} : \mathbb{T}(|A_1|_1 \times |A_2|_2) \rightarrow |\mathbf{T}_1 A_1|_1 \times |\mathbf{T}_2 A_2|_2$  by

$$\sigma_{(A_1, A_2)} = \langle \sigma_{A_1}^1 \circ \mathbb{T}\pi_1, \sigma_{A_2}^2 \circ \mathbb{T}\pi_2 \rangle, \quad (12)$$

where  $\pi_1$  and  $\pi_2$  denote the projections from  $|A_1|_1 \times |A_2|_2$ .

The situation gets much simpler when  $\mathbb{C} = \mathbf{Set}$ ,  $|_{-}|_1 = \mathbf{C}_1(1_1, -)$  and  $|_{-}|_2 = \mathbf{C}_2(1_2, -)$ , where we assume that  $\mathbf{C}_1$  and  $\mathbf{C}_2$  have terminal objects,  $1_1$  and  $1_2$  respectively. Each object  $S \longrightarrow |(A_1, A_2)|$  in the subscone defines a binary relation (again noted  $S$ ) on global elements of  $A_1$  and  $A_2$ . Obviously  $\mathbf{Set}$  satisfies all requirements from previous sections, with surjections as pseudoepi and injections as relevant monos.

For a moment imagine that  $\mathbf{T}_1$  and  $\mathbf{T}_2$  are *strong* monads and that we are able to lift strong monads to subscones – this will be tackled in detail in the following Sections 6 and 7. Given two CCCs  $\mathbf{C}_1$  and  $\mathbf{C}_2$  with respective strong monads  $\mathbf{T}_1$  and  $\mathbf{T}_2$ , the fact that  $\mathbf{Comp}$  is the free CCC with strong monad on the set  $B$  of base types means that there are two representations of CCCs-with-strong-monads,  $\llbracket - \rrbracket_1$  and  $\llbracket - \rrbracket_2$ , from  $\mathbf{Comp}$  to  $\mathbf{C}_1$  and  $\mathbf{C}_2$  respectively: they are the natural meaning functions for monadic types and computational  $\lambda$ -terms.

Our construction of a lifting together with standard constructions on subscones (Mitchell and Scedrov, 1993) yield another representation of CCCs-with-strong-monads  $\llbracket - \rrbracket$  from  $\mathbf{Comp}$  to  $(\mathbf{Set} \Downarrow |_{-}|)$ . That  $\llbracket - \rrbracket$  is a lifting means that  $U \circ \llbracket - \rrbracket = \langle \llbracket - \rrbracket_1, \llbracket - \rrbracket_2 \rangle$ , i.e., the diagram on the right commutes. When  $\mathbf{C}_1$  and  $\mathbf{C}_2$  are concrete categories, this means that

The diagram shows a commutative square. At the bottom left is  $\mathbf{Comp}$ . At the top right is  $(\mathbf{Set} \Downarrow |_{-}|)$ . At the bottom right is  $\mathbf{C}_1 \times \mathbf{C}_2$ . An arrow labeled  $\llbracket - \rrbracket$  points from  $\mathbf{Comp}$  to  $(\mathbf{Set} \Downarrow |_{-}|)$ . An arrow labeled  $U$  points from  $(\mathbf{Set} \Downarrow |_{-}|)$  to  $\mathbf{C}_1 \times \mathbf{C}_2$ . An arrow labeled  $\langle \llbracket - \rrbracket_1, \llbracket - \rrbracket_2 \rangle$  points from  $\mathbf{Comp}$  to  $\mathbf{C}_1 \times \mathbf{C}_2$ .

$$\forall a_1 \in \llbracket \Gamma \rrbracket_1, a_2 \in \llbracket \Gamma \rrbracket_2. (a_1, a_2) \in \llbracket \Gamma \rrbracket \Rightarrow (\llbracket t \rrbracket_1(a_1), \llbracket t \rrbracket_2(a_2)) \in \llbracket \tau \rrbracket \quad (13)$$

for all terms  $t$  of type  $\tau$  in the context  $\Gamma = x_1 : \tau_1, \dots, x_n : \tau_n$ ; representations of  $\Gamma$  are taken to be products of the representations of  $\tau_1, \dots, \tau_n$ ;  $\llbracket \tau \rrbracket$  is a relation between  $\llbracket \tau \rrbracket_1$

and  $\llbracket \tau \rrbracket_2$ , defined by induction on types  $\tau$  (the case where  $\tau$  is a base type is arbitrary):

$$\begin{aligned} (f_1, f_2) \in \llbracket \tau \rightarrow \tau' \rrbracket &\iff \forall (a_1, a_2) \in \llbracket \tau \rrbracket . (f_1(a_1), f_2(a_2)) \in \llbracket \tau' \rrbracket \\ ((a_1, a'_1), (a_2, a'_2)) \in \llbracket \tau \times \tau' \rrbracket &\iff (a_1, a_2) \in \llbracket \tau \rrbracket \wedge (a'_1, a'_2) \in \llbracket \tau' \rrbracket \\ (B_1, B_2) \in \llbracket \mathbf{T}\tau \rrbracket &\iff (B_1, B_2) \in \tilde{T}\llbracket \tau \rrbracket \end{aligned}$$

These equations (except possibly the last one) are the standard definition of a *logical relation*. (13) is the already cited *Basic Lemma*.

Further simplification is gained when  $\mathbf{C}_1 = \mathbf{C}_2 = \mathbf{Set}$ , the three monads  $\mathbf{T}_1$ ,  $\mathbf{T}_2$  and  $\mathbf{T}$  are identical and both  $|-|_1$  and  $|-|_2$  are identity functors. The monad morphism  $\sigma$  reduces to distributivity of the monad  $\mathbf{T}$  over binary product, and (12) rewrites to

$$\sigma_{(A_1, A_2)} = \langle \mathbf{T}\pi_1, \mathbf{T}\pi_2 \rangle : \mathbf{T}(A_1 \times A_2) \rightarrow \mathbf{T}A_1 \times \mathbf{T}A_2$$

where by  $\mathbf{T}$  we denote a given single monad on  $\mathbf{Set}$ . This is a particularly important special case, so we study it in more detail.

Every binary relation  $S \subseteq A_1 \times A_2$  has a representation  $S \xrightarrow{\langle \pi^{S_1}, \pi^{S_2} \rangle} A_1 \times A_2$  where the arrow is the inclusion induced by two projections  $\pi^{S_1} : S \rightarrow A_1$  and  $\pi^{S_2} : S \rightarrow A_2$ . In fact, the full subcategory of the subscone consisting exclusively of inclusions instead of all injections is equivalent to the whole subscone, so without loss of generality we consider only inclusions in the rest of this section.

Recall the action of a lifted monad  $\tilde{T}$  on a relation  $S \xrightarrow{\langle \pi^{S_1}, \pi^{S_2} \rangle} A_1 \times A_2$ :

$$\begin{array}{ccc} \mathbf{T}S & \xrightarrow{\mathbf{T}\langle \pi^{S_1}, \pi^{S_2} \rangle} & \mathbf{T}(A_1 \times A_2) \\ \downarrow & & \downarrow \sigma_{(A_1, A_2)} \\ \tilde{S} & \xrightarrow{\quad} & \mathbf{T}A_1 \times \mathbf{T}A_2 \end{array}$$

The functor  $\tilde{T}$  maps a relation  $S$  to the relation between sets  $\mathbf{T}A_1$  and  $\mathbf{T}A_2$  defined as the direct image of the function  $\langle \mathbf{T}\pi^{S_1}, \mathbf{T}\pi^{S_2} \rangle : \mathbf{T}S \rightarrow \mathbf{T}A_1 \times \mathbf{T}A_2$ , since the middle (dashed) triangle in the following diagram commutes by the universal property of product (together with all other triangles):

$$\begin{array}{ccccc} & & \mathbf{T}S & & \\ & & | & & \\ & \mathbf{T}\pi^{S_1} & \downarrow \langle \mathbf{T}\pi^{S_1}, \mathbf{T}\pi^{S_2} \rangle & \mathbf{T}\pi^{S_2} & \\ & & \mathbf{T}(A_1 \times A_2) & & \\ & \mathbf{T}\pi_1 & \downarrow \langle \mathbf{T}\pi_1, \mathbf{T}\pi_2 \rangle & \mathbf{T}\pi_2 & \\ \mathbf{T}A_1 & \xleftarrow{\pi'_1} & \mathbf{T}A_1 \times \mathbf{T}A_2 & \xrightarrow{\pi'_2} & \mathbf{T}A_2 \end{array}$$

where by  $\pi'_1$  and  $\pi'_2$  we denote two projections from  $\mathbf{T}A_1 \times \mathbf{T}A_2$ .

It is instructive to look at some concrete example before going to more technical Sections 6 and 7. Further examples are presented in more detail in Section 10. Consider  $\mathbf{T}A = \mathbb{P}_{\text{fin}}(A)$ , the finite powerset monad on  $\mathbf{Set}$ . If we assume for simplicity that  $\pi^{S_1}$  and  $\pi^{S_2}$  are simply inclusions, then the function  $\mathbf{T}\pi^{S_1}$  takes a finite relation  $R \subset S$  to its domain, i.e.  $\{x \mid \exists y . (x, y) \in R\}$ , and  $\mathbf{T}\pi^{S_2}$  takes  $R$  to its codomain, i.e.,  $\{y \mid \exists x . (x, y) \in R\}$ .

Hence, the image of the function  $\langle \mathbb{T}\pi^{S_1}, \mathbb{T}\pi^{S_2} \rangle$  is a relation  $\tilde{S}$  between finite subsets of  $A_1$  and  $A_2$  that contains precisely domain-codomain pairs of finite relations  $R \subseteq S$ . Hence  $(B_1, B_2) \in \tilde{S}$  iff

$$\forall b_1 \in B_1. \exists b_2 \in B_2. (b_1, b_2) \in S \quad \wedge \quad \forall b_2 \in B_2. \exists b_1 \in B_1. (b_1, b_2) \in S.$$

## 6. Lifting Monoidal Structures

### 6.1. Monoidal Categories

In this section, and the following ones, we assume that each of the categories  $\mathbf{C}$  and  $\mathbb{C}$  is equipped with a monoidal structure. In other words, we assume that  $(\mathbf{C}, \otimes, \mathbf{I}, \alpha, \ell, \mathbf{r})$  and  $(\mathbb{C}, \otimes, \mathbb{I}, \alpha, \mathbb{l}, \mathbb{r})$  are *monoidal categories* (Mac Lane, 1971). This will allow us to extend our lifting of a monad to one of a strong monad in the following Section 7.

This means that  $\mathbb{I}$  is an object of  $\mathbb{C}$ ,  $\otimes$  is a functor from  $\mathbb{C} \times \mathbb{C}$  to  $\mathbb{C}$ , and  $\alpha_{A,B,C} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$ ,  $\mathbb{l}_A : \mathbb{I} \otimes A \rightarrow A$ ,  $\mathbb{r}_A : A \otimes \mathbb{I} \rightarrow A$  are natural isomorphisms called the associativity, the left unit and the right unit laws respectively, making the following squares commute.

$$\begin{array}{ccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A \otimes B, C, D}} (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A, B, C \otimes D}} A \otimes (B \otimes (C \otimes D)) \\ \downarrow \alpha_{A, B, C} \otimes \text{id}_D & & \uparrow \text{id}_A \otimes \alpha_{B, C, D} \\ (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A, B \otimes C, D}} & A \otimes ((B \otimes C) \otimes D) \end{array} \quad (14)$$

$$\begin{array}{ccc} (A \otimes \mathbb{I}) \otimes B & \xrightarrow{\alpha_{A, \mathbb{I}, B}} & A \otimes (\mathbb{I} \otimes B) \\ \searrow \mathbb{r}_A \otimes \text{id}_B & & \swarrow \text{id}_A \otimes \mathbb{l}_B \\ & A \otimes B & \end{array} \quad (15)$$

And similarly for  $\mathbf{I}, \otimes, \alpha, \ell, \mathbf{r}$ .

We prefer to work in a slightly more general setting compared to (Mitchell and Scedrov, 1993), where cartesian structure was assumed. In Section 9 we show how this added generality can be exploited for a fragment of linear lambda calculus. Typically, our categories will have finite products, then  $\mathbb{I}$  will be a terminal object,  $\otimes$  will be binary product, and  $\alpha, \mathbb{l}$  and  $\mathbb{r}$  will be the obvious isomorphisms.

We also assume that  $|-|$  is a *monoidal functor* (Eilenberg and Kelly, 1966). That is, there is a *mediating pair*  $(\mathbb{O}, \mathbb{I})$ , composed of a natural transformation  $\mathbb{O}_{A,B} : |A| \otimes |B| \rightarrow$

$|A \otimes B|$  and a morphism  $\mathfrak{i} : \mathbb{I} \rightarrow |\mathbf{I}|$  satisfying the following *coherence conditions*:

$$\begin{array}{ccc}
(|A| \otimes |B|) \otimes |C| & \xrightarrow{\alpha_{|A|,|B|,|C|}} & |A| \otimes (|B| \otimes |C|) \\
\downarrow \theta_{A,B} \otimes \text{id}_{|C|} & & \downarrow \text{id}_{|A|} \otimes \theta_{B,C} \\
|A \otimes B| \otimes |C| & & |A| \otimes |B \otimes C| \\
\downarrow \theta_{A \otimes B, C} & & \downarrow \theta_{A, B \otimes C} \\
|(A \otimes B) \otimes C| & \xrightarrow{|\alpha_{A,B,C}|} & |A \otimes (B \otimes C)|
\end{array} \tag{16}$$

$$\begin{array}{ccc}
\mathbb{I} \otimes |A| & & \\
\downarrow \mathfrak{i} \otimes \text{id}_{|A|} & \searrow \mathfrak{l}_{|A|} & \\
|\mathbf{I}| \otimes |A| & & |A| \\
\downarrow \theta_{\mathbf{I}, A} & \nearrow |\ell_A| & \\
|\mathbf{I} \otimes A| & & 
\end{array} \tag{17}$$

$$\begin{array}{ccc}
|A| \otimes \mathbb{I} & & \\
\downarrow \text{id}_{|A|} \otimes \mathfrak{i} & \searrow \mathfrak{r}_{|A|} & \\
|A| \otimes |\mathbf{I}| & & |A| \\
\downarrow \theta_{A, \mathbf{I}} & \nearrow |\mathfrak{r}_A| & \\
|A \otimes \mathbf{I}| & & 
\end{array} \tag{18}$$

Finally, we assume that pseudoepis and relevant monos form a so-called *monoidal mono factorization system*, i.e., for every two pseudoepis  $e_1, e_2$ , then  $e_1 \otimes e_2$  is again a pseudoepi. This name stems from (Ambler, 1991), Definition 5.2.1, p.91.

We define below a lifting of the monoidal structure to the subscone: we show that the subscone is a monoidal category  $((\mathbb{C} \downarrow |_{-}|), \tilde{\otimes}, \tilde{\mathbf{I}}, \tilde{\alpha}, \tilde{\ell}, \tilde{r})$  in such a way that  $U(\tilde{A} \tilde{\otimes} \tilde{B}) = U\tilde{A} \otimes U\tilde{B}$ ,  $U\tilde{\mathbf{I}} = \mathbf{I}$ ,  $U\tilde{\alpha}_{\tilde{A}, \tilde{B}, \tilde{C}} = \alpha_{U\tilde{A}, U\tilde{B}, U\tilde{C}}$ ,  $U\tilde{\ell}_{\tilde{A}} = \ell_{U\tilde{A}}$ ,  $U\tilde{r}_{\tilde{A}} = \mathfrak{r}_{U\tilde{A}}$ . Lifting to scones is omitted here but can be easily extracted from diagrams below by dropping all factorizations.

### 6.2. Unit element $\tilde{\mathbf{I}}$ .

Let  $\tilde{\mathbf{I}}$  be the triple  $(\tilde{\mathbf{I}}, \tilde{\mathfrak{i}}, \mathbf{I})$  as built from the diagram on the right, obtained from a factorization of  $\mathfrak{i}$ .

$$\begin{array}{ccc}
\mathbb{I} & & \\
\downarrow e_{\mathbf{I}} & \searrow \mathfrak{i} & \\
\tilde{\mathbf{I}} & \xrightarrow{\tilde{\mathfrak{i}}} & |\mathbf{I}|
\end{array} \tag{19}$$

### 6.3. Tensor product $\tilde{\otimes}$ .

We build the tensor product  $\langle S_1, m_1, A_1 \rangle \tilde{\otimes} \langle S_2, m_2, A_2 \rangle$  in the obvious way: compose  $m_1 \otimes m_2$  with the mediating natural transformation  $\theta$ , and factorize.

The tensor product is then given by  $\langle S_{12}, m_{12}, A_1 \otimes A_2 \rangle$  on the right. This is similar to the construction of  $\tilde{\mathbf{I}}$ .

$$\begin{array}{ccc}
S_1 \otimes S_2 & \xrightarrow{m_1 \otimes m_2} & |A_1| \otimes |A_2| \\
\downarrow e_{12} & & \downarrow \theta_{A_1, A_2} \\
S_{12} & \xrightarrow{m_{12}} & |A_1 \otimes A_2|
\end{array} \tag{20}$$



#### 6.4. Associativity $\tilde{\alpha}$ .

This is more involved, but basically similar to the construction of the multiplication of the monad in the subscone. In the diagram below,  $e_{12} \otimes \text{id}_{S_3}$  is pseudoepi because both  $e_{12}$  (given by Diagram 20) and  $\text{id}_{S_3}$  are pseudoepis, and because our mono factorization system is monoidal. The two front faces and the two back faces are derived from the definition of  $\tilde{\otimes}$ , the top face is a naturality square for  $\alpha$ , the right face is the coherence condition (16). The dashed arrow, and then the dotted arrow  $\hat{\alpha}$ , are by the diagonal fill-in property of our factorization system. The desired associativity morphism is then the pair  $(\hat{\alpha}, \alpha_{A_1, A_2, A_3})$  (bottom face).

$$\begin{array}{ccc}
 (S_1 \otimes S_2) \otimes S_3 & \xrightarrow{(m_1 \otimes m_2) \otimes m_3} & (|A_1| \otimes |A_2|) \otimes |A_3| \\
 \downarrow e_{12} \otimes \text{id}_{S_3} & \searrow \alpha_{S_1, S_2, S_3} & \downarrow \alpha_{|A_1|, |A_2|, |A_3|} \\
 S_{12} \otimes S_3 & \xrightarrow{m_{12} \otimes m_3} & |A_1 \otimes A_2| \otimes |A_3| \\
 \downarrow e_{(12)3} & \searrow \text{id}_{S_1} \otimes e_{23} & \downarrow \theta_{A_1, A_2} \otimes \text{id}_{A_3} \\
 S_{(12)3} & \xrightarrow{m_{(12)3}} & |(A_1 \otimes A_2) \otimes A_3| \\
 \downarrow \hat{\alpha} & \searrow e_{1(23)} & \downarrow \theta_{A_1, A_2} \otimes A_3 \\
 S_{1(23)} & \xrightarrow{m_{1(23)}} & |A_1 \otimes (A_2 \otimes A_3)|
 \end{array}
 \quad (21)$$

The inverse is given by a very similar diagram (below).

$$\begin{array}{ccc}
 (S_1 \otimes S_2) \otimes S_3 & \xrightarrow{(m_1 \otimes m_2) \otimes m_3} & (|A_1| \otimes |A_2|) \otimes |A_3| \\
 \downarrow e_{12} \otimes \text{id}_{S_3} & \searrow \alpha_{S_1, S_2, S_3}^{-1} & \downarrow \alpha_{|A_1|, |A_2|, |A_3|}^{-1} \\
 S_{12} \otimes S_3 & \xrightarrow{m_{12} \otimes m_3} & |A_1 \otimes A_2| \otimes |A_3| \\
 \downarrow e_{(12)3} & \searrow \text{id}_{S_1} \otimes e_{23} & \downarrow \theta_{A_1, A_2} \otimes \text{id}_{A_3} \\
 S_{(12)3} & \xrightarrow{m_{(12)3}} & |(A_1 \otimes A_2) \otimes A_3| \\
 \downarrow \hat{\alpha}^{-1} & \searrow e_{1(23)} & \downarrow \theta_{A_1, A_2} \otimes A_3 \\
 S_{1(23)} & \xrightarrow{m_{1(23)}} & |A_1 \otimes (A_2 \otimes A_3)|
 \end{array}
 \quad (22)$$

### 6.5. Left unit $\tilde{\ell}$ .

Let  $\langle S, m, A \rangle$  be any object of the subscone. We build the diagram below. The left triangle in the upper back face is the definition of  $\tilde{I}$ , the rest of this face corresponds to two ways of writing  $\mathfrak{i} \otimes m$ , the lower back face is the definition of  $\tilde{I} \otimes \langle S, m, A \rangle$ . The upper, slanted face is a naturality square for  $\mathbb{I}$ .

Finally, the rightmost triangle is the coherence condition (17). As usual, we first derive the dashed, then the dotted arrow  $\hat{\mathbb{I}}$  by diagonal fill-ins. The desired left unit is  $(\hat{\mathbb{I}}, \ell_A)$ .

$$\begin{array}{ccc}
 \mathbb{I} \otimes S & \xrightarrow{\text{id}_{\mathbb{I}} \otimes m} & \mathbb{I} \otimes |A| \\
 \downarrow e_I \otimes \text{id}_S & \searrow \mathfrak{i} \otimes \text{id}_S & \downarrow \mathfrak{i} \otimes \text{id}_{|A|} \\
 \tilde{I} \otimes S & \xrightarrow{\tilde{\mathfrak{i}} \otimes \text{id}_S} & |I| \otimes S \xrightarrow{\text{id}_{|I|} \otimes m} |I| \otimes |A| \\
 \downarrow e_{12} & \searrow \mathbb{I}_S & \downarrow \theta_{I,A} \\
 S_{12} & \xrightarrow{m_{12}} & |I \otimes A| \\
 \downarrow \hat{\mathbb{I}} & & \downarrow \ell_A \\
 S & \xrightarrow{m} & |A|
 \end{array} \quad (23)$$

The inverse to  $\hat{\mathbb{I}}$  is also given by a diagonal fill-in. Start from  $S$ , then go to  $S$  (again) by the identity morphism—this is a pseudoepi—, then follow  $m$ ,  $|\ell_A^{-1}|$  to  $|I \otimes A|$ ; or start from  $S$ , climb along  $\mathbb{I}_S^{-1}$ , then follow  $e_I \otimes \text{id}_S$ ,  $e_{12}$ ,  $m_{12}$  (a mono) to  $|I \otimes A|$ . The diagonal fill-in is then an arrow from  $S$  to  $S_{12}$ , which is inverse to  $\hat{\mathbb{I}}$  by Fact 2.

### 6.6. Right unit $\tilde{r}$ .

This works exactly as for the left unit, see diagram on the right. The right triangle is the coherence condition (18). The desired right unit is given by  $(\hat{r}, r_A)$ .

The inverse of  $\hat{r}$  is built as for  $\hat{\mathbb{I}}$ .

$$\begin{array}{ccc}
 S \otimes \mathbb{I} & \xrightarrow{m \otimes \text{id}_{\mathbb{I}}} & |A| \otimes \mathbb{I} \\
 \downarrow \text{id}_S \otimes e_I & \searrow \text{id}_S \otimes \mathfrak{i} & \downarrow \text{id}_{|A|} \otimes \mathfrak{i} \\
 S \otimes \tilde{I} & \xrightarrow{\text{id}_S \otimes \tilde{\mathfrak{i}}} & S \otimes |I| \xrightarrow{m \otimes \text{id}_{|I|}} |A| \otimes |I| \\
 \downarrow e_{12} & \searrow r_S & \downarrow \theta_{A,I} \\
 S_{12} & \xrightarrow{m_{12}} & |A \otimes I| \\
 \downarrow \hat{r} & & \downarrow r_A \\
 S & \xrightarrow{m} & |A|
 \end{array} \quad (24)$$

Finally, all required naturality, isomorphism, and coherence conditions hold by Fact 2.

We recap what we need to lift monoidal structure to the subscone:

- 
- (i.a) monoidal categories  $(\mathcal{C}, \otimes, I, \alpha, \ell, r)$  and  $(\mathcal{C}, \otimes, \mathbb{I}, \alpha, \mathbb{I}, r)$ ,  
and a monoidal functor  $|-| : \mathcal{C} \rightarrow \mathcal{C}$ ,
  - (iii.a) a monoidal mono factorization system on  $\mathcal{C}$ .
-

### 6.7. Symmetric Monoidal Categories

We now assume that we have got, and want to preserve *symmetric* monoidal structure. Recall that a symmetric monoidal category  $(\mathbb{C}, \otimes, \mathbb{I}, \alpha, \mathbb{1}, \mathbb{r}, \mathfrak{c})$  is a monoidal category  $(\mathbb{C}, \otimes, \mathbb{I}, \alpha, \mathbb{1}, \mathbb{r})$ , together with a *commutativity* natural transformation  $\mathfrak{c}_{A,B} : A \otimes B \rightarrow B \otimes A$  obeying the following coherence conditions.

The first coherence condition is  $\mathfrak{c}_{B,A} \circ \mathfrak{c}_{A,B} = \text{id}_{A \otimes B}$ , which implies that  $\mathfrak{c}$  is actually a natural isomorphism. The others are:

$$\begin{array}{ccc}
 (A \otimes B) \otimes C & \xrightarrow{\mathfrak{c}_{A,B} \otimes \text{id}_C} & (B \otimes A) \otimes C \\
 \alpha_{A,B,C} \downarrow & & \downarrow \alpha_{B,A,C} \\
 A \otimes (B \otimes C) & & B \otimes (A \otimes C) \\
 \mathfrak{c}_{A,B \otimes C} \downarrow & & \downarrow \text{id}_B \otimes \mathfrak{c}_{A,C} \\
 (B \otimes C) \otimes A & \xrightarrow{\alpha_{B,C,A}} & B \otimes (C \otimes A)
 \end{array} \tag{25}$$

$$\begin{array}{ccc}
 \mathbb{I} \otimes A & \xrightarrow{\mathfrak{c}_{\mathbb{I},A}} & A \otimes \mathbb{I} \\
 \downarrow \mathbb{1}_A & & \downarrow \mathbb{r}_A \\
 & A & 
 \end{array} \tag{26}$$

We now need  $|-|$  to be a *symmetric* monoidal functor, that is, it should be monoidal and satisfy the extra coherence condition:

$$\begin{array}{ccc}
 |A_1| \otimes |A_2| & \xrightarrow{\mathfrak{c}_{|A_1|,|A_2|}} & |A_2| \otimes |A_1| \\
 \theta_{A_1,A_2} \downarrow & & \downarrow \theta_{A_2,A_1} \\
 |A_1 \otimes A_2| & \xrightarrow{|\mathfrak{c}_{A_1,A_2}|} & |A_2 \otimes A_1|
 \end{array} \tag{27}$$

### 6.8. Commutativity $\tilde{\ell}$ .

We lift the commutativity to the subscone, assuming  $\mathcal{C}$  and  $\mathbb{C}$  are symmetric monoidal, as follows. The back and front face are the definition of the two tensor products of  $\langle S_1, m_1, A_1 \rangle$  and  $\langle S_2, m_2, A_2 \rangle$ , the top face is by naturality of commutativity, the right

face is coherence (27). Finally  $\widehat{\mathbf{c}}$  is given by a diagonal fill-in, and all expected diagrams commute by Fact 2.

$$\begin{array}{ccccc}
S_1 \otimes S_2 & \xrightarrow{m_1 \otimes m_2} & |A_1| \otimes |A_2| & & \\
\downarrow e_{12} & \searrow c_{S_1, S_2} & \downarrow \theta_{A_1, A_2} & \searrow c_{|A_1|, |A_2|} & \\
S_2 \otimes S_1 & \xrightarrow{m_2 \otimes m_1} & |A_2| \otimes |A_1| & & \\
\downarrow e_{21} & \searrow c_{S_2, S_1} & \downarrow \theta_{A_2, A_1} & \searrow c_{|A_2|, |A_1|} & \\
S_{12} & \xrightarrow{m_{12}} & |A_1| \otimes |A_2| & & \\
\downarrow \widehat{\mathbf{c}} & \searrow c_{S_{12}, S_{21}} & \downarrow \theta_{A_1, A_2} & \searrow c_{|A_1|, |A_2|} & \\
S_{21} & \xrightarrow{m_{21}} & |A_2| \otimes |A_1| & & 
\end{array} \tag{28}$$

We recap what we need to lift *symmetric* monoidal structure to the subscone:

- 
- (i.b) symmetric monoidal categories  $(\mathbf{C}, \otimes, \mathbf{I}, \alpha, \ell, \mathbf{r}, \mathbf{c})$   
and  $(\mathbb{C}, \otimes, \mathbb{I}, \alpha, \ell, \mathbf{r}, \mathbf{c})$ ,  
and a symmetric monoidal functor  $|\_|\ : \mathbf{C} \rightarrow \mathbb{C}$ ,
  - (iii.a) a monoidal mono factorization system on  $\mathbb{C}$ .
- 

### 6.9. Lifting Cartesian Products

An important special case of symmetric monoidal structure is that given by finite products. This is given by one *terminal object*  $\mathbf{1}$  such that, for every object  $A$  in  $\mathbf{C}$ , there is a unique morphism  $A \xrightarrow{!} \mathbf{1}$ , and by a *binary product* operation as explained in Section 5. For any two morphisms  $f$  from  $A$  to  $A'$ ,  $g$  from  $B$  to  $B'$ , we write  $f \times g$  for the morphism  $\langle f \circ \pi_1, g \circ \pi_2 \rangle$  from  $A \times B$  to  $A' \times B'$ .

It is well-known that binary product can be turned into a functor from  $\mathbf{C} \times \mathbf{C}$  to  $\mathbf{C}$ , which is symmetric monoidal with unit  $\mathbf{1}$ , associativity  $\langle \pi_1 \circ \pi_1, \langle \pi_2 \circ \pi_1, \pi_2 \rangle \rangle$ , left unit  $\pi_2$ , right unit  $\pi_1$ , and commutativity  $\langle \pi_2, \pi_1 \rangle$ .

When  $\mathbf{C}$  and  $\mathbb{C}$  are both equipped with finite products, and  $|\_|\$  is a functor from  $\mathbf{C}$  to  $\mathbb{C}$  that is monoidal with respect to these products, then the construction of Sections 6.1 and 6.7 yields a symmetric monoidal structure on  $(\mathbb{C} \downarrow |\_|\ )$  that we claim stems from a finite product structure on the subscone.

To this end, we assume that  $|\_|\$  satisfies the coherence condition on the right, for  $i \in \{1, 2\}$ . We shall say that such a functor is *cartesian monoidal*. (Then  $|\_|\$  is automatically symmetric monoidal.)

$$\begin{array}{ccc}
|A_1| \times |A_2| & & \\
\downarrow \theta_{A_1, A_2} & \searrow \pi_i & \\
|A_1| \times |A_2| & \xrightarrow{|\pi_i|} & |A_i|
\end{array} \tag{29}$$

### 6.10. Terminal object $\tilde{\mathbf{1}}$ .

Let  $\tilde{\mathbf{1}}$  be the object  $(\tilde{T}, \tilde{\mathbb{1}}, \mathbf{1})$  given by diagram (19). Specializing this diagram to the case at hand, this is given as the unique object up to iso making the following diagram commute:

$$\begin{array}{ccc} \mathbb{1} & & \\ e_I \downarrow & \searrow \tilde{\mathbb{1}} & \\ \tilde{T} & \xrightarrow{\quad} & |\mathbf{1}| \end{array} \quad (30)$$

For any object  $\langle S, m, A \rangle$  of the subscone, there is a unique arrow  $\langle u, v \rangle$  from  $\langle S, m, A \rangle$  to  $(\tilde{T}, \tilde{\mathbb{1}}, \mathbf{1})$ . Indeed,  $v$  is the unique arrow  $!$  from  $A$  to  $\mathbf{1}$ , and  $u$  is given by  $e_I \circ !$ ;  $u$  is also unique, by Fact 2.

$$\begin{array}{ccc} & & \mathbb{1} \\ & & e_I \downarrow \\ & & \tilde{T} \\ & \nearrow ! & \xrightarrow{\quad} \\ S & \xrightarrow{m} & |A| \end{array} \quad \begin{array}{ccc} & & |\mathbf{1}| \\ & \searrow \tilde{\mathbb{1}} & \\ & & \downarrow |v| \end{array}$$

### 6.11. Binary product $\tilde{\times}$ .

Specializing the definition (20) of the lifted tensor product  $\tilde{\otimes}$  to the case at hand yields the object  $\langle S_{12}, m_{12}, A_1 \otimes A_2 \rangle = \langle S_1, m_1, A_1 \rangle \tilde{\times} \langle S_2, m_2, A_2 \rangle$  defined by the diagram on the right.

$$\begin{array}{ccc} S_1 \times S_2 & \xrightarrow{m_1 \times m_2} & |A_1| \times |A_2| \\ e_{12} \downarrow & & \theta_{A_1, A_2} \downarrow \\ S_{12} & \xrightarrow{m_{12}} & |A_1 \times A_2| \end{array} \quad (31)$$

The  $i$ th projection  $\langle \tilde{\pi}_i, \pi_i \rangle$  is then given by the diagram on the right. The back square is a copy of (31), the right triangle is an instance of the coherence condition (29), while the top, slanted face is by standard properties of  $\pi_i$ . From two routes from  $S_1 \times S_2$  to  $|A_i|$ , we get  $\tilde{\pi}_i$  by a diagonal fill-in.

$$\begin{array}{ccc} S_1 \times S_2 & \xrightarrow{m_1 \times m_2} & |A_1| \times |A_2| \\ e_{12} \downarrow & \searrow \pi_i & \downarrow \theta_{A_1, A_2} \\ S_{12} & \xrightarrow{m_{12}} & |A_1 \times A_2| \\ & \searrow \tilde{\pi}_i & \downarrow \pi_i \\ & & S_i & \xrightarrow{m_i} & |A_i| \end{array}$$

It remains to show that whenever we have two subscone morphisms  $\langle \tilde{f}_1, f_1 \rangle$  from  $\langle S, m, A \rangle$  to  $\langle S_1, m_1, A_1 \rangle$  and  $\langle \tilde{f}_2, f_2 \rangle$  from  $\langle S, m, A \rangle$  to  $\langle S_2, m_2, A_2 \rangle$ , there is a unique morphism  $\langle \tilde{h}, h \rangle$  from  $\langle S, m, A \rangle$  to the product  $\langle S_{12}, m_{12}, A_1 \times A_2 \rangle$  such that  $\langle \tilde{\pi}_i, \pi_i \rangle \circ \langle \tilde{h}, h \rangle = \langle \tilde{f}_i, f_i \rangle$  ( $i \in \{1, 2\}$ ). Existence is assured: take  $h = \langle f_1, f_2 \rangle$ ,  $\tilde{h} = e_{12} \circ \langle \tilde{f}_1, \tilde{f}_2 \rangle$ , which satisfies the claim: this is an easy consequence of the diagram above. Uniqueness follows from the uniqueness of  $h$  given by the definition of binary product in  $\mathcal{C}$ , and from Fact 2 guaranteeing the uniqueness of  $\tilde{h}$ .

As is now usual, we recap what we need to lift products to the subscone:

- 
- (i.c) categories  $\mathcal{C}$  and  $\mathbb{C}$  with finite products,
  - and a cartesian monoidal functor  $|-| : \mathcal{C} \rightarrow \mathbb{C}$ ,
  - (iii.a) a monoidal mono factorization system on  $\mathbb{C}$ .
-

## 7. Lifting Strong, Monoidal, and Commutative Monads to a Scone and a Subscone

Once we have got a monoidal structure on  $\mathbf{C}$  and  $\mathbb{C}$ , we may consider *strong* monads  $\mathbf{T}$  and  $\mathbb{T}$  instead of just monads on each category. This is what we need to develop a theory of logical relations for Moggi's monadic  $\lambda$ -calculus. We shall also consider the more demanding cases of *monoidal* monads, and of *commutative* monads.

### 7.1. Lifting Strong Monads

That  $\mathbb{T}$  is a strong monad means that a *strength* natural transformation  $\mathbb{t}_{A,B} : A \otimes \mathbb{T}B \rightarrow \mathbb{T}(A \otimes B)$  is given such that the diagrams in Definition 3.2 in (Moggi, 1991) commute, that is:

$$\begin{array}{ccc} \mathbb{I} \otimes \mathbb{T}B & \xrightarrow{\mathbb{t}_{\mathbb{I},B}} & \mathbb{T}(\mathbb{I} \otimes B) \\ & \searrow \downarrow \mathbb{t}_{\mathbb{T}B} & \downarrow \mathbb{T}\eta_B \\ & & \mathbb{T}B \end{array} \quad (32) \qquad \begin{array}{ccc} A \otimes B & \xrightarrow{\text{id}_A \otimes \eta_B} & A \otimes \mathbb{T}B \\ & \searrow \downarrow \eta_{A \otimes B} & \downarrow \mathbb{t}_{A,B} \\ & & \mathbb{T}(A \otimes B) \end{array} \quad (33)$$

$$\begin{array}{ccc} (A \otimes B) \otimes \mathbb{T}C & \xrightarrow{\mathbb{t}_{A \otimes B, C}} & \mathbb{T}((A \otimes B) \otimes C) \\ \downarrow \alpha_{A,B,\mathbb{T}C} & & \downarrow \mathbb{T}\alpha_{A,B,C} \\ A \otimes (B \otimes \mathbb{T}C) & \xrightarrow{\text{id}_A \otimes \mathbb{t}_{B,C}} & A \otimes \mathbb{T}(B \otimes C) \xrightarrow{\mathbb{t}_{A,B \otimes C}} \mathbb{T}(A \otimes (B \otimes C)) \end{array} \quad (34)$$

$$\begin{array}{ccc} A \otimes \mathbb{T}^2 B & \xrightarrow{\mathbb{t}_{A,\mathbb{T}B}} & \mathbb{T}(A \otimes \mathbb{T}B) \xrightarrow{\mathbb{T}\mathbb{t}_{A,B}} \mathbb{T}^2(A \otimes B) \\ \downarrow \text{id}_A \otimes \mu_B & & \downarrow \mu_{A \otimes B} \\ A \otimes \mathbb{T}B & \xrightarrow{\mathbb{t}_{A,B}} & \mathbb{T}(A \otimes B) \end{array} \quad (35)$$

Formally, a *strong monad* is a four-tuple  $(\mathbb{T}, \eta, \mu, \mathbb{t})$  where  $(\mathbb{T}, \eta, \mu)$  is a monad and  $\mathbb{t}$  is a strength making the above diagrams commute.

By lifting of  $\mathbf{T}$  to  $(\mathbb{C} \downarrow | \_ |)$  we now mean a strong monad, i.e. a monad  $(\tilde{\mathbb{T}}, \tilde{\eta}, \tilde{\mu})$  together with a strength  $\tilde{t}_{X,Y} : X \otimes \tilde{\mathbb{T}}Y \rightarrow \tilde{\mathbb{T}}(X \otimes Y)$ , such that diagram (2) commutes, equations (3) hold and

$$U\tilde{t}_{X,Y} = \mathbf{t}_{UX,UY},$$

i.e.,  $U$  preserves strength.

To be able to give an appropriate lifting, we extend the monad morphism to a *strong monad* morphism, i.e., a monad morphism making the following additional diagram

commute, which relates the strengths  $\mathbf{t}_{A_1, A_2}$  and  $\mathbb{k}_{|A_1|, |A_2|}$ :

$$\begin{array}{ccc}
|A_1| \otimes \mathbb{T}|A_2| & \xrightarrow{\mathbb{k}_{|A_1|, |A_2|}} & \mathbb{T}(|A_1| \otimes |A_2|) \\
\text{id}_{|A_1|} \otimes \sigma_{A_2} \downarrow & & \downarrow \mathbb{T}(\theta_{A_1, A_2}) \\
|A_1| \otimes |\mathbf{T}A_2| & & \mathbb{T}|A_1| \otimes |A_2| \\
\theta_{A_1, \mathbf{T}A_2} \downarrow & & \downarrow \sigma_{A_1 \otimes A_2} \\
|A_1| \otimes \mathbf{T}A_2 & \xrightarrow{|\mathbf{t}_{A_1, A_2}|} & |\mathbf{T}(A_1 \otimes A_2)|
\end{array} \tag{36}$$

Having lifted  $\mathbf{T}$  to scones and subscones in previous sections, we only need to give a lifting of the strength  $\mathbf{t}$ . For scones this is straightforward—define  $\tilde{t}$  pointwise by

$$\tilde{t}_{\langle S_1, m_1, A_1 \rangle, \langle S_2, m_2, A_2 \rangle} = \langle \mathbb{k}_{S_1, S_2}, \mathbf{t}_{A_1, A_2} \rangle.$$

Verifying that this is well-defined amounts to pasting together a naturality square for  $\mathbb{k}$  and a diagram (36):

$$\begin{array}{ccccc}
S_1 \otimes \mathbb{T}S_2 & \xrightarrow{m_1 \otimes \mathbb{T}m_2} & |A_1| \otimes \mathbb{T}|A_2| & \xrightarrow{\theta_{A_1, \mathbf{T}A_2} \circ (\text{id}_{|A_1|} \otimes \sigma_{A_2})} & |A_1| \otimes \mathbf{T}A_2 \\
\downarrow \mathbb{k}_{S_1, S_2} & & \downarrow \mathbb{k}_{|A_1|, |A_2|} & & \downarrow |\mathbf{t}_{A_1, A_2}| \\
\mathbb{T}(S_1 \otimes S_2) & \xrightarrow{\mathbb{T}(m_1 \otimes m_2)} & \mathbb{T}(|A_1| \otimes |A_2|) & \xrightarrow{\sigma_{A_1 \otimes A_2} \circ \mathbb{T}(\theta_{A_1, A_2})} & |\mathbf{T}(A_1 \otimes A_2)|
\end{array}$$

The upper side of this diagram is precisely  $\langle S_1, m_1, A_1 \rangle \tilde{\otimes} \tilde{\mathbb{T}}\langle S_2, m_2, A_2 \rangle$  in the scone while the lower side is  $\tilde{\mathbb{T}}(\langle S_1, m_1, A_1 \rangle \tilde{\otimes} \langle S_2, m_2, A_2 \rangle)$ . (We let the interested reader define for herself the tensor product  $\tilde{\otimes}$  in the scone.) Checking naturality of  $\tilde{t}$  and strength laws is immediate since  $\tilde{t}$ ,  $\tilde{\alpha}$ ,  $\tilde{\tau}$ ,  $\tilde{\eta}$  and  $\tilde{\mu}$  are all defined pointwise.

Now we move to subscones. Call  $\tilde{T}$  the lifted monad defined in (7), (9), (10) and (11) in Section 4.

As in previous sections,  $\tilde{t}$  in subscones will differ from the case of scones only in its  $\mathbb{C}$ -component  $\hat{\mathbb{k}}$ , and this component will be induced as a unique diagonal guaranteed by diagram (6) in the diagram below.

$$\begin{array}{ccccc}
S_1 \otimes \mathbb{T}S_2 & \xrightarrow{\text{id}_{S_1} \otimes \mathbb{T}m_2} & S_1 \otimes \mathbb{T}|A_2| & \xrightarrow{m_1 \otimes \text{id}_{\mathbb{T}|A_2|}} & |A_1| \otimes \mathbb{T}|A_2| \\
\downarrow \mathbb{k}_{S_1, S_2} & \searrow & \downarrow \text{id}_{S_1} \otimes \sigma_{A_2} & \downarrow \text{id}_{|A_1|} \otimes \sigma_{A_2} & \downarrow \mathbb{k}_{|A_1|, |A_2|} \\
\mathbb{T}(S_1 \otimes S_2) & \xrightarrow{\mathbb{T}(m_1 \otimes m_2)} & \mathbb{T}(|A_1| \otimes |A_2|) & & \\
\downarrow \text{id}_{S_1} \otimes e'_2 & & \downarrow \mathbb{T}e_{12} & & \downarrow \mathbb{T}(\theta_{A_1, A_2}) \\
S_1 \otimes \tilde{S}_2 & \xrightarrow{\text{id}_{S_1} \otimes m'_2} & S_1 \otimes |\mathbf{T}A_2| & \xrightarrow{m_1 \otimes \text{id}_{|\mathbf{T}A_2|}} & |A_1| \otimes |\mathbf{T}A_2| \\
\downarrow e'_{12} & \searrow & \downarrow \mathbb{T}m_{12} & \downarrow \theta_{A_1, \mathbf{T}A_2} & \downarrow \mathbb{T}(\theta_{A_1, A_2}) \\
\mathbb{T}S_{12} & \xrightarrow{\mathbb{T}m_{12}} & \mathbb{T}|A_1| \otimes |A_2| & & \\
\downarrow e' & & \downarrow \sigma_{A_1 \otimes A_2} & & \downarrow \sigma_{A_1 \otimes A_2} \\
S'_{12} & \xrightarrow{m'_{12}} & |A_1| \otimes \mathbf{T}A_2 & \xrightarrow{|\mathbf{t}_{A_1, A_2}|} & |\mathbf{T}(A_1 \otimes A_2)| \\
\downarrow \hat{\mathbb{k}} & & \downarrow m' & & \\
\tilde{S}_{12} & \xrightarrow{m'} & |\mathbf{T}(A_1 \otimes A_2)| & & 
\end{array} \tag{37}$$

As ingredients of this diagram we have used:

— An instance  $\mathbb{T}S_2 \xrightarrow{\mathbb{T}m_2} \mathbb{T}|A_2|$  of Diagram (7) defining  $\tilde{T}\langle S_2, m_2, A_2 \rangle$ ; this is tensored

$$\begin{array}{ccc} \mathbb{T}S_2 & \xrightarrow{\mathbb{T}m_2} & \mathbb{T}|A_2| \\ e'_2 \downarrow & & \downarrow \sigma_{A_2} \\ \widetilde{S}_2 & \xrightarrow{m'_2} & |TA_2| \end{array}$$

by  $S_1$  on the left to get the upper left square of the back face. Notice that  $\text{id}_{S_1} \otimes e'_2$  is pseudoepi because our mono factorization system is monoidal.

— An instance  $S_1 \otimes \widetilde{S}_2 \xrightarrow{m_1 \otimes m'_2} |A_1| \otimes |TA_2|$  of Diagram (20) defining the tensor

$$\begin{array}{ccc} S_1 \otimes \widetilde{S}_2 & \xrightarrow{m_1 \otimes m'_2} & |A_1| \otimes |TA_2| \\ e'_{12} \downarrow & & \downarrow \theta_{A_1, TA_2} \\ S'_{12} & \xrightarrow{m'_{12}} & |A_1 \otimes TA_2| \end{array}$$

product of  $\langle S_1, m_1, A_1 \rangle$  with  $\tilde{T}\langle S_2, m_2, A_2 \rangle = \langle \widetilde{S}_2, m'_2, TA_2 \rangle$ ; this is the lower square of the back face. (Note that the upper right square of the back face commutes trivially.)

— Another instance  $S_1 \otimes S_2 \xrightarrow{m_1 \otimes m_2} |A_1| \otimes |A_2|$  of Diagram (20) defining the tensor

$$\begin{array}{ccc} S_1 \otimes S_2 & \xrightarrow{m_1 \otimes m_2} & |A_1| \otimes |A_2| \\ e_{12} \downarrow & & \downarrow \theta_{A_1, A_2} \\ S_{12} & \xrightarrow{m_{12}} & |A_1 \otimes A_2| \end{array}$$

product  $\langle S_{12}, m_{12}, A_1 \otimes A_2 \rangle$  of  $\langle S_1, m_1, A_1 \rangle$  with  $\langle S_2, m_2, A_2 \rangle$ ; we apply  $\mathbb{T}$  to this square to get the upper half of the front face.

— Another instance of Diagram (7) defining the application of  $\tilde{T}$  to the just mentioned tensor product  $\langle S_{12}, m_{12}, A_1 \otimes A_2 \rangle$ : this is the lower half of the front face.

— A naturality square for  $\mathbb{k}$  (top face), and

— An instance of Diagram (36), which defines the right face.

As usual, the dashed and the dotted arrows are given by diagonal fill-ins, therefore  $\tilde{t} = (\hat{\mathbb{k}}, \mathbf{t})$  is well-defined. Again, checking naturality of  $\tilde{t}$  and strength laws is immediate by Fact 2.

Here is the final set of ingredients for lifting a strong monad  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{t})$  on category  $\mathbf{C}$  to  $(\mathbb{C} \Downarrow |_-|)$ :

- 
- (i.a) monoidal categories  $(\mathbf{C}, \otimes, \mathbf{I}, \boldsymbol{\alpha}, \boldsymbol{\ell}, \mathbf{r})$  and  $(\mathbb{C}, \otimes, \mathbb{I}, \boldsymbol{\alpha}, \boldsymbol{\ell}, \mathbf{r})$ , and a monoidal functor  $|_-| : \mathbf{C} \rightarrow \mathbb{C}$ ,
  - (ii.a) a strong monad  $(\mathbb{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbb{k})$  on  $\mathbb{C}$ , related to  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{t})$  by a strong monad morphism  $(|_-|, \sigma)$  defined in (4) and (36),
  - (iii.a) a monoidal mono factorization system on  $\mathbb{C}$ .
  - (iv)  $\mathbb{T}$  maps relevant pseudoepis to pseudoepis.
- 

## 7.2. Monoidal Monads

Several strong monads are in fact monoidal—in fact all the monads of Section 10 are monoidal. While this notion is not needed in Moggi’s account of computation (Moggi, 1991),



this occurs naturally, and will be used in Section 8.4. A *monoidal* monad is a four-tuple  $(\mathbb{T}, \eta, \mu, \mathfrak{d})$ , where  $\mathfrak{d}_{A,B} : \mathbb{T}A \otimes \mathbb{T}B \rightarrow \mathbb{T}(A \otimes B)$  is a *mediator* natural transformation, making the following diagrams commute:

$$\begin{array}{ccc}
\mathbb{I} \otimes \mathbb{T}B & & \mathbb{T}A \otimes \mathbb{I} \\
\eta_{\mathbb{I}} \otimes \text{id}_{\mathbb{T}B} \downarrow & & \text{id}_{\mathbb{T}A} \otimes \eta_{\mathbb{I}} \downarrow \\
\mathbb{T}\mathbb{I} \otimes \mathbb{T}B & \xrightarrow{\mathfrak{l}_{\mathbb{T}B}} & \mathbb{T}A \otimes \mathbb{T}\mathbb{I} \\
\mathfrak{d}_{\mathbb{I},B} \downarrow & & \mathfrak{d}_{A,\mathbb{I}} \downarrow \\
\mathbb{T}(\mathbb{I} \otimes B) & \xrightarrow{\mathfrak{T}_{\mathbb{I}B}} & \mathbb{T}(A \otimes \mathbb{I}) \xrightarrow{\mathfrak{T}_{r_A}} \mathbb{T}A
\end{array} \quad (38) \qquad (39)$$

$$\begin{array}{ccc}
A \otimes B & \xrightarrow{\eta_A \otimes \eta_B} & \mathbb{T}A \otimes \mathbb{T}B \\
& \searrow \eta_{A \otimes B} & \downarrow \mathfrak{d}_{A,B} \\
& & \mathbb{T}(A \otimes B)
\end{array} \quad (40)$$

$$\begin{array}{ccc}
(\mathbb{T}A \otimes \mathbb{T}B) \otimes \mathbb{T}C & \xrightarrow{\mathfrak{d}_{A,B} \otimes \text{id}_{\mathbb{T}C}} & \mathbb{T}(A \otimes B) \otimes \mathbb{T}C \xrightarrow{\mathfrak{d}_{A \otimes B, C}} \mathbb{T}((A \otimes B) \otimes C) \\
\downarrow \alpha_{\mathbb{T}A, \mathbb{T}B, \mathbb{T}C} & & \downarrow \mathbb{T}\alpha_{A, B, C} \\
\mathbb{T}A \otimes (\mathbb{T}B \otimes \mathbb{T}C) & \xrightarrow{\text{id}_{\mathbb{T}A} \otimes \mathfrak{d}_{B,C}} & \mathbb{T}A \otimes \mathbb{T}(B \otimes C) \xrightarrow{\mathfrak{d}_{A, B \otimes C}} \mathbb{T}(A \otimes (B \otimes C))
\end{array} \quad (41)$$

$$\begin{array}{ccc}
\mathbb{T}^2 A \otimes \mathbb{T}^2 B & \xrightarrow{\mathfrak{d}_{\mathbb{T}A, \mathbb{T}B}} & \mathbb{T}(\mathbb{T}A \otimes \mathbb{T}B) \xrightarrow{\mathbb{T}\mathfrak{d}_{A,B}} \mathbb{T}^2(A \otimes B) \\
\downarrow \mu_A \otimes \mu_B & & \downarrow \mu_{A \otimes B} \\
\mathbb{T}A \otimes \mathbb{T}B & \xrightarrow{\mathfrak{d}_{A,B}} & \mathbb{T}(A \otimes B)
\end{array} \quad (42)$$

Diagrams (38), (39), (41) state that  $\mathbb{T}$  is a monoidal functor with mediating pair  $(\mathfrak{d}, \eta_{\mathbb{I}})$ . Diagram (40) states that  $\eta$  is a so-called monoidal natural transformation, while Diagram (42) states that  $\mu$  is another monoidal natural transformation.

Given any monoidal monad  $(\mathbb{T}, \eta, \mu, \mathfrak{d})$  on  $\mathbb{C}$ , it is easy to check that  $(\mathbb{T}, \eta, \mu, \mathfrak{l})$  is a strong monad, where  $\mathfrak{l}_{A,B} = \mathfrak{d}_{A,B} \circ (\eta_A \otimes \text{id}_{\mathbb{T}B})$ . Furthermore,  $\mathfrak{l}'_{A,B}$  defined as  $\mathfrak{d}_{A,B} \circ (\text{id}_{\mathbb{T}A} \otimes \eta_B)$  is a *dual strength*, that is, a natural transformation  $\mathfrak{l}'_{A,B} : \mathbb{T}A \otimes B \rightarrow \mathbb{T}(A \otimes B)$  obeying the obvious duals of the strength laws (32), (33), (34), (35). (Formally, a dual strength is a strength on the dual monoidal category  $(\mathbb{C}, \mathbb{I}, \otimes^{op}, \alpha^{-1}, \mathfrak{r}, \mathbb{I})$ , where  $A \otimes^{op} B$  is defined as  $B \otimes A$ .)

Moreover, the strength  $\mathfrak{l}$  and the dual strength  $\mathfrak{l}'$  are compatible with the associativity, in the sense that the diagram below commutes.

$$\begin{array}{ccc}
(A \otimes \mathbb{T}B) \otimes C & \xrightarrow{\mathfrak{l}_{A,B} \otimes \text{id}_C} & \mathbb{T}(A \otimes B) \otimes C \xrightarrow{\mathfrak{l}'_{A \otimes B, C}} \mathbb{T}((A \otimes B) \otimes C) \\
\downarrow \alpha_{A, \mathbb{T}B, C} & & \downarrow \mathbb{T}\alpha_{A, B, C} \\
A \otimes (\mathbb{T}B \otimes C) & \xrightarrow{\text{id}_A \otimes \mathfrak{l}'_{B,C}} & A \otimes \mathbb{T}(B \otimes C) \xrightarrow{\mathfrak{l}_{A, B \otimes C}} \mathbb{T}(A \otimes (B \otimes C))
\end{array} \quad (43)$$

Finally, the strength and the dual strength commute, in the sense that the diagram on the right commutes. In fact, the common diagonal from  $\mathbb{T}A \otimes \mathbb{T}B$  to  $\mathbb{T}(A \otimes B)$  is just  $\mathfrak{d}_{A,B}$ .

$$\begin{array}{ccc}
\mathbb{T}A \otimes \mathbb{T}B & \xrightarrow{\mathfrak{t}_{\mathbb{T}A,B}} & \mathbb{T}(\mathbb{T}A \otimes B) \\
\downarrow \mathfrak{t}'_{A,\mathbb{T}B} & & \downarrow \mathbb{T}\mathfrak{t}'_{A,B} \\
& & \mathbb{T}^2(A \otimes B) \\
& & \downarrow \mathfrak{p}_{A \otimes B} \\
\mathbb{T}(A \otimes \mathbb{T}B) & \xrightarrow{\mathbb{T}\mathfrak{t}_{A,B}} \mathbb{T}^2(A \otimes B) \xrightarrow{\mathfrak{p}_{A \otimes B}} & \mathbb{T}(A \otimes B)
\end{array} \quad (44)$$

In general, a monoidal monad can be defined equivalently as a monad with a strength and a dual strength that make the diagrams (43) and (44) commute. See Appendix A, in particular Appendix A.1 and Appendix A.2, for a proof.

It is natural to define a *monoidal monad morphism* from  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{d})$  to  $(\mathbb{T}, \eta, \mu, \mathfrak{d})$  as a monad morphism  $\sigma$  from  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu})$  to  $(\mathbb{T}, \eta, \mu)$  making the following diagram commute:

$$\begin{array}{ccc}
\mathbb{T}|A_1| \otimes \mathbb{T}|A_2| & \xrightarrow{\mathfrak{d}_{|A_1|,|A_2|}} & \mathbb{T}(|A_1| \otimes |A_2|) \\
\sigma_{A_1} \otimes \sigma_{A_2} \downarrow & & \downarrow \mathbb{T}(\theta_{A_1, A_2}) \\
|\mathbf{T}A_1| \otimes |\mathbf{T}A_2| & & \mathbb{T}|A_1 \otimes A_2| \\
\theta_{\mathbf{T}A_1, \mathbf{T}A_2} \downarrow & & \downarrow \sigma_{A_1 \otimes A_2} \\
|\mathbf{T}A_1 \otimes \mathbf{T}A_2| & \xrightarrow{|\mathfrak{d}_{A_1, A_2}|} & |\mathbf{T}(A_1 \otimes A_2)|
\end{array} \quad (45)$$

Every monoidal monad morphism  $\sigma$  is also a strong monad morphism from  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{t})$  to  $(\mathbb{T}, \eta, \mu, \mathfrak{t})$ , and also from  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{t}')$  to  $(\mathbb{T}, \eta, \mu, \mathfrak{t}')$ , where  $\mathbf{t}_{A,B} = \mathbf{d}_{A,B} \circ (\boldsymbol{\eta}_A \otimes \text{id}_{\mathbf{T}B})$ ,  $\mathfrak{t}_{A,B} = \mathfrak{d}_{A,B} \circ (\eta_A \otimes \text{id}_{\mathbb{T}B})$ ,  $\mathbf{t}'_{A,B} = \mathbf{d}_{A,B} \circ (\text{id}_{\mathbf{T}A} \otimes \boldsymbol{\eta}_B)$ ,  $\mathfrak{t}'_{A,B} = \mathfrak{d}_{A,B} \circ (\text{id}_{\mathbb{T}A} \otimes \eta_B)$ . In Appendix A.3, we show that the monoidal monad morphisms are exactly the natural transformations  $\sigma$  that are both a strong monad morphism and a dual strong monad morphism.

One may think that lifting monoidal monads to scones and subscones is easy: get the strength and the dual strength from the mediator, and lift them as in Section 7.1. However, it is not immediately clear that what one would get would arise as strength and dual strengths of a mediator, in particular that (44) would commute. Let us therefore state the construction explicitly. This mimicks the construction of the lifted strength from Section 7.1. For scones, the lifted mediator is again defined pointwise by

$$\tilde{\mathfrak{d}}_{\langle S_1, m_1, A_1 \rangle, \langle S_2, m_2, A_2 \rangle} = \langle \mathbf{d}_{S_1, S_2}, \mathfrak{d}_{A_1, A_2} \rangle$$

For subscones, we mimick Diagram (37) in Diagram (46) below. We let the reader check

all commutations.

$$\begin{array}{ccccc}
\mathbb{T}S_1 \otimes \mathbb{T}S_2 & \xrightarrow{\mathbb{T}m_1 \otimes \mathbb{T}m_2} & \mathbb{T}|A_1| \otimes \mathbb{T}|A_2| & \xrightarrow{\mathbb{T}d_{|A_1|, |A_2|}} & \mathbb{T}(|A_1| \otimes |A_2|) \\
\downarrow e'_1 \otimes e'_2 & \searrow d_{S_1, S_2} & \downarrow \sigma_{A_1} \otimes \sigma_{A_2} & & \downarrow \sigma_{|A_1|, |A_2|} \\
\mathbb{T}(S_1 \otimes S_2) & \xrightarrow{\mathbb{T}(m_1 \otimes m_2)} & \mathbb{T}(|A_1| \otimes |A_2|) & \xrightarrow{\mathbb{T}(\theta_{A_1, A_2})} & \mathbb{T}(|A_1| \otimes |A_2|) \\
\downarrow \mathbb{T}e_{12} & \xrightarrow{m'_1 \otimes m'_2} & \downarrow \theta_{\mathbf{T}A_1, \mathbf{T}A_2} & & \downarrow \theta_{A_1, A_2} \\
\widetilde{S}_1 \otimes \widetilde{S}_2 & \xrightarrow{\mathbb{T}e_{12}} & |\mathbf{T}A_1| \otimes |\mathbf{T}A_2| & \xrightarrow{\mathbb{T}m_{12}} & \mathbb{T}|A_1| \otimes |A_2| \\
\downarrow e'_{12} & \searrow e' & \downarrow \sigma_{A_1} \otimes \sigma_{A_2} & & \downarrow \sigma_{|A_1|, |A_2|} \\
S'_{12} \subset \widetilde{S}_{12} & \xrightarrow{m'_{12}} & |\mathbf{T}A_1| \otimes |\mathbf{T}A_2| & \xrightarrow{\mathbb{T}d_{A_1, A_2}} & \mathbb{T}|A_1| \otimes |A_2| \\
\downarrow \widehat{d} & \searrow m' & \downarrow \sigma_{A_1} \otimes \sigma_{A_2} & & \downarrow \sigma_{|A_1|, |A_2|} \\
\widetilde{S}_{12} \subset S_{12} & \xrightarrow{m'} & |\mathbf{T}(A_1 \otimes A_2)| & \xrightarrow{\sigma_{A_1 \otimes A_2}} & \mathbb{T}|A_1 \otimes A_2|
\end{array} \tag{46}$$

So, to lift a monoidal monad  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{d})$  on category  $\mathbf{C}$  to  $(\mathbb{C} \downarrow |_{-})$ , we require:

- 
- (i.a) monoidal categories  $(\mathbf{C}, \otimes, \mathbf{I}, \boldsymbol{\alpha}, \boldsymbol{\ell}, \boldsymbol{r})$  and  $(\mathbb{C}, \otimes, \mathbb{I}, \boldsymbol{\alpha}, \boldsymbol{\ell}, \boldsymbol{r})$ , and a monoidal functor  $|_{-} : \mathbf{C} \rightarrow \mathbb{C}$ ,
  - (ii.b) a monoidal monad  $(\mathbb{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{d})$  on  $\mathbb{C}$ , related to the monoidal monad  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{d})$  on  $\mathbf{C}$  by a monoidal monad morphism  $(|_{-}, \sigma)$  defined in (4) and (45),
  - (iii.a) a monoidal mono factorization system on  $\mathbb{C}$ .
  - (iv)  $\mathbb{T}$  maps relevant pseudoepis to pseudoepis.
- 

### 7.3. Commutative Monads

In the case of *symmetric* monoidal categories  $\mathbf{C}$  and  $\mathbb{C}$ , recall that if we also want to make the subscone a symmetric monoidal category, it suffices to replace (i.a) by (i.b), which requires  $|_{-}$  to be a *symmetric* monoidal functor. This case occurs notably if we want to lift a *commutative* monad to the subscone.

Recall that a strong monad  $(\mathbb{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{\ell})$  is commutative if and only if, letting  $\mathbb{t}'_{A,B}$  be the dual strength  $\mathbb{T}\mathbb{c}_{B,A} \circ \mathbb{t}_{B,A} \circ \mathbb{c}_{\mathbb{T}A,B}$ , then Diagram (44) commutes.

Let  $\mathbb{d}_{A,B}$  be the common diagonal  $\mathbb{m}_{A \otimes B} \circ$

$\mathbb{T}\mathbb{t}'_{A,B} \circ \mathbb{t}_{\mathbb{T}A,B} = \mathbb{m}_{A \otimes B} \circ \mathbb{T}\mathbb{t}_{A,B} \circ \mathbb{t}'_{A, \mathbb{T}B}$ .

We can check that  $\mathbb{d}$  is then a mediator, whence every commutative monad is monoidal. In fact, a monoidal monad is commutative if and only if the following additional diagram commutes (see Appendix A.4).

$$\begin{array}{ccc}
\mathbb{T}A \otimes \mathbb{T}B & \xrightarrow{\mathbb{d}_{A,B}} & \mathbb{T}(A \otimes B) \\
\mathbb{c}_{\mathbb{T}A, \mathbb{T}B} \downarrow & & \downarrow \mathbb{T}\mathbb{c}_{A,B} \\
\mathbb{T}B \otimes \mathbb{T}A & \xrightarrow{\mathbb{d}_{B,A}} & \mathbb{T}(B \otimes A)
\end{array} \tag{47}$$

For convenience, we shall now understand commutative monads as monoidal monads satisfying (47). The lifting of monoidal monads of Section 7.2 then yields a lifting of

commutative monads, by Fact 2. Therefore, to lift a commutative monad  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{t})$  on category  $\mathbf{C}$  to  $(\mathbb{C} \Downarrow |\_|)$ , we require:

- 
- (i.b) symmetric monoidal categories  $(\mathbf{C}, \otimes, \mathbf{I}, \boldsymbol{\alpha}, \boldsymbol{\ell}, \boldsymbol{r}, \mathbf{c})$  and  $(\mathbb{C}, \otimes, \mathbb{I}, \boldsymbol{\alpha}, \boldsymbol{\ell}, \boldsymbol{r}, \mathbf{c})$ , and a symmetric monoidal functor  $|\_| : \mathbf{C} \rightarrow \mathbb{C}$ ,
  - (ii.c) a commutative monad  $(\mathbb{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{d})$  on  $\mathbb{C}$ , related to the commutative monad  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{d})$  on  $\mathbf{C}$  by a monoidal monad morphism  $(|\_|, \sigma)$  defined in (4) and (45),
  - (iii.a) a monoidal mono factorization system on  $\mathbb{C}$ .
  - (iv)  $\mathbb{T}$  maps relevant pseudoepis to pseudoepis.
- 

When  $\mathbb{C}$  has all finite products and we consider the induced symmetric monoidal structure, we might require that the monoidal monad  $(\mathbb{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{d})$  is not just commutative but even *cartesian*, by which we mean that  $\mathbb{T}$  is a cartesian monoidal functor with mediating pair  $(\mathbf{d}, \boldsymbol{\eta}_{\mathbb{I}})$ . (Recall that  $\mathbb{T}$  is always a monoidal functor with this very mediating pair.) This means that  $\mathbb{T}\pi_i \circ \mathbf{d}_{A,B} = \pi_i$ ,  $i \in \{1, 2\}$ . We just do not need this in our constructions; but it is often easier to prove that a monad is cartesian and infer that it is commutative than to prove that it is commutative directly: we shall see examples of cartesian monads in Section 10.

## 8. Building Monad Morphisms from Adjunctions

It is often the case that we have a (strong) monad on  $\mathbf{C}$ , and wish to build another one on  $\mathbb{C}$  related to the latter by a monad morphism. The following results are then of some help.

Recall that, given two categories  $\mathcal{C}$  and  $\mathcal{D}$ , a pair of functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $U : \mathcal{D} \rightarrow \mathcal{C}$  is an *adjunction*  $F \dashv U$  if and only if there are natural transformations  $\eta : \cdot \rightarrow UF$ . (the *unit* of the adjunction) and  $\epsilon : FU \rightarrow \cdot$ . (the *counit*) such that  $\epsilon_{F(A)} \circ F\eta_A = \text{id}_{F(A)}$  and  $U\epsilon_A \circ \eta_{U(A)} = \text{id}_{U(A)}$ .  $F$  is said to be *left adjoint* to  $U$ ,  $U$  is *right adjoint* to  $F$ .

Then any adjunction  $F \dashv U$  gives rise to a monad  $(UF, \boldsymbol{\eta}, U\epsilon_F)$  on  $\mathcal{C}$ . Conversely, there are two standard ways of retrieving an adjunction from a monad  $(T, \boldsymbol{\eta}, \boldsymbol{\mu})$  on  $\mathcal{C}$ , from Eilenberg-Moore algebras, or from the Kleisli category of the monad.

### 8.1. Eilenberg-Moore algebras.

A *T-algebra* is a morphism  $T(A) \xrightarrow{s} A$ , for some object  $A$  of  $\mathcal{C}$ , satisfying the commutativity conditions:

$$\begin{array}{ccc}
 & & T^2(A) \\
 & \swarrow \mu_A & \downarrow T(s) \\
 & T(A) & T(A) \\
 \eta_A \nearrow & & \downarrow s \\
 A & \xrightarrow{\text{id}_A} & A \\
 & & \downarrow s \\
 & & A \xlongequal{\quad} A
 \end{array} \tag{48}$$

$A$  is called the *support* of the algebra,  $s$  its *structure map*. A morphism from  $T(A) \xrightarrow{s} A$  to  $T(B) \xrightarrow{u} B$  is a morphism  $f : A \rightarrow B$  in  $\mathbb{C}$  that commutes with structure maps, i.e., such that  $f \circ s = u \circ T(f)$ .  $T$ -algebras together with these morphisms forms a category  $T\text{-Alg}$ .

Then  $F^T \dashv U^T$  is an adjunction, where  $U^T : T\text{-Alg} \rightarrow \mathcal{C}$  maps objects  $T(A) \xrightarrow{s} A$  to  $A$  and morphisms  $f$  from  $T(A) \xrightarrow{s} A$  to  $T(B) \xrightarrow{u} B$  to the underlying morphism  $f$  from  $A$  to  $B$  in  $\mathcal{C}$ ; and where  $F^T : \mathcal{C} \rightarrow T\text{-Alg}$  maps the object  $A$  to the  $T$ -algebra  $T^2(A) \xrightarrow{\mu_A} T(A)$ , and the morphisms  $A \xrightarrow{f} B$  to  $f$  seen as morphism from  $F^T(A)$  to  $F^T(B)$ . The unit of the adjunction is  $\eta$ , while the counit  $\epsilon$  is given on each  $T$ -algebra  $T(A) \xrightarrow{s} A$  as the morphism  $s$  itself, from  $F^T U^T ( T(A) \xrightarrow{s} A ) = T^2(A) \xrightarrow{\mu_A} T(A)$  to  $T(A) \xrightarrow{s} A$ .

Moreover, the monad of this adjunction is the original monad  $(T, \eta, \mu)$ .

## 8.2. Kleisli category.

The objects of  $\mathbf{Kleisli}(T)$  are the objects of  $\mathcal{C}$ , while the morphisms  $A \xrightarrow{f} B$  of  $\mathbf{Kleisli}(T)$  are the morphisms  $A \xrightarrow{f} T(B)$  of  $\mathcal{C}$ . To avoid confusion, we write  $\bar{f}$  the morphisms  $f$  seen as a morphism in  $\mathbf{Kleisli}(T)$ . The identity  $\overline{\text{id}_A}$  on the object  $A$  in  $\mathbf{Kleisli}(T)$  is  $\eta_A$ , while composition  $\bar{g} \circ \bar{f}$  is  $\overline{\mu_C \circ T(g) \circ f}$ , where  $A \xrightarrow{f} T(B)$  and  $B \xrightarrow{g} T(C)$  in  $\mathcal{C}$ .

Define  $F_T : \mathcal{C} \rightarrow \mathbf{Kleisli}(T)$  as mapping the object  $A$  to  $A$ , and the morphism  $A \xrightarrow{f} B$  to the morphism from  $A$  to  $B$  in  $\mathbf{Kleisli}(T)$  defined as  $\eta_B \circ f$ . Define  $U_T : \mathbf{Kleisli}(T) \rightarrow \mathcal{C}$  as mapping the object  $A$  to  $T(A)$ , and the morphism  $\bar{f}$  from  $A$  to  $B$  in  $\mathbf{Kleisli}(T)$  to  $\mu_B \circ T(f)$ . Then  $F_T \dashv U_T$  is an adjunction, whose unit is  $\eta$ , and whose counit  $\epsilon$  is the identity morphism from  $F_T U_T(A)$  to  $T(A)$  in  $\mathcal{C}$ , seen as a morphism from  $F_T U_T(A)$  to  $A$  in  $\mathbf{Kleisli}(T)$ . The monad of  $F_T \dashv U_T$  is again  $(T, \eta, \mu)$ .

## 8.3. Monad Morphisms from Adjunctions

**Proposition 1.** Let  $(\mathbf{T}, \eta, \mu)$  be a monad on a category  $\mathcal{C}$ ,  $|\_ : \mathcal{C} \rightarrow \mathbb{C}$  be a functor with a left adjoint  $\mathbf{D} : \mathbb{C} \rightarrow \mathcal{C}$ . Let  $\epsilon_A : \mathbf{D}|A| \rightarrow A$  be the counit of the adjunction,  $\eta_E : E \rightarrow |\mathbf{D}(E)|$  be the unit of the adjunction.

Define  $\mathbb{T} = |\_ \circ \mathbf{T} \circ \mathbf{D} = |\mathbf{T}\mathbf{D}|$ ,  $\eta_E = |\eta_{\mathbf{D}(E)}| \circ \dot{\eta}_E$ ,  $\mu_E = |\mu_{\mathbf{D}(E)} \circ \mathbf{T}\epsilon_{\mathbf{T}\mathbf{D}(E)}|$ . Finally, let  $\sigma_A = |\mathbf{T}\epsilon_A| : \mathbb{T}|A| \rightarrow |\mathbf{T}A|$ . Then  $(\mathbb{T}, \eta, \mu)$  is a monad on  $\mathbb{C}$  and  $(|\_, \sigma)$  is a monad morphism from  $\mathbf{T}$  to  $\mathbb{T}$ .

*Proof.* Let  $F \dashv U$  be any adjunction generating the monad, i.e., such that  $UF = \mathbf{T}$ , whose unit is  $\eta$ , and whose counit  $\epsilon$  is such that  $\mu = U\epsilon_F$ . We may choose, e.g.,  $F^T \dashv U^T$  or  $F_T \dashv U_T$ . Compose the adjunction  $\mathbf{D} \dashv |\_$  with  $F \dashv U$ , yielding an adjunction

$FD \dashv |U|$ . The unit of this adjunction (on object  $E$ ) is  $|\eta_{D(E)}| \circ \dot{\eta}_E$ , its counit (on object  $A$ ) is  $\epsilon_A \circ F(\dot{\epsilon}_{UA})$ .

The monad of this adjunction is  $(|UFD|, |\eta_{D(\cdot)}| \circ \dot{\eta}, |U(\epsilon_{FD(\cdot)} \circ F(\dot{\epsilon}_{UFD(\cdot)}))|)$ . But the monad of  $F \dashv U$  is  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu})$ , so  $UF = \mathbf{T}$  and  $\boldsymbol{\mu} = U\epsilon_F$ . It follows that the monad of  $FD \dashv |U|$  is  $(|\mathbf{T}D|, |\eta_{D(\cdot)}| \circ \dot{\eta}, |\boldsymbol{\mu}_{D(\cdot)} \circ \mathbf{T}(\dot{\epsilon}_{\mathbf{T}D(\cdot)})|)$ . This is  $(\mathbb{T}, \eta, \mu)$ , which is therefore a monad.

It remains to show that  $\sigma = |\mathbf{T}\dot{\epsilon}|$  is a monad morphism from  $\mathbf{T}$  to  $\mathbb{T}$ . This is checked using the following diagrams. In the left diagram, the top triangle is one of the adjunction laws, the bottom square is by naturality of  $|\eta|$ ; so  $\sigma_A$  (bottom arrow) composed with  $\eta_{|A|}$  (leftmost vertical path) equals  $|\eta_A|$  (rightmost path from  $|A|$  to  $|\mathbf{T}A|$ ).

In the right diagram, the top square is by naturality of  $|\mathbf{T}\dot{\epsilon}|$ , the bottom square is by naturality of  $|\boldsymbol{\mu}|$ , so  $\sigma_A$  (bottom arrow) composed with  $\mu_{|A|}$  (leftmost vertical path) equals  $|\boldsymbol{\mu}_A| \circ \sigma_{\mathbf{T}A} \circ \mathbb{T}\sigma_A$  (the other path from top left to bottom right).  $\square$

$$\begin{array}{ccc}
|A| & & |\mathbf{T}D|\mathbf{T}D|A| \xrightarrow{|\mathbf{T}D|\mathbf{T}\dot{\epsilon}_A|} |\mathbf{T}D|\mathbf{T}A| \\
\dot{\eta}_{|A|} \downarrow & \searrow \text{id}_{|A|} & |\mathbf{T}\dot{\epsilon}_{\mathbf{T}D|A|} \downarrow & & |\mathbf{T}\dot{\epsilon}_{\mathbf{T}A|} \downarrow \\
|D|A| & \xrightarrow{|\epsilon_A|} & |A| & & |\mathbf{T}D|A| \xrightarrow{|\mathbf{T}\mathbf{T}\dot{\epsilon}_A|} |\mathbf{T}A| \\
|\eta_{D|A|} \downarrow & & |\eta_A| \downarrow & & |\boldsymbol{\mu}_{D|A|} \downarrow & & |\boldsymbol{\mu}_A| \downarrow \\
|\mathbf{T}D|A| & \xrightarrow{|\mathbf{T}\dot{\epsilon}_A|} & |\mathbf{T}A| & & |\mathbf{T}D|A| & \xrightarrow{|\mathbf{T}\dot{\epsilon}_A|} & |\mathbf{T}A|
\end{array}$$

Note that we could have checked the required diagrams directly; the proof would be longer than going through adjunctions, as we did.

#### 8.4. Monoidal, and Strong Monad Morphisms from Monoidal Adjunctions

We first reproduce the argument of Proposition 1 in the monoidal case. While monads correspond to adjunctions in well-defined ways, only *monoidal* monads can be linked to so-called *monoidal* adjunctions. This is the reason why we deal with monoidal monads first.

Let  $(\mathcal{C}, I^{\mathcal{C}}, \otimes^{\mathcal{C}}, \alpha^{\mathcal{C}}, \ell^{\mathcal{C}}, r^{\mathcal{C}})$  and  $(\mathcal{D}, I^{\mathcal{D}}, \otimes^{\mathcal{D}}, \alpha^{\mathcal{D}}, \ell^{\mathcal{D}}, r^{\mathcal{D}})$  be two monoidal categories. Let  $F \dashv U$  be an adjunction, where  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $U : \mathcal{D} \rightarrow \mathcal{C}$ , with unit  $\eta$ , and counit  $\epsilon$ . This is a *monoidal adjunction* if and only if  $F$  and  $U$  are monoidal functors (with respective mediating pairs  $(\theta^F, \iota^F)$  and  $(\theta^U, \iota^U)$ ), and the unit  $\eta$  and the counit  $\epsilon$  are *monoidal natural transformations*, by which we mean that the following diagrams commute:

$$\begin{array}{ccc}
I^{\mathcal{C}} & \xrightarrow{\iota^U} & U(I^{\mathcal{D}}) \\
\eta_{I^{\mathcal{C}}} \searrow & & \downarrow U(\iota^F) \\
& & UF(I^{\mathcal{C}})
\end{array} \quad (49)$$

$$\begin{array}{ccc}
A \otimes^{\mathcal{C}} B & \xrightarrow{\eta_A \otimes^{\mathcal{C}} \eta_B} & UF(A) \otimes^{\mathcal{C}} UF(B) \\
\eta_{A \otimes^{\mathcal{C}} B} \downarrow & & \swarrow \theta_{F(A), F(B)}^U \\
& & U(F(A) \otimes^{\mathcal{D}} F(B)) \\
& & \swarrow U(\theta_{A, B}^F) \\
UF(A \otimes^{\mathcal{C}} B) & & 
\end{array} \quad (50)$$

$$\begin{array}{ccc}
I^{\mathcal{D}} & & FU(A) \otimes^{\mathcal{D}} FU(B) \quad (52) \\
\downarrow \iota^F & \swarrow \epsilon_{I^{\mathcal{D}}} & \downarrow \theta_{U(A), U(B)}^F \\
F(I^{\mathcal{C}}) & \xrightarrow{F(\iota^U)} & F(U(A) \otimes^{\mathcal{C}} U(B)) \\
& & \downarrow \epsilon_{A \otimes^{\mathcal{D}} B} \\
& & A \otimes^{\mathcal{D}} B \\
& & \downarrow \epsilon_{A \otimes^{\mathcal{D}} B} \\
& & FU(A \otimes^{\mathcal{D}} B) \xrightarrow{\epsilon_{A \otimes^{\mathcal{D}} B}} A \otimes^{\mathcal{D}} B
\end{array}$$

The value of monoidal adjunctions is their relation with monoidal monads (Section 7.2). Recall that a monoidal monad on  $\mathcal{C}$  is a tuple  $(T, \eta, \mu, d)$ , where  $(T, \eta, \mu)$  is a monad on  $\mathcal{C}$  and  $d_{A,B}$  is a natural transformation from  $TA \otimes^{\mathcal{C}} TB$  to  $T(A \otimes^{\mathcal{C}} B)$  such that the following diagrams commute:

$$\begin{array}{ccc}
I^{\mathcal{C}} \otimes^{\mathcal{C}} TB & & TA \otimes^{\mathcal{C}} I^{\mathcal{C}} \quad (39) \\
\downarrow \eta_{I^{\mathcal{C}}} \otimes^{\mathcal{C}} \text{id}_{TB} & \searrow \ell_{TB}^{\mathcal{C}} & \downarrow \text{id}_{TA} \otimes^{\mathcal{C}} \eta_{I^{\mathcal{C}}} \\
TI^{\mathcal{C}} \otimes^{\mathcal{C}} TB & & TA \otimes^{\mathcal{C}} TI^{\mathcal{C}} \\
\downarrow d_{I^{\mathcal{C}}, B} & & \downarrow d_{A, I^{\mathcal{C}}} \\
T(I^{\mathcal{C}} \otimes^{\mathcal{C}} B) & \xrightarrow{T\ell_B^{\mathcal{C}}} & TB \\
& & \downarrow \text{Tr}_A^{\mathcal{C}} \\
& & T(A \otimes^{\mathcal{C}} I^{\mathcal{C}}) \xrightarrow{\text{Tr}_A^{\mathcal{C}}} TA
\end{array}$$

$$\begin{array}{ccc}
A \otimes^{\mathcal{C}} B & \xrightarrow{\eta_A \otimes^{\mathcal{C}} \eta_B} & TA \otimes^{\mathcal{C}} TB \quad (40) \\
& \searrow \eta_{A \otimes^{\mathcal{C}} B} & \downarrow d_{A,B} \\
& & T(A \otimes^{\mathcal{C}} B)
\end{array}$$

$$\begin{array}{ccc}
(TA \otimes^{\mathcal{C}} TB) \otimes^{\mathcal{C}} TC & \xrightarrow{d_{A,B} \otimes^{\mathcal{C}} \text{id}_{TC}} & T(A \otimes^{\mathcal{C}} B) \otimes^{\mathcal{C}} TC \xrightarrow{d_{A \otimes^{\mathcal{C}} B, C}} T((A \otimes^{\mathcal{C}} B) \otimes^{\mathcal{C}} C) \quad (41) \\
\downarrow \alpha_{TA, TB, TC}^{\mathcal{C}} & & \downarrow T\alpha_{A, B, C}^{\mathcal{C}} \\
TA \otimes^{\mathcal{C}} (TB \otimes^{\mathcal{C}} TC) & \xrightarrow{\text{id}_{TA} \otimes^{\mathcal{C}} d_{B, C}} & TA \otimes^{\mathcal{C}} T(B \otimes^{\mathcal{C}} C) \xrightarrow{d_{A, B \otimes^{\mathcal{C}} C}} T(A \otimes^{\mathcal{C}} (B \otimes^{\mathcal{C}} C))
\end{array}$$

$$\begin{array}{ccc}
T^2 A \otimes^{\mathcal{C}} T^2 B & \xrightarrow{d_{TA, TB}} & T(TA \otimes^{\mathcal{C}} TB) \xrightarrow{Td_{A, B}} & T^2(A \otimes^{\mathcal{C}} B) \quad (42) \\
\downarrow \mu_A \otimes^{\mathcal{C}} \mu_B & & & \downarrow \mu_{A \otimes^{\mathcal{C}} B} \\
TA \otimes^{\mathcal{C}} TB & \xrightarrow{\quad \quad \quad} & & T(A \otimes^{\mathcal{C}} B)
\end{array}$$

The following lemmas show respectively that every monoidal adjunction gives rise to a monoidal monad, that every monoidal monad yields a monoidal adjunction between the base category and the Kleisli category of the monad, and that monoidal adjunctions compose to yield monoidal adjunctions. Except for the first, the arguments are tedious computations, and therefore relegated to appendices<sup>†</sup>.

<sup>†</sup> These results are folklore, and were confirmed in discussions with Paul-André Mellès, François Lamarche, and Albert Burroni notably. However, we have been unable to find references on this. We have often been directed to the pioneering paper (Street, 1972), but could not find the expected results therein.

**Lemma 1.** Let  $F \dashv U$  be a monoidal adjunction, with unit  $\eta$  and counit  $\epsilon$ , where  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $U : \mathcal{D} \rightarrow \mathcal{C}$ . Let  $T$  be  $UF$ ,  $\mu_A$  be  $U\epsilon_{F(A)}$ , and  $d_{A,B}$  be  $U\theta_{A,B}^F \circ \theta_{F(A),F(B)}^U$ . Then  $(T, \eta, \mu, d)$  is a monoidal monad on  $\mathcal{C}$ .

*Proof.*  $(T, \eta, \mu)$  is a monad by Proposition 1. We check the mediator laws (38), (39), (40), (41), (42) for  $d$ .

Diagram (38) is obtained by considering the following diagram. The top left triangle is a copy of (49), tensored by  $UF(B)$  on the right. The square next to it on its right is a naturality square for  $\theta^U$ . The next trapezoid on the right (the top right trapezoid) is  $U$  applied to a coherence square (17) for  $\theta^F$ ,  $\ell^C$  and  $\ell^D$ . The bottom face, atop the curved arrow  $\ell_{UF(B)}^C$ , is another instance of a coherence square (17) for  $\theta^F$ ,  $\ell^C$  and  $\ell^D$ .

$$\begin{array}{ccccc}
I^C \otimes^C UF(B) & \xrightarrow{\eta_{I^C} \otimes^C \text{id}_{UF(B)}} & UF(I^C) \otimes^C UF(B) & \xrightarrow{\theta_{F(I^C), F(B)}^U} & U(F(I^C) \otimes^D F(B)) & \xrightarrow{U(\theta_{I^C, B}^F)} & UF(I^C \otimes^C B) \\
& \searrow \scriptstyle U \otimes^C \text{id}_{UF(B)} & \uparrow \scriptstyle U(\iota^F) \otimes^C \text{id}_{UF(B)} & & \uparrow \scriptstyle U(\iota^F \otimes^D \text{id}_{F(B)}) & & \downarrow \scriptstyle UF(\ell_B^C) \\
& & U(I^D) \otimes^C UF(B) & \xrightarrow[\theta_{I^D, F(B)}^U]{} & U(I^D \otimes^D F(B)) & & \downarrow \scriptstyle U(\ell_{F(B)}^D) \\
& & & \searrow \scriptstyle \ell_{UF(B)}^C & & \searrow \scriptstyle U(\ell_{F(B)}^D) & \downarrow \scriptstyle UF(\ell_B^C) \\
& & & & & & UF(B)
\end{array}$$

Now the topmost composition of arrows is  $t_{I^C, B}$ , the bottommost arrow from  $I^C \otimes^C UF(B)$  to  $UF(B)$  is  $\ell_{TB}^C$ , and the rightmost vertical arrow is  $T\ell_B^C$ .

Diagram (39). This is checked by similar arguments, replacing the coherence square (17) by (18).

Diagram (40). This is the diagram on the right, an instance of Diagram (50), stating that  $\eta$  is a monoidal natural transformation. We recognize  $d_{A,B}$  as the rightmost composition of vertical arrows, hence the desired Diagram (33).

$$\begin{array}{ccc}
A \otimes^C B & \xrightarrow{\eta_A \otimes^C \eta_B} & UF(A) \otimes^C UF(B) \\
& \searrow \scriptstyle \eta_{A \otimes^C B} & \downarrow \scriptstyle \theta_{(F(A), F(B))}^U \\
& & U(F(A) \otimes^D F(B)) \\
& & \downarrow \scriptstyle U\theta_{A, B}^F \\
& & UF(A \otimes^C B)
\end{array}$$

Diagram (41). For space reasons, we flip the diagram so that arrows involving strengths are vertical, and arrows involving associativities are horizontal. Also, we drop most sub-



scripts, which are inferrable from context.

$$\begin{array}{ccc}
(UF(A) \otimes^C UF(B)) & \xrightarrow{\alpha^C} & UF(A) \otimes^C \\
\otimes^C UF(C) & & (UF(B) \otimes^C UF(C)) \\
\theta^U \otimes^C \text{id}_{UF(C)} \downarrow & & \downarrow \text{id}_{UF(A)} \otimes^C \theta^U \\
U(F(A) \otimes^D F(B)) & & UF(A) \otimes^C \\
\otimes^C UF(C) & \text{(coherence (16) for } \theta^U) & U(F(B) \otimes^D F(C)) \\
U\theta^F \otimes^C \text{id}_{UF(C)} \downarrow & \swarrow \theta^U & \swarrow \theta^U \\
UF(A \otimes^C B) & \text{(naturality of } \theta^U) & U(F(A) \otimes^D \\
\otimes^C UF(C) & \text{ } & (F(B) \otimes^D \\
& \xrightarrow{U\alpha^D} & F(C)) \\
& & \text{(naturality of } \theta^U) \\
& & UF(A) \otimes^C \\
& & UF(B \otimes^C C) \\
& & \downarrow \text{id}_{UF(A)} \otimes^C U\theta^F \\
& & \downarrow \theta^U \\
& & U(F(A) \otimes^D \\
& & F(B \otimes^C C)) \\
& & \downarrow U\theta^F \\
& & UF(A \otimes^C \\
& & (B \otimes^C C)) \\
\theta^U \downarrow & \swarrow U(\theta^F \otimes^D \text{id}_{F(C)}) & \swarrow U(\text{id}_{F(A)} \otimes^D \theta^F) \\
U(F(A \otimes^C B) & \text{(U applied to coherence (16) for } \theta^F) & U(F(A) \otimes^D \\
\otimes^D F(C)) & & F(B \otimes^C C)) \\
U\theta^F \downarrow & & \downarrow U\theta^F \\
UF((A \otimes^C B) & \xrightarrow{UF\alpha^C} & UF(A \otimes^C \\
\otimes^C C) & & (B \otimes^C C))
\end{array}$$

The vertical arrows on the left compose to form  $d_{A \otimes^C B, C} \circ (d_{A, B} \otimes^C \text{id}_{UF(C)})$ , while the vertical arrows on the right compose to form  $d_{A \otimes^C B, C} \circ (\text{id}_{UF(A)} \otimes^C d_{B, C})$ , whence the result.

Diagram (42). Similarly, we flip the diagram so that vertical arrows become horizontal and conversely:

$$\begin{array}{ccc}
UFUF(A) \otimes^C UFUF(B) & \xrightarrow{U\epsilon_{F(A)} \otimes^C U\epsilon_{F(B)}} & UF(A) \otimes^C UF(B) \\
\theta^U \downarrow & & \downarrow \theta^U \\
U(FUF(A) \otimes^D FUF(B)) & \text{(naturality of } \theta^U) & \\
U\theta^F \downarrow & \swarrow U(\epsilon_{F(A)} \otimes^D \epsilon_{F(B)}) & \\
UF(UF(A) \otimes^C UF(B)) & \text{(U applied to (52))} & U(F(A) \otimes^D F(B)) \\
UF\theta^U \downarrow & \swarrow U\epsilon_{F(A) \otimes^D F(B)} & \downarrow U\theta^F \\
UFU(F(A) \otimes^D F(B)) & \text{(naturality of } U\epsilon) & \\
UFU\theta^F \downarrow & & \\
UFUF(A \otimes^C B) & \xrightarrow{U\epsilon_{F(A \otimes^C B)}} & UF(A \otimes^C B)
\end{array}$$

We recognize  $Td_{A, B} \circ d_{TA, TB}$  as the leftmost composition of vertical arrows, and the rightmost vertical composition is  $d_{A, B}$ . Also, the top horizontal arrow is  $\mu_A \otimes^C \mu_B$ , while the bottom arrow is  $\mu_{A \otimes^C B}$ .  $\square$

**Lemma 2.** Let  $(\mathcal{C}, I^C, \otimes^C, \alpha^C, \ell^C, r^C)$  be a monoidal category, and let  $(T, \eta, \mu, d)$  be a monoidal monad on  $\mathcal{C}$ .

Let  $\mathcal{D}$  be the Kleisli category of  $T$ ,  $I^D = I^C$ ,  $\otimes^D$  be defined on objects by  $A \otimes^D B = A \otimes^C B$  and on morphisms by letting  $f \otimes^D g$  (in  $\mathcal{D}$ ) be the morphism  $d \circ (f \otimes^C g)$  in  $\mathcal{C}$ ; let  $\alpha^D = \eta \circ \alpha^C$ ,  $\ell^D = \eta \circ \ell^C$ ,  $r^D = \eta \circ r^C$ . Then  $(\mathcal{D}, I^D, \otimes^D, \alpha^D, \ell^D, r^D)$ , is a monoidal category.

Moreover,  $F_T \dashv U_T$  is a monoidal adjunction. The mediating pairs of  $F_T$  and  $U_T$  are

$(\theta^{F_T}, \iota^{F_T})$  and  $(\theta^{U_T}, \iota^{U_T})$  respectively, where  $\theta_{A,B}^{F_T} : F_T(A) \otimes^{\mathcal{D}} F_T(B) \rightarrow F_T(A \otimes^{\mathcal{C}} B)$  (in  $\mathbf{Kleisli}(T)$ ) is the morphism  $\eta_{A \otimes^{\mathcal{C}} B}$  in  $\mathcal{C}$ ,  $\iota^{F_T} : I^{\mathcal{D}} \rightarrow F_T(I^{\mathcal{C}})$  (in  $\mathbf{Kleisli}(T)$ ) is  $\eta_{I^{\mathcal{C}}}$ ,  $\theta_{A,B}^{U_T} : U_T(A) \otimes^{\mathcal{C}} U_T(B) \rightarrow U_T(A \otimes^{\mathcal{D}} B)$  (in  $\mathcal{C}$ ) is  $d_{A,B}$ , and  $\iota^{U_T} : I^{\mathcal{C}} \rightarrow U_T(I^{\mathcal{D}})$  (in  $\mathcal{C}$ ) is  $\eta_{I^{\mathcal{C}}}$ .

Finally,  $F_T \dashv U_T$  generates the monoidal monad, in the sense that  $U_T F_T = T$ ,  $\eta$  is the unit of the adjunction and of  $T$ ,  $\mu_A = U_T \epsilon_{F(A)}$  where  $\epsilon$  is the counit of the adjunction, and  $d_{A,B} = U_T \theta_{A,B}^{F_T} \circ \theta_{F_T(A), F_T(B)}^{U_T}$ .

*Proof.* Tedious. See Appendix B.  $\square$

**Lemma 3.** Let  $\mathbb{C} \begin{array}{c} \xrightarrow{D} \\ \xleftarrow{U} \\ \text{|\_}| \end{array} \mathbb{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{U} \\ \text{|\_}| \end{array} \mathbb{D}$  be a diagram of functors. Assume that these func-

tors are monoidal; let  $(\theta, \mathfrak{i})$  be the mediating pair of  $|\_|$ ,  $(\theta, \mathfrak{i})$  that of  $\mathbb{D}$ ,  $(\theta^U, \iota^U)$  that of  $U$ ,  $(\theta^F, \iota^F)$  that of  $F$ .

Then  $F\mathbb{D}$  and  $|U|$  are monoidal functors, with respective mediating pairs  $(F\theta \circ \theta^F, F\mathfrak{i} \circ \iota^F)$  and  $(|\theta^U| \circ \theta, |\iota^U| \circ \mathfrak{i})$ .

Furthermore, if  $\mathbb{D} \dashv |\_|$  and  $F \dashv U$  are monoidal adjunctions, then  $F\mathbb{D} \dashv |U|$  is a monoidal adjunction, too.

*Proof.* Straightforward. See Appendix C.  $\square$

The following proposition is then both similar and proved similarly to Proposition 1.

**Proposition 2.** Let  $(\mathbb{C}, \otimes, I, \alpha, \ell, \mathfrak{r})$  and  $(\mathbb{C}, \otimes, \mathbb{I}, \alpha, \mathbb{l}, \mathfrak{r})$  be monoidal categories.

Let  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{d})$  be a monoidal monad on  $\mathbb{C}$ ,  $|\_| : \mathbb{C} \rightarrow \mathbb{C}$  and  $\mathbf{D} : \mathbb{C} \rightarrow \mathbb{C}$  be monoidal functors, yielding a monoidal adjunction  $\mathbf{D} \dashv |\_|$ . Let  $\acute{\epsilon}_A : \mathbf{D}|A| \rightarrow A$  be the counit of the adjunction,  $\acute{\eta}_E : E \rightarrow |\mathbf{D}(E)|$  be the unit of the adjunction. Let  $(\theta, \mathfrak{i})$  be the mediating pair of  $|\_|$ ,  $(\theta, \mathfrak{i})$  be the mediating pair of  $\mathbf{D}$ .

Define  $\mathbb{T} = |\_| \circ \mathbf{T} \circ \mathbf{D} = |\mathbf{T}\mathbf{D}|$ ,  $\eta_E = |\boldsymbol{\eta}_{\mathbf{D}(E)}| \circ \acute{\eta}_E$ ,  $\mu_E = |\boldsymbol{\mu}_{\mathbf{D}(E)} \circ \mathbf{T}\acute{\epsilon}_{\mathbf{T}\mathbf{D}(E)}|$ ,  $\mathfrak{d}_{E,F} = |\mathbf{T}\theta_{E,F} \circ \mathbf{d}_{\mathbf{D}E, \mathbf{D}F}| \circ \theta_{\mathbf{T}\mathbf{D}E, \mathbf{T}\mathbf{D}F}$ . Finally, let  $\sigma_A = |\mathbf{T}\acute{\epsilon}_A| : \mathbb{T}|A| \rightarrow |\mathbf{T}A|$ . Then  $(\mathbb{T}, \eta, \mu, \mathfrak{d})$  is a monoidal monad on  $\mathbb{C}$  and  $(|\_|, \sigma)$  is a monoidal monad morphism from  $\mathbf{T}$  to  $\mathbb{T}$ .

*Proof.* Let  $F = F_{\mathbf{T}}$ ,  $U = U_{\mathbf{T}}$ . By Lemma 2,  $F \dashv U$  is a monoidal adjunction which generates the monoidal monad  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{d})$ . Compose the monoidal adjunction  $\mathbf{D} \dashv |\_|$  with the monoidal adjunction  $F \dashv U$ , yielding the adjunction  $F\mathbf{D} \dashv |U|$ . This is also a monoidal adjunction by Lemma 3.

By Lemma 2, this monoidal adjunction generates a monoidal monad, and this is  $(\mathbb{T}, \eta, \mu, \mathfrak{d})$  as stated in the Proposition. Indeed, all cases except the mediator have been dealt with in Proposition 1, and the mediator is by definition  $|\mathbf{T}\theta \circ \theta^F| \circ |\theta^U| \circ \theta = |\mathbf{T}\theta| \circ |\theta^F \circ \theta^U| \circ \theta = |\mathbf{T}\theta| \circ |\mathbf{d}| \circ \theta$ .

It remains to check the monoidal monad morphism Diagram (45). This is given by the

following diagram:

$$\begin{array}{ccc}
\begin{array}{c} |TD|A_1| \\ \otimes |TD|A_2| \end{array} & \xrightarrow{\theta} & \begin{array}{c} |TD|A_1| \\ \otimes |TD|A_2| \end{array} & \xrightarrow{|d|} & \begin{array}{c} |T \left( \begin{array}{c} D|A_1| \\ \otimes D|A_2| \end{array} \right)| \\ (52) \end{array} & \xrightarrow{|T\theta|} & |TD(|A_1| \otimes |A_2|)| \\
\downarrow |T\epsilon_{A_1}| \otimes |T\epsilon_{A_2}| & & \downarrow & & \downarrow |T\theta| & & \downarrow |TD\theta| \\
|TA_1| \otimes |TA_2| & \xrightarrow{\text{(naturality of } |d| \circ \theta)} & & & |T(\epsilon_{A_1} \otimes \epsilon_{A_2})| & & |TD|A_1 \otimes A_2| \\
\downarrow \phi & & & & \downarrow |T\epsilon_{A_1} \otimes \epsilon_{A_2}| & & \downarrow |T\epsilon_{A_1} \otimes \epsilon_{A_2}| \\
|TA_1 \otimes TA_2| & \xrightarrow{|d|} & & & |T(A_1 \otimes A_2)| & & |T(A_1 \otimes A_2)|
\end{array}$$

□

We can in fact prove something similar with just strong monads. Unfortunately, it seems that we cannot use the nice trick of going through some adjunction generating the strong monad. The proof therefore goes through extremely tedious diagram checking.

**Proposition 3.** Let  $(\mathcal{C}, \otimes, I, \alpha, \ell, r)$  and  $(\mathbb{C}, \otimes, \mathbb{I}, \alpha, \mathbb{I}, \tau)$  be monoidal categories.

Let  $|-|$  be a monoidal functor from  $\mathcal{C}$  to  $\mathbb{C}$ , with mediating pair  $(\theta, \mathfrak{i})$ ,  $D$  be a monoidal functor from  $\mathbb{C}$  to  $\mathcal{C}$ , with mediating pair  $(\theta, \mathfrak{i})$ , and assume that  $D \dashv |-|$  is a monoidal adjunction.

Define  $\mathbb{T} = |-| \circ T \circ D = |TD|$ ,  $\eta_E = |\eta_{D(E)}| \circ \dot{\eta}_E$ ,  $\mu_E = |\mu_{D(E)} \circ T\epsilon_{TD(E)}|$ ,  $\sigma_A = |T\epsilon_A| : \mathbb{T}|A| \rightarrow |TA|$ .

Define also  $\mathfrak{k}_{E,F}$  as the composite  $|TD\theta_{E,F} \circ \mathfrak{t}_{D,E,D,F}| \circ \theta_{D,E,TD,F} \circ (\dot{\eta}_E \otimes \text{id}_{\mathbb{T}F})$ .

Then  $(\mathbb{T}, \eta, \mu, \mathfrak{k})$  is a strong monad on  $\mathbb{C}$  and  $(|-|, \sigma)$  is a strong monad morphism from  $T$  to  $\mathbb{T}$ .

*Proof.* Because of Proposition 1, we only have to check the strength laws (32), (33), (34), (35) for  $\mathfrak{k}$ , and the strong monad morphism law (36). As we said, this is tedious, hence relegated to Appendix D. □

## 9. Lifting Closed Structures to the Subscone

If we are to lift the whole structure of a cartesian-closed category together with a strong monad on it, to the subscone, the only thing that remains is to lift exponential objects. As this is essentially the subject of (Mitchell and Scedrov, 1993) (together with the fact that subscones generalize logical relations), it would be legitimate to skip over this construction, knowing that it has been dealt with elsewhere.

However, we notice that the standard lifting construction of exponentials to the subscone requires  $|-|$  to preserve products, at least up to natural isos. That is, it requires  $|\mathbf{1}| \cong \mathbb{1}$ ,  $|A \times B| \cong |A| \times |B|$ . This is certainly the case for the functor  $|-| = \mathcal{C}(\mathbf{1}, -)$ , which is the standard choice in scoping constructions (Mitchell and Scedrov, 1993).

Until now, we have only assumed that  $|-|$  was a monoidal (Section 6.1), resp. a symmetric monoidal (Section 6.7), resp. a cartesian monoidal (Section 6.9) functor. It would therefore be nice if we could dispense with the stringent requirement that  $|-|$  preserved monoidal or cartesian structure exactly. This would also afford us some added generality.

It turns out that having a monoidal mono factorization system is all we need: exponentials lift to the subscone without any additional requirements compared to Section 6. This only requires a slight adjustment of the standard exponential lifting diagrams. This is the topic of Section 9.1. We deal with fragments of the linear  $\lambda$ -calculus in Section 9.2.

### 9.1. Lifting Exponentials

Recall that an *exponential*, or *internal hom object* (on the right), in a monoidal category  $(\mathbb{C}, \otimes, \mathbb{I}, \alpha, \mathbb{l}, \mathbb{r})$ , is an object  $B^A$  together with a morphism  $\mathbf{App}$  from  $B^A \otimes A$  to  $B$  and, for every morphism  $u$  from  $C \otimes A$  to  $B$ , a morphism  $\Lambda(u)$  from  $C$  to  $B^A$  satisfying the two equations

$$\begin{array}{ccc} C \otimes A & \xrightarrow{\Lambda(u) \otimes \text{id}_A} & B^A \otimes A \\ & \searrow u & \downarrow \mathbf{App} \\ & & B \end{array} \quad (53)$$

for every morphism  $u$  from  $C \otimes A$  to  $B$  ( $\beta$ -equivalence), and

$$\Lambda \left( C \otimes A \xrightarrow{v \otimes \text{id}_A} B^A \otimes A \xrightarrow{\mathbf{App}} B \right) = v \quad (54)$$

for every morphism  $v$  from  $C$  to  $B^A$  ( $\eta$ -equivalence).  $\Lambda(u)$  is called the *currification* or the *abstraction* of  $u$ .

A more traditional definition is to require the existence of a *unique* morphism  $\Lambda(u)$  as in Diagram (53). Uniqueness is indeed implied by (54): if there were two morphisms  $v$  and  $v'$  such that  $u = \mathbf{App} \circ (v \otimes \text{id}_A) = \mathbf{App} \circ (v' \otimes \text{id}_A)$ , then  $v = \Lambda(\mathbf{App} \circ (v \otimes \text{id}_A)) = \Lambda(u) = \Lambda(\mathbf{App} \circ (v' \otimes \text{id}_A)) = v'$ . Conversely, uniqueness of  $\Lambda(u)$  implies Diagram (54): take  $u = \mathbf{App} \circ (v \otimes \text{id}_A)$  in Diagram (53).

Exponentials on the right are unique up to iso when they exist. A monoidal category is said to be *monoidal closed* (on the right) if and only if the exponential  $B^A$  exists for all objects  $A$  and  $B$ . Similarly, we call exponential on the left any object  ${}^A B$  with a morphism  $\mathbf{qqA}$  from  $A \otimes {}^A B$  to  $B$  such that, for every morphism  $u$  from  $A \otimes C$  to  $B$ , there is a unique morphism  $(u)\Lambda$  from  $C$  to  ${}^A B$  such that  $u = \mathbf{qqA} \circ (\text{id}_A \otimes (u)\Lambda)$ . In a *symmetric* monoidal category, it is equivalent to require the existence of exponentials on the right or on the left, and they coincide up to iso. A category with finite products that is also monoidal closed (for the monoidal structure induced by the product) is called *cartesian closed*.

Note that, in a monoidal closed category, there is a functor  ${}_-\!^A$  for each object  $A$ , which maps every object  $B$  to  $B^A$ , and every morphism  $B \xrightarrow{f} B'$  to  $f^A = \Lambda(f \circ \mathbf{App})$ , from  $B^A$  to  $B'^A$ . In fact, there is a bifunctor from  $\mathbb{C}^{op} \times \mathbb{C}$  to  $\mathbb{C}$  mapping  $A, B$  to  $B^A$ .

Moreover, the functor  ${}_-\!^A$  preserves monos: if  $m$  is mono, then so is  $m^A = \Lambda(m \circ \mathbf{App})$ . It suffices to show that there is at most one morphism  $f$  such that  $\Lambda(m \circ \mathbf{App}) \circ f = h$  where  $h$  is given. Indeed,  $\mathbf{App} \circ (h \otimes \text{id}) = \mathbf{App} \circ (\Lambda(m \circ \mathbf{App}) \otimes \text{id}) \circ (f \otimes \text{id}) = m \circ \mathbf{App} \circ (f \otimes \text{id})$  by (53); if there were two such morphisms  $f$  and  $f'$ , then  $m \circ \mathbf{App} \circ (f \otimes \text{id}) = m \circ \mathbf{App} \circ (f' \otimes \text{id})$ ,

so  $\mathbf{App} \circ (f \otimes \text{id}) = \mathbf{App} \circ (f' \otimes \text{id})$  since  $m$  is mono. Applying  $\Lambda$  on both sides implies  $f = f'$  by (54).

Once this is known, the standard way of lifting exponentials to the subscone is to require that  $\mathbf{C}$  and  $\mathbb{C}$  are cartesian closed,  $\mathbb{C}$  has pullbacks, and that  $|-|$  preserves finite products (exactly, or up to natural iso). This standard construction actually does not require cartesian closedness, and works equally well with monoidal closed categories, assuming  $\mathbb{C}$  has pullbacks and  $|-|$  preserves unit and tensor.

We recall this construction now. As  $|-|$  preserves unit, the unit (terminal object in the cartesian closed case)  $\tilde{I}$  is  $(\mathbb{I}, \text{id}_{\mathbb{I}}, \mathbf{I})$  witness by the arrow  $\mathbb{I} \xrightarrow{\text{id}_{\mathbb{I}}} \mathbf{I} = \mathbb{I}$ , tensor product (binary product in the cartesian closed case) is given by  $\langle S_1, m_1, A_1 \rangle \tilde{\otimes} \langle S_2, m_2, A_2 \rangle = \langle S_1 \otimes S_2, m_1 \otimes m_2, A_1 \otimes A_2 \rangle$ , and the exponential  $\langle \tilde{S}_2^1, \tilde{m}_2^1, A_2^{A_1} \rangle = \langle S_2, m_2, A_2 \rangle^{(S_1, m_1, A_1)}$  is given by the square on the left below.

$$\begin{array}{ccc}
\tilde{S}_2^1 \mathbb{C} & \xrightarrow{\tilde{m}_2^1} & |A_2^{A_1}| \\
\downarrow \ell_2^1 & \lrcorner & \downarrow \Lambda \left( |\mathbf{App}| \circ \left( \text{id}_{|A_2^{A_1}|} \otimes m_1 \right) \right) \\
S_2^{S_1} \mathbb{C} & \xrightarrow{m_2^{S_1}} & |A_2|^{S_1}
\end{array}
\quad
\begin{array}{ccc}
|A_2^{A_1}| \otimes S_1 & & \\
\downarrow \text{id}_{|A_2^{A_1}|} \otimes m_1 & & \\
|A_2^{A_1}| \otimes |A_1| & & \\
\parallel & & \\
|A_2^{A_1}| \otimes A_1 & & \\
\downarrow |\mathbf{App}| & & \\
|A_2| & &
\end{array}$$

where the vertical morphism  $\Lambda \left( |\mathbf{App}| \circ \left( \text{id}_{|A_2^{A_1}|} \otimes m_1 \right) \right)$  is  $\Lambda$  applied to the composition of vertical morphisms on the right, the morphism  $m_2^{S_1}$  is mono because  $|-|$  preserves monos, and  $\tilde{m}_2^1$  is mono because pullbacks preserve monos. (We temporarily revert to the notation  $\xrightarrow{\subset}$  to denote all monos.)

Application from  $\langle \tilde{S}_2^1, \tilde{m}_2^1, A_2^{A_1} \rangle \tilde{\otimes} \langle S_1, m_1, A_1 \rangle$  in the subscone is given by

$$\begin{array}{ccccc}
\tilde{S}_2^1 \otimes S_1 & \xrightarrow{\tilde{m}_2^1 \otimes \text{id}_{S_1}} & |A_2^{A_1}| \otimes S_1 & \xrightarrow{\text{id} \otimes m_1} & |A_2^{A_1}| \otimes |A_1| \\
\downarrow \ell_2^1 \otimes \text{id}_{S_1} & & \downarrow \Lambda \left( |\mathbf{App}| \circ \left( \text{id}_{|A_2^{A_1}|} \otimes m_1 \right) \right) \otimes \text{id}_{S_1} & & \parallel \\
S_2^{S_1} \otimes S_1 & \xrightarrow{m_2^{S_1} \otimes \text{id}_{S_1}} & |A_2|^{S_1} \otimes S_1 & & \\
\downarrow \mathbf{App} & & \downarrow \mathbf{App} & & \\
S_2 \mathbb{C} & \xrightarrow{m_2} & |A_2| & \xleftarrow{|\mathbf{App}|} & |A_2^{A_1}| \otimes |A_1|
\end{array}$$

where the top left square is the definition of the exponential  $\langle \tilde{S}_2^1, \tilde{m}_2^1, A_2^{A_1} \rangle$  tensor  $S_1$  on the right. The bottom left square and the right square commute by (53). Then, application in the subscone is  $\langle \mathbf{App} \circ (\ell_2^1 \otimes \text{id}), |\mathbf{App}| \rangle$ .

Abstraction of the morphism  $\langle u, v \rangle$  in the subscone from  $\langle S, m, A \rangle \tilde{\otimes} \langle S_1, m_1, A_1 \rangle$  to

$\langle S_2, m_2, A_2 \rangle$  is then the pair  $\langle \tilde{u}, \Lambda(v) \rangle$  given by the diagram

$$\begin{array}{ccc}
S\mathcal{C} & \xrightarrow{m} & |A| \\
\Lambda(u) \searrow & \tilde{u} \dashrightarrow & \downarrow |\Lambda(v)| \\
& \tilde{S}_2^1 & \xrightarrow{\tilde{m}_2^1} & |A_2^{A_1}| \\
& \downarrow \ell_2^1 & \lrcorner & \downarrow \Lambda(|\mathbf{App}| \circ (\text{id} \otimes m_1)) \\
S_2^{S_1} \mathcal{C} & \xrightarrow{m_2^{S_1}} & |A_2|^{S_1}
\end{array}$$

where  $\tilde{u}$  is given by the universal property of pullbacks. To this end, we must first check that the outer contour of the diagram commutes. We leave this to the reader.

The equations (53) and (54) then hold in the subscone, because they hold in  $\mathcal{C}$ , and using Fact 2.

In the case we are interested in here,  $|-|$  does not preserve unit and tensor. Rather, we have required  $|-|$  to be a monoidal functor, a strictly weaker notion. We have already seen in Section 6 that this was enough to lift monoidal structure to the subscone, using a monoidal mono factorization. We now realize that this is also enough to lift monoidal *closed* structure to the subscone. This requires only minor adjustments to the constructions above.

First, we require that the functor  $_{-}^A$  preserves *relevant* monos, for all  $A$ . While it always preserves monos, it is not clear that it should preserve relevant monos, hence the added assumption. (We return to our convention that  $\xrightarrow{\quad} \mathcal{C}$  denotes relevant monos only.) We might also require that pullbacks preserve relevant monos, but this is not necessary, as we can use the mono factorization instead. Summing up, the exponential  $\langle \tilde{S}_2^1, \tilde{m}_2^1, A_2^{A_1} \rangle = \langle S_2, m_2, A_2 \rangle^{\langle S_1, m_1, A_1 \rangle}$  is given by the diagram on the left below.

$$\begin{array}{ccc}
\widehat{S}_2^1 \xrightarrow{\tilde{e}_2^1} \tilde{S}_2^1 \mathcal{C} \xrightarrow{\tilde{m}_2^1} |A_2^{A_1}| & & |A_2^{A_1}| \otimes S_1 \\
\lrcorner & & \downarrow \text{id}_{|A_2^{A_1}|} \otimes m_1 \\
\widehat{S}_2^1 \xrightarrow{\ell_2^1} S_2^{S_1} \mathcal{C} \xrightarrow{m_2^{S_1}} |A_2|^{S_1} & \xrightarrow{\Lambda(|\mathbf{App}| \circ \theta \circ (\text{id} \otimes m_1))} & |A_2^{A_1}| \otimes |A_1| \\
& & \downarrow \theta_{A_2^{A_1}, A_1} \\
& & |A_2^{A_1}| \otimes |A_1| \\
& & \downarrow |\mathbf{App}| \\
& & |A_2|
\end{array} \tag{55}$$

where the vertical morphism  $\Lambda(|\mathbf{App}| \circ \theta \circ (\text{id} \otimes m_1))$  is  $\Lambda$  applied to the composition of vertical morphisms on the right, the morphism  $m_2^{S_1}$  is a relevant mono by our assumption that  $_{-}^{S_1}$  preserves relevant monos, the topmost horizontal composition of arrows (from  $\widehat{S}_2^1$  to  $|A_2^{A_1}|$ ) is given by pullback, and is factored as  $\tilde{e}_2^1$  followed by  $\tilde{m}_2^1$ ; and finally the morphism  $\ell_2^1$  is given by a diagonal fill-in, from two paths from  $\widehat{S}_2^1$  to  $|A_2|^{S_1}$ .

Application from  $\langle \tilde{S}_2^1, \tilde{m}_2^1, A_2^{A_1} \rangle \otimes \langle S_1, m_1, A_1 \rangle$  is then given by  $\langle \widehat{\mathbf{App}}, |\mathbf{App}| \rangle$  as given

in the diagram below.

$$\begin{array}{ccccc}
\tilde{S}_2^1 \otimes S_1 & \xrightarrow{\tilde{m}_2^1 \otimes \text{id}_{S_1}} & |A_2^{A_1}| \otimes S_1 & \xrightarrow{\text{id} \otimes m_1} & |A_2^{A_1}| \otimes |A_1| & (56) \\
\ell_2^1 \otimes \text{id}_{S_1} \downarrow & & \Lambda(|\text{App}| \circ \theta \circ (\text{id} \otimes m_1)) \otimes \text{id}_{S_1} \downarrow & & \downarrow \theta_{A_2^{A_1}, A_1} \\
S_2^{S_1} \otimes S_1 & \xrightarrow{m_2^{S_1} \otimes \text{id}_{S_1}} & |A_2|^{S_1} \otimes S_1 & & \\
\text{App} \downarrow & & \downarrow \text{App} & & \\
S_2 \hookrightarrow & \xrightarrow{m_2} & |A_2| & \xleftarrow{|\text{App}|} & |A_2^{A_1}| \otimes |A_1| \\
\widehat{\text{App}} \nearrow & & \nearrow m_{\text{App}} & & \\
S_{\text{App}} & & & & 
\end{array}$$

The top left square comes from the definition of the exponential  $\langle \tilde{S}_2^1, \tilde{m}_2^1, A_2^{A_1} \rangle$ , tensored by  $S_1$  on the right. The bottom left square commutes because  $\text{App} \circ (m_2^{S_1} \otimes \text{id}_{S_1}) = \text{App} \circ (\Lambda(m_2 \circ \text{App}) \otimes \text{id}_{S_1}) = m_2 \circ \text{App}$  by (53). The right square commutes by (53) again. The outer contour, defined by  $\tilde{m}_2^1 \otimes m_1$  on the top,  $\theta_{A_2^{A_1}, A_1}$  on the right,  $e_{\text{App}}$  on the left,  $m_{\text{App}}$  at the bottom, is the definition of the tensor product  $\langle \tilde{S}_2^1, \tilde{m}_2^1, A_2^{A_1} \rangle \otimes \langle S_1, m_1, A_1 \rangle$  in the subscone.  $\widehat{\text{App}}$  is given by a diagonal fill-in, considering the two paths  $e_{\text{App}}$  followed by  $|\text{App}| \circ m_{\text{App}}$  and  $\text{App} \circ (\ell_2^1 \otimes \text{id}_{S_1})$  followed by  $m_2$  from  $\tilde{S}_2^1 \otimes S_1$  to  $|A_2|$ .

Given any morphism  $\langle u, v \rangle$  in the subscone from  $\langle S, m, A \rangle \otimes \langle S_1, m_1, A_1 \rangle$  to  $\langle S_2, m_2, A_2 \rangle$ , by definition the following diagram commutes. The top square is the definition of  $\langle S, m, A \rangle \otimes \langle S_1, m_1, A_1 \rangle$ .

$$\begin{array}{ccc}
S \otimes S_1 & \xrightarrow{m \otimes m_1} & |A| \otimes |A_1| & (57) \\
e_{.1} \downarrow & & \theta_{A, A_1} \downarrow & \\
S_1 \hookrightarrow & \xrightarrow{m_{.1}} & |A| \otimes |A_1| & \\
u \downarrow & & \downarrow |v| & \\
S_2 & \xrightarrow{m_2} & |A_2| & 
\end{array}$$

Abstraction of the morphism  $\langle u, v \rangle$  in the subscone from  $\langle S, m, A \rangle \otimes \langle S_1, m_1, A_1 \rangle$  to  $\langle S_2, m_2, A_2 \rangle$  is then the pair  $\langle \tilde{e}_2^1 \circ \tilde{u}, \Lambda(v) \rangle$  given by the diagram:

$$\begin{array}{ccc}
S \hookrightarrow & \xrightarrow{m} & |A| & (58) \\
\tilde{u} \searrow & & \downarrow |\Lambda(v)| & \\
\tilde{S}_2^1 & \xrightarrow{\tilde{e}_2^1} & \tilde{S}_2^1 \hookrightarrow & \xrightarrow{\tilde{m}_2^1} & |A_2^{A_1}| \\
\Lambda(u \circ e_{.1}) \searrow & & \downarrow \ell_2^1 & & \downarrow \Lambda(|\text{App}| \circ \theta \circ (\text{id} \otimes m_1)) \\
S_2 \hookrightarrow & \xrightarrow{m_2^{S_1}} & |A_2|^{S_1} & 
\end{array}$$

In this diagram,  $\tilde{u}$  is given by the universal property of pullbacks, and this will be justified by the fact that the two outer paths from  $S$  to  $|A_2|^{S_1}$  are equal, which we have to check. First, we note the identity

$$\Lambda(s) \circ t = \Lambda(s \circ (t \otimes \text{id}_C)) \quad (59)$$

whenever  $t$  is a morphism from  $A$  to  $B$ , and  $s$  from  $B \otimes C$  to  $D$ . Indeed,  $\Lambda(s) \circ t = \Lambda(\text{App} \circ ((\Lambda(s) \circ t) \otimes \text{id}_C))$  (by (54))  $= \Lambda(\text{App} \circ (\Lambda(s) \otimes \text{id}_C) \circ (t \otimes \text{id}_C)) = \Lambda(s \circ (t \otimes \text{id}_C))$  (by

(53)). It follows that the lower path from  $S$  to  $|A_2|^{S_1}$  in Diagram 58 is

$$\begin{aligned}
m_2^{S_1} \circ \Lambda(u \circ e_1) &= \Lambda(m_2 \circ \mathbf{App}) \circ \Lambda(u \circ e_1) \\
&= \Lambda(m_2 \circ \mathbf{App} \circ (\Lambda(u \circ e_1) \otimes \text{id}_{S_1})) \quad (\text{by (59)}) \\
&= \Lambda(m_2 \circ u \circ e_1) \quad (\text{by (53)})
\end{aligned}$$

while the upper one is

$$\begin{aligned}
&\Lambda(|\mathbf{App}| \circ \Theta \circ (\text{id} \otimes m_1)) \circ |\Lambda(v)| \circ m \\
&= \Lambda(|\mathbf{App}| \circ \Theta \circ (\text{id} \otimes m_1) \circ ((|\Lambda(v)| \circ m) \otimes \text{id}_{S_1})) \quad (\text{by (59)}) \\
&= \Lambda(|\mathbf{App}| \circ \Theta \circ (|\Lambda(v)| \otimes \text{id}_{|A_1|}) \circ (m \otimes m_1)) \\
&= \Lambda(|\mathbf{App}| \circ |\Lambda(v)| \otimes \text{id}_{A_1} \circ \Theta \circ (m \otimes m_1)) \quad (\text{by naturality of } \Theta) \\
&= \Lambda(|v| \circ \Theta \circ (m \otimes m_1)) \quad (\text{by (53)})
\end{aligned}$$

and these two quantities are equal by Diagram (57).

Finally, the equations (53) and (54) then hold in the subscone, because they hold in  $\mathcal{C}$ , and using Fact 2.

We sum up what we need to lift exponentials to the subscone, as usual. To lift exponentials on the right:

- 
- (i.a.r) monoidal closed (on the right) categories  $(\mathcal{C}, \otimes, I, \alpha, \ell, r)$  and  $(\mathbb{C}, \otimes, \mathbb{I}, \alpha, \ell, r)$ , and a monoidal functor  $|-| : \mathcal{C} \rightarrow \mathbb{C}$ ,
  - (iii.a) a monoidal mono factorization system on  $\mathbb{C}$ .
  - (v)  $\mathbb{C}$  has pullbacks.
  - (vi.r)  $-^A$  preserves relevant monos.
- 

Exponentials are given by (55), application by (56), abstraction by (58).

The construction works equally well to lift exponentials on the left, reversing arguments to tensor products, so we require in this case:

- 
- (i.a.l) monoidal closed (on the left) categories  $(\mathcal{C}, \otimes, I, \alpha, \ell, r)$  and  $(\mathbb{C}, \otimes, \mathbb{I}, \alpha, \ell, r)$  and a monoidal functor  $|-| : \mathcal{C} \rightarrow \mathbb{C}$ ,
  - (iii.a) a monoidal mono factorization system on  $\mathbb{C}$ .
  - (v)  $\mathbb{C}$  has pullbacks.
  - (vi.l)  ${}^A_-$  preserves relevant monos.
- 

Clearly, in the symmetric monoidal closed case we require the following to get a symmetric monoidal closed subscone:

- 
- (i.b) symmetric monoidal closed categories  $(\mathcal{C}, \otimes, I, \alpha, \ell, r, c)$  and  $(\mathbb{C}, \otimes, \mathbb{I}, \alpha, \ell, r, c)$ , and a symmetric monoidal functor  $|-| : \mathcal{C} \rightarrow \mathbb{C}$ ,
  - (iii.a) a monoidal mono factorization system on  $\mathbb{C}$ .
  - (v)  $\mathbb{C}$  has pullbacks.
  - (vi)  $-^A$  preserves relevant monos.
- 

And in the cartesian closed case, we require:



- 
- (i.c) cartesian closed categories  $\mathbf{C}$  and  $\mathbb{C}$ ,  
and a cartesian monoidal functor  $|-| : \mathbf{C} \rightarrow \mathbb{C}$ ,
  - (iii.a) a monoidal mono factorization system on  $\mathbb{C}$ .
  - (v)  $\mathbb{C}$  has pullbacks.
  - (vi)  $_-^A$  preserves relevant monos.
- 

Putting all conditions together, we get a notion of subscone for categorical models of Moggi's meta-language, i.e., for cartesian closed categories with a strong monad, we require:

- 
- (i.c) cartesian closed categories  $\mathbf{C}$  and  $\mathbb{C}$ ,  
and a cartesian monoidal functor  $|-| : \mathbf{C} \rightarrow \mathbb{C}$ ,
  - (ii.a) a strong monad  $(\mathbb{T}, \eta, \mu, \mathbb{t})$  on  $\mathbb{C}$ , related to  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{t})$  by  
a strong monad morphism  $(|-|, \sigma)$  defined in (4) and (36),
  - (iii.a) a monoidal mono factorization system on  $\mathbb{C}$ .
  - (iv)  $\mathbb{T}$  maps relevant pseudoepis to pseudoepis.
  - (v)  $\mathbb{C}$  has pullbacks.
  - (vi)  $_-^A$  preserves relevant monos.
- 

## 9.2. Linear $\lambda$ -Calculus

Similarly, we may deal with fragments of the linear  $\lambda$ -calculus. In particular, the intuitionistic multiplicative fragment has symmetric monoidal closed categories as models. (We base ourselves loosely on (Bierman, 1995).) We sketch how this can be enriched with a strong monad '?'. (Said otherwise, we take models of multiplicative-exponential intuitionistic linear logic where the comonad '!' is the identity.) This will be the framework that we shall use in Section 10.12, for example.

Syntactically, consider the intuitionistic multiplicative fragment with a strong monad. Types  $\tau$  are given by:

$$\tau ::= b | 1 | \tau \otimes \tau | \tau \multimap \tau | ?\tau$$

where  $b$  ranges over a given collection of base types. Linear  $\lambda$ -terms are given by:

$$\begin{aligned} s, t, u, v, \dots & ::= x | * | \text{let } * = s \text{ in } t | s \otimes t | \text{let } x \otimes y = s \text{ in } t | st | \lambda x \cdot s \\ & | \text{val } s | \text{let val } x = s \text{ in } t \end{aligned}$$

Typing rules are given as follows, where contexts  $\Gamma$  are multisets of bindings  $x : \tau$ , with

pairwise distinct variable parts  $x$ .

$$\begin{array}{c}
\frac{}{x : A \vdash x : A} \text{ (Var)} \\
\frac{\Gamma \vdash s : 1 \quad \Delta, x : 1 \vdash t : \tau}{\Gamma, \Delta \vdash \text{let } * = s \text{ in } t : \tau} \text{ (1E)} \\
\frac{\Gamma \vdash s : \tau \otimes \tau' \quad \Delta, x : \tau, y : \tau' \vdash t : \tau''}{\Gamma, \Delta \vdash \text{let } x \otimes y = s \text{ in } t : \tau''} \text{ (}\otimes\text{E)} \\
\frac{\Gamma \vdash s : \tau \multimap \tau' \quad \Delta \vdash t : \tau}{\Gamma, \Delta \vdash st : \tau'} \text{ (App)} \\
\frac{\Gamma \vdash s : ?\tau \quad \Delta, x : \tau \vdash t : ?\tau'}{\Gamma, \Delta \vdash \text{let val } x = s \text{ in } t : ?\tau'} \text{ (Let)} \\
\frac{}{\vdash * : 1} \text{ (1)} \\
\frac{\Gamma \vdash s : \tau \quad \Delta \vdash t : \tau'}{\Gamma, \Delta \vdash s \otimes t : \tau \otimes \tau'} \text{ (Pair)} \\
\frac{\Gamma, x : \tau \vdash s : \tau'}{\Gamma \vdash \lambda x \cdot s : \tau \multimap \tau'} \text{ (Abs)} \\
\frac{\Gamma \vdash s : \tau}{\Gamma \vdash \text{val } s : ?\tau} \text{ (Val)}
\end{array}$$

Conversion rules are given by judgments  $\Gamma \vdash s \rightarrow t : \tau$  (which we write  $s \rightarrow t$  when  $\Gamma$  and  $\tau$  are clear or irrelevant), of which the most important are:

$$\begin{array}{ll}
\text{let } * = * \text{ in } s \rightarrow s & \vdash s \rightarrow * : 1 \\
\text{let } x \otimes y = s \otimes t \text{ in } u \rightarrow u[x := s, y := t] & \text{let } x \otimes y = s \text{ in } x \otimes y \rightarrow s \\
(\lambda x \cdot s)t \rightarrow s[x := t] & \lambda x \cdot sx \rightarrow s \quad (x \text{ not free in } s) \\
\text{let val } x = \text{val } s \text{ in } t \rightarrow t[x := s] & \text{let val } x = s \text{ in val } x \rightarrow s \\
\text{let val } x = \text{let val } y = s \text{ in } t \text{ in } u \rightarrow \text{let val } y = s \text{ in let val } x = t \text{ in } u
\end{array}$$

There are other rules, notably the infamous commutative conversion rules—which originate in (Prawitz, 1965)—and which we won't state. The interpretation in symmetric monoidal closed categories follows the same lines as those of the ordinary  $\lambda$ -calculus with a monadic type, replacing cartesian products by tensor products. Our constructions then yield a notion of subscone for this kind of calculus and models, requiring:

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- (i.b) symmetric monoidal closed categories  $(\mathbf{C}, \otimes, \mathbf{I}, \alpha, \ell, \mathbf{r}, \mathbf{c})$  and  $(\mathbb{C}, \otimes, \mathbb{I}, \alpha, \ell, \mathbf{r}, \mathbf{c})$ , and a symmetric monoidal functor  $|-| : \mathbf{C} \rightarrow \mathbb{C}$ ,
  - (ii.a) a strong monad  $(\mathbb{T}, \eta, \mu, \flat)$  on  $\mathbb{C}$ , related to  $(\mathbf{T}, \eta, \mu, \mathbf{t})$  by a strong monad morphism  $(|-|, \sigma)$  defined in (4) and (36),
  - (iii.a) a monoidal mono factorization system on  $\mathbb{C}$ .
  - (iv)  $\mathbb{T}$  maps relevant pseudoepis to pseudoepis.
  - (v)  $\mathbb{C}$  has pullbacks.
  - (vi)  $|-|^A$  preserves relevant monos.
- 

We let the reader do similar mixing-and-matching to handle his/her own favorite model or logic.

## 10. Examples

At this point, we suspect the reader is relatively fed up with categorical diagrams and general abstract nonsense. It is therefore time to instantiate our constructions. We start with fairly easy cases in the category **Set** of sets in Section 10.1. We examine in more

detail the non-determinism monad in Section 10.7. A more demanding case is the name creation monad, which involves presheaves, and which we deal with in Section 10.8. We terminate with the subtle case of the probability monad, and its numerous variations in Section 10.9.

### 10.1. *Lift, Exceptions, State, Non-Determinism, and Continuations in **Set***

As in Section 5, suppose  $\mathbf{C}_1 = \mathbf{C}_2 = \mathbf{C} = \mathbf{Set}$ , and  $|-|_1$  and  $|-|_2$  are identities. In particular,  $|-|$  is the cartesian product functor from  $\mathbf{Set} \times \mathbf{Set}$  to  $\mathbf{Set}$ . Below we summarize the action of  $\tilde{T}$  on a relation  $S \hookrightarrow A_1 \times A_2$ , for different computational monads  $\mathbb{T}$  of Moggi (Moggi, 1991). This is parameterized by a binary relation  $R_E$  on exceptions in  $E$  in the exception monad  $A + E$ , by a binary relation  $R_{\text{St}}$  on states in the state monad  $(A \times \text{St})^{\text{St}}$ , and by a binary relation  $R_{\mathcal{R}}$  in the continuation monad  $\mathcal{R}^A$ .

Monad $\mathbb{T}$	relation $\tilde{S} \subseteq \mathbb{T}A_1 \times \mathbb{T}A_2$
$\mathbb{T}A = A_{\perp} = A \cup \{\perp\}$	$\tilde{S} = S \cup \{(\perp, \perp)\}$
$\mathbb{T}A = A + E$	$(v_1, v_2) \in \tilde{S} \iff (v_1, v_2) \in S \vee (v_1, v_2) \in R_E$
$\mathbb{T}A = (A \times \text{St})^{\text{St}}$	$(f, g) \in \tilde{S} \iff \forall s_1, s_2 \in \text{St}.$ $(s_1, s_2) \in R_{\text{St}} \Rightarrow (\pi_1(fs_1), \pi_1(gs_2)) \in S \wedge (\pi_2(fs_1), \pi_2(gs_2)) \in R_{\text{St}}$
$\mathbb{T}A = \mathbb{P}_{\text{fin}}(A)$	$(B_1, B_2) \in \tilde{S} \iff$ $\forall b_1 \in B_1. \exists b_2 \in B_2. (b_1, b_2) \in S \wedge$ $\forall b_2 \in B_2. \exists b_1 \in B_1. (b_1, b_2) \in S$
$\mathbb{T}A = \mathcal{R}^A$	$(\alpha_1, \alpha_2) \in \tilde{S} \iff ($ $\forall k_1, k_2. (\forall a_1, a_2. (a_1, a_2) \in S \Rightarrow (k_1(a_1), k_2(a_2)) \in R_{\mathcal{R}}) \Rightarrow$ $\left. (\alpha_1(k_1), \alpha_2(k_2)) \in R_{\mathcal{R}} \right)$

We examine each case in more detail. We take surjections as pseudoepi, injections as relevant monos. This is the canonical choice for an epi-mono factorization system on  $\mathbf{Set}$ . Note that condition **(iv)**, that  $\mathbb{T}$  maps relevant pseudoepi to pseudoepi, is always satisfied when  $\mathbf{C} = \mathbf{Set}$ . In fact,  $\mathbb{T}$  preserves pseudoepi: every pseudoepi (surjective function)  $e : A \rightarrow B$  has a section  $m : B \rightarrow A$  (i.e.,  $e \circ m = \text{id}_B$ ), by the Axiom of Choice. Then  $\mathbb{T}e \circ \mathbb{T}m = \text{id}_{\mathbb{T}B}$ , showing that  $\mathbb{T}e$  is surjective, hence pseudoepi.

### 10.2. *Lift monad $A_{\perp}$ .*

The monad morphism  $\sigma$  from  $(A \times B)_{\perp}$  to  $A_{\perp} \times B_{\perp}$  maps  $\perp$  to  $(\perp, \perp)$ , and every pair  $(x, y) \in A \times B$  to itself. This is a commutative monad, hence monoidal, hence strong. The mediator  $\mathfrak{d}_{A,B}$  from  $A_{\perp} \times B_{\perp}$  to  $(A \times B)_{\perp}$  maps  $(\perp, y)$  and  $(x, \perp)$  to  $\perp$ , and  $(x, y)$  where  $x \in A$  and  $y \in B$  to itself. Note that it is not a cartesian monad, because  $(\pi_2)_{\perp} \circ \mathfrak{d}_{A,B}$  maps  $(\perp, y)$  to  $\perp$ , while  $\pi_2(\perp, y) = y$ . Nonetheless, all required conditions are satisfied to lift  $\mathbb{T}$  to a commutative monad on the subscone.

### 10.3. Exception monad $A + E$ .

The monad morphism  $\sigma$  from  $(A \times B) + E$  to  $(A + E) \times (B + E)$  maps the pair  $(x, y) \in A \times B$  to itself, and the exception  $e \in E$  to the pair  $(e, e)$ . The strength  $\mathbb{k}_{A,B}$  from  $A \times (B + E)$  to  $(A \times B) + E$  maps  $(x, y)$  with  $x \in A$  and  $y \in B$  to  $(x, y)$ , and  $(x, e)$  with  $x \in A$  and  $e \in E$  to  $e$ . (This is *not* a commutative monad.) Therefore this lifts to a monoidal monad on the subscone. As shown in the table above, the lifted monad relates two values  $v_1$  and  $v_2$  if and only if both are non-exceptional values (in  $A$ ) and are related by  $S$ , or both are exceptions related by some a priori relation  $R_E$ .

### 10.4. State transformer monad $(A \times \text{St})^{\text{St}}$ .

The monad morphism  $\sigma$  from  $((A \times B) \times \text{St})^{\text{St}}$  to  $(A \times \text{St})^{\text{St}} \times (B \times \text{St})^{\text{St}}$  maps  $f : \text{St} \rightarrow (A \times B) \times \text{St}$  to the pair of functions mapping  $s \in \text{St}$  to  $(v_1, s')$  and to  $(v_2, s')$  respectively, where  $f(s) = ((v_1, v_2), s')$ . The strength maps  $(x, f) \in A \times (B \times \text{St})^{\text{St}}$  to the function mapping  $s \in \text{St}$  to  $((x, y), s')$ , where  $(y, s') = f(s)$ . (Again, this is not a commutative monad.)

It follows that our lifting constructions apply, yielding the lifted monad described in the table above. Note that  $f : \text{St} \rightarrow A_1 \times \text{St}$ ,  $g : \text{St} \rightarrow A_2 \times \text{St}$  can be read as the transition functions of deterministic transition systems, which go from a state  $s$  to another state  $s'$  and emit a value in  $A_1$  (resp. in  $A_2$ ). These transition functions are in relation by  $\tilde{S}$  if and only if, for any two states that are in relation via  $R_{\text{St}}$ , the values emitted by firing the transitions by  $f$  and  $g$  are in relation by  $S$ , and the target states after the transition are in relation via  $R_{\text{St}}$  again. This states that  $f$  and  $g$  are in relation by  $\tilde{S}$  if and only if  $R_{\text{St}}$  is a *bisimulation* between states.

### 10.5. Finite powerset (non-determinism) monad $\mathbb{P}_{\text{fin}}(A)$ .

The monad morphism from  $\mathbb{P}_{\text{fin}}(A \times B)$  to  $\mathbb{P}_{\text{fin}}(A) \times \mathbb{P}_{\text{fin}}(B)$  maps every relation  $R \subseteq A \times B$  to the pair consisting of its domain  $\{x | \exists y \cdot (x, y) \in R\}$  and its codomain  $\{y | \exists x \cdot (x, y) \in R\}$ . The mediator  $\mathfrak{d}_{A,B}$  maps  $X \subseteq A$  and  $Y \subseteq B$  to the relation  $X \times Y \subseteq A \times B$ , and makes  $\mathbb{T}$  a commutative monad.

Our construction in the case of the finite powerset monad  $\mathbb{P}_{\text{fin}}()$  in fact expands to:  $(B_1, B_2) \in \tilde{S}$  iff  $B_1 = \{x | (x, y) \in R\}$  and  $B_2 = \{y | (x, y) \in R\}$  for some  $R \subseteq S$ . (Recall that  $\tilde{T}$  maps relations  $S$  to the direct image  $\tilde{S}$  of  $\langle \mathbb{T}\pi_1, \mathbb{T}\pi_2 \rangle : TS \rightarrow \mathbb{T}A_1 \times \mathbb{T}A_2$ , see the end of Section 5.) This is equivalent to the condition given in the table above, which is the more usual way of defining bisimulations.

Indeed, if  $B_1 = \{x | (x, y) \in R\}$  and  $B_2 = \{y | (x, y) \in R\}$  for some  $R \subseteq S$  then for every  $b_1 \in B_1$  by construction there is some  $b_2 \in B_2$  such that  $(b_1, b_2) \in R$ , therefore  $(b_1, b_2) \in S$  since  $R \subseteq S$ , and symmetrically for every  $b_2 \in B_2$  there is some  $b_1 \in B_1$  such that  $(b_1, b_2) \in S$ :  $B_1$  and  $B_2$  are bisimilar.

Conversely, if  $B_1$  and  $B_2$  are bisimilar (in the sense just given), then let  $R$  be the restriction of  $S$  to  $B_1 \times B_2$ . For every  $b_1 \in B_1$ , by bisimilarity there is some  $b_2 \in B_2$  such that  $(b_1, b_2) \in S$ , so  $(b_1, b_2) \in R$ , therefore  $b_1 \in \{x | (x, y) \in R\}$ ; so  $B_1 \subseteq \{x | (x, y) \in R\}$ .

The reverse inclusion is obvious, so  $B_1 = \{x \mid (x, y) \in R\}$ . The other equality  $B_2 = \{y \mid (x, y) \in R\}$  is by symmetry.

That logical relations on powersets define bisimulations was conjectured in (Lazić and Nowak, 2000) and, for pre-logical relations, in (Honsell and Sannella, 2002).

Note that there is nothing special with the *finite* powerset monad here. We might have taken the set of all subsets instead, or the set of all subsets of cardinality at least  $\alpha$  and strictly less than  $\beta$ , where  $\alpha < \beta$  are two cardinal numbers such that  $\alpha \leq 1$ , and every finite product of cardinals between  $\alpha$  (inclusive) and  $\beta$  (exclusive) is again so. The finite powerset monad is the case  $\alpha = 0$ ,  $\beta = \aleph_0$ ; the lift monad is the case  $\alpha = 0$ ,  $\beta = 2$ .

Note also that although this monad is always commutative, it is a cartesian monad if and only if  $\alpha \geq 1$ . Indeed,  $\mathbb{T}$  is a cartesian monad if and only if the domain of the relation  $X \times Y$  is  $X$ , and its codomain is  $Y$ . This is wrong if  $Y$  or  $X$  is allowed to be empty, but holds as soon as  $X$  and  $Y$  are non-empty. In particular, the finite-and-non-empty powerset monad (for serial non-determinism—no state is final) is cartesian.

#### 10.6. The continuation monad $\mathcal{R}^{\mathcal{R}^A}$ .

The monad morphism  $\sigma$  from  $\mathcal{R}^{\mathcal{R}^{A \times B}}$  to  $\mathcal{R}^{\mathcal{R}^A} \times \mathcal{R}^{\mathcal{R}^B}$  maps  $\alpha : \mathcal{R}^{A \times B} \rightarrow \mathcal{R}$  to the pair  $(\alpha_1, \alpha_2)$ , where  $\alpha_1$  maps  $k_1 \in \mathcal{R}^A$  to  $\alpha(k_1 \circ \pi_1)$  and  $\alpha_2$  maps  $k_2 \in \mathcal{R}^B$  to  $\alpha(k_2 \circ \pi_2)$ . The strength  $\mathbb{k}_{A,B}$  maps  $(x, \alpha) \in A \times \mathcal{R}^{\mathcal{R}^B}$  to the function mapping  $k \in \mathcal{R}^{A \times B}$  to  $\alpha(\lambda y \in B \cdot k(x, y))$ . This monad is not commutative.

Our construction yields the rather opaque condition in the table above, where  $a_1, a_2$  are values,  $k_1, k_2$  are continuations, and  $\alpha_1, \alpha_2$  are programs, taking continuations to answers in  $\mathcal{R}$ . Intuitively, think of continuations as computation environments (a toplevel loop, a shell) that take the result of a program and print something (called an answer, in  $\mathcal{R}$ ) on a computer terminal. To evaluate a program  $\alpha$  in continuation (environment)  $k$ , apply  $\alpha$  to  $k$ . For simplicity, assume that the relation  $R_{\mathcal{R}}$  on answers is equality. The condition then states that two programs are related by  $\tilde{S}$  if and only if they give identical answers when evaluated in related continuations (environments), where two environments are related if and only if they print the same answer on values that are related by  $S$ , i.e., if and only if they do not make any difference between  $S$ -related values. This is a form of observational equivalence.

#### 10.7. Labelled transition systems and bisimulations

The case  $\mathbb{T}A = \mathbb{P}_{\text{fin}}(A)$  allows one to define labelled transition systems as elements of  $(\mathbb{T}A)^{A \times L}$ , with labels in  $L$  and states in  $A$ , as functions mapping states  $a$  and labels  $\ell$  to the set of states  $a'$  such that  $a \xrightarrow{\ell} a'$ . Our monad lifting  $\tilde{S}$  in this case is parameterized by a binary relation on  $R_L$  on labels and is defined by:

$$(f_1, f_2) \in \tilde{S} \iff (\forall a_1, a_2, \ell_1, \ell_2 \cdot (a_1, a_2) \in S \wedge (\ell_1, \ell_2) \in R_L \Rightarrow \left\{ \begin{array}{l} \forall b_1 \in f_1(a_1, \ell_1). \exists b_2 \in f_2(a_2, \ell_2). (b_1, b_2) \in S \wedge \\ \forall b_2 \in f_2(a_2, \ell_2). \exists b_1 \in f_1(a_1, \ell_1). (b_1, b_2) \in S \end{array} \right. )$$

In case  $R_L$  is the equality relation, the relation  $\tilde{S}$  relates  $f_1$  and  $f_2$  iff  $S$  is a *strong bisimulation* between the labelled transition systems  $f_1$  and  $f_2$ .

### 10.8. Logical relations for dynamic name creation

Consider the categorical model of dynamic name creation defined in (Stark, 1996). Let  $\mathcal{I}$  be the category of finite sets and injective functions, and  $\mathbf{Set}^{\mathcal{I}}$  be the category of functors from  $\mathcal{I}$  to  $\mathbf{Set}$  and natural transformations (the category of *covariant presheaves* over  $\mathcal{I}$ ). For short, write  $\mathbf{T}A$ s for  $\mathbf{T}(A)(s)$  and similarly for other notations. Let  $+$  denote disjoint union in  $\mathcal{I}$ .<sup>‡</sup>

We define the strong monad  $(\mathbf{T}, \eta, \mu, \mathbf{t})$  on  $\mathbf{Set}^{\mathcal{I}}$  by:

—  $\mathbf{T}A = \text{colim}_{s'} A(\_ + s') : \mathcal{I} \rightarrow \mathbf{Set}$ .

On objects, this is given by  $\mathbf{T}A$ s =  $\text{colim}_{s'} A(s + s')$ , i.e.,  $\mathbf{T}A$ s is the set of all equivalence classes of pairs  $(s', a)$  with  $s' \in \mathcal{I}$  and  $a \in A(s + s')$  modulo the smallest equivalence relation  $\equiv$  such that  $(s', a) \equiv (s'', A(\text{id}_s + j)a)$  for every morphism  $s' \xrightarrow{j} s''$  in  $\mathcal{I}$  (intuitively, given a set of *names*  $s$ , elements of  $\mathbf{T}A$ s are formal expressions  $(\nu s')a$  where all names in  $s'$  are bound and every name free in  $a$  is in  $s + s'$ —modulo the fact that  $(\nu s', s'')a \equiv (\nu s')a$  for any additional set of new names  $s''$  not free in  $a$ ). We shall write  $[s', a]$  the equivalence class of  $(s', a)$ .

On morphisms  $s_1 \xrightarrow{i} s_2$ ,  $\mathbf{T}A$ i maps  $[s', a]$  to the equivalence class of  $[s', A(i + \text{id}_{s'})a]$ .

— For any  $f : A \rightarrow B$  in  $\mathbf{Set}^{\mathcal{I}}$ ,  $\mathbf{T}f$ s :  $\mathbf{T}A$ s  $\rightarrow$   $\mathbf{T}B$ s is defined by  $\mathbf{T}f$ s $[s', a] = [s', f(s + s')a]$ . This is compatible with  $\equiv$  because  $f$  is natural.

—  $\eta_A$ s :  $A$ s  $\rightarrow$   $\mathbf{T}A$ s is defined by  $\eta_A$ s $a = [\emptyset, a]$ .

—  $\mu_A$ s :  $\mathbf{T}^2$ As  $\rightarrow$   $\mathbf{T}A$ s is defined by  $\mu_A$ s $[s', [s'', a]] = [s' + s'', a]$ .

—  $\mathbf{t}_{A,B}$ s :  $A$ s  $\times$   $\mathbf{T}B$ s  $\rightarrow$   $\mathbf{T}(A \times B)$ s is defined by  $\mathbf{t}_{A,B}$ s $(a, [s', b]) = [s', (A\text{inl}s's'a, b)]$  where  $\text{inl}s's' : s \rightarrow s + s'$  is the canonical injection.

Furthermore,  $\mathbf{T}$  is a commutative monad, whose mediator  $\mathbf{d}_{A,B}$ s :  $\mathbf{T}A$ s  $\times$   $\mathbf{T}B$ s  $\rightarrow$   $\mathbf{T}(A \times B)$ s maps  $([s', a], [s'', b])$  to  $[s' + s'', (A(\text{id}_s + \text{inl}s''s'a), A(\text{id}_s + \text{inr}s''s'b))]$ , where  $\text{inr}s''s' : s'' \rightarrow s' + s''$  is the other canonical injection. In fact,  $\mathbf{T}$  is a cartesian monad. Indeed,  $(\mathbf{T}\pi_1 \circ \mathbf{d}_{A,B})$ s maps  $([s', a], [s'', b])$  to  $[s' + s'', A(\text{id}_s + \text{inl}s''s'a)] = [s', a]$ , so  $\mathbf{T}\pi_1 \circ \mathbf{d}_{A,B} = \pi_1$ , and similarly  $\mathbf{T}\pi_2 \circ \mathbf{d}_{A,B} = \pi_2$ .

It is important to note how  $\equiv$  works. The category  $\mathcal{I}$  has pushouts: in particular, if  $s_0 \xrightarrow{i_1} s_1$  and  $s_0 \xrightarrow{i_2} s_2$  are two morphisms in  $\mathcal{I}$ , then there is a finite set  $s_1 +_{s_0} s_2$  and two morphisms  $s_1 \xrightarrow{j_1} s_1 +_{s_0} s_2$ ,  $s_2 \xrightarrow{j_2} s_1 +_{s_0} s_2$  such that  $j_1 \circ i_1 = j_2 \circ i_2$ —take  $s_1 +_{s_0} s_2$  to be the disjoint sum  $s_1 + s_2$  modulo the equivalence relation relating  $i_1(a_0) = i_2(a_0)$  for every  $a_0 \in s_0$ .

It follows that  $(*)$  for every  $a_1 \in A(s + s_1)$ ,  $a_2 \in A(s + s_2)$ ,  $(s_1, a_1) \equiv (s_2, a_2)$  if and only if there is a finite set  $s_{12}$  and two arrows  $s_1 \xrightarrow{j_1} s_{12}$  and  $s_2 \xrightarrow{j_2} s_{12}$  such that  $A(\text{id}_{s_1} + j_1)a_1 = A(\text{id}_{s_2} + j_2)a_2$ .

We take  $\mathbf{C}_1 = \mathbf{C}_2 = \mathbb{C} = \mathbf{Set}^{\mathcal{I}}$ , hence objects in the subscone give rise to  $\mathcal{I}$ -indexed

<sup>‡</sup> Note that  $+$  is not a coproduct in  $\mathcal{I}$ . In fact,  $\mathcal{I}$  does not have a coproduct. However  $+$  is functorial in both components, associative, and has a neutral element, and this is all we need.

*Kripke logical relations.* Furthermore,  $|-|_1 = |-|_2 = |-|$  is the identity functor and  $\mathbb{T}$  is just  $\mathbf{T}$ . The category  $\mathbf{Set}^{\mathcal{I}}$  has a mono factorization consisting of pointwise surjections and pointwise injections.

As in Section 5, the monad morphism  $\sigma_{\langle A_1, A_2 \rangle} s : \mathbb{T}(A_1 \times A_2)s \rightarrow \mathbf{T}A_1s \times \mathbf{T}A_2s$  is equal to  $\langle \mathbb{T}\pi_1, \mathbb{T}\pi_2 \rangle s$ . That is to say

$$\sigma_{\langle A_1, A_2 \rangle} s[s', (a_1, a_2)] = ([s', a_1], [s', a_2])$$

where  $s' \in \mathcal{I}$ ,  $a_1 \in A_1(s + s')$ ,  $a_2 \in A_2(s + s')$ .

Again as in  $\mathbf{Set}$ , every functor  $S$  from  $\mathcal{I}$  to  $A_1 \times A_2$  has a representation

$$\langle \pi^S_1, \pi^S_2 \rangle : S \hookrightarrow A_1 \times A_2$$

where each arrow  $\langle \pi^S_1, \pi^S_2 \rangle s : Ss \hookrightarrow A_1s \times A_2s$  is an inclusion.

Hence,  $S$  is a family of relations  $Ss$  between  $A_1s$  and  $A_2s$ , functorial in  $s$  (for each  $s' \xrightarrow{i} s''$ ,  $Si$  is the appropriate restriction of  $A_1i \times A_2i$ ). Recall from Section 5 that  $\tilde{S}$  is defined as the direct image of  $\langle \mathbb{T}\pi^S_1, \mathbb{T}\pi^S_2 \rangle$ . So,  $(\dagger)$   $[s_1, a_1] \tilde{S} [s_2, a_2]$  if and only if for some  $s' \in \mathcal{I}$ ,  $a'_1 \in A_1s'$ ,  $a'_2 \in A_2s'$ ,  $(s_1, a_1) \equiv (s', a'_1)$ ,  $(s_2, a_2) \equiv (s', a'_2)$  and  $a'_1 S(s + s') a'_2$ .

Using  $(*)$  above,  $(s_1, a_1) \equiv (s', a'_1)$  means that there is a finite set  $s'_1$  and two arrows  $s_1 \xrightarrow{j_1} s'_1$ ,  $s' \xrightarrow{j'_1} s'_1$  in  $\mathcal{I}$  such that: (a)  $A_1(\text{id}_s + j_1)a_1 = A_1(\text{id}_s + j'_1)a'_1$ . Similarly,  $(s_2, a_2) \equiv (s', a'_2)$  means there is a finite set  $s'_2$  and two arrows  $s_2 \xrightarrow{j_2} s'_2$ ,  $s' \xrightarrow{j'_2} s'_2$  in  $\mathcal{I}$  such that: (b)  $A_2(\text{id}_s + j_2)a_2 = A_2(\text{id}_s + j'_2)a'_2$ . Consider arrows  $j'_1$  and  $j'_2$ , which both have  $s'$  as source, and build their pushout  $s_0 = s'_1 +_{s'} s'_2$ , with arrows  $s'_1 \xrightarrow{j''_1} s_0$ ,  $s'_2 \xrightarrow{j''_2} s_0$ . Let  $j$  be  $j''_1 \circ j'_1 = j''_2 \circ j'_2$ . By applying  $A_1(\text{id}_s + j''_1)$  to both sides of (a),  $A_1(\text{id}_s + (j''_1 \circ j'_1))a_1 = A_1(\text{id}_s + j)a'_1$ . By applying  $A_2(\text{id}_s + j''_2)$  to both sides of (b),  $A_2(\text{id}_s + (j''_2 \circ j'_2))a_2 = A_2(\text{id}_s + j)a'_2$ . Since  $a'_1 S(s + s') a'_2$  and  $S$  is functorial,  $A_1(\text{id}_s + j)a'_1 S(s + s_0) A_2(\text{id}_s + j)a'_2$ , so  $A_1(\text{id}_s + (j''_1 \circ j'_1))a_1 S(s + s_0) A_2(\text{id}_s + (j''_2 \circ j'_2))a_2$ .

So if  $(s_1, a_1) \tilde{S} (s_2, a_2)$  then there are arrows  $s_1 \xrightarrow{i_1} s_0$  and  $s_2 \xrightarrow{i_2} s_0$ , namely  $i_1 = j''_1 \circ j'_1$  and  $i_2 = j''_2 \circ j'_2$ , such that  $A_1(\text{id}_s + i_1)a_1 S(s + s_0) A_2(\text{id}_s + i_2)a_2$ .

Conversely, if the latter holds, then  $(\dagger)$  above clearly holds for  $s' = s_0$ ,  $a'_1 = A_1(\text{id}_s + i_1)a_1$  and  $s'_2 = A_2(\text{id}_s + i_2)a_2$ .

$\tilde{S}s \hookrightarrow \mathbf{T}A_1s \times \mathbf{T}A_2s$  is thus given by

$$[s_1, a_1] \tilde{S} [s_2, a_2] \iff \exists s_0 \in \mathcal{I} \cdot \exists i_1 : s_1 \rightarrow s_0 \in \mathcal{I} \cdot \exists i_2 : s_2 \rightarrow s_0 \in \mathcal{I} \cdot (A_1(\text{id}_s + i_1)a_1) S(s + s_0) (A_2(\text{id}_s + i_2)a_2) \quad (60)$$

where  $a_1 \in A_1(s + s_1)$  and  $a_2 \in A_2(s + s_2)$ .

From (60) we define a logical relation for Moggi's metalanguage, as suggested in Section 5, by induction on types  $\tau$ . Each relation  $\llbracket \tau \rrbracket$  is a functor from  $\mathcal{I}$  to  $\mathbf{Set} \times \mathbf{Set}$ , so

that  $\llbracket \tau \rrbracket s$  is a binary relation for each type  $\tau$  and each finite set  $s$ . We have:

$$\begin{aligned}
(f_1, f_2) \in \llbracket \tau \rightarrow \tau' \rrbracket s &\iff \forall s', i : s \rightarrow s' \in \mathcal{I}, (a_1, a_2) \in \llbracket \tau \rrbracket s' \cdot \\
&\quad (f_1 s'(i, a_1), f_2 s'(i, a_2)) \in \llbracket \tau' \rrbracket s' \\
((a_1, a'_1), (a_2, a'_2)) \in \llbracket \tau \times \tau' \rrbracket s &\iff (a_1, a_2) \in \llbracket \tau \rrbracket s \wedge (a'_1, a'_2) \in \llbracket \tau' \rrbracket s \\
([s_1, a_1], [s_2, a_2]) \in \llbracket \mathbf{T}\tau \rrbracket s &\iff \exists s_0, i_1 : s_1 \rightarrow s_0, i_2 : s_2 \rightarrow s_0 \in \mathcal{I} \cdot \\
&\quad (A_1(\text{id}_s + i_1)a_1, A_2(\text{id}_s + i_2)a_2) \in \llbracket \tau \rrbracket (s + s_0)
\end{aligned}$$

This is similar to the logical relations of (Pitts and Stark, 1993; Stark, 1998). While (60) is roughly similar to the notion of logical relation of (Pitts and Stark, 1993), this paper does not rest on Moggi's computational  $\lambda$ -calculus. On the other hand (Stark, 1998) does rest on the computational  $\lambda$ -calculus but does not define a suitable notion of logical relation.

Zhang and Nowak show in (Zhang and Nowak, 2003) that the logical relation above is in fact strictly weaker than Pitts and Stark's logical relation (Pitts and Stark, 1993) when restricted to the latter's nu-calculus; Zhang and Nowak also claim that by reinstantiating our construction of the subscone with  $\mathbf{C}_1 = \mathbf{C}_2 = \mathbf{Set}^{\mathcal{I}}$  as above but  $\mathbf{C} = \mathbf{Set}^{\mathcal{I}^\rightarrow}$ , where  $\mathcal{I}^\rightarrow$  is the comma category whose objects are the morphisms of  $\mathcal{I}$ , another Kripke logical relation is obtained that coincides with Pitts and Stark's on the nu-calculus up to first order. This is in fact wrong, but Zhang (Zhang, 2005) shows that this can be repaired by replacing  $\mathcal{I}^\rightarrow$  by the category  $\mathcal{P}\mathcal{I}^\rightarrow$  whose objects are, again, morphisms of  $\mathcal{I}$ , but whose morphisms are pullback diagrams in  $\mathcal{I}$ . It extends it to the full monadic meta-language, and rests purely on semantic principles, while Pitts and Stark's definition of their logical relation relies on normalization properties of the nu-calculus.

### 10.9. Monads of Measures and Probabilities

Defining  $\mathbf{T}A$  to be space of all probability distributions over the space  $A$  allows us to define probabilistic transition systems as objects of  $(\mathbf{T}A)^A$ . In principle, this should work just like ordinary transition systems (Section 10.7). In actual practice, defining what the right spaces should be, and ensuring that the required monads exist and have the required properties, is much subtler.

As regards the right category of spaces, the obvious choice is  $\mathbf{Mes}$ , the category of measurable sets (sets equipped with a  $\sigma$ -algebra) and measurable functions (such that the inverse image of any measurable subset is measurable). F. W. Lawvere, then M. Giry (Lawvere, 1962; Giry, 1981) showed that  $\mathbf{Mes}$  indeed admits a monad  $(\mathbf{T}, \eta, \mu) : \mathbf{T}A$  is the set of (probability) measures over  $A$ , together with the smallest  $\sigma$ -algebra such that, for every measurable subset  $X$  of  $A$ , the map  $p_X$  mapping  $\nu \in \mathbf{T}A$  to  $\nu(X)$  is measurable.

$\mathbf{Mes}$  has bad properties. It is very unlikely to be cartesian closed<sup>§</sup>, and the monad above is not strong. In fact, the category  $\mathbf{C}$  that is best suited for our purposes here is the category  $\mathbf{Cpo}$  of dcpos and continuous maps, using the continuous valuation monad of Jones (Jones, 1990). We examine this case in Section 10.10, taking  $\mathbf{Set}$  as the observation

<sup>§</sup> Although we have been unable to find a proof of this negative assertion.



category  $\mathbb{C}$ . We shall see that we retrieve Larsen and Skou’s notion of probabilistic bisimulation (Larsen and Skou, 1991) in this case, at least when specialized to finite sets and discrete probabilities. In the case of continuous (non-discrete) state spaces, this notion is too weak for most purposes, and we extend it: we present in Section 10.11 the most precise notion of logical relation that our construction apparently affords us. This is rather technical, and the main obstacle is condition **(iv)** stating that  $\mathbb{T}$  should map relevant pseudoepis to pseudoepis. Another different route is taken in Section 10.12, where we examine the category  $\mathbf{C} = \mathbf{Met}$  of metric spaces and non-expansive maps, which has a monad of probability measures; observed from  $\mathbb{C} = \mathbf{Met}$  again, our logical relations define a notion of families of metric spaces indexed by types, extending (Desharnais et al., 2000; Desharnais et al., 1999), modulo some technicalities.

#### 10.10. Continuous Valuations on Dcpo, observed from $\mathbf{Set}$

Let us consider a natural CCC equipped with a notion of measure:  $\mathbf{Cpo}$ , the category of *directed complete partial orders*, a.k.a., *dcpo*s. The objects (dcpo) are partial orders such that every directed subset (every non-empty set such that any two elements have an upper bound) has a least upper bound; the morphisms are continuous functions. This is cartesian-closed, and has pullbacks.

10.10.1. *Dcpo and the Continuous Valuation Monad.* For every dcpo  $A$ , Jones (Jones, 1990) observed that the set  $\mathbf{TA}$  of all continuous valuations, to be defined next, was again a dcpo, and that this construction could be used to define a probabilistic powerdomain monad. (Jones calls dcpo *ipos*, i.e., *inductive* partial orders, and calls continuous valuations *evaluations*. We follow the majority of authors in using “dcpo” and “valuation”.) Recall that a Scott open of a dcpo  $A$  is an upper closed subset  $O$  such that every directed family whose sup is in  $O$  intersects  $O$ , that Scott opens form the *Scott topology* on  $A$ , so that we can look at dcpo as particular topological spaces.

Let  $(A, \mathcal{O})$  be a topological space. A (bounded) *valuation* on  $A$  is a map  $\nu : \mathcal{O} \rightarrow \mathbb{R}_+$  such that  $\nu(\emptyset) = 0$  (*strictness*);  $U \subseteq V$  implies  $\nu(U) \leq \nu(V)$  for every opens  $U, V \in \mathcal{O}$  (*monotonicity*); and  $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$  for every opens  $U, V \in \mathcal{O}$  (*modularity*). A *continuous* valuation in addition satisfies  $\nu(\bigcup_i U_i) = \sup_i \nu(U_i)$  for every directed family  $(U_i)_{i \in I}$  of opens—such a family is *directed* if and only if  $I \neq \emptyset$  and for every  $i, j \in I$ , there is  $k \in I$  such that  $U_i, U_j \subseteq U_k$ . All the valuations we shall consider are bounded, i.e.,  $\nu(A) < +\infty$ ; we shall not mention this any longer. Continuous valuations are a concept close to that of measure; while measures are defined on  $\sigma$ -algebras, valuations are naturally defined on topological spaces.

The set  $\mathbf{TA}$  of continuous valuations on the topological space  $A$  is a dcpo, with ordering  $\nu \leq \xi$  if and only if  $\nu(O) \leq \xi(O)$  for every open  $O$  in  $A$ . When we restrict  $A$  to be a dcpo,  $\mathbf{T}$  is functorial: for every  $A \xrightarrow{f} B$ ,  $\mathbf{T}f$  maps the continuous valuation  $\nu \in \mathbf{TA}$  to the continuous valuation mapping every open  $O$  in  $B$  to  $\nu(f^{-1}(O))$ . This gives rise to a monad, whose unit  $\eta_A$  maps  $x \in A$  to the *Dirac valuation*  $\delta_x$  on  $A$ , such that  $\delta_x(O) = 1$  for every open  $O$  of  $A$  containing  $x$ ,  $\delta_x(O) = 0$  otherwise. Its multiplication  $\mu_A$  maps continuous valuations  $\nu$  on  $\mathbf{TA}$  to the continuous valuation on  $A$  mapping every open

$O \subseteq A$  to  $\int_{\xi \in \mathbf{T}A} \xi(O) d\nu$  (the average of all possible evaluations  $\xi(O)$  of  $O$ , weighted by the probability of each valuation  $\xi$ ), and the strength  $\mathbf{t}_{A,B}$  maps  $a \in A$  and  $\nu \in \mathbf{T}B$  to the continuous valuation on  $A \times B$  mapping every open  $O \subseteq A \times B$  to  $\nu(\{y \in B \mid (a, y) \in O\})$ . Then  $(\mathbf{T}, \eta, \mu, \mathbf{t})$  is a strong monad on  $\mathbf{Cpo}$  (Jones, 1990).

There is no real need to recall the definition of the Jones integral used above. There are several definitions, the original one by Jones (Jones, 1990), and extensions by Kirch (Kirch, 1993), by Tix (Tix, 1995), by Heckmann (Heckmann, 1996). They are all equivalent on  $\text{dcpos}$ , and define a function mapping every continuous function  $f : A \rightarrow (\mathbb{R}_+ \cup \{+\infty\})$ , where  $(\mathbb{R}_+ \cup \{+\infty\})$  is seen as a  $\text{dcpo}$  (such Scott continuous functions are usually called *upper continuous*; note that the Scott topology on  $(\mathbb{R}_+ \cup \{+\infty\})$  is strictly coarser than its metric topology) and every continuous valuation  $\nu$  on  $A$  to  $\int_{x \in A} f(x) d\nu \in (\mathbb{R}_+ \cup \{+\infty\})$ . Its main properties on  $\mathbf{Cpo}$  are (see (Heckmann, 1996, Theorem 7.1)):

- 1 the map  $f \in \mathbb{R}_+^A, \nu \in \mathbf{T}A \mapsto \int_{x \in A} f(x) d\nu$  is continuous in each argument separately (hence jointly, since joint continuity and separate continuity coincide in  $\mathbf{Cpo}$ ), and linear in  $f$  and  $\nu$ ;
- 2  $\int_{x \in A} f(x) d\delta_{x_0} = f(x_0)$ ;
- 3 (*Change of Variables formula.*) For every continuous function  $g : A \rightarrow B$ ,  $\int_{x \in A} (f \circ g)(x) d\nu = \int_{y \in B} f(y) d\mathbf{T}g(\nu)$ ;
- 4 for every open  $O$  of  $A$ , letting  $\chi_O$  be the characteristic function of  $O$  in  $A$ ,  $\int_{x \in A} \chi_O(x) d\nu = \nu(O)$ .

For the sake of completeness, note that  $\mathbf{T}$  is in fact a monoidal monad on  $\mathbf{Cpo}$ , whose mediator  $\mathbf{d}_{A,B}$  maps  $\nu \in \mathbf{T}A$  and  $\xi \in \mathbf{T}B$  to the continuous valuation that Jones ((Jones, 1990), Section 3.10) writes  $\nu \otimes \xi$ , which maps every Scott open subset  $O$  of the  $\text{dcpo}$   $A \times B$  to  $\int_{x \in A} \left( \int_{y \in B} \chi_O(x, y) d\xi \right) d\nu$ . On the subcategory of continuous  $\text{dcpos}$ , this yields a commutative monad by Fubini's Theorem on valuations (Jones, 1990, Theorem 3.17). On  $\mathbf{Cpo}$ , it is unknown whether Fubini's Theorem holds, hence whether the monad is commutative; in case it is not,  $\mathbf{T}$  would be an example of a monoidal monad that is not commutative.

10.10.2. *Constructing a Logical Relation for the Continuous Valuation Monad.* Contrarily to the previous examples, we do not take  $\mathbf{C}_1 = \mathbf{C}_2 = \mathbb{C} = \mathbf{Cpo}$ ; we have been unable to show that  $\mathbf{T}$  preserved pseudoepi for any notion of pseudoepi in  $\mathbf{Cpo}$ . We take here a simple solution, which we shall refine in Section 10.11.

Take  $\mathbb{C} = \mathbf{Set}$ , and use Proposition 2. Equip  $\mathbf{C} = \mathbf{Cpo}$  and  $\mathbb{C} = \mathbf{Set}$  with the standard (symmetric) monoidal structure given by finite products. There is a forgetful functor  $|-|$  from  $\mathbf{Cpo}$  to  $\mathbf{Set}$  sending every  $\text{dcpo}$   $A$  to its set of elements  $|A|$ , and every continuous function  $f : A \rightarrow B$  to the underlying set-theoretic function. It has a left adjoint  $\mathbf{D}$  mapping every set  $E$  to  $E$  seen as a  $\text{dcpo}$  with equality as ordering. This is called the *discretization* functor, and  $\mathbf{D}E$  is called a *discrete*  $\text{dcpo}$ . Note that all subsets of  $\mathbf{D}E$  are open, so that as a topological space,  $\mathbf{D}E$  has the discrete topology.

Proposition 2 applies:  $|-|$  and  $\mathbf{D}$  preserve products exactly, hence are monoidal (with trivial mediating pairs). The unit  $\eta_E$  is the identity on  $E$ , while the counit  $\epsilon_A$  is the

identity function from the dcpo  $\mathbf{D}|A|$  ( $A$  with equality as ordering) to  $A$ , and are clearly monoidal. This yields a monoidal monad  $(\mathbb{T}, \eta, \mu, \mathfrak{d})$  on **Set**. For every set  $E$ ,  $\mathbb{T}E$  is the set of all continuous valuations on  $E$  equipped with the discrete topology; such a continuous valuation  $\nu$  is entirely determined by the values  $\nu\{x\}$ ,  $x \in E$ , since then for every  $X \subseteq E$ ,  $\nu(X)$  must equal  $\sum_{x \in X} \nu\{x\}$ . This implies that  $\nu$  has countable support, and is a countable linear combination  $\sum_{x \in E} a_x \delta_x$  of Dirac valuations; such valuations are called *discrete* in (Goubault-Larrecq, 2004). The unit  $\eta_E$  maps  $x \in E$  to  $\delta_x$ , multiplication simplifies to  $\mu_E(\sum_{\xi \in \mathbb{T}X} a_\xi \delta_\xi) = \sum_{\xi \in \mathbb{T}X} a_\xi \xi$ . The mediator  $\mathfrak{d}_{E,F}$  maps  $(\sum_{x \in X} a_x \delta_x, \sum_{y \in Y} b_y \delta_y)$  to  $\sum_{x \in X, y \in Y} a_x b_y \delta_{(x,y)}$ ; the induced strength  $\mathfrak{k}_{E,F}$  maps  $(x, \sum_{y \in Y} b_y \delta_y)$  to  $\sum_{y \in Y} b_y \delta_{(x,y)}$ . Finally, the monad morphism maps the discrete valuation  $\nu \in \mathbb{T}|A|$  to  $\nu$  seen as a continuous valuation in  $|TA|$ .

In the binary relation case, take  $\mathbf{C} = \mathbf{C}_1 \times \mathbf{C}_2$ , where  $\mathbf{C}_1 = \mathbf{C}_2 = \mathbf{Cpo}$ .  $\mathbf{C}$  is cartesian closed, with all products and exponentials taken componentwise. Also, there is a monoidal monad  $(\mathbf{T} \times \mathbf{T}, (\boldsymbol{\eta}, \boldsymbol{\eta}), (\boldsymbol{\mu}, \boldsymbol{\mu}), (\mathbf{d}, \mathbf{d}))$ , given componentwise, on  $\mathbf{C}$ . The product functor from  $\mathbf{C}$  to  $\mathbf{C}_1$  has the diagonal functor as left adjoint. Composing this adjunction with the adjunction  $\mathbf{D} \dashv |\cdot|$  above yields a monoidal adjunction again, so that Proposition 2 applies. This yields a slightly different monoidal monad  $\mathbb{T}'$  on **Set**, where  $\mathbb{T}'A = \mathbb{T}A \times \mathbb{T}A$ ; it is easily seen to be isomorphic to  $\mathbb{T}$ .

Whatever route we take, letting  $|\cdot|$  be the product functor and  $\sigma_{A_1, A_2}$  map  $\nu \in \mathbb{T}(|A_1| \times |A_2|)$  to  $(\nu_1, \nu_2) \in |\mathbb{T}A_1| \times |\mathbb{T}A_2|$ , where  $\nu_i = \mathbb{T}\pi_i \nu$  ( $i = 1, 2$ ) yields a monad morphism from  $\mathbf{T} \times \mathbf{T}$  to  $\mathbb{T}$ .

As before, we take injections as pseudoepis and surjections as relevant monos;  $\mathbb{T}$  preserves pseudoepis, because every functor does so in **Set**.

Let us spell out the resulting construction: for any relation  $S \subseteq |A_1| \times |A_2|$ , let  $\langle \pi_1^S, \pi_2^S \rangle$  be the inclusion  $S \hookrightarrow |A_1| \times |A_2|$ , then the diagram defining the lifting of the monad is:

$$\begin{array}{ccccc}
\mathbb{T}S & \xrightarrow{\mathbb{T}\langle \pi_1^S, \pi_2^S \rangle} & \mathbb{T}(|A_1| \times |A_2|) & & \\
\parallel & & \parallel & & \\
|\mathbb{T}DS| & \xrightarrow{|\mathbb{T}D\langle \pi_1^S, \pi_2^S \rangle|} & |\mathbb{T}D(|A_1| \times |A_2|)| & & \\
\downarrow & \searrow f & \downarrow \langle |\mathbb{T}D\pi_1|, |\mathbb{T}D\pi_2| \rangle & \searrow \sigma_{A_1, A_2} & \\
\tilde{S} & \xrightarrow{m} & |\mathbb{T}D|A_1|| \times |\mathbb{T}D|A_2|| & \hookrightarrow & |\mathbb{T}A_1| \times |\mathbb{T}A_2|
\end{array}$$

The inclusion arrow that ends in the bottom-right corner, corresponds to the fact that any discrete valuation  $\nu_i \in |\mathbb{T}D|A_i||$  can be seen as a continuous valuation in  $A_i$ , i.e., an element of  $|\mathbb{T}A_i|$ , for  $i = 1, 2$ .

So  $\tilde{S}$  is the range of the function  $f = \langle |\mathbb{T}D\pi_1|, |\mathbb{T}D\pi_2| \rangle \circ |\mathbb{T}D\langle \pi_1^S, \pi_2^S \rangle|$ . Now  $|\mathbb{T}D\langle \pi_1^S, \pi_2^S \rangle|$  maps every discrete valuation  $\nu$  on  $\mathbf{D}S$  to a discrete valuation  $\nu'$  on  $\mathbf{D}(|A_1| \times |A_2|)$  mapping any subset  $X$  of  $|A_1| \times |A_2|$  to  $\nu(X \cap S)$ . Further,  $|\mathbb{T}D\pi_1|$  maps a discrete valuation  $\nu'$  on  $\mathbf{D}(|A_1| \times |A_2|)$  to a discrete valuation  $\nu_1$  on  $\mathbf{D}|A_1|$  that maps the subset  $O_1$  of  $A_1$  to  $\nu'(O_1 \times A_2)$ . Similarly,  $|\mathbb{T}D\pi_2|$  maps a discrete valuation  $\nu'$  on  $\mathbf{D}(|A_1| \times |A_2|)$  to a discrete valuation  $\nu_2$  on  $\mathbf{D}|A_2|$  that maps the subset  $O_2$  of  $A_2$  to  $\nu'(A_1 \times O_2)$ . Therefore  $f$

maps  $\nu \in \mathbb{T}S$  to  $(\nu_1, \nu_2)$  such that for every subset  $O_1$  of  $A_1$ ,  $\nu_1(O_1) = \nu((O_1 \times |A_2|) \cap S)$ , and for every subset  $O_2$  of  $A_2$ ,  $\nu_2(O_2) = \nu(|A_1| \times O_2 \cap S)$ .

As  $\tilde{S}$  is the range of this function  $f$ , the lifted relation  $\tilde{S}$  between continuous valuations  $\nu_1 \in |\mathbf{T}A_1|$  and  $\nu_2 \in |\mathbf{T}A_2|$  is given by:

$$(\nu_1, \nu_2) \in \tilde{S} \iff \left( \exists \nu \in \mathbb{T}S. \begin{cases} \forall O_1 \subseteq A_1. \nu_1(O_1) = \nu((O_1 \times A_2) \cap S) \wedge \\ \forall O_2 \subseteq A_2. \nu_2(O_2) = \nu((A_1 \times O_2) \cap S) \end{cases} \right)$$

10.10.3. *Probabilistic Transition Systems.* Interestingly, and analogously with Section 10.7, we may define a *probabilistic* labelled transition system as an element of  $(\mathbf{T}A)^{A \times L}$ . Then two such transition systems  $f_1$  and  $f_2$  are in relation if and only if:

$$\forall a_1 \in |A_1|, a_2 \in |A_2| \quad , \quad \ell_1, \ell_2 \in L. (a_1, a_2) \in S \wedge (\ell_1, \ell_2) \in R_L \Rightarrow \quad (61)$$

$$\left( \exists \nu \in \mathbb{T}S \quad . \quad \begin{cases} \forall O_1 \subseteq A_1. f_1(a_1, \ell_1)(O_1) = \nu((O_1 \times A_2) \cap S) \wedge \\ \forall O_2 \subseteq A_2. f_2(a_2, \ell_2)(O_2) = \nu((A_1 \times O_2) \cap S) \end{cases} \right)$$

Note that this all works with the monad of probability valuations (where the measure of whole spaces must be 1), or of subprobability evaluations (where the measure of whole spaces must be at most 1), for example.

Look at the special case of finite sets  $A_1, A_2$  seen as discrete dcpos, and with discrete probability distributions, and the relation on labels  $R_L$  is equality. View the disjoint union  $A_1 + A_2$  as the state space of a unique combined probabilistic transition system. Write  $p_\ell(a, a')$  the probability of the transition  $a \xrightarrow{\ell} a'$ : this is  $f_1(a, \ell)(\{a'\})$  if  $a, a' \in A_1$ ,  $f_2(a, \ell)(\{a'\})$  if  $a, a' \in A_2$ , zero otherwise. Let  $\equiv$  be the smallest equivalence relation on  $A_1 + A_2$  containing  $S$ . For every  $C \in (A_1 + A_2)/\equiv$ :

$$((C \cap A_1) \times A_2) \cap S = (A_1 \times (C \cap A_2)) \cap S \quad (62)$$

Indeed, the inclusion of the left-hand side in the right-hand side means that for every  $a_1 \in C \cap A_1$  and for every  $a_2 \in A_2$  such that  $(a_1, a_2) \in S$ , then  $a_2$  is in  $C$ . The converse inclusion means that for every  $a_2 \in C \cap A_2$  and every  $a_1 \in A_1$  such that  $(a_1, a_2) \in S$  then  $a_1 \in C$ . Both inclusions hold because in each case  $a_1 \equiv a_2$  and  $C$  is an equivalence class for  $\equiv$ . So, if (61) holds, letting  $\nu_i$  be  $f_i(a_i, \ell)$ ,  $O_1 = C \cap A_1$ ,  $O_2 = C \cap A_2$ , then  $\nu_1(C \cap A_1) = \nu_2(C \cap A_2)$ . Since  $\nu_i(C \cap A_i) = \sum_{a \in C} p_\ell(a_i, a)$ , (61) implies:

$$\forall a_1, a_2 \in A_1 + A_2. (a_1, a_2) \in S \Rightarrow \forall \ell \in L. \forall C \in (A_1 + A_2)/\equiv . \quad (63)$$

$$\sum_{a \in C} p_\ell(a_1, a) = \sum_{a \in C} p_\ell(a_2, a)$$

Since  $\equiv$  is the reflexive symmetric transitive closure of  $S$ , iterating (63) entails:

$$\forall a_1 \in A_1, a_2 \in A_2. a_1 \equiv a_2 \Rightarrow \forall \ell \in L. \forall C \in (A_1 + A_2)/\equiv . \quad (64)$$

$$\sum_{a \in C} p_\ell(a_1, a) = \sum_{a \in C} p_\ell(a_2, a)$$

This is exactly Larsen and Skou's condition that  $\equiv$  be a *probabilistic bisimulation* (Larsen and Skou, 1991).

Conversely, (64), i.e., that  $\nu_1(C \cap A_1) = \nu_2(C \cap A_2)$  for every equivalence class  $C$  of

$\equiv$  where  $\nu_1$  and  $\nu_2$  are defined as above, implies (61) in case  $S$  is an equivalence relation already ( $S = \equiv$ ): take  $\nu(\{(a'_1, a'_2)\}) = \nu_1(\{a'_1\}) \times \nu_2(\{a'_2\}) / \nu_1(C \cap A_1) = \nu_1(\{a'_1\}) \times \nu_2(\{a'_2\}) / \nu_2(C \cap A_2)$  if  $a'_1 \equiv a'_2$ , where  $C$  is the equivalence class of  $a'_1$  and  $a'_2$ , and provided  $\nu_1(C \cap A_1) = \nu_2(C \cap A_2) \neq 0$ ; zero otherwise. Indeed, first observe that this is a probability distribution:  $\sum_{a'_1 \in A_1, a'_2 \in A_2} \nu(\{(a'_1, a'_2)\}) = \sum_C \sum_{a'_1 \in C \cap A_1} \sum_{a'_2 \in C \cap A_2} \nu_1(\{a'_1\}) \times \nu_2(\{a'_2\}) / \nu_2(C \cap A_2) = \sum_C \sum_{a'_1 \in C \cap A_1} \nu_1(\{a'_1\}) = \sum_C \nu_1(C \cap A_1) = 1$ .

Second,  $\nu(\{(a'_1\} \times A_2) \cap S) = \sum_{a'_2/a'_1 \equiv a'_2} \nu(\{(a'_1, a'_2)\}) = \sum_{a'_2 \in C \cap A_2} \nu(\{(a'_1, a'_2)\})$  (where  $C$  is the equivalence class of  $a'_1$ )  $= \sum_{a'_2 \in C \cap A_2} \nu_1(\{a'_1\}) \times \nu_2(\{a'_2\}) / \nu_2(C \cap A_2) = \nu_1(\{a'_1\})$ . Summing over all  $a'_1 \in O_1$ , where  $O_1 \subseteq A_1$ , we obtain  $\nu((O_1 \times A_2) \cap S) = \nu_1(O_1)$ . Symmetrically,  $\nu((A_1 \times O_2) \cap S) = \nu_2(O_2)$ , and this is exactly condition (61). Recall that  $\nu_i = f_i(a_i, \ell)$ .

So the two notions are equivalent, in the discrete case: relations  $S$  as described by our subscone construction have probabilistic bisimulations as reflexive symmetric transitive closures, and probabilistic bisimulations are equivalence relations  $S$  as described by our construction—in the discrete probability case.

### 10.11. Continuous Valuations on Dcpo, observed from **Cpo**

The construction of the previous section is somewhat limited, since the only valuations in  $\mathbb{T}\mathcal{S}$  are discrete. In particular, and this can be seen on condition (61), the only transition systems that can be related by the construction of the previous section are themselves discrete transition systems.

In the case of continuous transition systems, a more expressive construction must be sought. Here we take  $\mathbb{C} = \mathbf{Cpo}$ . Our first concern is to define a relevant mono factorization system on  $\mathbf{Cpo}$ . One that works here is as follows. First, for every dcpo  $A$  and every  $X \subseteq A$ , the *sup-closure*  $\overline{X}$  of  $X$  in  $A$  is the smallest sup-closed subset of  $A$  containing  $X$  (i.e., the smallest sub-dcpo of  $A$  containing  $X$ ). Note two pitfalls here: first,  $\overline{X}$  is in general not just the set of sups in  $A$  of directed subsets of  $X$ , and the process of taking sups may have to be iterated transfinitely; second, the sup-closure is in general much smaller than the Scott-topological closure of  $X$  in  $A$ , which is  $\downarrow \overline{X}$ . Now  $e : A \twoheadrightarrow C$  is a pseudoepi if and only if  $e(\overline{A}) = C$ , i.e., every element in  $C$  is in the sup-closure of the range of  $e$ . And  $m : C \twoheadrightarrow B$  is a relevant mono if and only if  $m$  is a *dcpo embedding*, meaning a Scott-continuous function such that  $m(x) \leq m(y)$  if and only if  $x \leq y$ . If this is the case, then the range  $m(C)$  of  $m$  is a dcpo,  $m$  is injective, and the inverse  $m^{-1} : m(C) \rightarrow C$  is continuous. This yields a mono factorization, even an epi-mono factorization on  $\mathbf{Cpo}$ , as can be checked easily. In fact what we called pseudoepis are just the epis in  $\mathbf{Cpo}$ , and the relevant monos are exactly the *extremal monos*; the latter correspond exactly to sub-dcpo up to iso.

It is tempting to define the monad  $\mathbb{T}$  just as  $\mathbf{T}$ , since the categories  $\mathbf{C}$  and  $\mathbb{C}$  are the same. However, condition (iv) would be problematic. Indeed, we have not been able to show that  $\mathbf{T}$  preserved pseudoepis in  $\mathbf{Cpo}$ ; we conjecture that it does not, although we have been unable to find any counterexample. Moreover, the structure of relevant pseudoepis seems intricate, and showing that  $\mathbf{T}$  maps relevant pseudoepis to pseudoepis seems arduous, too.

Let us discuss this a bit more. Following Jones (Jones, 1990), call a valuation *simple* if and only if it is a finite linear combination  $\sum_{i=1}^n a_i \delta_{x_i}$  of Dirac valuations. Call a valuation *quasi-simple* if and only if it is a sup of a directed family of simple valuations, and *accessible* if and only if it is in the sup-closure of the set of all simple valuations. Every discrete valuation is quasi-simple, and every quasi-simple valuation is accessible, but converses do not hold in general (Goubault-Larrecq, 2004). Now it seems likely that there should be a dcpo  $A$  on which some continuous valuation  $\nu$  is not in the sup-closure of the set of simple valuations, i.e., containing a continuous, but not accessible valuation. We have no example of this, but Alvarez-Manilla (Alvarez-Manilla, 2000, Section 2.7, p.73) shows an example of a dcpo and a continuous valuation on it that is not quasi-simple, which comes close; also, all accessible valuations are point-continuous valuations since the space of point-continuous valuations forms a dcpo containing all simple valuations (Heckmann, 1996). So any dcpo on which there is a continuous, but not point-continuous valuation  $\nu$  would yield an example of a continuous valuation  $\nu$  that is not accessible. In any case, if  $A$  and  $\nu$  exist so that  $\nu$  is continuous but not accessible, then let  $B$  be the dcpo have the same underlying set as  $A$  but with equality as ordering, and let  $i : B \rightarrow A$  the continuous function mapping  $x \in B$  to  $x \in A$ . Clearly  $i$  is surjective, hence pseudoepi. On the other hand, every continuous valuation on the discrete space  $B$  is discrete (Goubault-Larrecq, 2004, Proposition 3), and the image  $\sum_{x \in B} a_x \delta_{i(x)}$  of any discrete valuation  $\sum_{x \in B} a_x \delta_x$  by  $\mathbb{T}i$  is again discrete, so  $\nu$  cannot be in the sup-closure of the range of  $\mathbb{T}i$ , showing that  $\mathbb{T}i$  is not pseudoepi.

Instead of defining  $\mathbb{T}A$  as the dcpo of all continuous valuations on  $A$ , we let  $\mathbb{T}A$  be the dcpo of all accessible valuations on  $A$ . Note that by Theorem 5.2 of (Jones, 1990), if  $A$  is a *continuous* dcpo, every continuous valuation on  $A$  is quasi-simple, in particular accessible. So there would be no difference between  $\mathbb{T}A$  and  $\mathbf{T}A$  if we restricted ourselves to a category of continuous dcpos. Moreover, if  $A$  is a continuous dcpo, so is  $\mathbf{T}A$  (Jones, 1990, Corollary 5.4). Unfortunately, no cartesian-closed subcategory of  $\mathbf{Cpo}$  is known that consists only of continuous dcpos and to which the monad  $\mathbf{T}$  has a restriction (Jung and Tix, 1998).

All this forces us to be content with accessible valuations. We start with a few required technical lemmas. Recall that  $\overline{X}$  denotes the sup-closure of  $X$ .

**Lemma 4.** Let  $X$  be any subset of the dcpo  $A$ . Every sup of a directed family of elements of the sup-closure  $\overline{X}$  is in  $\overline{X}$ .

*Proof.* Obvious. □

**Corollary 1.** Every sup of a directed family of accessible valuations is accessible.

**Lemma 5.** Let  $X$  be any subset of  $\mathbf{T}A$  that is closed under linear combinations, i.e., such that whenever  $\nu_1, \dots, \nu_n \in X$ , and  $a_1, \dots, a_n \in \mathbb{R}_+$ , then  $\sum_{i=1}^n a_i \nu_i \in X$ . Then  $\overline{X}$  is closed under linear combinations.

*Proof.* Define  $\overline{X}_\alpha$  for each ordinal  $\alpha$  by  $\overline{X}_0 = X$ , and if  $\alpha \neq 0$  then  $\overline{X}_\alpha$  is the set of all sups of directed families  $(\nu_i)_{i \in I}$  of continuous valuations  $\nu_i \in \overline{X}_{\alpha_i}$ ,  $\alpha_i < \alpha$ . Clearly  $\overline{X} = \bigcup_\alpha \overline{X}_\alpha$ .

We first claim that if  $\nu$  is in  $\overline{X}_\alpha$  then  $a\nu$  is in  $\overline{X}_\alpha$ , by induction on  $\alpha$ . If  $\alpha = 0$ , then by assumption  $a\nu \in X = \overline{X}_0$ . Otherwise,  $\nu = \sup_{i \in I} \nu_i$  for some directed family  $(\nu_i)_{i \in I}$  of valuations in  $\overline{X}_{\alpha_i}$ . Then the family  $(a\nu_i)_{i \in I}$  is also directed,  $a\nu_i$  is in  $\overline{X}_{\alpha_i}$  by induction hypothesis, and  $a\nu = \sup_{i \in I} a\nu_i$ .

We then claim that if  $\nu$  is in  $\overline{X}_\alpha$  and  $\nu'$  is in  $\overline{X}_{\alpha'}$ , then  $\nu + \nu'$  is in  $\overline{X}$ . This is by induction on  $\alpha, \alpha'$  ordered lexicographically. If  $\alpha = \alpha' = 0$ , then both  $\nu$  and  $\nu'$  are in  $X$ , hence  $\nu + \nu'$  is in  $X$  by assumption, therefore in  $\overline{X}$ . If  $\alpha' \neq 0$ , then  $\nu'$  is the sup of some directed family  $(\nu'_j)_{j \in J}$  of valuations in  $\overline{X}_{\alpha'_j}$ ,  $\alpha'_j < \alpha'$ . By induction hypothesis  $\nu + \nu'_j$  is in  $\overline{X}$ ; the family  $(\nu + \nu'_j)_{j \in J}$  is clearly directed, and  $\sup_{j \in J} \nu + \nu'_j = \nu + \nu'$ , so  $\nu + \nu'$  is in  $\overline{X}$ . If  $\alpha \neq 0$ , then  $\nu$  is the sup of some directed family  $(\nu_i)_{i \in I}$  of valuations in  $\overline{X}_{\alpha_i}$ ,  $\alpha_i < \alpha$ . By induction hypothesis,  $\nu_i + \nu'$  is in  $\overline{X}$ , so similarly  $\nu + \nu'$  is in  $\overline{X}$ .  $\square$

**Corollary 2.** Every finite linear combination of accessible valuations is accessible.

*Proof.* Take  $X$  the set of simple valuations on  $A$ .  $\square$

In the sequel, call a valuation  $\alpha$ -accessible if and only if it is in  $\overline{X}_\alpha$  where  $X$  is the set of simple valuations. In other words, the 0-accessible valuations are the simple valuations; if  $(\nu_i)_{i \in I}$  is a directed family of valuations, where  $\nu_i$  is  $\alpha_i$ -accessible for some  $\alpha_i < \alpha$ , then  $\sup_{i \in I} \nu_i$  is  $\alpha$ -accessible. The accessible valuations are those that are  $\alpha$ -accessible for some ordinal  $\alpha$ .

Define  $\eta_A$  as mapping  $x \in A$  to  $\delta_x$  in  $\mathbb{T}A$ . This is clearly natural in  $A$ . Define  $\mu_A$  as mapping every accessible valuation  $\nu$  on  $A$  to the continuous valuation  $\int_{\xi \in \mathbb{T}A} \xi(O) d\nu$ .

**Lemma 6.** For every accessible valuation  $\nu$ ,  $\mu_A(\nu)$  is an accessible valuation.

*Proof.* We show that  $\mu_A(\nu)$  is accessible whenever  $\nu$  is  $\alpha$ -accessible, by induction on  $\alpha$ . If  $\alpha = 0$ , then  $\nu$  is simple, that is,  $\nu$  is of the form  $\sum_{i=1}^n a_i \delta_{\xi_i}$ . Then  $\mu_A(\nu) = \sum_{i=1}^n a_i \xi_i$ , which is accessible by Corollary 2. If  $\alpha \neq 0$ , then  $\nu$  is the directed sup of a family of  $\alpha_i$ -accessible valuations  $\nu_i$ ,  $\alpha_i < \alpha$ . Since integration is continuous in its valuation argument,  $\mu_A(\nu) = \sup_{i \in I} \mu_A(\nu_i)$ , hence is accessible again by Corollary 1.  $\square$

So  $\mu_A$  is a map from  $\mathbb{T}^2 A$  to  $\mathbb{T}A$ . It is also continuous, since integration is continuous in its function argument, and the map  $\xi \mapsto \xi(O)$  is Scott-continuous for every open  $O$ . Moreover,  $\mu_A$  is natural in  $A$ : if  $f$  is continuous from  $A$  to  $B$ , then  $\mu_B(\mathbb{T}^2 f(\nu))(O) = \int_{\eta \in \mathbb{T}B} \eta(O) d\mathbb{T}^2 f(\nu) = \int_{\xi \in \mathbb{T}A} \mathbb{T}f(\xi)(O) d\nu$  (using the Change of Variables formula) =  $\int_{\xi \in \mathbb{T}A} \xi(f^{-1}(O)) d\nu$  (by definition of  $\mathbb{T}f$ ) =  $\mu_A(\nu)(f^{-1}(O)) = \mathbb{T}f(\mu_A(\nu))(O)$  (by definition of  $\mathbb{T}f$ ).

We define  $\mathfrak{d}_{A,B}$  as mapping  $\nu \in \mathbb{T}A$  and  $\xi \in \mathbb{T}B$  to the continuous valuation  $\nu \otimes \xi$  that takes the open subset of the dcpo  $A \times B$  to  $\int_{x \in A} \left( \int_{y \in B} \chi_O(x, y) d\xi \right) d\nu$ .

**Lemma 7.** For every accessible valuations  $\nu$  and  $\xi$ ,  $\nu \otimes \xi$  is an accessible valuation.

*Proof.* We show the claim under the assumption that  $\nu$  is  $\alpha$ -accessible and  $\xi$  is  $\alpha'$ -accessible, by induction on  $\alpha, \alpha'$  ordered lexicographically. If  $\alpha = \alpha' = 0$ , then  $\nu$  and  $\xi$  are simple, say  $\nu = \sum_{i=1}^m a_i \delta_{x_i}$  and  $\xi = \sum_{j=1}^n b_j \delta_{y_j}$ , so  $\nu \otimes \xi = \sum_{i=1}^m \sum_{j=1}^n a_i b_j \delta_{(x_i, y_j)}$  is simple, hence accessible. If  $\alpha' \neq 0$ , then  $\xi$  is the sup of some directed family  $(\xi_j)_{j \in J}$ .

of  $\alpha'_j$ -accessible valuations,  $\alpha'_j < \alpha'$ . Since integration is continuous,  $\mathfrak{d}_{A,B}(\nu, \xi)(O) = \sup_{j \in J} \mathfrak{d}_{A,B}(\nu, \xi_j)(O)$ , where  $\mathfrak{d}_{A,B}(\nu, \xi_j)$  is accessible by induction hypothesis; so  $\mathfrak{d}_{A,B}(\nu, \xi)$  is accessible. By a similar argument,  $\mathfrak{d}_{A,B}(\nu, \xi)$  is accessible when  $\alpha \neq 0$ .  $\square$

Clearly  $\mathfrak{d}_{A,B}$  is continuous, since  $\mathbf{d}_{A,B}$  is, and the two coincide on accessible valuations. Also,  $\mathfrak{d}_{A,B}$  is natural in  $A$  and  $B$ : if  $f$  is a continuous map from  $A$  to  $A'$  and  $g$  is a continuous map from  $B$  to  $B'$ ,  $\mathfrak{d}_{A',B'}(\mathbb{T}f(\nu), \mathbb{T}g(\xi))(O) = \int_{x' \in A'} \int_{y' \in B'} \chi_O(x', y') d\mathbb{T}g(\xi) d\mathbb{T}f(\nu) = \int_{x \in A} \int_{y \in B} \chi_O(f(x), g(y)) d\xi d\nu$  (using the Change of Variables formula). This is equal to  $\int_{x \in A} \int_{y \in B} \chi_{(f \times g)^{-1}(O)}(x, y) d\xi d\nu = \mathfrak{d}_{A,B}(\nu, \xi)((f \times g)^{-1}(O)) = \mathbb{T}(f \times g)(\mathfrak{d}_{A,B}(\nu, \xi))$ .

All this leads to:

**Lemma 8.**  $(\mathbb{T}, \eta, \mathbb{I}, \mathfrak{d})$  is a monoidal monad on **Cpo**, the *accessible valuation monad*. Letting  $|\_$  be the identity endofunctor on **Cpo**, and  $\sigma_A : \mathbb{T}|A| \rightarrow |TA|$  map every accessible valuation  $\nu$  on  $A$  to itself, seen as a continuous valuation on  $A$ ,  $(|\_ , \sigma)$  is a monoidal monad morphism from **T** to  $\mathbb{T}$ .

*Proof.* It remains to show that  $(|\_ , \sigma)$  is a monoidal monad morphism. But since both  $|\_$  and  $\sigma$  are identities, Diagrams (4) and (45) are obvious. (Note that the mediating pair  $(\mathbb{0}, \mathbb{I})$  of  $|\_$  consists of identities exclusively.)  $\square$

So conditions **(i.c)**, **(ii.a)** hold. (In fact even condition **(ii.b)**.) Condition **(iii.a)** holds, too: our mono factorization system on **Cpo** is monoidal, since for every two pseudoepi  $f$  and  $g$ ,  $f \times g$  is pseudoepi, for sups are taken componentwise. As we said earlier, the main difficulty with probabilistic powerdomains is in showing that condition **(iv)** holds:

**Lemma 9.** If  $f : A \rightarrow B$  is pseudoepi in **Cpo**, then so is  $\mathbb{T}f$ .

*Proof.* Let  $X$  be the set of valuations of the form  $\sum_{i=1}^m a_i \delta_{f(x_i)}$ , where  $m \in \mathbb{N}$ ,  $a_i \in \mathbb{R}_+$ ,  $x_i \in A$ . Observe that  $\sum_{i=1}^m a_i \delta_{f(x_i)} = \mathbb{T}f(\sum_{i=1}^m a_i \delta_{x_i})$ , so  $X$  is included in the range of  $f$ . We show that every accessible valuation  $\xi$  on  $B$  is in  $\overline{X}$ , which will imply that it is in the sup-closure of the range of  $f$ , hence that  $\mathbb{T}f$  is pseudoepi.

It suffices to show that every  $\alpha$ -accessible valuation  $\xi$  in  $\mathbb{T}B$  is in  $\overline{X}$ , by induction on  $\alpha$ . If  $\alpha = 0$ , then  $\xi$  is simple, hence can be written under the form  $\sum_{j=1}^n b_j \delta_{y_j}$ , for some finite family of elements  $y_j$  in  $B$ . Since  $f$  is pseudoepi,  $y_j$  is in the sup-closure of the range of  $f$ . Since the map  $x \mapsto \delta_x$  (i.e.,  $\eta_A$ ) is Scott continuous,  $\delta_{y_j}$  is in the sup-closure of the set of all Dirac maps of the form  $\delta_{f(x)}$ ,  $x \in A$ . (Note indeed that, for every Scott continuous map  $g$ ,  $g$  maps the sup-closure of any subset to the sup-closure of its direct image.) So  $\delta_{y_j}$  is in  $\overline{X}$ , therefore  $\xi = \sum_{j=1}^n b_j \delta_{y_j}$  is in  $\overline{X}$ , too, by Lemma 5. If  $\alpha \neq 0$ , then  $\nu$  is the sup of some directed family  $(\nu_i)_{i \in I}$  of  $\alpha_i$ -accessible valuations  $\nu_i$ ,  $\alpha_i < \alpha$ . By induction hypothesis,  $\nu_i$  is in  $\overline{X}$ , so  $\nu$  is, too.  $\square$

Next, condition **(v)**, that **Cpo** has pullbacks, holds: if  $f$  is a continuous map from  $A$  to  $C$ , and  $g$  is continuous from  $B$  to  $C$ , their pullback is the set  $\{(x, y) \in A \times B \mid f(x) = g(y)\}$  with the product ordering. Finally, we show condition **(vi)**:

**Lemma 10.** For every dcpo  $A$ ,  $\_{}^A$  preserves relevant monos in **Cpo**.

*Proof.* For every dcpo embedding (relevant mono)  $m$  from  $B$  to  $C$ ,  $m^A$  is the continuous



map sending every continuous map  $f \in B^A$  to  $m \circ f \in C^A$ . Then  $m^A(f) \leq m^A(g)$  if and only if  $m \circ f \leq m \circ g$ , if and only if  $m(f(x)) \leq m(g(x))$  for every  $x \in A$ , if and only if  $f(x) \leq g(x)$  for every  $x \in A$  (since  $m$  is a dcpo embedding), if and only if  $f \leq g$ . Since  $m^A$  is continuous, it is a dcpo embedding.  $\square$

So all conditions **(i.c)**, **(ii.a)**, **(iii.a)**, **(iv)**, **(v)**, **(vi)** necessary to define a notion of subscone in **Cpo** for Moggi's meta-language are met. It is fair to call such a subscone a *cpo logical relation*, extending Plotkin's cpo logical relations for the (non-monadic) lambda-calculus (Mitchell, 1996).

All this yields the following notion of binary logical relation  $\llbracket \tau \rrbracket$  between dcpos  $\llbracket \tau \rrbracket_1$  and  $\llbracket \tau \rrbracket_2$  indexed by types of Moggi's meta-language, where the interpretations  $\llbracket \_ \rrbracket_i$  are defined so that  $\llbracket \tau \rightarrow \tau' \rrbracket_i$  is the dcpo of all continuous functions from  $\llbracket \tau \rrbracket_i$  to  $\llbracket \tau' \rrbracket_i$  and  $\llbracket \mathbf{T}\tau \rrbracket_i = \mathbf{T} \llbracket \tau \rrbracket_i$ . Call a binary relation  $R$  between partial orders a *dcpo relation* if and only if  $R$ , as a set of pairs, is a dcpo. The family of dcpo relations  $\llbracket \tau \rrbracket$  indexed by types  $\tau$  is a *dcpo logical relation* if and only if:

$$\begin{aligned} (f_1, f_2) \in \llbracket \tau \rightarrow \tau' \rrbracket &\iff \forall a_1 \in \llbracket \tau \rrbracket_1, a_2 \in \llbracket \tau \rrbracket_2 \cdot \\ &\quad (a_1, a_2) \in \llbracket \tau \rrbracket \Rightarrow (f_1(a_1), f_2(a_2)) \in \llbracket \tau' \rrbracket \\ ((a_1, a'_1), (a_2, a'_2)) \in \llbracket \tau \times \tau' \rrbracket &\iff (a_1, a_2) \in \llbracket \tau \rrbracket \wedge (a'_1, a'_2) \in \llbracket \tau' \rrbracket \end{aligned}$$

and finally  $(\nu_1, \nu_2) \in \llbracket \mathbf{T}\tau \rrbracket$  if and only if  $(\nu_1, \nu_2)$  is in the sup-closure inside  $\mathbf{T} \llbracket \tau \rrbracket_1 \times \mathbf{T} \llbracket \tau \rrbracket_2$  of the set of all pairs of continuous valuations  $(\nu'_1, \nu'_2)$  such that there is an accessible valuation  $\nu$  on the dcpo  $\llbracket \tau \rrbracket$  with

$$\begin{cases} \forall O_1 \text{ open of } \llbracket \tau \rrbracket_1 \cdot \nu'_1(O_1) = \nu((O_1 \times \llbracket \tau \rrbracket_2) \cap \llbracket \tau \rrbracket) \wedge \\ \forall O_2 \text{ open of } \llbracket \tau \rrbracket_2 \cdot \nu'_2(O_2) = \nu((\llbracket \tau \rrbracket_1 \times O_2) \cap \llbracket \tau \rrbracket) \end{cases}$$

In other words,  $\nu_1$  and  $\nu_2$  are related provided they are transfinitely iterated directed sups of valuations  $\nu'_1$  and  $\nu'_2$  related by some accessible valuation  $\nu$  on  $\llbracket \tau \rrbracket$  as in the formula above. (We do not need to mention any sup-closure in the clauses for  $\llbracket \tau \rightarrow \tau' \rrbracket$  and  $\llbracket \tau \times \tau' \rrbracket$  because the definitions above are already sup-closed.)

This definition is not quite perfect. First, it is complex. Second, as shown above,  $\llbracket \mathbf{T}\tau \rrbracket$  is only able to relate accessible valuations. If there are any dcpos on which not all continuous valuations are accessible, then  $\llbracket \mathbf{T}\tau \rrbracket$  will not be reflexive, which is a bit surprising. We have not been able to generalize the construction further. On the other hand, most dcpos of interest are continuous, and on continuous dcpos every continuous valuation is quasi-simple, hence accessible.

## 10.12. Probability Measures on Metric Spaces

The standard approach to probability theory is not through valuations, as used until now, but through measures. As we said in Section 10.9, the category of measurable spaces is ill-suited to this task. A much better-behaved category is the category **Pol** of Polish spaces and continuous maps. A *Polish space* is the topological space underlying a complete separable metric space. (Separable means that the space has a countable dense subset.) It was shown by Giry (Giry, 1981) that there is a probability monad  $\mathbf{T}$  on **Pol**, where  $\mathbf{T}A$

is the set of all probability measures on the Borel  $\sigma$ -algebra of  $A$ , topologized with the *weak topology*, that is, the smallest topology making the function  $\lambda \nu \in \mathbf{TA} \cdot \int_{x \in A} f(x) d\nu$  continuous, for every bounded continuous function  $f$  from  $A$  to  $\mathbb{R}_+$ .

The closest category to **Pol** that we know of, and which is symmetric monoidal closed, is the category **CMet** of complete metric spaces, with non-expansive maps as morphisms. The separability condition has to be dropped, which is not preserved by exponentials. We shall also drop the completeness requirement, which we simply do not need—the interested reader may check that the whole construction works directly on complete spaces, too.

So consider the category **Met** of metric spaces and non-expansive maps. We write  $d_A$  for the distance on  $A$ , and allow distances to take the value  $+\infty$ . (This is as in (Lawvere, 1973), who considers additional relaxations on the notion of distance.) In other words, a distance  $d_A$  on  $A$  is any function from  $A$  to  $\mathbb{R}_+ \cup \{+\infty\}$ , such that  $d_A(x, y) = 0$  if and only if  $x = y$ , for every  $x, y \in A$ ;  $d_A(x, y) = d_A(y, x)$  for every  $x, y \in A$ ; and  $d_A(x, z) \leq d_A(x, y) + d_A(y, z)$  for every  $x, y, z \in A$ . A map  $f$  is *non-expansive* from  $A$  to  $B$  if and only if  $d_B(f(x), f(x')) \leq d_A(x, x')$  for every  $x, x' \in A$ .

Every metric space  $A$  has a topology, generated by its open balls  $\mathcal{B}(x, \epsilon) = \{y \mid d_A(x, y) < \epsilon\}$ , for which it is Hausdorff. There is a cartesian product on **Met**:  $A \times B$  is the set of pairs  $(x, y)$ ,  $x \in A$ ,  $y \in B$ , with distance  $d_{A \times B}$  defined by  $d_{A \times B}((x, y), (x', y')) = \max(d_A(x, x'), d_B(y, y'))$ . But **Met** is not cartesian closed, as there is no corresponding notion of exponential.

On the other hand, **Met** is symmetric monoidal closed. The tensor product  $A \otimes B$  is the set of pairs  $(x, y)$ ,  $x \in A$ ,  $y \in B$ , with distance  $d_{A \otimes B}$  defined by  $d_{A \otimes B}((x, y), (x', y')) = d_A(x, x') + d_B(y, y')$ . The exponential  $C^B$  is the set of non-expansive maps  $f$  from  $B$  to  $C$ , with distance  $d_{C^B}(f, f') = \sup_{x \in B} d_C(f(x), f'(x))$ . (This is well-defined because we allow distances to take the value  $+\infty$ .) The underlying topology of  $A \otimes B$ , as well as of  $A \times B$ , is the product topology of  $A$  and  $B$ .

A *measure* on  $A$  is by convention a measure on the Borel  $\sigma$ -algebra of its topology. A *probability* measure maps  $A$  to 1. A natural choice for the probability monad **T** on **Met** is to let **TA** be the set of all probability measures on  $A$ , equipped with the *Hutchinson metric*:

$$d_{\mathbf{TA}}(\nu, \xi) = \sup_{g \in \mathbf{Met}(A, [0, 1])} \left| \int_{x \in A} g(x) d\nu - \int_{x \in A} g(x) d\xi \right|$$

making **TA** a metric space. It can be checked that **T** gives rise to a monoidal monad.

It is then tempting to observe **T** from **Met** again. That is, we are tempted to let  $\mathbb{C} = \mathbf{Met}$  again, and **T** to be the same monad as **T**. However, we do not know whether this preserves pseudoepis. Let us make this clear.

There is a natural epi-mono factorization on **Met**: relevant monos are *isometric embeddings*, that is, maps  $m : C \rightarrow B$  such that  $d_B(m(x), m(y)) = d_C(x, y)$  for all  $x, y \in C$ . Pseudoepis from  $A$  to  $C$  are *surjective* non-expansive maps from  $A$  to  $C$ . It is fairly easy to see that this yields a monoidal mono factorization system on **Met**. (By the way, we *cannot* take  $\mathbb{C}$  to be **CMet**, even if we had chosen **C** to be **CMet**. In that case, we could not insist that pseudoepis be surjective. Then it would be natural to take pseudoepis as

non-expansive maps with dense range, but this would not be enough for the construction to come in Lemma 11 below, which requires surjectivity.)

Does  $\mathbf{T}$  defined above preserve surjective non-expansive maps? A standard result (Bourbaki, 1969, 2.4, Lemma 1) states that if  $f$  is continuous surjective from  $A$  to  $B$ , and both  $A$  and  $B$  are compact Hausdorff, then  $\mathbf{T}f$  is surjective. (Since  $\mathbf{T}$  is a functor on  $\mathbf{Met}$ ,  $\mathbf{T}f$  is always non-expansive as soon as  $f$  is.) Similarly if the underlying topological space of  $A$  is an analytic space (Bourbaki, 1969, 2.4, Proposition 9). However, we cannot restrict to compact metric spaces, or to metric analytic spaces—there is no reason why the monoidal closed structure on  $\mathbf{Met}$  would survive this restriction.

Instead, we use another monad  $\mathbb{T}$  on  $\mathbf{Met}$ . Let  $\mathcal{C}_b(A)$  be the vector space of all bounded non-expansive functions from the metric space  $A$  to  $\mathbb{R}$ , with the sup norm: for all  $g \in \mathcal{C}_b(A)$ ,  $\|g\| = \sup_{x \in A} |g(x)|$ . Let  $\mathcal{L}_A$  be the vector space of all continuous linear forms on  $\mathcal{C}_b(A)$ , that is, of all continuous linear functions  $F : \mathcal{C}_b(A) \rightarrow \mathbb{R}$ . Recall that a linear function  $F$  from  $\mathcal{C}_b(A)$  to  $\mathbb{R}$  is continuous if and only if it is Lipschitz, i.e., if and only if  $\|F\| = \sup_{\|g\|=1} |F(g)|$  is a well-defined real (i.e., not  $+\infty$ ). Then  $\|F\|$  is called the *norm* of  $F$ . Let  $\mathcal{L}_A^1$  be the subset of those  $F$  in  $\mathcal{L}_A$  of norm 1. Finally, let  $\mathbb{T}A$  be the subset of those  $F$  in  $\mathcal{L}_A^1$  which are positive, i.e., such that  $g \geq 0$  implies  $F(g) \geq 0$ . Both  $\mathcal{L}_A^1$  and  $\mathbb{T}A$  inherit the distance on  $\mathcal{L}_A$  that is induced by norm. Equivalently,  $\mathbb{T}A$  is a metric space whose distance is defined by  $d_{\mathbb{T}A}(F, G) = \sup_{g \in \mathbf{Met}(A, [-1, 1])} |F(g) - G(g)|$ .  $\mathbb{T}$  then defines an endofunctor on  $\mathbf{Met}$  by letting  $\mathbb{T}f$  map  $F \in \mathbb{T}A$  to  $\lambda g \in \mathcal{C}_b(B) \cdot F(g \circ f)$ , for every non-expansive map  $f$  from  $A$  to  $B$ .

The space of all probability measures on  $A$  embeds in  $\mathbb{T}A$  by  $\nu \mapsto \lambda g \in \mathcal{C}_b(A) \cdot \int_{x \in A} g(x) d\nu$ . In fact, the distance on probability measures on  $A$  inherited from the metric structure on  $\mathbb{T}A$  is a slight variant of the Hutchinson metric introduced above. Moreover, if  $A$  is compact, then every element of  $\mathbb{T}A$  arises from a probability measure in this way, by the Riesz Representation Theorem. This makes  $\mathbb{T}A$  a metric space that is arguably close enough to a space of probability measures.

The point of this definition of  $\mathbb{T}$  is that  $\mathbb{T}$  preserves pseudoepis in  $\mathbf{Met}$ . The proof is similar to that of (Bourbaki, 1969, 2.4, Lemma 1).

**Lemma 11.** *If  $f$  is any surjective non-expansive map from  $A$  to  $B$ , then  $\mathbb{T}f$  is surjective from  $\mathbb{T}A$  to  $\mathbb{T}B$ .*

*Proof.* Let  $\lambda_f$  be the function mapping  $g \in \mathcal{C}_b(B)$  to  $g \circ f$ . Note that  $\lambda_f(g)$  is bounded and non-expansive, hence in  $\mathcal{C}_b(A)$ . Also,  $\|\lambda_f(g)\| = \sup_{x \in A} |g(f(x))| = \sup_{y \in B} |g(y)|$  (since  $f$  is surjective)  $= \|g\|$ , so that  $\lambda_f$  is an isometric embedding of  $\mathcal{C}_b(B)$  into  $\mathcal{C}_b(A)$ .

Let  $H$  be the range of  $\lambda_f$  in  $\mathcal{C}_b(A)$ .  $H$  is a linear subspace of  $\mathcal{C}_b(A)$ , which is by construction isometrically isomorphic to  $\mathcal{C}_b(B)$ . In particular,  $\lambda_f^{-1}$  is a continuous linear map from  $H$  to  $\mathcal{C}_b(B)$ .

Given  $G \in \mathbb{T}B$ ,  $G \circ \lambda_f^{-1}$  is therefore a continuous linear form on  $H$ . By the Hahn-Banach Theorem,  $G \circ \lambda_f^{-1}$  can be extended to a continuous linear form  $F$  on  $\mathcal{C}_b(A)$ , with the same norm, i.e.,  $\|F\| = \|G \circ \lambda_f^{-1}\|$ . The latter means that  $\|F\| = \sup_{h \in \mathcal{C}_b(A), \|h\|=1} |F(h)| = \sup_{h \in H, \|h\|=1} |G(\lambda_f^{-1}(h))| = \sup_{g \in \mathcal{C}_b(B), \|g \circ f\|=1} |G(g)| = \|G\| = 1$ .

As far as positivity is concerned, we first note that  $\|G\| = G(\lambda y \in B \cdot 1)$ . Indeed, first,

$\|\lambda y \in B \cdot 1\| = 1$ , so  $\|G\| \geq G(\lambda y \in B \cdot 1)$ . To show the converse inequality, note first that for every  $g \in \mathcal{C}_b(B)$ , we may write  $g$  as the difference  $g_+ - g_-$  of two positive functions, so  $|G(g)| = |G(g_+) - G(g_-)| \leq \max(G(g_+), G(g_-))$  (since  $G(g_+), G(g_-) \geq 0$ ); if  $\|g\| = 1$ , then  $g_+(y) \leq 1$  and  $g_-(y) \leq 1$  for all  $y \in B$ , so  $|G(g)| \leq G(\lambda y \in B \cdot 1)$ ; taking the sup over all  $g$ ,  $\|G\| \leq G(\lambda y \in B \cdot 1)$ .

Then  $F(\lambda x \in A \cdot 1) = F(\lambda_f(\lambda y \in B \cdot 1)) = G(\lambda y \in B \cdot 1)$ . So  $F(\lambda x \in A \cdot 1) = \|G\| = \|F\|$ . If  $F$  was not positive, there would be some  $g \in \mathcal{C}_b(A)$ ,  $g \geq 0$ , with  $F(g) < 0$ . We may assume without loss of generality that  $g \leq 1$ , so  $\lambda x \in A \cdot 1 - g(x)$  is of norm at most 1. But then  $\|F\| \geq F(\lambda x \in A \cdot 1 - g(x)) = F(\lambda x \in A \cdot 1) - F(g) > F(\lambda x \in A \cdot 1)$ , a contradiction. So  $F$  is positive.

Finally,  $\mathbb{T}f(F) = \lambda g \in \mathcal{C}_b(B) \cdot F(g \circ f) = \lambda g \in \mathcal{C}_b(B) \cdot G(\eta(g \circ f)) = \lambda g \in \mathcal{C}_b(B) \cdot G(g) = G$ . Since  $G$  is arbitrary,  $\mathbb{T}f$  is surjective.  $\square$

$\mathbb{T}$  can be used to form a monoidal monad on **Met**. Define  $\eta_A$  as the function mapping  $x \in A$  to  $\lambda g \in \mathcal{C}_b(A) \cdot g(x)$ . This is natural in  $A$ , and non-expansive in  $x$ . Define  $\mu_A$  as mapping  $F \in \mathbb{T}^2 A$  to  $\lambda g \in \mathcal{C}_b(\mathbb{T}A) \cdot F(\lambda G \in \mathbb{T}A \cdot G(g))$ . It requires a bit more work to show that  $\mu_A$  is well-defined and non-expansive in  $F$ ; it is however clear that this is natural in  $A$ . Furthermore, the monad laws are satisfied, so that  $(\mathbb{T}, \eta, \mu)$  is a monad on **Met**. (One might remark that the formulae for  $\eta_A$  and  $\mu_A$  are formally analogous to those of the continuation monad.) Finally, define  $d_{A,B}$  as mapping  $(F, G) \in \mathbb{T}A \otimes \mathbb{T}B$  to  $\lambda h \in \mathcal{C}_b(A \otimes B) \cdot F(\lambda x \in A \cdot G(\lambda y \in B \cdot h(x, y)))$ . This is a mediator, which is not in general commutative. The underlying strength  $\mathbb{k}_{A,B}$  maps  $(x, G) \in A \otimes \mathbb{T}B$  to  $\lambda h \in \mathcal{C}_b(A \otimes B) \cdot G(\lambda y \in B \cdot h(x, y))$ .

To define a notion of binary metric logical relations, we take  $\mathbf{C} = \mathbf{Met} \times \mathbf{Met}$ , define a monad  $\mathbf{T}$  on  $\mathbf{C}$  pointwise (i.e.,  $\mathbf{T}(A_1, A_2) = (\mathbb{T}A_1, \mathbb{T}A_2)$ ). It remains to define a monad morphism from  $\mathbf{T}$  to  $\mathbb{T}$ .

Define  $|A_1, A_2|$  as the cartesian product  $A_1 \times A_2$ , much as in **Set**. Note that the only difference between cartesian product and tensor product is that their distances differ:  $d_{A_1 \times A_2}((x, y), (x', y'))$  is the max of the distances  $d_{A_1}(x, x')$  and  $d_{A_2}(y, y')$ , while  $d_{A_1 \otimes A_2}((x, y), (x', y'))$  is the sum of the same distances.

Similarly as in **Set**, define the monad morphism  $\sigma_{A_1, A_2}$  as  $\langle \mathbb{T}\pi_1, \mathbb{T}\pi_2 \rangle$ . That is, for every  $F \in \mathbb{T}(A_1 \times A_2)$ ,  $\sigma_{A_1, A_2}(F) = (\lambda g_1 \in \mathcal{C}_b(A_1) \cdot F(\lambda(x, y) \cdot g_1(x)), \lambda g_2 \in \mathcal{C}_b(A_2) \cdot F(\lambda(x, y) \cdot g_2(y)))$ . The functor  $|-|$  is symmetric monoidal. Its mediating pair  $(\mathbb{0}, \mathbb{i})$  consists of  $\mathbb{i} : \mathbb{I} \rightarrow |\mathbf{I}|$  (where  $\mathbb{I}$  is the one-element complete separable metric space  $\{*\}$ , and  $\mathbf{I}$  is the pair  $\mathbb{I}, \mathbb{I}$ ) mapping  $*$  to  $(*, *)$ ; and of the natural transformation  $\mathbb{0}_{(A_1, A_2), (B_1, B_2)} : |A_1, A_2| \otimes |B_1, B_2| \rightarrow |A_1 \otimes B_1, A_2 \otimes B_2|$  which maps  $((x_1, x_2), (y_1, y_2))$  to  $((x_1, y_1), (x_2, y_2))$ . The latter is non-expansive, because we have chosen  $\sigma_{A_1, A_2}$  to be  $A_1 \times A_2$  and not

$A_1 \otimes A_2$  for example. Indeed,

$$\begin{aligned}
& d_{|A_1 \otimes B_1, A_2 \otimes B_2|}(((x_1, y_1), (x_2, y_2)), ((x'_1, y'_1), (x'_2, y'_2))) \\
&= \max(d_{A_1 \otimes B_1}((x_1, y_1), (x'_1, y'_1)), d_{A_2 \otimes B_2}((x_2, y_2), (x'_2, y'_2))) \\
&= \max(d_{A_1}(x_1, x'_1) + d_{B_1}(y_1, y'_1), d_{A_2}(x_2, x'_2) + d_{B_2}(y_2, y'_2)) \\
&\leq \max(d_{A_1}(x_1, x'_1), d_{A_2}(x_2, x'_2)) + \max(d_{B_1}(y_1, y'_1), d_{B_2}(y_2, y'_2)) \\
&= d_{|A_1, A_2|}((x_1, x_2), (x'_1, x'_2)) + d_{|B_1, B_2|}((y_1, y_2), (y'_1, y'_2)) \\
&= d_{|A_1, A_2| \otimes |B_1, B_2|}(((x_1, x_2), (y_1, y_2)), ((x'_1, x'_2), (y'_1, y'_2)))
\end{aligned}$$

It is clear that  $\Theta$  is natural, and that the coherence conditions (16)–(18) as well as (27) are satisfied.

Then  $(|-|, \sigma)$  is a monoidal monad morphism, as Diagram (45) commutes. It is also easy to check that **Met** has pullbacks: the pullback of  $f : A \rightarrow C$  and  $g : B \rightarrow C$  is the subspace of  $A \times B$  consisting of all  $(x, y)$  such that  $f(x) = g(y)$ ; and  $_-^A$  preserves relevant monos:  $_-^A$  maps  $f : B \rightarrow C$  to  $\lambda g \in B^A \cdot f \circ g$ , which is an isometric embedding as soon as  $f$  is.

This yields the following notion of binary metric logical relation  $\llbracket \tau \rrbracket$  between metric spaces  $\llbracket \tau \rrbracket_1$  and  $\llbracket \tau \rrbracket_2$  indexed by types in the linear version of Moggi's meta-language adumbrated in Section 9.2:

$$\tau ::= b|\tau \otimes \tau| \tau \multimap \tau| \tau$$

Note that  $\llbracket \tau \otimes \tau' \rrbracket_i = \llbracket \tau \rrbracket_i \otimes \llbracket \tau' \rrbracket_i$ ,  $\llbracket \tau \multimap \tau' \rrbracket_i = \llbracket \tau' \rrbracket_i^{\llbracket \tau \rrbracket_i}$ ,  $\llbracket ?\tau \rrbracket_i = \mathbb{T} \llbracket \tau \rrbracket_i$ . Then the definition of the subscone specializes to:

$$\begin{aligned}
(f_1, f_2) \in \llbracket \tau \multimap \tau' \rrbracket &\iff \forall a_1 \in \llbracket \tau \rrbracket_1, a_2 \in \llbracket \tau' \rrbracket_2 \cdot \\
&\quad (a_1, a_2) \in \llbracket \tau \rrbracket \Rightarrow (f_1(a_1), f_2(a_2)) \in \llbracket \tau' \rrbracket \\
((a_1, a'_1), (a_2, a'_2)) \in \llbracket \tau \otimes \tau' \rrbracket &\iff (a_1, a_2) \in \llbracket \tau \rrbracket \wedge (a'_1, a'_2) \in \llbracket \tau' \rrbracket
\end{aligned}$$

Finally,  $(F_1, F_2) \in \llbracket ?\tau \rrbracket$  if and only if there is  $F \in \mathbb{T} \llbracket \tau \rrbracket$  such that

$$\begin{aligned}
F_1(g_1) &= F(\lambda(x, y) \in \llbracket \tau \rrbracket \cdot g_1(x)) \text{ for every } g_1 \in \mathcal{C}_b(\llbracket \tau \rrbracket_1) \\
F_2(g_2) &= F(\lambda(x, y) \in \llbracket \tau \rrbracket \cdot g_2(y)) \text{ for every } g_2 \in \mathcal{C}_b(\llbracket \tau \rrbracket_2)
\end{aligned}$$

(Up to isomorphism, relevant monos are inclusions, so we may consider that  $\llbracket \tau \rrbracket \subseteq \llbracket \tau \rrbracket_1 \times \llbracket \tau \rrbracket_2$ .) This condition implies that, if  $(F_1, F_2) \in \llbracket ?\tau \rrbracket$ , then for every  $g_1 \in \mathcal{C}_b(\llbracket \tau \rrbracket_1)$  and  $g_2 \in \mathcal{C}_b(\llbracket \tau \rrbracket_2)$  such that  $g_1(a_1) = g_2(a_2)$  for every  $(a_1, a_2) \in \llbracket \tau \rrbracket$ , then  $F_1(g_1) = F_2(g_2)$ . It turns out that this is equivalent to it, just as in the case of the continuation monad:

**Lemma 12.**  $(F_1, F_2) \in \llbracket ?\tau \rrbracket$  if and only if, for every  $g_1 \in \mathcal{C}_b(\llbracket \tau \rrbracket_1)$  and  $g_2 \in \mathcal{C}_b(\llbracket \tau \rrbracket_2)$  such that  $g_1(a_1) = g_2(a_2)$  for every  $(a_1, a_2) \in \llbracket \tau \rrbracket$ , then  $F_1(g_1) = F_2(g_2)$ .

*Proof.* The only if direction is clear. Conversely, assume that  $F_1(g_1) = F_2(g_2)$  for every  $g_1 \in \mathcal{C}_b(\llbracket \tau \rrbracket_1)$  and  $g_2 \in \mathcal{C}_b(\llbracket \tau \rrbracket_2)$  such that  $g_1(a_1) = g_2(a_2)$  for every  $(a_1, a_2) \in \llbracket \tau \rrbracket$ . Let  $S$  be  $\llbracket \tau \rrbracket$ , and  $H$  be the subspace of  $\mathcal{C}_b(S)$  of those functions of the form  $\lambda(x, y) \in S \cdot g_1(x) + g_2(y)$ , where  $g_1$  ranges over  $\mathcal{C}_b(\llbracket \tau \rrbracket_1)$ ,  $g_2$  ranges over  $\mathcal{C}_b(\llbracket \tau \rrbracket_2)$ . Let  $F_0(g)$  be defined as  $F_1(g_1) + F_2(g_2)$  for every  $g = \lambda(x, y) \in S \cdot g_1(x) + g_2(y)$  in  $H$ . This is independent

of the choice of  $g_1$  and  $g_2$ , since if  $g$  can also be written as  $\lambda(x, y) \in S \cdot g'_1(x) + g'_2(y)$ , then by construction  $g_1(a_1) - g'_1(a_1) = g'_2(a_2) - g_2(a_2)$  for every  $(a_1, a_2) \in S = \llbracket \tau \rrbracket$ , so  $F_1(g_1 - g'_1) = F_2(g'_2 - g_2)$ , which implies  $F_1(g_1) + F_2(g_2) = F_1(g'_1) + F_2(g'_2)$ . It is easy to see that  $F_0$  is a continuous linear form, so it extends to a continuous linear form  $F$  with the same norm on the whole of  $\mathcal{C}_b(S)$ , by the Hahn-Banach Theorem. By the same argument as in Lemma 11,  $F$  is a positive linear form. Clearly  $F_1(g_1) = F(\lambda(x, y) \in \llbracket \tau \rrbracket \cdot g_1(x))$  for every  $g_1 \in \mathcal{C}_b(\llbracket \tau \rrbracket_1)$ , and  $F_2(g_2) = F(\lambda(x, y) \in \llbracket \tau \rrbracket \cdot g_2(y))$  for every  $g_2 \in \mathcal{C}_b(\llbracket \tau \rrbracket_2)$ , whence the claim.  $\square$

It is reasonable to represent the set of probabilistic transition systems on a metric space  $A$  as the metric space  $\mathbb{T}A^A$  of all non-expansive maps from the set of states  $A$  to the set of probability measures on  $A$ . The subscone construction provides a condition when two such probabilistic transition systems are related. Given any relation  $S$  on  $A$  (defining a metric subspace of  $A \times A$ ), define the relation  $\tilde{S}$  on  $\mathbb{T}A^A$  by: for any non-expansive maps  $f_1, f_2$  from  $A$  to  $\mathbb{T}A$ ,  $(f_1, f_2) \in \tilde{S}$  if and only if, for every  $(a_1, a_2) \in S$ , there is  $F \in \mathbb{T}S$  such that

$$\begin{aligned} f_1(a_1)(g_1) &= F(\lambda(x, y) \in S \cdot g_1(x)) \text{ for every } g_1 \in \mathcal{C}_b(A) \\ f_2(a_2)(g_2) &= F(\lambda(x, y) \in S \cdot g_2(y)) \text{ for every } g_2 \in \mathcal{C}_b(A) \end{aligned}$$

or, equivalently, for every  $g_1, g_2 \in \mathcal{C}_b(A)$  such that  $g_1(x) = g_2(y)$  for every  $(x, y) \in S$ , then  $f_1(a_1)(g_1) = f_2(a_2)(g_2)$ . (Use Lemma 12.) In case  $A$  is compact, by the Riesz Representation Theorem, we may replace positive continuous linear functionals on  $\mathcal{C}_b(A)$  by probability measures. Rephrasing the above, we get: for any non-expansive maps  $f_1, f_2$  from  $A$  to the space of probability measures on  $A$ ,  $(f_1, f_2) \in \tilde{S}$  if and only if, for every  $(a_1, a_2) \in S$ , there is a probability measure  $\nu$  on  $S$  such that

$$\begin{aligned} f_1(a_1)(X_1) &= \nu((X_1 \times A) \cap S) \text{ for every measurable subset } X_1 \text{ of } A \\ f_2(a_2)(X_2) &= \nu((A \times X_2) \cap S) \text{ for every measurable subset } X_2 \text{ of } A \end{aligned}$$

We retrieve a notion of probabilistic bisimulation similar to that of Section 10.11, yet simpler:  $S$  is a *bisimulation* on the space  $A$  of states between the transition systems  $f_1$  and  $f_2$  if and only if  $\tilde{S}$  relates  $f_1$  and  $f_2$ .

This notion is formally analogous to the notion of probabilistic bisimulation of (Desharnais et al., 2002); i.e., the formulas we use and theirs for defining bisimulations is the same. A difference is that the latter define bisimulations on the category of analytic spaces; this is awkward in our setting, since we do not know any monoidal closed structure on the category of analytic spaces. (But there is a monoidal monad of probability measures on completely regular analytic spaces, where the space of probability measures is equipped with the weak topology.) This is also formally analogous to metric bisimulations as introduced by de Vink and Rutten (de Vink and Rutten, 1999) for ultrametric spaces, and extended by Worrell (Worrell, 2000) to generalized metric spaces. A difference is that the latter authors use a coalgebraic approach; in particular, their construction requires a  $\mathbb{T}$  functor that preserves isometric embeddings (our relevant monos), while we require it to preserve pseudoepis (surjective non-expansive maps). There seems to be no connection between the approach of this section and the metric defined in (Desharnais et al., 1999), where the

goal is to define a distance between processes that vanishes exactly on bisimilar processes; our construction only yields relations as sets of pairs equipped with a distance.

We won't pursue this comparison. After all, our purpose in this paper was not to define any new approach to the theory of probabilistic bisimulations per se, but to adapt logical relations to the case of monads. It can be considered a nice byproduct that our approach is able to define extensions of some bisimulations to all higher orders.

## 11. Conclusions

The main contribution of this paper is a natural extension of logical relations able to deal with monadic types. We illustrate its naturality and its practical value by demonstrating that various notions of bisimulations and a non-trivial notion of logical relation for dynamic name creation are instances of our construction. Besides, our construction provides a natural integration between notions of simulations for transition systems (possibly probabilistic), higher-order computation (the import of the  $\lambda$ -calculus), and limited forms of side-effects (e.g., dynamic names), yielding streamlined criteria for observational equivalence of those combined systems.

## Acknowledgments

We are grateful to Masahito Hasegawa, John Power, Ian Stark, Paul-André Melliès, François Lamarche, Vincent Danos, and Albert Burroni for their useful comments.

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## Appendix A. Monoidal Monads, Commutative Monads

### A.1. From Mediators to Strengths and Dual Strengths

Let  $(\mathbb{T}, \eta, \mu, \mathfrak{d})$  be a monoidal monad. Let  $\mathfrak{k}_{A,B} = \mathfrak{d}_{A,B} \circ (\eta_A \otimes \text{id}_{\mathbb{T}B})$ ,  $\mathfrak{k}'_{A,B} = \mathfrak{d}_{A,B} \circ (\text{id}_{\mathbb{T}A} \otimes \eta_B)$ . We show that  $\mathfrak{k}$  is a strength,  $\mathfrak{k}'$  a dual strength, and that Diagrams (43) and (44) commute.

Diagram (32). This is exactly Diagram (38).

Diagram (33). This is exactly Diagram (40).

Diagram (34). This is shown by the diagram below. The topmost row is  $\mathfrak{k}_{A \otimes B, C}$ , while the bottommost composition of morphisms from  $\mathbb{T}((A \otimes B) \otimes C)$  to  $\mathbb{T}(A \otimes (B \otimes C))$  is  $\text{id}_A \otimes \mathfrak{k}_{B, C}$  followed by  $\mathfrak{k}_{A, B \otimes C}$ .

$$\begin{array}{ccccc}
(A \otimes B) \otimes \mathbb{T}C & \xrightarrow{\eta_{A \otimes B} \otimes \text{id}} & \mathbb{T}(A \otimes B) \otimes \mathbb{T}C & \xrightarrow{d_{A \otimes B, C}} & \mathbb{T}((A \otimes B) \otimes C) \\
\downarrow \alpha_{A, B, \mathbb{T}C} & \searrow (\eta_A \otimes \eta_B) \otimes \text{id} & \uparrow d_{A, B} \otimes \text{id} & & \downarrow \mathbb{T}\alpha_{A, B, C} \\
& & (\mathbb{T}A \otimes \mathbb{T}B) \otimes \mathbb{T}C & \xrightarrow{\text{mediator law (41)}} & \\
& & \downarrow \alpha_{\mathbb{T}A, \mathbb{T}B, \mathbb{T}C} & & \\
& & \mathbb{T}A \otimes (\mathbb{T}B \otimes \mathbb{T}C) & \xrightarrow{\text{functoriality of } \otimes} & \mathbb{T}A \otimes \mathbb{T}(B \otimes C) & \xrightarrow{d_{A, B \otimes C}} & \mathbb{T}(A \otimes (B \otimes C)) \\
& & \uparrow \eta_A \otimes (\eta_B \otimes \text{id}) & & \uparrow \eta_A \otimes \text{id} & & \\
A \otimes (B \otimes \mathbb{T}C) & \xrightarrow{\text{id} \otimes (\eta_B \otimes \text{id})} & A \otimes (\mathbb{T}B \otimes \mathbb{T}C) & \xrightarrow{\text{id} \otimes d_{B, C}} & A \otimes \mathbb{T}(B \otimes C) & & \\
& & \downarrow \text{id} \otimes d_{B, C} & & & & 
\end{array}$$

Diagram (35).

$$\begin{array}{ccccccc}
A \otimes \mathbb{T}^2 B & \xrightarrow{\eta_A \otimes \text{id}} & \mathbb{T}A \otimes \mathbb{T}^2 B & \xrightarrow{d_{A, \mathbb{T}^2 B}} & \mathbb{T}(A \otimes \mathbb{T}B) & \xrightarrow{\mathbb{T}(\eta_A \otimes \text{id})} & \mathbb{T}(\mathbb{T}A \otimes \mathbb{T}B) & \xrightarrow{\mathbb{T}d_{A, B}} & \mathbb{T}^2(A \otimes B) \\
\downarrow \text{id} \otimes \mu_B & & \downarrow \mathbb{T}\eta_A \otimes \mathbb{T}\text{id} & & \downarrow d_{\mathbb{T}A, \mathbb{T}B} & & \downarrow \mu_{\mathbb{T}A \otimes \mathbb{T}B} & & \downarrow \mu_{A \otimes B} \\
A \otimes \mathbb{T}B & \xrightarrow{\eta_A \otimes \text{id}} & \mathbb{T}A \otimes \mathbb{T}B & \xrightarrow{d_{A, B}} & \mathbb{T}(A \otimes B) & & & & \\
& & \downarrow \mu_{A \otimes \mathbb{T}B} & & & & & & 
\end{array}$$

So  $\theta$  is a strength. That  $\theta'$  is a dual strength is proved similarly. We use Diagram (39) instead of Diagram (38) to prove the dual form of Diagram (32).

Diagram (43). In the diagram below, the top row is  $\mathbb{k}_{A,B} \otimes \text{id}$  followed by  $\mathbb{k}'_{A \otimes B, C}$ , while the bottom row is  $\text{id} \otimes \mathbb{k}'_{B,C}$  followed by  $\mathbb{k}_{A, B \otimes C}$ .

$$\begin{array}{ccccccc}
(A \otimes TB) \otimes C & \xrightarrow{(\eta_A \otimes \text{id}) \otimes \text{id}} & (TA \otimes TB) \otimes C & \xrightarrow{d_{A,B} \otimes \text{id}} & T(A \otimes B) \otimes C & \xrightarrow{\text{id} \otimes \eta_C} & T(A \otimes B) \otimes TC & \xrightarrow{d_{A \otimes B, C}} & T((A \otimes B) \otimes C) \\
\downarrow \alpha_{A, TB, C} & \searrow & \downarrow (\eta_A \otimes \text{id}) \otimes \eta_C & \searrow & \downarrow \text{(functoriality of } \otimes) & \searrow & \downarrow d_{A,B} \otimes \text{id} & \searrow & \downarrow T\alpha_{A,B,C} \\
A \otimes (TB \otimes C) & \xrightarrow{\text{id} \otimes \eta_C} & A \otimes (TB \otimes TC) & \xrightarrow{\text{id} \otimes d_{B,C}} & T(A \otimes (TB \otimes TC)) & \xrightarrow{\eta_A \otimes \text{id}} & T(A \otimes (TB \otimes C)) & \xrightarrow{d_{A, B \otimes C}} & T((A \otimes (TB \otimes C))) \\
& & \downarrow \eta_A \otimes (\text{id} \otimes \eta_C) & \searrow & \downarrow \alpha_{TA, TB, TC} & \searrow & \downarrow \text{(mediator law (41))} & \searrow & \\
& & A \otimes (TB \otimes C) & \xrightarrow{\text{id} \otimes d_{B,C}} & T(A \otimes (TB \otimes C)) & \xrightarrow{\eta_A \otimes \text{id}} & T(A \otimes (TB \otimes C)) & \xrightarrow{d_{A, B \otimes C}} & T((A \otimes (TB \otimes C)))
\end{array}$$

Diagram (44). This is the diagram below, where the top row is  $\mathbb{k}_{TA, B}$ , the leftmost vertical arrows form  $\mathbb{k}'_{A, TB}$ , the rightmost ones form  $\mathbb{T}\mathbb{k}'_{A, B}$  followed by  $\mu_{A \otimes B}$ , and the bottom row is  $\mathbb{T}\mathbb{k}_{A, B}$  followed by  $\mu_{A \otimes B}$ . Note that, as announced, the common diagonal from the top left corner to the bottom right corner is  $\mathfrak{d}_{A, B}$ .

$$\begin{array}{ccccc}
TA \otimes TB & \xrightarrow{\eta_{TA} \otimes \text{id}} & T^2 A \otimes TB & \xrightarrow{d_{TA, B}} & T(TA \otimes B) \\
\downarrow \text{id} \otimes \eta_{TB} & \searrow & \downarrow \text{id} \otimes T\eta_B & \searrow & \downarrow T(\text{id} \otimes \eta_B) \\
TA \otimes T^2 B & \xrightarrow{T\eta_A \otimes \text{id}} & T^2 A \otimes T^2 B & \xrightarrow{d_{TA, TB}} & T(TA \otimes TB) \\
\downarrow d_{A, TB} & \searrow & \downarrow \mu_A \otimes \mu_B & \searrow & \downarrow Td_{A, B} \\
T(A \otimes TB) & \xrightarrow{T(\eta_A \otimes \text{id})} & T(TA \otimes TB) & \xrightarrow{Td_{A, B}} & T^2(A \otimes B) \\
& & \downarrow \mu_{A \otimes B} & \searrow & \downarrow \mu_{A \otimes B} \\
& & T(A \otimes B) & \xrightarrow{\mu_{A \otimes B}} & T(A \otimes B)
\end{array}$$

## A.2. From Strengths and Dual Strengths to Mediators

Let  $(\mathbb{T}, \eta, \mu)$  be a monad such that  $\mathbb{k}$  is a strength and  $\mathbb{k}'$  is a dual strength making Diagrams (43) and (44) commute. Let  $\mathfrak{d}$  be the common diagonal of Diagram (44), i.e.,  $\mathfrak{d}_{A, B} = \mu_{A \otimes B} \circ \mathbb{T}\mathbb{k}'_{A, B} \circ \mathbb{k}_{TA, B} = \mu_{A \otimes B} \circ \mathbb{T}\mathbb{k}_{A, B} \circ \mathbb{k}'_{A, TB}$ . We show that  $\mathfrak{d}$  is a mediator, which means that we check Diagrams (38), (39), (40), (41), and (42). By convention, call (32'), (33'), (34'), (35') the dual diagrams satisfied by the dual strength  $\mathbb{k}'$ .

First, notice that  $\mathbb{k}_{A, B} = \mathfrak{d}_{A, B} \circ (\eta_A \otimes \text{id}_{TB})$ . Indeed, using the definition  $\mathfrak{d}_{A, B} = \mu_{A \otimes B} \circ \mathbb{T}\mathbb{k}'_{A, B} \circ \mathbb{k}_{TA, B}$ ,  $\mathfrak{d}_{A, B} \circ (\eta_A \otimes \text{id}_{TB})$  is the top row of the following diagram, while



Diagram (42). First, we notice that

$$\mu_{A \otimes B} \circ \mathbb{T}d_{A,B} \circ \mathbb{T}\mathfrak{t}'_{TA, TB} = d_{A,B} \circ (\mu_A \otimes \text{id}_{TB}) \quad (65)$$

This is shown using the diagram on the right. The left-hand side is the path from the top left corner to the bottom right corner on the right that goes down then right, and the right-hand side is the path going right then down.

$$\begin{array}{ccc}
\mathbb{T}^2 A \otimes \mathbb{T}B & \xrightarrow{\mu_A \otimes \text{id}} & \mathbb{T}A \otimes \mathbb{T}B \\
\mathfrak{t}'_{TA, TB} \downarrow & & \downarrow \mathfrak{t}'_{A, TB} \\
\mathbb{T}(\mathbb{T}A \otimes \mathbb{T}B) & \xrightarrow{\text{(dual strength law (35'))}} & \mathbb{T}(A \otimes \mathbb{T}B) \\
\mathbb{T}\mathfrak{t}_{A, TB} \downarrow & \nearrow \mu_{A \otimes \mathbb{T}B} & \downarrow \mathbb{T}\mathfrak{t}_{A, B} \\
\mathbb{T}^2(A \otimes \mathbb{T}B) & \xrightarrow{\text{(naturality of } \mu)} & \mathbb{T}^2(A \otimes B) \\
\mathbb{T}^2\mathfrak{t}_{A, B} \downarrow & \nearrow \mu_{\mathbb{T}(A \otimes B)} & \downarrow \mu_{A \otimes B} \\
\mathbb{T}^3(A \otimes B) & \xrightarrow{\text{(monad law)}} & \mathbb{T}^2(A \otimes B) \\
\mathbb{T}\mu_{A \otimes B} \downarrow & & \downarrow \mu_{A \otimes B} \\
\mathbb{T}^2(A \otimes B) & \xrightarrow{\mu_{A \otimes B}} & \mathbb{T}(A \otimes B)
\end{array}$$

By symmetry, the following also holds:

$$\mu_{A \otimes B} \circ \mathbb{T}d_{A,B} \circ \mathbb{T}\mathfrak{t}_{TA, TB} = d_{A,B} \circ (\text{id}_{\mathbb{T}A} \otimes \mu_B) \quad (66)$$

Diagram (42) then follows. The top row below is  $d_{\mathbb{T}A, \mathbb{T}B}$ , while the leftmost vertical arrows form  $\mu_A \otimes \mu_B$ .

$$\begin{array}{ccccc}
\mathbb{T}^2 A \otimes \mathbb{T}^2 B & \xrightarrow{\mathfrak{t}_{\mathbb{T}^2 A, \mathbb{T}^2 B}} & \mathbb{T}(\mathbb{T}^2 A \otimes \mathbb{T}B) & \xrightarrow{\mathbb{T}\mathfrak{t}'_{TA, TB}} & \mathbb{T}^2(\mathbb{T}A \otimes \mathbb{T}B) & \xrightarrow{\mu_{\mathbb{T}A \otimes \mathbb{T}B}} & \mathbb{T}(\mathbb{T}A \otimes \mathbb{T}B) \\
\downarrow \mu_A \otimes \text{id}_{\mathbb{T}^2 B} & & \downarrow \mathbb{T}(\mu_A \otimes \text{id}_{\mathbb{T}B}) & & \downarrow \mathbb{T}^2 d_{A, B} & \text{(naturality of } \mu) & \downarrow \mathbb{T}d_{A, B} \\
& & \text{(naturality of } \mathfrak{t}) & & \mathbb{T}^3(A \otimes B) & \xrightarrow{\mu_{\mathbb{T}(A \otimes B)}} & \mathbb{T}^2(A \otimes B) \\
& & & & \downarrow \mathbb{T}\mu_{A \otimes B} & \text{(monad law)} & \downarrow \mu_{A \otimes B} \\
\mathbb{T}A \otimes \mathbb{T}^2 B & \xrightarrow{\mathfrak{t}_{\mathbb{T}A, \mathbb{T}^2 B}} & \mathbb{T}(\mathbb{T}A \otimes \mathbb{T}B) & \xrightarrow{\mathbb{T}d_{A, B}} & \mathbb{T}^2(A \otimes B) & & \mathbb{T}^2(A \otimes B) \\
\downarrow \text{id}_{\mathbb{T}A} \otimes \mu_B & & & & \searrow \mu_{A \otimes B} & & \downarrow \mu_{A \otimes B} \\
\mathbb{T}A \otimes \mathbb{T}B & \xrightarrow{\mathfrak{t}_{A, B}} & \mathbb{T}(A \otimes \mathbb{T}B) & \xrightarrow{\mathfrak{t}_{A, B}} & \mathbb{T}(A \otimes B) & & \mathbb{T}(A \otimes B) \\
& & & & \downarrow d_{A, B} & & 
\end{array}$$

### A.3. Monoidal Monad Morphisms

We first show that every monoidal monad morphism from  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{d})$  to  $(\mathbb{T}, \eta, \mu, \mathfrak{d})$  is both a strong monad morphism from  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{t})$  to  $(\mathbb{T}, \eta, \mu, \mathfrak{t})$  and (by symmetry) a dual strong monad morphism from  $(\mathbf{T}, \boldsymbol{\eta}, \boldsymbol{\mu}, \mathbf{t}')$  to  $(\mathbb{T}, \eta, \mu, \mathfrak{t}')$ , where  $\mathfrak{t}_{A, B} = d_{A, B} \circ (\eta_A \otimes \text{id}_{\mathbb{T}B})$ ,  $\mathfrak{t}'_{A, B} = d_{A, B} \circ (\text{id}_{\mathbb{T}A} \otimes \eta_B)$ . This is by the following diagram.

$$\begin{array}{ccccc}
|A_1| \otimes \mathbb{T}|A_2| & \xrightarrow{\eta_{|A_1|} \otimes \text{id}} & \mathbb{T}|A_1| \otimes \mathbb{T}|A_2| & \xrightarrow{d_{|A_1|, |A_2|}} & \mathbb{T}(|A_1| \otimes |A_2|) \\
\text{id} \otimes \sigma_{A_2} \downarrow & \searrow & \downarrow \sigma_{A_1} \otimes \sigma_{A_2} & \text{(monoidal} & \downarrow \mathbb{T}(\theta_{A_1, A_2}) \\
|A_1| \otimes |TA_2| & \xrightarrow{\eta_{|A_1|} \otimes \sigma_{A_2}} & |TA_1| \otimes |TA_2| & \text{monad} & \mathbb{T}|A_1| \otimes |A_2| \\
\downarrow \theta_{A_1, TA_2} & \searrow & \downarrow \theta_{TA_1, TA_2} & \text{morphism} & \downarrow \sigma_{A_1 \otimes A_2} \\
|A_1| \otimes |TA_2| & \xrightarrow{|\eta_{A_1} \otimes \text{id}|} & |TA_1| \otimes |TA_2| & \text{(45)} & \\
\downarrow \theta_{A_1, TA_2} & \searrow & \downarrow \theta_{TA_1, TA_2} & & \\
|A_1| \otimes |TA_2| & \xrightarrow{|\eta_{A_1} \otimes \text{id}|} & |TA_1| \otimes |TA_2| & \xrightarrow{|\mathbf{d}_{A_1, A_2}|} & |T(A_1 \otimes A_2)|
\end{array}$$

Conversely, let  $\sigma$  be both a strong monad morphism from  $(\mathbf{T}, \eta, \mu, \mathbf{t})$  to  $(\mathbb{T}, \eta, \mu, \mathbb{t})$  and a dual strong monad morphism from  $(\mathbf{T}, \eta, \mu, \mathbf{t}')$  to  $(\mathbb{T}, \eta, \mu, \mathbb{t}')$ . Let  $d$  be the common diagonal of Diagram (44), i.e.,  $d_{A,B} = \mu_{A \otimes B} \circ \mathbb{T}\mathbb{t}'_{A,B} \circ \mathbb{t}_{\mathbb{T}A,B} = \mu_{A \otimes B} \circ \mathbb{T}\mathbb{t}_{A,B} \circ \mathbb{t}'_{A, \mathbb{T}B}$ . Define  $\mathbf{d}$  similarly.

In the diagram below, the top row is  $d_{|A_1|, \mathbb{T}|A_1|}$ , while the leftmost vertical arrows from  $\sigma_{A_1} \otimes \sigma_{A_2}$  followed by  $\theta_{\mathbf{T}A_1, \mathbf{T}A_2}$ . The bottom row is  $|\mathbf{d}_{A_1, A_2}|$  followed by  $|\mu_{A_1 \otimes A_2}|$ .

$$\begin{array}{ccccccc}
\mathbb{T}|A_1| \otimes \mathbb{T}|A_2| & \xrightarrow{\mathbb{t}'_{|A_1|, \mathbb{T}|A_2|}} & \mathbb{T}(|A_1| \otimes \mathbb{T}|A_2|) & \xrightarrow{\mathbb{T}d_{|A_1|, |A_2|}} & \mathbb{T}^2(|A_1| \otimes |A_2|) & \xrightarrow{\mu_{|A_1| \otimes |A_2|}} & \mathbb{T}(|A_1| \otimes |A_2|) \\
\text{id} \otimes \sigma_{A_2} \downarrow & \text{(naturality} & \mathbb{T}(\text{id} \otimes \sigma_{A_2}) \downarrow & \text{(strong} & \mathbb{T}^2 \theta_{A_1, A_2} \downarrow & \text{(naturality} & \mathbb{T} \theta_{A_1, A_2} \downarrow \\
& \text{of } \mathbb{t}') & & \text{monad} & & \text{of } \mu) & \\
\mathbb{T}|A_1| \otimes |TA_2| & \xrightarrow{\mathbb{t}'_{|A_1|, |TA_2|}} & \mathbb{T}(|A_1| \otimes |TA_2|) & \xrightarrow{\mathbb{T}d_{|A_1|, |TA_2|}} & \mathbb{T}^2|A_1| \otimes |A_2| & \xrightarrow{\mu_{|A_1| \otimes |A_2|}} & \mathbb{T}|A_1| \otimes |A_2| \\
\sigma_{A_1} \otimes \text{id} \downarrow & \text{(dual} & \mathbb{T} \theta_{A_1, TA_2} \downarrow & \text{(45))} & \mathbb{T} \sigma_{A_1} \otimes \sigma_{A_2} \downarrow & & \downarrow \sigma_{A_1 \otimes A_2} \\
& \text{strong} & & & & & \\
|TA_1| \otimes |TA_2| & \xrightarrow{\mathbb{t}'_{|TA_1|, |TA_2|}} & |TA_1| \otimes |TA_2| & \xrightarrow{\mathbb{T}|\mathbf{t}_{A_1, A_2}|} & |T(A_1 \otimes A_2)| & \xrightarrow{\sigma_{A_1 \otimes A_2}} & |T(A_1 \otimes A_2)| \\
\theta_{TA_1, TA_2} \downarrow & \text{(monad} & \downarrow \theta_{TA_1, TA_2} & \text{morphism} & \text{(4)} & & \\
& \text{morphism} & & & & & \\
& \text{(45')} & & & & & \\
|TA_1| \otimes |TA_2| & \xrightarrow{|\mathbf{t}'_{A_1, TA_2}|} & |T(A_1 \otimes TA_2)| & \xrightarrow{|\mathbf{t}_{TA_1, A_2}|} & |T^2(A_1 \otimes A_2)| & \xrightarrow{|\mu_{A_1 \otimes A_2}|} & |T(A_1 \otimes A_2)|
\end{array}$$

#### A.4. Commutative Monads are Monoidal

Let  $(\mathbb{T}, \eta, \mu, \theta)$  be a strong monad on a symmetrical monoidal category. Let  $\mathbb{t}'_{A,B}$  be the dual strength  $\mathbb{T}c_{B,A} \circ \mathbb{t}_{B,A} \circ c_{\mathbb{T}A,B}$ . That  $\mathbb{T}$  is a commutative monad means that Diagram (44) commutes. To show that  $d_{A,B} = \mu_{A \otimes B} \circ \mathbb{T}\mathbb{t}'_{A,B} \circ \mathbb{t}_{\mathbb{T}A,B} = \mu_{A \otimes B} \circ \mathbb{T}\mathbb{t}_{A,B} \circ \mathbb{t}'_{A, \mathbb{T}B}$  is a mediator, it then suffices to show that Diagram (43) commutes. In the diagram below, the top row is  $\mathbb{t}_{A,B} \otimes \text{id}_C$  followed by  $\mathbb{t}'_{A \otimes B, C}$ , the bottom row is  $\text{id}_A \otimes \mathbb{t}'_{B,C}$  followed by  $\mathbb{t}_{A, B \otimes C}$ . The two triangles involving  $c$  on the left side are instances of the identity  $c \circ c = \text{id}$ .

$$\begin{array}{ccccccc}
(A \otimes TB) \otimes C & \xrightarrow{\mathfrak{t}_{A,B} \otimes \text{id}} & T(A \otimes B) \otimes C & \xrightarrow{c} & C \otimes T(A \otimes B) & \xrightarrow{\mathfrak{t}_{C,A \otimes B}} & T(C \otimes (A \otimes B)) \xrightarrow{Tc} T((A \otimes B) \otimes C) \\
\downarrow \text{id} & \searrow c & \text{(naturality of } c) & \nearrow \text{id} \otimes \mathfrak{t}_{A,B} & & \uparrow T\alpha_{C,A,B} & \uparrow \\
(A \otimes TB) \otimes C & \xleftarrow{c} & C \otimes (A \otimes TB) & \xrightarrow{\mathfrak{t}_{C \otimes A, B}} & T((C \otimes A) \otimes B) & \xrightarrow{T(c \otimes \text{id})} & T((A \otimes C) \otimes B) \xrightarrow{T\alpha_{A,C,B}} T(A \otimes (C \otimes B)) \\
\downarrow \alpha_{A,TB,C} & \text{(coherence (25))} & \uparrow \alpha_{C,A,TB} & \text{(strength law (34))} & \downarrow T(c \otimes \text{id}) & \text{(T of coherence (25))} & \downarrow T\alpha_{A,C,B} \\
(A \otimes TB) \otimes C & \xrightarrow{\alpha_{A,TB,C}} & A \otimes (TB \otimes C) & \xrightarrow{\text{id} \otimes c} & A \otimes (C \otimes TB) & \xrightarrow{\mathfrak{t}_{A,C \otimes B}} & T(A \otimes (C \otimes B)) \xrightarrow{T(\text{id} \otimes c)} T(A \otimes (B \otimes C)) \\
\downarrow \text{id} & \nearrow \text{id} \otimes c & \downarrow \alpha_{A,C,TB} & \text{(strength law (34))} & \downarrow \alpha_{A,C,TB} & \text{(naturality of } \mathfrak{t}) & \downarrow T(\text{id} \otimes c) \\
A \otimes (TB \otimes C) & \xrightarrow{\text{id} \otimes c} & A \otimes (C \otimes TB) & \xrightarrow{\text{id} \otimes \mathfrak{t}_{C,B}} & T(C \otimes B) & \xrightarrow{\text{id} \otimes Tc} & T(B \otimes C) \xrightarrow{\mathfrak{t}_{A,B \otimes C}} T(B \otimes C) \\
\downarrow \text{id} & \nearrow \text{id} \otimes c & \downarrow \alpha_{A,C,TB} & \text{(strength law (34))} & \downarrow \alpha_{A,C,TB} & \text{(naturality of } \mathfrak{t}) & \downarrow T(\text{id} \otimes c) \\
A \otimes (TB \otimes C) & \xrightarrow{\text{id} \otimes c} & A \otimes (C \otimes TB) & \xrightarrow{\text{id} \otimes \mathfrak{t}_{C,B}} & T(C \otimes B) & \xrightarrow{\text{id} \otimes Tc} & T(B \otimes C) \xrightarrow{\mathfrak{t}_{A,B \otimes C}} T(B \otimes C)
\end{array}$$

We now observe that Diagram (47) commutes:

$$\begin{array}{ccccccccccc}
TA \otimes TB & \xrightarrow{c} & TB \otimes TA & \xrightarrow{\mathfrak{t}_{TB,A}} & T(TB \otimes A) & \xrightarrow{Tc} & T(A \otimes TB) & \xrightarrow{T\mathfrak{t}_{A,B}} & T^2(A \otimes B) & \xrightarrow{\mu^{A \otimes B}} & T(A \otimes B) \\
\downarrow c & \nearrow \text{id} & \nearrow \mathfrak{t}_{TB,A} & \nearrow \mathfrak{t}_{TB,A} & \nearrow T\mathfrak{t}_{A,B} & \nearrow \text{id} & \nearrow \text{id} & \text{(naturality of } \mu) & \downarrow Tc & \downarrow Tc & \downarrow Tc \\
TB \otimes TA & \xrightarrow{\mathfrak{t}_{TB,A}} & T(TB \otimes A) & \xrightarrow{Tc} & T(A \otimes TB) & \xrightarrow{T\mathfrak{t}_{A,B}} & T^2(A \otimes B) & \xrightarrow{T^2c} & T^2(B \otimes A) & \xrightarrow{\mu^{B \otimes A}} & T(B \otimes A)
\end{array}$$

Indeed, the leftmost triangle is by  $c \circ c = \text{id}$ , the inner parallelogram obviously commutes, and the top and bottom rows are the two ways of writing  $\mathfrak{d}_{A,B}$ .

Conversely, given any monoidal monad making Diagram (47) commute, we claim that the derived strength  $\mathfrak{t}$  and dual strength  $\mathfrak{t}'$  are related by  $\mathfrak{t}'_{A,B} = Tc_{B,A} \circ \mathfrak{t}_{B,A} \circ c_{TA,B}$ . Equivalently, since  $c \circ c = \text{id}$ , we check that  $\mathfrak{t}'_{A,B} \circ c_{B,TA} = Tc_{B,A} \circ \mathfrak{t}_{B,A}$ . This is obvious:

$$\begin{array}{ccc}
B \otimes TA & \xrightarrow{c_{B,TA}} & TA \otimes B \\
\eta_B \otimes \text{id}_{TA} \downarrow & \text{(naturality of } c) & \downarrow \text{id}_{TA} B \\
TB \otimes TA & \xrightarrow{\mathfrak{t}_{B,A}} & T(B \otimes A) \\
c_{TB,TA} \downarrow & \text{(47)} & \downarrow Tc_{B,A} \\
TA \otimes TB & \xrightarrow{\mathfrak{d}_{A,B}} & T(A \otimes B)
\end{array}$$

## Appendix B. Proof of Lemma 2

We first check that  $(\mathcal{D}, I^{\mathcal{D}}, \otimes^{\mathcal{D}}, \alpha^{\mathcal{D}}, \ell^{\mathcal{D}}, r^{\mathcal{D}})$  is a monoidal category. Write  $\bar{\circ}$  the notion of composition in  $\mathcal{D} = \mathbf{Kleisli}(T)$ , i.e.,  $g\bar{\circ}f = \mu \circ Tg \circ f$ .



— Functoriality of  $\otimes^{\mathcal{D}}$ . First,  $\otimes^{\mathcal{D}}$  applied to the identity on  $(A, B)$  is  $\text{id}_A \otimes^{\mathcal{D}} \text{id}_B = \text{id}_{F_T(A)} \otimes^{\mathcal{D}} \text{id}_{F_T(B)} = F_T(\text{id}_A) \otimes^{\mathcal{D}} F_T(\text{id}_B) = F_T(\text{id}_A \otimes^{\mathcal{C}} \text{id}_B) = F_T(\text{id}_{A \otimes^{\mathcal{C}} B}) = \text{id}_{F_T(A \otimes^{\mathcal{C}} B)} = \text{id}_{A \otimes^{\mathcal{D}} B}$ .

Second,  $\otimes^{\mathcal{D}}$  preserves composition. Indeed, let  $f, f', g, g'$  be morphisms from  $A$  to  $B$ , from  $A'$  to  $B'$ , from  $B$  to  $C$  and from  $B'$  to  $C'$  respectively in  $\mathcal{D}$ . Then the composition of  $g \otimes^{\mathcal{D}} g'$  with  $f \otimes^{\mathcal{D}} f'$  in  $\mathcal{D}$  is the morphism  $\mu_{C \otimes^{\mathcal{C}} C'} \circ T(d_{C, C'} \circ (g \otimes^{\mathcal{C}} g')) \circ d_{B, B'} \circ (f \otimes^{\mathcal{C}} f')$  in  $\mathcal{C}$ . The following diagram then commutes:

$$\begin{array}{ccccc}
A \otimes^{\mathcal{C}} A' & \xrightarrow{f \otimes^{\mathcal{C}} f'} & TB \otimes^{\mathcal{C}} TB' & \xrightarrow{d_{B, B'}} & T(B \otimes^{\mathcal{C}} B') \\
& & \downarrow Tg \otimes^{\mathcal{C}} Tg' & \text{(naturality of } d) & \downarrow T(g \otimes^{\mathcal{C}} g') \\
& & T^2 C \otimes^{\mathcal{C}} T^2 C' & \xrightarrow{d_{TC, TC'}} & T(TC \otimes^{\mathcal{C}} TC') \\
& & & \text{(42)} & \downarrow Td_{C, C'} \\
& & & \searrow \mu_C \otimes^{\mathcal{C}} \mu_{C'} & T^2(C \otimes^{\mathcal{C}} C') \\
& & & & \downarrow \mu_{C \otimes^{\mathcal{C}} C'} \\
& & & & T(C \otimes^{\mathcal{C}} C') \\
& \searrow (g \bar{\circ} f) \otimes^{\mathcal{C}} (g' \bar{\circ} f') & & & 
\end{array}$$

The top right route from  $A \otimes^{\mathcal{C}} A'$  to  $T(C \otimes^{\mathcal{C}} C')$  is  $(g \otimes^{\mathcal{D}} g') \bar{\circ} (f \otimes^{\mathcal{D}} f')$ , while the straight line diagonal is  $(g \bar{\circ} f) \otimes^{\mathcal{D}} (g' \bar{\circ} f')$ .

— Naturality of  $\alpha^{\mathcal{D}}$ . Let  $f, g, h$  be morphisms from  $A$  to  $A'$ ,  $B$  to  $B'$  and  $C$  to  $C'$  respectively in  $\mathcal{D}$ . Then  $\alpha^{\mathcal{D}}$  composed in  $\mathcal{D}$  with  $(f \otimes^{\mathcal{D}} g) \otimes^{\mathcal{D}} h$  is  $\mu \circ T(\eta \circ \alpha^{\mathcal{C}}) \circ d \circ ((d \circ (f \otimes^{\mathcal{C}} g)) \otimes^{\mathcal{C}} h) = T\alpha^{\mathcal{C}} \circ d \circ (d \otimes^{\mathcal{C}} \text{id}) \circ ((f \otimes^{\mathcal{C}} g) \otimes^{\mathcal{C}} h)$  (the route going all the way down then right from the top left corner to the bottom right corner below), while  $f \otimes^{\mathcal{D}} (g \otimes^{\mathcal{D}} h)$  composed with  $\alpha^{\mathcal{D}}$  in  $\mathcal{D}$  is  $\mu \circ T(d \circ (f \otimes^{\mathcal{C}} (d \circ (g \otimes^{\mathcal{C}} h)))) \circ \eta \circ \alpha^{\mathcal{C}} = \mu \circ Td \circ T(\text{id} \otimes^{\mathcal{C}} d) \circ T(f \otimes^{\mathcal{C}} (g \otimes^{\mathcal{C}} h)) \circ \eta \circ \alpha^{\mathcal{C}}$  (the other route, going right then down).

$$\begin{array}{ccccc}
(A \otimes^{\mathcal{C}} B) \otimes^{\mathcal{C}} C & \xrightarrow{\alpha^{\mathcal{C}}} & A \otimes^{\mathcal{C}} (B \otimes^{\mathcal{C}} C) & \xrightarrow{\eta} & T(A \otimes^{\mathcal{C}} (B \otimes^{\mathcal{C}} C)) \\
\downarrow (f \otimes^{\mathcal{C}} g) \otimes^{\mathcal{C}} h & \text{(naturality of } \alpha^{\mathcal{C}}) & \downarrow f \otimes^{\mathcal{C}} (g \otimes^{\mathcal{C}} h) & & \downarrow T(f \otimes^{\mathcal{C}} (g \otimes^{\mathcal{C}} h)) \\
(TA' \otimes^{\mathcal{C}} TB') \otimes^{\mathcal{C}} TC' & \xrightarrow{\alpha^{\mathcal{C}}} & TA' \otimes^{\mathcal{C}} (TB' \otimes^{\mathcal{C}} TC') & & T(TA' \otimes^{\mathcal{C}} (TB' \otimes^{\mathcal{C}} TC')) \\
\downarrow d_{A', B'} \otimes^{\mathcal{C}} \text{id}_{TC'} & & \downarrow \text{id}_{TA'} \otimes^{\mathcal{C}} d_{B', C'} & \text{(naturality of } \eta) & \downarrow T(\text{id}_{TA'} \otimes^{\mathcal{C}} d_{B', C'}) \\
T(A' \otimes^{\mathcal{C}} B') \otimes^{\mathcal{C}} TC' & \text{(mediator law (41))} & TA' \otimes^{\mathcal{C}} T(B' \otimes^{\mathcal{C}} C') & & T(TA' \otimes^{\mathcal{C}} T(B' \otimes^{\mathcal{C}} C')) \\
\downarrow d_{A' \otimes^{\mathcal{C}} B', C'} & & \downarrow d_{A', B'} \otimes^{\mathcal{C}} C' & & \downarrow Td_{A', B'} \otimes^{\mathcal{C}} C' \\
T((A' \otimes^{\mathcal{C}} B') \otimes^{\mathcal{C}} C') & \xrightarrow{T\alpha^{\mathcal{C}}} & T(A' \otimes^{\mathcal{C}} (B' \otimes^{\mathcal{C}} C')) & \xrightarrow{\eta} & T^2(A' \otimes^{\mathcal{C}} (B' \otimes^{\mathcal{C}} C')) \\
& & \searrow \text{(monad law)} & & \downarrow \mu_{A' \otimes^{\mathcal{C}} (B' \otimes^{\mathcal{C}} C')} \\
& & & & T(A' \otimes^{\mathcal{C}} (B' \otimes^{\mathcal{C}} C'))
\end{array}$$

— Naturality of  $\ell^{\mathcal{D}}$ . Let  $f$  be any morphism from  $B$  to  $B'$  in  $\mathcal{D}$ . Then the composition of

$f$  with  $\ell^{\mathcal{D}}$  in  $\mathcal{D}$  is  $\mu \circ T f \circ \eta \circ \ell^{\mathcal{C}}$  (route going right then down), while the composition of  $\ell^{\mathcal{D}}$  with  $(\text{id}^{\mathcal{D}} \otimes^{\mathcal{D}} f)$  is  $\mu \circ T(\eta \circ \ell^{\mathcal{C}}) \circ d \circ (\eta \otimes^{\mathcal{C}} f)$  (dented route going down then right).

$$\begin{array}{ccccc}
I^{\mathcal{C}} \otimes^{\mathcal{C}} B & \xrightarrow{\ell^{\mathcal{C}}} & B & \xrightarrow{\eta_B} & TB \\
\text{id}_{I^{\mathcal{C}}} \otimes^{\mathcal{C}} f \downarrow & \text{(naturality of } \ell^{\mathcal{C}}) & f \downarrow & \text{(naturality of } \eta) & \downarrow T f \\
I^{\mathcal{C}} \otimes^{\mathcal{C}} TB' & \xrightarrow{\ell_{TB'}^{\mathcal{C}}} & TB' & \xrightarrow{\eta_{TB'}} & T^2 B' \\
\eta_{I^{\mathcal{C}}} \otimes^{\mathcal{C}} \text{id}_{TB'} \downarrow & \text{(38)} & \downarrow T \eta_{B'} & \text{(monad law)} & \downarrow \mu_B \\
TI^{\mathcal{C}} \otimes^{\mathcal{C}} TB' & \xrightarrow{T \ell_{B'}^{\mathcal{C}}} & TTB' & \xrightarrow{\text{id}_{TB'}} & T^2 B' \\
d_{I^{\mathcal{C}}, B'} \downarrow & & \downarrow \mu_{B'} & & \downarrow \mu_B \\
T(I^{\mathcal{C}} \otimes^{\mathcal{C}} B') & & T^2 B' & \xrightarrow{\mu_{B'}} & TB'
\end{array}$$

- Naturality of  $r^{\mathcal{D}}$  is obtained similarly, using (39) instead of (38).
- $\alpha^{\mathcal{D}}$ ,  $\ell^{\mathcal{D}}$ ,  $r^{\mathcal{D}}$  are isomorphisms: indeed  $\alpha^{\mathcal{D}} = F_T(\alpha^{\mathcal{C}})$ ,  $\alpha^{\mathcal{C}}$  is an isomorphism, and every functor preserves isomorphisms. Similarly for  $\ell^{\mathcal{D}}$  and  $r^{\mathcal{D}}$ . Since they are natural, they are natural isomorphisms.
- Pentagon identity (14). We have to check that (dropping subscripts)  $\alpha^{\mathcal{D}} \bar{\alpha}^{\mathcal{D}} = (\text{id}^{\mathcal{D}} \otimes^{\mathcal{D}} \alpha^{\mathcal{D}}) \bar{\alpha}^{\mathcal{D}} \bar{\alpha}^{\mathcal{D}} (\alpha^{\mathcal{D}} \otimes^{\mathcal{D}} \text{id}^{\mathcal{D}})$ . Note that the left-hand side is  $\mu \circ T(\eta \circ \alpha^{\mathcal{C}}) \circ \eta \circ \alpha^{\mathcal{C}} = T\alpha^{\mathcal{C}} \circ \eta \circ \alpha^{\mathcal{C}} = \eta \circ \alpha^{\mathcal{C}} \circ \alpha^{\mathcal{C}}$  (by naturality of  $\eta$ ,  $Tf \circ \eta = \eta \circ f$  for any  $f$ ). Now notice that by (40),  $d \circ (\eta \otimes^{\mathcal{C}} \eta) = \eta$ . Also,  $\mu \circ T\eta = \text{id}$ . So the right-hand side is:

$$\begin{aligned}
& \mu \circ T(d \circ (\eta \otimes^{\mathcal{C}} (\eta \circ \alpha^{\mathcal{C}}))) \circ [\alpha^{\mathcal{D}} \bar{\alpha}^{\mathcal{D}} (\alpha^{\mathcal{D}} \otimes^{\mathcal{D}} \text{id}^{\mathcal{D}})] \\
&= \mu \circ T(d \circ (\eta \otimes^{\mathcal{C}} \eta)) \circ T(\text{id} \otimes^{\mathcal{C}} \alpha^{\mathcal{C}}) \circ [\alpha^{\mathcal{D}} \bar{\alpha}^{\mathcal{D}} (\alpha^{\mathcal{D}} \otimes^{\mathcal{D}} \text{id}^{\mathcal{D}})] \\
&= \mu \circ T\eta \circ T(\text{id} \otimes^{\mathcal{C}} \alpha^{\mathcal{C}}) \circ [\alpha^{\mathcal{D}} \bar{\alpha}^{\mathcal{D}} (\alpha^{\mathcal{D}} \otimes^{\mathcal{D}} \text{id}^{\mathcal{D}})] \quad \text{(by (40))} \\
&= T(\text{id} \otimes^{\mathcal{C}} \alpha^{\mathcal{C}}) \circ [\alpha^{\mathcal{D}} \bar{\alpha}^{\mathcal{D}} (\alpha^{\mathcal{D}} \otimes^{\mathcal{D}} \text{id}^{\mathcal{D}})] \\
&= T(\text{id} \otimes^{\mathcal{C}} \alpha^{\mathcal{C}}) \circ \mu \circ T(\eta \circ \alpha^{\mathcal{C}}) \circ (\alpha^{\mathcal{D}} \otimes^{\mathcal{D}} \text{id}^{\mathcal{D}}) \\
&= T(\text{id} \otimes^{\mathcal{C}} \alpha^{\mathcal{C}}) \circ T\alpha^{\mathcal{C}} \circ (\alpha^{\mathcal{D}} \otimes^{\mathcal{D}} \text{id}^{\mathcal{D}}) \\
&= T(\text{id} \otimes^{\mathcal{C}} \alpha^{\mathcal{C}}) \circ T\alpha^{\mathcal{C}} \circ d \circ ((\eta \circ \alpha^{\mathcal{C}}) \otimes^{\mathcal{C}} \eta) \\
&= T(\text{id} \otimes^{\mathcal{C}} \alpha^{\mathcal{C}}) \circ T\alpha^{\mathcal{C}} \circ d \circ (\eta \otimes^{\mathcal{C}} \eta) \circ (\alpha^{\mathcal{C}} \otimes^{\mathcal{C}} \text{id}) \\
&= T(\text{id} \otimes^{\mathcal{C}} \alpha^{\mathcal{C}}) \circ T\alpha^{\mathcal{C}} \circ \eta \circ (\alpha^{\mathcal{C}} \otimes^{\mathcal{C}} \text{id}) \quad \text{(by (40))} \\
&= \eta \circ (\text{id} \otimes^{\mathcal{C}} \alpha^{\mathcal{C}}) \circ \alpha^{\mathcal{C}} \circ (\alpha^{\mathcal{C}} \otimes^{\mathcal{C}} \text{id}) \quad \text{(by naturality of } \eta)
\end{aligned}$$

and we are done, since by the pentagon identity (14) for  $\alpha^{\mathcal{C}}$ ,  $\alpha^{\mathcal{C}} \circ \alpha^{\mathcal{C}} = (\text{id} \otimes^{\mathcal{C}} \alpha^{\mathcal{C}}) \circ \alpha^{\mathcal{C}} \circ (\alpha^{\mathcal{C}} \otimes^{\mathcal{C}} \text{id})$ .

- Triangle identity (15). We must check that  $(\text{id}^{\mathcal{D}} \otimes^{\mathcal{D}} \ell^{\mathcal{D}}) \bar{\alpha}^{\mathcal{D}} = r^{\mathcal{D}}$ . The left-hand side is  $\mu \circ T(d \circ (\eta \otimes^{\mathcal{C}} (\eta \circ \ell^{\mathcal{C}}))) \circ \eta \circ \alpha^{\mathcal{C}} = \mu \circ T(d \circ (\eta \otimes^{\mathcal{C}} \eta) \circ (\text{id} \otimes^{\mathcal{C}} \ell^{\mathcal{C}})) \circ \eta \circ \alpha^{\mathcal{C}} = \mu \circ T(\eta \circ (\text{id} \otimes^{\mathcal{C}} \ell^{\mathcal{C}})) \circ \eta \circ \alpha^{\mathcal{C}}$  (by (40))  $= T(\text{id} \otimes^{\mathcal{C}} \ell^{\mathcal{C}}) \circ \eta \circ \alpha^{\mathcal{C}}$  (by the monad laws)  $= \eta \circ (\text{id} \otimes^{\mathcal{C}} \ell^{\mathcal{C}}) \circ \alpha^{\mathcal{C}}$  (by naturality of  $\eta$ )  $= \eta \circ r^{\mathcal{C}}$  (by the triangle identity (15) in  $\mathcal{C}$ )  $= r^{\mathcal{D}}$ , which is the right-hand side.

We must now check that  $F_T \dashv U_T$  is a monoidal adjunction. We already know that it is an adjunction.

- $(\theta^{F_T}, \iota^{F_T})$  is mediating for  $F$ . Indeed, both  $\theta_{A,B}^{F_T}$  and  $\iota^{F_T}$  are just identity morphisms in  $\mathcal{D}$ , from which the coherence conditions (16), (17) and (18) are clear. The first

reduces to showing that  $F_T(\alpha^{\mathcal{C}}) = \alpha^{\mathcal{D}}$ , the second to  $F_T(\ell^{\mathcal{C}}) = \ell^{\mathcal{D}}$ , the third to  $F_T(r^{\mathcal{C}}) = r^{\mathcal{D}}$ , which hold by construction.

- $(\theta^{U_T}, \iota^{U_T})$  is mediating for  $U_T$ . Indeed, the first coherence condition (16) is just Diagram (41), since  $U_T \alpha^{\mathcal{D}} = U_T F_T \alpha^{\mathcal{C}} = T \alpha^{\mathcal{C}}$ . The second coherence condition (17) is exactly Diagram (38), and the third, (18), is exactly Diagram (39).
- The unit  $\eta$  and the counit  $\epsilon$  are monoidal natural transformations. (Recall that  $\epsilon_A$  is the identity morphism from  $TA$  to  $TA$  in  $\mathcal{C}$ , seen as a morphism from  $TA$  to  $A$  in  $\mathcal{D}$ .) Indeed, Diagram (49) (in  $\mathcal{C}$ ) means that  $\eta_{I^{\mathcal{C}}} = U_T(\iota^{F_T}) \circ \iota^{U_T} = U_T(\text{id}) \circ \eta_{I^{\mathcal{C}}}$ , which is obvious. Diagram (50) states that  $U_T(\theta_{A,B}^{F_T}) \circ \theta^{U_T} \circ (\eta_A \otimes^{\mathcal{C}} \eta_B) = \eta_{A \otimes^{\mathcal{C}} B}$ , i.e.,  $d_{A,B} \circ (\eta_A \otimes^{\mathcal{C}} \eta_B) = \eta_{A \otimes^{\mathcal{C}} B}$ , which is just Diagram (40). Turning to diagrams in  $\mathcal{D}$ , Diagram (51) means that  $\epsilon_{I^{\mathcal{D}}} \circ F_T(\iota^{U_T}) \circ \iota^{F_T}$  is the identity in  $\mathcal{D}$ ; seen as a morphism in  $\mathcal{C}$ , this is

$$\begin{aligned} \mu \circ T(\text{id}) \circ [F_T(\iota^{U_T}) \circ \iota^{F_T}] &= \mu \circ [F_T(\iota^{U_T}) \circ \iota^{F_T}] \\ &= \mu \circ \mu \circ T(F_T(\iota^{U_T})) \circ \iota^{F_T} \\ &= \mu \circ \mu \circ T(\eta_{T I^{\mathcal{C}}} \circ \eta_{I^{\mathcal{C}}}) \circ \eta_{I^{\mathcal{C}}} = \eta_{I^{\mathcal{C}}} \end{aligned}$$

(by the monad laws), and the latter is just the identity on  $I^{\mathcal{D}}$  in  $\mathcal{D}$ , as required. Finally, Diagram (52) means that  $\epsilon_{A \otimes^{\mathcal{D}} B} \circ F_T \theta^{U_T} \circ \theta^{F_T} = \epsilon_A \otimes^{\mathcal{D}} \epsilon_B$ . Seen as a morphism in  $\mathcal{C}$ , the left-hand side is

$$\begin{aligned} &\mu_{A \otimes^{\mathcal{C}} B} \circ T \epsilon_{A \otimes^{\mathcal{D}} B} \circ [F_T \theta^{U_T} \circ \theta^{F_T}] \\ &= \mu_{A \otimes^{\mathcal{C}} B} \circ T \text{id} \circ [F_T \theta^{U_T} \circ \theta^{F_T}] \\ &= \mu_{A \otimes^{\mathcal{C}} B} \circ [F_T \theta^{U_T} \circ \theta^{F_T}] \\ &= \mu_{A \otimes^{\mathcal{C}} B} \circ \mu_{T(A \otimes^{\mathcal{C}} B)} \circ T(F_T \theta^{U_T}) \circ \theta^{F_T} \\ &= \mu_{A \otimes^{\mathcal{C}} B} \circ \mu_{T(A \otimes^{\mathcal{C}} B)} \circ T(\eta_{T(A \otimes^{\mathcal{C}} B)} \circ d_{A,B}) \circ \eta_{A \otimes^{\mathcal{C}} B} \end{aligned}$$

which we recognize as the long trip in the diagram below from the top left corner that goes down, right, and up to the top right corner.

$$\begin{array}{ccccc} TA \otimes^{\mathcal{C}} TB & \xrightarrow{d_{A,B}} & T(A \otimes^{\mathcal{C}} B) & \xrightarrow{\text{id}_{T(A \otimes^{\mathcal{C}} B)}} & T(A \otimes^{\mathcal{C}} B) \\ \eta_{A \otimes^{\mathcal{C}} B} \downarrow & \text{(naturality of } \eta) & \downarrow \eta_{T(A \otimes^{\mathcal{C}} B)} & \text{(monad law)} & \uparrow \mu_{A \otimes^{\mathcal{C}} B} \\ T(TA \otimes^{\mathcal{C}} TB) & \xrightarrow{Td_{A,B}} & T^2(A \otimes^{\mathcal{C}} B) & \xrightarrow{\text{id}_{T^2(A \otimes^{\mathcal{C}} B)}} & T^2(A \otimes^{\mathcal{C}} B) \\ & & \searrow T\eta_{T(A \otimes^{\mathcal{C}} B)} & \text{(monad law)} & \nearrow \mu_{T(A \otimes^{\mathcal{C}} B)} \\ & & & & T^3(A \otimes^{\mathcal{C}} B) \end{array}$$

Then the composition of arrows from the top left corner to the top right corner is  $d_{A,B}$ . This is exactly the desired morphism  $\epsilon_A \otimes^{\mathcal{D}} \epsilon_B$  of  $\mathcal{D}$ , which, seen as a morphism of  $\mathcal{C}$ , is  $d_{A,B} \circ (\text{id}_{TA} \otimes^{\mathcal{C}} \text{id}_{TB})$  by definition.

We check finally that  $F_T \dashv U_T$  generates the monoidal monad. The only thing to check is  $d_{A,B} = U_T \theta_{A,B}^{F_T} \circ \theta_{F_T(A), F_T(B)}^{U_T}$ . This is clear, since the right-hand side is  $U_T(\text{id}_{A \otimes^{\mathcal{D}} B}) \circ d_{A,B} = d_{A,B}$ .  $\square$

### Appendix C. Composition of Monoidal Adjunctions

We first show that  $|U|$  is a monoidal functor, with mediating pair  $(|\theta^U| \circ \theta, |\iota^U| \circ \mathfrak{i})$ .

We must check all three coherence conditions (16), (17), (18).

Diagram (16).

$$\begin{array}{ccc}
 \begin{array}{c} (|U(A)| \otimes |U(B)|) \\ \otimes |U(C)| \\ \downarrow \theta \otimes \text{id}_{|U(C)|} \\ |U(A) \otimes U(B)| \\ \otimes |U(C)| \\ \downarrow |\theta^U| \otimes \text{id}_{|U(C)|} \\ |U(A \otimes^{\mathcal{D}} B)| \\ \otimes |U(C)| \\ \downarrow \theta \\ |U(A \otimes^{\mathcal{D}} B) \otimes U(C)| \\ \downarrow |\theta^U| \\ |U((A \otimes^{\mathcal{D}} B) \otimes^{\mathcal{D}} C)| \end{array} & \xrightarrow{\alpha} & \begin{array}{c} |U(A)| \otimes \\ (|U(B)| \otimes |U(C)|) \\ \downarrow \text{id}_{|U(A)|} \otimes \theta \\ |U(A)| \otimes \\ |U(B) \otimes U(C)| \\ \downarrow \text{id}_{|U(A)|} \otimes |\theta^U| \\ |U(A)| \otimes \\ |U(B \otimes^{\mathcal{D}} C)| \\ \downarrow \theta \\ |U(A) \otimes \\ U(B \otimes^{\mathcal{D}} C)| \\ \downarrow |\theta^U| \\ |U(A \otimes^{\mathcal{D}} \\ (B \otimes^{\mathcal{D}} C))| \end{array} \\
 \downarrow \theta & \text{(coherence (16) for } \theta) & \downarrow \theta \\
 \begin{array}{c} |U(A) \otimes \\ \otimes U(B)| \\ \otimes U(C) \\ \downarrow |\alpha| \\ |U(A) \otimes \\ U(B) \otimes \\ U(C)| \end{array} & \xrightarrow{|\alpha|} & \begin{array}{c} |U(A) \otimes \\ U(B) \otimes \\ U(C)| \\ \downarrow \text{id}_{U(A)} \otimes \theta^U \\ |U(A) \otimes \\ U(B \otimes^{\mathcal{D}} C)| \end{array} \\
 \downarrow \theta^U \otimes \text{id}_{U(C)} & \text{(naturality of } \theta) & \downarrow \theta \\
 \begin{array}{c} |U(A \otimes^{\mathcal{D}} B) \otimes U(C)| \\ \downarrow \theta \\ |U(A \otimes^{\mathcal{D}} B) \otimes U(C)| \\ \downarrow |\theta^U| \\ |U((A \otimes^{\mathcal{D}} B) \otimes^{\mathcal{D}} C)| \end{array} & \xrightarrow{|\alpha^{\mathcal{D}}|} & \begin{array}{c} |U(A) \otimes \\ U(B \otimes^{\mathcal{D}} C)| \\ \downarrow |\theta^U| \\ |U(A \otimes^{\mathcal{D}} \\ (B \otimes^{\mathcal{D}} C))| \end{array} \\
 \downarrow \theta^U & \text{(|-| applied to coherence (16) for } \theta^U) & \downarrow \theta^U \\
 \begin{array}{c} |U(A \otimes^{\mathcal{D}} B) \otimes U(C)| \\ \downarrow |\theta^U| \\ |U((A \otimes^{\mathcal{D}} B) \otimes^{\mathcal{D}} C)| \end{array} & \xrightarrow{|\alpha^{\mathcal{D}}|} & \begin{array}{c} |U(A) \otimes \\ U(B \otimes^{\mathcal{D}} C)| \\ \downarrow |\theta^U| \\ |U(A \otimes^{\mathcal{D}} \\ (B \otimes^{\mathcal{D}} C))| \end{array}
 \end{array}$$

Diagram (17).

$$\begin{array}{ccc}
 \mathbb{I} \otimes |UA| & & \\
 \downarrow \mathfrak{i} \otimes \text{id}_{|UA|} & & \searrow \mathfrak{l}_{|UA|} \\
 |I| \otimes |UA| & & \\
 \downarrow |\iota^U| \otimes \text{id}_{|UA|} & \xrightarrow{\theta_{I,UA}} \text{(coherence condition (17))} & \\
 |UI^{\mathcal{D}}| \otimes |UA| & \xrightarrow{\text{(naturality of } \theta)} & |I \otimes UA| \xrightarrow{|\ell_{UA}|} |UA| \\
 \downarrow \theta_{UI^{\mathcal{D}},UA} & \xrightarrow{\text{(coherence condition (17))}} & \\
 |UI^{\mathcal{D}} \otimes UA| & \xrightarrow{|\iota^U \otimes \text{id}_{UA}|} & \\
 \downarrow \theta_{I^{\mathcal{D}},A}^U & \xrightarrow{|\ell_A^{\mathcal{D}}|} & \\
 |U(I^{\mathcal{D}} \otimes^{\mathcal{D}} A)| & & 
 \end{array}$$

Diagram (18). This is checked by a diagram similar to the latter.

By the same reasoning,  $F\mathbf{D}$  is also a monoidal functor, with mediating pair  $(F\theta \circ \theta^F, F\mathfrak{i} \circ \iota^F)$ .

We now check the monoidal adjunction conditions (49), (50), (51), (52). Let  $\epsilon$  the counit, and  $\dot{\eta}$  the unit of  $\mathbf{D} \dashv |-|$ , and  $\epsilon$  the counit, and  $\eta$  the unit of  $F \dashv U$ .

Diagram (49).

$$\begin{array}{ccccc}
\mathbb{I} & \xrightarrow{\dot{\eta}} & |\mathbf{I}| & \xrightarrow{|\iota^U|} & |U\mathbf{I}^{\mathcal{D}}| \\
& \searrow^{((49) \text{ for } \mathbf{D} \dashv |-|)} & \downarrow |\dot{\eta}| & \searrow^{((49) \text{ for } F \dashv U)} & \downarrow |U\iota^F| \\
& & |\mathbf{D}\mathbb{I}| & \xrightarrow{|\eta_{\mathbf{I}}|} & |U\mathbf{F}\mathbf{I}| \\
& & & \text{(naturality of } |\eta| \text{)} & \downarrow |U\mathbf{F}\mathbf{I}^{\mathcal{D}}| \\
& & & \searrow^{|\eta_{\mathbf{D}\mathbb{I}}|} & |U\mathbf{F}\mathbf{D}\mathbb{I}|
\end{array}$$

Diagram (50).

$$\begin{array}{ccccc}
A \otimes B & \xrightarrow{\dot{\eta}_A \otimes \dot{\eta}_B} & |\mathbf{D}A| \otimes |\mathbf{D}B| & \xrightarrow{|\eta_{\mathbf{D}A}| \otimes |\eta_{\mathbf{D}B}|} & |U\mathbf{F}\mathbf{D}A| \otimes |U\mathbf{F}\mathbf{D}B| \\
\downarrow \dot{\eta}_{A \otimes B} & \searrow^{((50) \text{ for } \mathbf{D} \dashv |-|)} & \downarrow \theta_{\mathbf{D}A, \mathbf{D}B} & \searrow^{\text{(naturality of } \theta \text{)}} & \downarrow \theta_{U\mathbf{F}\mathbf{D}A, U\mathbf{F}\mathbf{D}B} \\
& & |\mathbf{D}A \otimes \mathbf{D}B| & \xrightarrow{|\eta_{\mathbf{D}A \otimes \mathbf{D}B}|} & |U\mathbf{F}\mathbf{D}A \otimes U\mathbf{F}\mathbf{D}B| \\
& \searrow^{\theta_{A,B}} & \downarrow & \searrow^{((50) \text{ for } F \dashv U)} & \downarrow \theta_{U\mathbf{F}\mathbf{D}A, \mathbf{F}\mathbf{D}B}^U \\
& & |\mathbf{D}(A \otimes B)| & \xrightarrow{|\eta_{\mathbf{D}(A \otimes B)}|} & |U(\mathbf{F}\mathbf{D}A \otimes^{\mathcal{D}} \mathbf{F}\mathbf{D}B)| \\
& & \text{(naturality of } |\eta| \text{)} & \searrow^{U\theta_{\mathbf{D}A, \mathbf{D}B}^F} & \downarrow \theta_{U\mathbf{F}\mathbf{D}A, \mathbf{D}B}^F \\
& & |\mathbf{D}(A \otimes B)| & \xrightarrow{|\eta_{\mathbf{D}(A \otimes B)}|} & |U\mathbf{F}(\mathbf{D}A \otimes \mathbf{D}B)| \\
& & \downarrow & \searrow^{U\theta_{A,B}^F} & \downarrow \theta_{U\mathbf{F}\mathbf{D}A, B}^F \\
& & |U\mathbf{F}\mathbf{D}(A \otimes B)| & & |U\mathbf{F}\mathbf{D}(A \otimes B)|
\end{array}$$

Diagrams (51) and (52) are dealt with similar, and give rise to similar verifications.

### Appendix D. Proof of Proposition 3

Diagram (32) is obtained by considering the following diagram. We drop subscripts on natural transformations so as to save space; they can be inferred from context.

$$\begin{array}{ccccccc}
\mathbb{I} \otimes |\mathbf{T}\mathbf{D}\mathbf{F}| & \xrightarrow{\dot{\eta}_{\mathbb{I}} \otimes \text{id}} & |\mathbf{D}\mathbb{I}| \otimes |\mathbf{T}\mathbf{D}\mathbf{F}| & \xrightarrow{\theta} & |\mathbf{D}\mathbb{I} \otimes \mathbf{T}\mathbf{D}\mathbf{F}| & \xrightarrow{|\epsilon|} & |\mathbf{T}(\mathbf{D}\mathbb{I} \otimes \mathbf{D}\mathbf{F})| \xrightarrow{|\mathbf{T}\theta|} |\mathbf{T}\mathbf{D}(\mathbb{I} \otimes \mathbf{F})| \\
& \searrow^{(49)} & \uparrow |\dot{\eta}| \otimes \text{id} & \text{(naturality of } \theta \text{)} & \uparrow |\dot{\eta}| \otimes \text{id} & \text{(naturality of } |\mathbf{T}| \text{)} & \uparrow |\mathbf{T}(\dot{\eta} \otimes \text{id})| \\
& & |\mathbf{I}| \otimes |\mathbf{T}\mathbf{D}\mathbf{F}| & \xrightarrow{\theta} & |\mathbf{I} \otimes \mathbf{T}\mathbf{D}\mathbf{F}| & \xrightarrow{|\epsilon|} & |\mathbf{T}(\mathbf{I} \otimes \mathbf{D}\mathbf{F})| \xrightarrow{|\mathbf{T}(17)|} |\mathbf{T}\mathbf{D}\mathbf{I}_F| \\
& & \downarrow & \text{(17)} & \downarrow & \text{(32)} & \downarrow |\mathbf{T}\ell_{\mathbf{D}\mathbf{F}}| \\
& & & \searrow^{|\mathbf{T}\ell_{\mathbf{D}\mathbf{F}}|} & & \searrow^{|\ell_{\mathbf{T}\mathbf{D}\mathbf{F}}|} & |\mathbf{T}\mathbf{D}\mathbf{F}|
\end{array}$$

We have also decorated the interior of each face with the justification why its edges commute. For example, the top left triangle is a copy of (49), one of the diagrams stating that the adjunction is monoidal. The bottom face, whose bottom edge is a curved arrow, commutes by the coherence square (17) for  $\theta$ . The rightmost parallelogram commutes by  $|T|$  applied to another coherence square (17), this time for  $\theta$ . Now the topmost composition of arrows is  $\mathbb{t}_{\mathbb{I},F}$ , the bottommost arrow from  $\mathbb{I} \otimes |TDF|$  to  $|TDF|$  is  $\mathbb{t}_{\mathbb{T}F}$ , and the rightmost vertical arrow is  $\mathbb{T}\mathbb{t}_F$ .

Diagram (33). This is the diagram below. We recognize  $\mathbb{t}_{E,F}$  as the rightmost composition of vertical arrows,  $\text{id}_E \otimes \eta_F$  as the topmost composition of horizontal arrows, and  $\eta_{E \otimes F}$  is the diagonal from  $E \otimes F$  to  $|TD(E \otimes F)|$ .

$$\begin{array}{ccccc}
E \otimes F & \xrightarrow{\text{id} \otimes \dot{\eta}_F} & E \otimes |DF| & \xrightarrow{\text{id} \otimes |\eta_{DF}|} & E \otimes |TDF| \\
& \searrow^{\dot{\eta}_E \otimes \dot{\eta}_F} & & & \downarrow \eta_E \otimes \text{id} \\
& & |DE| \otimes |DF| & \xrightarrow{\text{id} \otimes |\eta_{DF}|} & |DE| \otimes |TDF| \\
& \text{(functoriality of } \otimes \text{)} & & & \downarrow \theta_{DE, TDF} \\
& \text{(monoidal} & \downarrow \theta_{DE, DF} \text{ (naturality of } \theta \text{)} & & \downarrow \theta_{DE, TDF} \\
& \text{adjunction} & |DE \otimes DF| & \xrightarrow{|\text{id} \otimes \eta_{DF}|} & |DE \otimes TDF| \\
& \text{(50))} & & & \downarrow \mathbb{t}_{DE, DF} \\
& \eta_{E \otimes F} & \downarrow |\theta_{E,F}| & \text{(strength law (33))} & \downarrow \mathbb{t}_{DE, DF} \\
& & |D(E \otimes F)| & \xrightarrow{|\eta_{DE \otimes DF}|} & |T(DE \otimes DF)| \\
& & \downarrow |\eta_{D(E \otimes F)}| & \text{(naturality of } |\eta| \text{)} & \downarrow |T\theta_{E,F}| \\
& & & & |TD(E \otimes F)|
\end{array}$$

Diagram (34). For reasons of space, we flip the diagram so that arrows involving strengths are vertical, and arrows involving associativities are horizontal. Again, we drop most

subscripts from natural transformations, which are inferrable from context.

$$\begin{array}{ccc}
\begin{array}{c} (E \otimes F) \\ \otimes |TDG| \end{array} & \xrightarrow{\alpha} & \begin{array}{c} E \otimes (F \otimes \\ |TDG|) \end{array} \\
\downarrow \eta_{E \otimes F} \otimes \text{id} & \searrow (\eta_E \otimes \eta_F) \otimes \text{id} & \downarrow \text{id} \otimes (\eta_F \otimes \text{id}) \\
\begin{array}{c} |D(E \otimes F)| \\ \otimes |TDG| \end{array} & \xrightarrow{\text{(monoidal adjunction (50))}} & \begin{array}{c} E \otimes (|DF| \otimes \\ |TDG|) \end{array} \\
\downarrow \theta & \downarrow \theta \otimes \text{id} & \downarrow \text{id} \otimes \theta \\
\begin{array}{c} |D(E \otimes F) \otimes DF| \\ \otimes |TDG| \end{array} & \xrightarrow{\text{(coherence (16) of } \theta \text{)}} & \begin{array}{c} |DE| \otimes \\ |DF| \otimes \\ |TDG| \end{array} \\
\downarrow |\mathbf{t}| & \downarrow |\mathbf{t}| & \downarrow \text{id} \otimes \theta \\
\begin{array}{c} |D(E \otimes F) \otimes DF \otimes TDG| \\ \otimes |TDG| \end{array} & \xrightarrow{\text{(strength law (34))}} & \begin{array}{c} |DE| \otimes \\ |DF \otimes \\ |TDG| \end{array} \\
\downarrow |\mathbf{T}\theta| & \downarrow |\mathbf{T}\alpha| & \downarrow \text{id} \otimes |\mathbf{T}\theta| \\
\begin{array}{c} |T \left( \begin{array}{c} D(E \otimes F) \\ \otimes DF \end{array} \right) \otimes TDG| \\ \otimes |TDG| \end{array} & \xrightarrow{\text{(coherence } |\mathbf{T}(16)| \text{ of } \theta \text{)}} & \begin{array}{c} |T \left( \begin{array}{c} DE \otimes \\ DF \otimes \\ DG \end{array} \right)| \end{array} \\
\downarrow |\mathbf{T}\theta| & \downarrow |\mathbf{T}(\text{id} \otimes \theta)| & \downarrow |\mathbf{T}(\text{id} \otimes \theta)| \\
\begin{array}{c} |TD \left( \begin{array}{c} E \otimes F \\ \otimes G \end{array} \right)| \end{array} & \xrightarrow{|\mathbf{T}D\alpha|} & \begin{array}{c} |TD \left( \begin{array}{c} E \otimes \\ F \otimes G \end{array} \right)| \end{array}
\end{array}
\end{array}$$

The vertical arrows on the left compose to form  $t_{A \otimes^c B, C}$ , while the vertical arrows on the right compose to form  $\text{id} \otimes^c t_{B, C}$  followed by  $t_{A, B \otimes^c C}$ , whence the result.

Diagram (35). This is the diagram below. Again, the diagram is flipped sideways, and certain subscripts have been dropped. The topmost arrow from  $E \otimes |\mathbf{TD}|\mathbf{TDF}||$  to  $E \otimes |\mathbf{TDF}|$  is  $\text{id}_E \otimes \mu_{DF}$  by definition, while the bottommost arrow from  $|\mathbf{TD}|\mathbf{TD}(E \otimes F)||$  to  $|\mathbf{TD}(E \otimes F)|$  is  $\mu_{E \otimes F}$ . The rightmost composition of vertical arrows, from  $E \otimes |\mathbf{TDF}|$  to  $|\mathbf{TD}(E \otimes F)|$  is  $\mu_{E \otimes F}$ , is  $\mathbb{t}_{E,F}$ , and the leftmost one is  $\mathbb{t}_{E,TF}$  followed by  $\mathbb{T}\mathbb{t}_{E,F}$ . (Recall that  $\mathbb{T} = |\mathbf{TD}|$ .)

$$\begin{array}{ccccc}
E \otimes |\mathbf{TD}|\mathbf{TDF}|| & \xrightarrow{\text{id}_E \otimes |\mathbf{T}\acute{\epsilon}_{\mathbf{TDF}}|} & E \otimes |\mathbf{T}^2\mathbf{DF}| & \xrightarrow{\text{id}_E \otimes |\mu_{\mathbf{DF}}|} & E \otimes |\mathbf{TDF}| \\
\downarrow \dot{\eta}_E \otimes \text{id} & & \text{(functoriality of } \otimes) & & \downarrow \dot{\eta}_E \otimes \text{id}_{|\mathbf{TDF}|} \\
|\mathbf{DE}| \otimes |\mathbf{TD}|\mathbf{TDF}|| & \xrightarrow{\text{id} \otimes |\mathbf{T}\acute{\epsilon}_{\mathbf{TDF}}|} & |\mathbf{DE}| \otimes |\mathbf{T}^2\mathbf{DF}| & & |\mathbf{DE}| \otimes |\mathbf{TDF}| \\
\downarrow \mathbb{\theta}_{\mathbf{DE}, \mathbf{TD}|\mathbf{TDF}|} & \text{=} & \downarrow \mathbb{\theta}_{\mathbf{DE}, \mathbf{T}^2\mathbf{DF}} & \searrow \text{=} & \downarrow \mathbb{\theta}_{\mathbf{DE}, \mathbf{TDF}} \\
|\mathbf{DE} \otimes \mathbf{TD}|\mathbf{TDF}|| & \xrightarrow{|\text{id} \otimes \mathbf{T}\acute{\epsilon}_{\mathbf{TDF}}|} & |\mathbf{DE} \otimes \mathbf{T}^2\mathbf{DF}| & \text{(naturality of } \mathbb{\theta}) & |\mathbf{DE}| \otimes |\mathbf{TDF}| \\
\downarrow |\mathbf{t}_{\mathbf{DE}, \mathbf{D}|\mathbf{TDF}}| & \text{(naturality of } |\mathbf{t}|) & \downarrow |\mathbf{t}_{\mathbf{DE}, \mathbf{TDF}}| & & \downarrow \mathbb{\theta}_{\mathbf{DE}, \mathbf{TDF}} \\
|\mathbf{T}(\mathbf{DE} \otimes \mathbf{D}|\mathbf{TDF})|| & \xrightarrow{|\mathbf{T}(\text{id} \otimes \acute{\epsilon}_{\mathbf{TDF}})|} & |\mathbf{T}(\mathbf{DE} \otimes \mathbf{TDF})| & \searrow \text{=} & |\mathbf{DE} \otimes \mathbf{TDF}| \\
\downarrow |\mathbf{T}\theta_{E, \mathbf{TDF}}| & \text{(adjunction law)} & \downarrow |\mathbf{T}t_{\mathbf{DE}, \mathbf{DF}}| & & \downarrow \mathbb{\theta}_{\mathbf{DE}, \mathbf{TDF}} \\
|\mathbf{TD}(E \otimes |\mathbf{TDF}|)|| & \xrightarrow{|\mathbf{T}(\mathbf{D}\dot{\eta}_E \otimes \text{id}_{|\mathbf{TDF}|})|} & \downarrow |\mathbf{T}t_{\mathbf{DE}, \mathbf{DF}}| & \text{(strength law |(35)|)} & |\mathbf{DE} \otimes \mathbf{TDF}| \\
\downarrow |\mathbf{TD}(\dot{\eta}_E \otimes \text{id}_{|\mathbf{TDF}|})| & \text{(naturality of } |\mathbf{T}\theta|) & \downarrow |\mathbf{T}t_{\mathbf{DE}, \mathbf{DF}}| & & \downarrow |\mathbf{t}_{\mathbf{DE}, \mathbf{DF}}| \\
|\mathbf{TD}(|\mathbf{DE}| \otimes |\mathbf{TDF}|)|| & \xrightarrow{|\mathbf{T}\theta_{|\mathbf{DE}|, |\mathbf{TDF}|}|} & |\mathbf{T}^2(\mathbf{DE} \otimes \mathbf{DF})| & \searrow \text{=} & |\mathbf{T}(\mathbf{DE} \otimes \mathbf{DF})| \\
\downarrow |\mathbf{TD}\mathbb{\theta}_{\mathbf{DE}, \mathbf{TDF}}| & \text{monoidal adjunction law |(52)|} & \downarrow |\mathbf{T}^2\theta_{E, F}| & \text{(naturality of } |\mu|) & \downarrow |\mathbf{T}\theta_{E, F}| \\
|\mathbf{TD}|\mathbf{DE} \otimes \mathbf{TDF}|| & \xrightarrow{|\mathbf{T}\acute{\epsilon}_{\mathbf{DE} \otimes \mathbf{TDF}}|} & \downarrow |\mathbf{T}^2\theta_{E, F}| & & \downarrow |\mathbf{T}\theta_{E, F}| \\
\downarrow |\mathbf{TD}|\mathbf{t}_{\mathbf{DE}, \mathbf{DF}}|| & \text{(naturality of } |\mathbf{T}\acute{\epsilon}|) & \downarrow |\mathbf{T}^2\theta_{E, F}| & & \downarrow |\mathbf{T}\theta_{E, F}| \\
|\mathbf{TD}|\mathbf{T}(\mathbf{DE} \otimes \mathbf{DF})|| & \xrightarrow{|\mathbf{T}\acute{\epsilon}_{\mathbf{T}(\mathbf{DE} \otimes \mathbf{DF})}|} & \downarrow |\mathbf{T}^2\theta_{E, F}| & & \downarrow |\mathbf{T}\theta_{E, F}| \\
\downarrow |\mathbf{TD}|\mathbf{T}\theta_{E, F}|| & \text{(naturality of } |\mathbf{T}\acute{\epsilon}|) & \downarrow |\mathbf{T}^2\theta_{E, F}| & & \downarrow |\mathbf{T}\theta_{E, F}| \\
|\mathbf{TD}|\mathbf{TD}(E \otimes F)|| & \xrightarrow{|\mathbf{T}\acute{\epsilon}_{\mathbf{TD}(E \otimes F)}|} & |\mathbf{T}^2\mathbf{D}(E \otimes F)| & \xrightarrow{|\mu_{\mathbf{D}(E \otimes F)}|} & |\mathbf{TD}(E \otimes F)|
\end{array}$$

Diagram (36). Again, we flip the diagram sideways. We recognize  $|\mathbf{t}_{A_1, A_2}|$  as the rightmost vertical arrow,  $\mathbb{t}_{|A_1|, |A_2|}$  as the leftmost composition of vertical arrows,  $\text{id}_{|A_1|} \otimes \sigma_{A_2}$  followed by  $\mathbb{\theta}_{A_1, \mathbf{T}A_2}$  on the top row, and  $\mathbb{T}(\mathbb{\theta}_{A_1, A_2})$  followed by  $\sigma_{A_1 \otimes A_2}$  on the bottom



