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Choquet-Kendall-Matheron Theorems for Non-Hausdorff Spaces

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We establish Choquet-Kendall-Matheron theorems on non-Hausdorff topological spaces. This typical result of random set theory is profitably recast in purely topological terms, using intuitions and tools from domain theory. We obtain three variants of the theorem, each one characterizing distributions, in the form of continuous valuations, over relevant powerdomains of demonic, resp. angelic, resp. erratic non-determinism.

1. Introduction

This paper is in the line of a development that—up to our knowledge—has its origin in a famous paper by G. Choquet (Choquet, 54). The motivation comes from the well-known fact that, on the unit interval, a positive measure μ is completely determined by its distribution function $F(x) = \mu([0, x])$ which is an upper semi-continuous monotone increasing function from the unit interval into the reals and every such function is the distribution function of a unique positive measure. The point is that distribution functions—certain real valued functions on the base space $[0, 1]$—are simpler objects than measures—certain real valued functions on the σ-algebra of Borel sets of the base space.

Choquet-Kendall-Matheron type theorems achieve a similar goal for hyperspaces $\mathcal{H}sp(X)$, that is, spaces of subsets of a topological space X, e.g., closed, open or compact subsets of X topologized in some natural way. The goal is to describe measures defined on the σ-algebra of Borel sets of $\mathcal{H}sp(X)$ by some kind of 'distribution function' defined on one of the spaces $\mathcal{H}sp(X)$ directly. The classical Choquet Theorem (Choquet, 54, Theorem 50.1) concerns the hyperspace $\mathcal{K}(X)$ of all compact subsets of a locally compact Hausdorff space X with the Vietoris topology: The Radon measures on the hyperspace $\mathcal{K}(X)$ are in one-to-one correspondence with the upper semi-continuous non-negative real valued functions defined on $\mathcal{K}(X)$ which are *monotone of infinite order* according

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to Choquet's terminology. Excellent references for Choquet-type theorems are the books by Matheron and Molchanov (Matheron, 1975; Molchanov, 2005) on the theory of random sets and integral geometry. In both books the authors restrict themselves to second countable locally compact Hausdorff spaces. Choquet-Kendall-Matheron type theorems are used also in economics, where they are conceived as a form of "completion of a misspecified model" (Gilboa and Schmeidler, 1992).

We establish Choquet-Kendall-Matheron type theorems over spaces that are not necessarily Hausdorff nor second countable; only local compactness is used, and in the third of our three cases to come, an additional hypothesis of coherence, which is trivially satisfied in Hausdorff spaces. In our setting, the classical Choquet Theorem on the representation of measures on the hyperspace $\mathcal{K}(X)$ of compact subsets of a locally compact space splits into two parts, a 'demonic' and an 'angelic' one: Indeed the Vietoris topology splits in a natural way in an upper and a lower Vietoris topology which both are far from being Hausdorff. A third case combining the demonic and the angelic versions arises—we call it the erratic case—that has no significance in the Hausdorff situation.

We are motivated by denotational semantics to take non-Hausdorff spaces into account. Semantic domains in the sense of D.S. Scott are far from being Hausdorff, see (Abramsky and Jung, 1994; Gierz et al., 2003). Quite some intuitions and tools are obtained from the domain-theoretic perspective. Indeed, the hyperspaces under discussion are typical examples of continuous lattices. Also, measures are profitably replaced with the essentially equivalent notion of continuous valuations. The notion of a valuation on a lattice goes back to G. Birkhoff (Birkhoff, 1940, Chapter X, Sec. 1). Valuations on distributive lattices and their integral representations are discussed by Choquet (Choquet, 54, §41). As a substitute for measures, valuations are abundantly used in geometric probability theory as witnessed by the nice monograph by D.A. Klain and G.-C. Rota (Klain and Rota, 1997). Continuous valuations were used as a convenient model of probabilistic choice in denotational semantics (Jones and Plotkin, 1989).

Concerning the use in denotational semantics, one may think of subsets A of X as specifications of non-deterministic choice processes—picking an element from A. And there are three classical forms of non-determinism, demonic, angelic, and erratic. Accordingly, we prove three theorems of the kind of Choquet, Kendall, and Matheron. In the demonic case (Section 4), we show that continuous valuations on the Smyth space of all non-empty compact saturated subsets of X are in one-to-one correspondence with continuous credibilities, as soon as X is locally compact. In the angelic case (Section 5), we show that continuous valuations on the Hoare space of all non-empty closed subsets are in one-to-one correspondence with continuous plausibilities whenever X is core-compact. Finally, in the erratic case (Section 6), we show that continuous valuations on the Plotkin space of all non-empty lenses are in one-to-one correspondence with a new notion that we call *sesqui-continuous estimates*. These measure crescents instead of opens, and the theorem is shown assuming X is locally compact and coherent.

Using domain theoretical ideas also sheds a new light on the classical results. One should however note that we are not the first to extend such theorems to the non-Hausdorff case: Norberg (Norberg, 1989) has a theorem that is essentially the same as our Theorem 5.12 relating continuous plausibilities with continuous valuations on spaces of closed subsets (the angelic case). We discuss the connection more precisely right after the proof of this theorem.

2. Preliminaries

We refer to (Abramsky and Jung, 1994; Gierz et al., 2003; Mislove, 1998) for background material on domain theory and topology, and to (Molchanov, 2005) for capacities and related concepts.

2.1. *Topology.*

A *topology* on X is a family of subsets, called the *opens*, such that any union and any finite intersection of opens is open. The complements of open subsets are called *closed*. The largest open contained in a subset A of X is its *interior* $int(A)$, while the smallest closed set containing A is its *closure* $cl(A)$.

Given any family of subsets $\mathcal{A}$ of X, there is a smallest topology on X *generated by* $\mathcal{A}$, i.e., making all elements of $\mathcal{A}$ open. Then every open in this topology is a union of finite intersections of elements of $\mathcal{A}$; $\mathcal{A}$ is a *subbase* of the topology. If every open is a union of elements of $\mathcal{A}$, then $\mathcal{A}$ is called a *base* of the topology.

A map $f : X \to Y$ is *continuous* iff $f^{-1}(U)$ is open in X for every open subset U of Y. We shall often use the fact that, if $\mathcal{A}$ is a subbase of the topology of Y, $f : X \to Y$ is continuous iff $f^{-1}(U)$ is open in X for all elements U of $\mathcal{A}$.

A subset Q of X is *compact* iff one can extract a finite subcover from every open cover of Q. It is *saturated* iff it is the intersection of all opens containing it, a.k.a. it is upward-closed in the *specialization quasi-ordering* $\leq$, defined by $x \leq y$ iff every open containing x contains y. The *saturation* $\uparrow A$ of a subset A of X is defined equivalently as the intersection of all opens U containing A, or as the upward-closure $\{x \in X \mid \exists y \in A \cdot y \leq x\}$. We write $\downarrow A$ for the downward-closure $\{x \in X \mid \exists y \in A \cdot x \leq y\}$. Every open subset is upward-closed, and every closed subset is downward-closed. In T_0 spaces X that are not T_1, such as dcpos (see below), there are compact subsets that are not saturated, e.g., $\{x\}$ where x is not maximal in X. However, for any compact subset K, $\uparrow K$ is both compact and saturated. In particular, the saturation of any finite set is compact.

We shall use *Alexander's Subbase Lemma*, which states that in a space X with subbase $\mathcal{A}$, K is compact if and only if one can extract a finite subcover from every cover of K consisting of elements of $\mathcal{A}$.

A topological space X is *locally compact* if and only if, whenever $x \in U$ with U open, there is a compact subset K such that $x \in int(K) \subseteq K \subseteq U$. In this definition, we might as well require K to be saturated, but this is unnecessary: if $x \in int(K) \subseteq K \subseteq U$, then $Q = \uparrow K$ is compact, saturated, and $x \in int(Q) \subseteq Q \subseteq U$. In any locally compact space, whenever Q is a compact subset of some open U, then there is a compact saturated subset Q_1 such that $Q \subseteq int(Q_1) \subseteq Q_1 \subseteq U$.

X is *coherent* if and only if the intersection of any two compact saturated subsets is again compact.

Our theorems will be concerned with locally compact spaces, and in one case, with locally compact, coherent spaces.

An important class of locally compact, coherent spaces is given by the *stably locally compact* spaces, which are those locally compact coherent spaces that are additionally T_0 and well-filtered. A space X is *well-filtered* if and only if, for every filtered family $(Q_i)_{i\in I}$ of compact saturated subsets in X, for every open U, if $\bigcap_{i\in I} Q_i \subseteq U$ then $Q_i \subseteq U$ already for some $i \in I$. We shall only need well-filteredness in relating our results, which are concerned with continuous valuations, to more classical formulations of the Choquet-Kendall-Matheron Theorems based on measures.

Among the stably locally compact spaces, we find the *stably compact* spaces, namely those that are additionally compact. Stable compactness has a long history, going back to Nachbin (1948; see (Jung, 2004)). To give a concrete example, $[0,1]_\sigma$, the set $[0,1]$ with opens of the form $(t,1]$, $0 \leq t \leq 1$, plus $[0,1]$ itself, is stably compact. This is just $[0,1]$ with the Scott topology of its natural ordering $\leq$, see below. Similarly, $[0,1]^I_\sigma$, is stably compact for any set I.

Stable (local) compactness is also usually defined by requiring sobriety instead of well-filteredness. A *sober* space is a T_0 space X where every irreducible closed set is the closure $\downarrow x$ of a point $x \in X$. A closed set F is irreducible iff it is non-empty, and whenever F is contained in the union of two closed subsets, then F is contained in one of them. As remarked by Jung (Jung, 2004, Section 2.3), referring to (Gierz et al., 2003, Theorem II-1.21), this is equivalent in the presence of local compactness. Sobriety alone implies well-filteredness, as a consequence of the Hofmann-Mislove Theorem.

A space is *Hausdorff*, or T_2, iff every two distinct points x, y can be separated by opens U, V, i.e., $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Every locally compact Hausdorff space is stably locally compact, e.g., $[0,1]$ with its usual metric topology; the converse fails, as for example $[0,1]_\sigma$ is stably compact but not T_2.

2.2. *Domain Theory.*

A set with a partial ordering is a *poset*. A *dcpo* is a poset in which every directed family $(x_i)_{i\in I}$ has a least upper bound (a.k.a., *sup*) $\sup_{i\in I} x_i$. A family $(x_i)_{i\in I}$ is *directed* iff it is non-empty, and any two elements have an upper bound in the family. We shall also use the dual notion of filteredness: the family $(x_i)_{i\in I}$ is *filtered* iff it is non-empty, and any two elements have an lower bound in the family. Any poset can be equipped with the *Scott topology*, whose opens are the upward closed sets U such that whenever $(x_i)_{i\in I}$ is a directed family that has a least upper bound in U, then some x_i is in U already. The Scott topology is always T_0, and its specialization ordering is the original partial ordering.

The *way-below* relation $\ll$ on a poset X is defined by $x \ll y$ iff, for every directed family $(z_i)_{i\in I}$ that has a least upper bound z such that $y \leq z$, then $x \leq z_i$ for some $i \in I$ already. Note that $x \ll y$ implies $x \leq y$, and that $x' \leq x \ll y \leq y'$ implies $x' \ll y'$. However, $\ll$ is not reflexive or irreflexive in general. Write $\twoheaduparrow E = \{y \in X \mid \exists x \in E \cdot x \ll y\}$, $\twoheaddownarrow E = \{y \in X \mid \exists x \in E \cdot y \ll x\}$. X is *continuous* iff, for every $x \in X$, $\twoheaddownarrow x$ is a directed family, and has x as least upper bound. One may be more precise: A *basis* is a subset B of

X such that any element $x \in X$ is the least upper bound of a directed family of elements way-below x *in* B. Then X is continuous if and only if it has a basis. In a continuous poset X with basis B, the *interpolation property* holds: whenever $x \ll z$, then $x \ll y \ll z$ for some $y \in B$ (Mislove, 1998, Lemma 4.16). It follows that, in any continuous poset X, $\Uparrow x$ is Scott-open for all x, and every Scott-open set U is a union of such sets, more precisely $U = \bigcup_{x \in U \cap B} \Uparrow x$.

A map $f : X \to Y$ between posets X, Y is continuous (with respect to the respective Scott topologies) if and only if it has the following two properties: (1) f is monotone, that is, whenever $x \leq x'$ in X, then $f(x) \leq f(x')$ in Y, and (2) whenever $(x_i)_{i \in I}$ is directed and $\sup_{i \in I} x_i$ exists in X, then $\sup_{i \in I} f(x_i)$ exists in Y and $f(\sup_{i \in I} x_i) = \sup_{i \in I} f(x_i)$. For dcpos X and Y, the second condition simplifies to (2'): $f(\sup_{i \in I} x_i) = \sup_{i \in I} f(x_i)$ for every directed family (x_i) in X. We will use the following standard lemma on the continuous extension of functions:

Extension Lemma. Let X be a continuous dcpo with a basis B and $f : B \to \mathbb{R}$ a bounded monotone function. Then the function $f^* : X \to \mathbb{R}$ defined by

$$f^*(x) = \sup\{f(b) \mid b \in B \text{ and } b \ll x\}$$

is continuous on X. It is the greatest among the continuous functions such that $f^*(x) \leq f(x)$ for all $x \in B$. If f^* is an extension of f, that is, if $f^*(b) = f(b)$ for all $b \in B$, then it is the unique continuous extension of f to all of X.

We stress that, throughout this paper, we consider $\mathbb{R}$ with its usual ordering as a poset with its Scott topology, the non-trivial open sets of which are the open infinite intervals $(r, +\infty[$. The subspace $\mathbb{R}^+$ of non-negative reals is equipped with the subspace topology, which is also the Scott topology. Functions into $\mathbb{R}$ or $\mathbb{R}^+$ that are continuous in our sense (with respect to the Scott topology) are called *lower semi-continuous* in general topology.

Every continuous dcpo X is sober, hence well-filtered, and locally compact. If additionally X is *pointed*, i.e., has a least element $\bot$, then X is compact. If finally X is also coherent, then X is stably compact. The stably compact dcpos are sometimes referred to in the literature as *Lawson-compact* dcpos, meaning that they are exactly the dcpos that are compact in their Lawson topology. $[0, 1]$ with its Scott topology is an example.

The lattice $\mathcal{O}(X)$ of any topological space is in particular a dcpo. The spaces X such that $\mathcal{O}(X)$ is a continuous dcpo are by definition the *core-compact* spaces. Every locally compact space is core-compact. Concretely, if X is locally compact, then the way-below relation on $\mathcal{O}(X)$, which we shall write $\Subset$ to distinguish it from the notation $\ll$ used in more mundane dcpos, is characterized by: $U \Subset V$ iff there is a compact subset Q such that $U \subseteq Q \subseteq V$. We shall also say that U is relatively compact in V, instead of U is way-below V, in agreement with terminology in general topology. We say that U is relatively compact, if it is relatively compact in the whole space X, that is, if it is contained in some compact subset of X. That $\mathcal{O}(X)$ is continuous when X is locally compact follows from the easily proved fact that any open U of X can be written as $\bigcup_Q \mathit{int}(Q)$, where Q ranges over all compact (saturated) subsets of U.

2.3. *Powerdomains.*

Powerdomains were introduced, by several people, to give denotational semantics to non-deterministic choice in higher-order programming languages. The three main such powerdomains are the Smyth powerdomain for demonic non-determinism, the Hoare powerdomain for angelic non-determinism, and the Plotkin powerdomain for erratic non-determinism, see (Abramsky and Jung, 1994, Section 6.2). This viewpoint traditionally stays with the category of dcpos, but is easily and profitably extended to general topological spaces, in the tradition of hyperspaces initiated by Hausdorff and Vietoris; see in particular (Abramsky and Jung, 1994, Sections 6.2.3, 6.2.4), and Schalk's PhD thesis (Schalk, 1993).

The (topological version of the) *Smyth powerdomain* $\mathcal{Q}_{\mathcal{V}}(X)$ of a space X is the set of all non-empty compact saturated subsets Q of X, with the *upper Vietoris* topology, which has a base given by subsets of the form $\Box U = \{Q \in \mathcal{Q}_{\mathcal{V}}(X) \mid Q \subseteq U\}$, U open in X. The specialization ordering of $\mathcal{Q}_{\mathcal{V}}(X)$ is reverse inclusion. It is more traditional in domain theory to define the poset $\mathcal{Q}(X)$ of all non-empty compact saturated subsets, ordered by reverse inclusion $\supseteq$. When X is locally compact and well-filtered, $\mathcal{Q}(X)$ is a continuous dcpo, and the way-below relation is given by $Q \ll Q'$ iff $Q' \subseteq int(Q)$, so $\Uparrow Q = \Box int(Q)$; in particular, the upper Vietoris and Scott topologies coincide in this case, and $\mathcal{Q}(X) = \mathcal{Q}_{\mathcal{V}}(X)$. If X is itself a continuous dcpo, a basis of $\mathcal{Q}(X)$ is given by the compacts of the form $\uparrow E$, E finite non-empty; $\uparrow E \ll \uparrow E'$ iff $E' \subseteq \Uparrow E$, i.e., iff for every $y \in E'$, there is an $x \in E$ such that $x \ll y$ in X.

The (topological version of the) *Hoare powerdomain* $\mathcal{H}_{\mathcal{V}}(X)$ of X is the set of all non-empty closed subsets F of X, with the *lower Vietoris* topology, which has a subbase (not a base) given by subsets of the form $\Diamond U = \{F \in \mathcal{H}_{\mathcal{V}}(X) \mid F \cap U \neq \emptyset\}$, U open in X. The specialization ordering of $\mathcal{H}_{\mathcal{V}}(X)$ is ordinary inclusion. It is more traditional in domain theory to define the dcpo $\mathcal{H}(X)$ of all non-empty closed subsets of X, ordered by $\subseteq$. The sup of a directed family $(F_i)_{i\in I}$ of non-empty closed subsets in $\mathcal{H}(X)$ is then $cl(\bigcup_{i\in I} F_i)$. When X is a continuous dcpo, so is $\mathcal{H}(X)$, and its way-below relation is given by $F \ll F'$ iff there is a non-empty finite subset E of (a given basis of) X such that $F \subseteq \downarrow E$ and $E \subseteq \Downarrow F'$. Then the subsets $\downarrow E$ themselves, with E non-empty and finite, form a basis of $\mathcal{H}(X)$, and $\downarrow E \ll F'$ in $\mathcal{H}(X)$ iff $E \subseteq \Downarrow F'$. It is then an easy exercise to show that the Scott and the lower Vietoris topologies coincide, i.e., that $\mathcal{H}(X) = \mathcal{H}_{\mathcal{V}}(X)$ as soon as X is a continuous dcpo.

Finally, the Plotkin powerdomain $\mathcal{P}\ell(X)$ over X is the space of (compact) lenses L of X. A *lens* L of X is the intersection $Q \cap F$ of a compact saturated subset Q of X and a closed subset F of X, provided this intersection is non-empty. Then L has a canonical presentation as $\uparrow L \cap cl(L)$, where $\uparrow L$ is compact saturated, and $cl(L)$ is closed. There is a domain-theoretic definition, as a dcpo $\mathcal{P}\ell(X)$ of lenses, ordered by the *topological Egli-Milner ordering* $\sqsubseteq_{\mathrm{EM}}$, defined by $L \sqsubseteq_{\mathrm{EM}} L'$ iff $\uparrow L \supseteq \uparrow L'$ and $cl(L) \subseteq cl(L')$ (Abramsky and Jung, 1994, Section 6.2.3). We shall again prefer the purely topological counterpart, which we write $\mathcal{P}\ell_{\mathcal{V}}(X)$: this is $\mathcal{P}\ell(X)$ with the *Vietoris topology*, generated by sets $\{L \in \mathcal{P}\ell(X) \mid L \subseteq U\}$, which we shall write $\Box U$ again, and $\{L \in \mathcal{P}\ell(X) \mid L \cap U \neq \emptyset\}$, which we shall write $\Diamond U$, for any subset U of X. The specialization ordering of $\mathcal{P}\ell_{\mathcal{V}}(X)$ is $\sqsubseteq_{\mathrm{EM}}$, and the Scott topology of $\mathcal{P}\ell(X)$ is always finer than the Vietoris topology.

The two topologies coincide when X is a stably compact, continuous dcpo. Indeed, in this case, $\mathcal{P}\ell(X)$ is a continuous dcpo, a basis is given by the finite lenses $\langle E\rangle = \downarrow E \cap \uparrow E$ (E a finite subset of X), and $\langle E\rangle$ is way below a lens L iff $E \subseteq \twoheaddownarrow cl(L)$ and $L \subseteq \twoheaduparrow E$. So the basic Scott-open $\twoheaduparrow\langle E\rangle$, where $E = \{x_1, \ldots, x_n\}$, is the Vietoris open $\Box\twoheaduparrow E \cap \bigcap_{i=1}^n \Diamond\twoheaduparrow x_i$. Conversely, it is easy to see that all subbasic open sets of the Vietoris topology are Scott-open.

2.4. *Measures and Valuations.*

It is traditional to define integration and probabilistic processes using measures. An *algebra* (of subsets) on a set X is a collection of subsets of X that contains the empty set and is closed under complements and finite unions. A *σ-algebra* is defined similarly, except with countable unions instead of finite unions. The smallest σ-algebra containing all the opens of a given topological space X is the *Borel σ-algebra* of X.

Given a σ-algebra $\mathcal{A}$ on X, a *measure* is a map $\mu : \mathcal{A} \to \mathbb{R}^+$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$ for every sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint elements of $\mathcal{A}$. Our measures are usually referred to as *bounded* measures, i.e., μ does not take the value $+\infty$. When X is a topological space, we shall always assume that measures are measures on the Borel σ-algebra of X.

An alternative to measures on topological spaces is given by continuous valuations, whose usefulness in semantics was strongly supported by Jones and Plotkin (Jones and Plotkin, 1989). Instead of measuring measurable subsets, they only measure opens. Continuous valuations are a special case of *capacities*, a notion we shall need as well, and therefore introduce right away. The latter take their roots in Choquet's work on capacities (Choquet, 54), and are instrumental in potential theory, probability theory, and economics (Gilboa and Schmeidler, 1992).

Generalizing slightly the case of a topological space X with the lattice $\mathcal{O}(X)$ of open subsets, we consider a set X with a lattice of subsets, i.e., a collection $\mathcal{L}$ of subsets of X with the properties that $\emptyset \in \mathcal{L}$, $X \in \mathcal{L}$ and $U, V \in \mathcal{L}$ implies $U \cap V, U \cup V \in \mathcal{L}$. Taking the naming conventions of (Goubault-Larrecq, 2007a), and following (Gilboa and Schmeidler, 1992), a *capacity* ν on $\mathcal{L}$ is a strict map ν from $\mathcal{L}$ to $\mathbb{R}^+$; strictness means that $\nu(\emptyset) = 0$. A *game* is a monotone capacity, i.e., $\nu(U) \leq \nu(V)$ whenever $U \subseteq V$. We shall consider later slight relaxations of these notions, where $\mathcal{L}$ is no longer assumed to be a lattice of subsets.

The game ν is *modular* (resp., *convex*, resp. *concave*) iff $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$ (resp. $\geq$, resp. $\leq$) for all $U, V \in \mathcal{L}$. The terms supermodular and submodular are sometimes used in lieu of convex, concave. A modular game is called a *valuation*.

If $\mathcal{L}$ is the lattice $\mathcal{O}(X)$ of all open sets in a topological space X, then a valuation on $\mathcal{O}(X)$ is also called a valuation on the space X.

On a space X, we say that a game is *continuous* iff $\nu(\bigcup_{i\in I} U_i) = \sup_{i\in I} \nu(U_i)$ for every directed family $(U_i)_{i\in I}$ of opens.

Any measure μ on the Borel σ-algebra of X restricts to a valuation ν on X. It is continuous as soon as X is locally compact and μ is what we might call a *weakly Radon* measure, i.e., a (bounded) measure such that, for every open U, for every $\epsilon > 0$, there is a compact saturated subset Q of U and a measurable subset B of Q such that $\mu(B) \geq \mu(U) - \epsilon$. This is a mild generalization of the usual definition of Radon measure, where the same property is required for all *measurable* U; in Hausdorff spaces, Q will always be saturated; in metrizable spaces, additionally, compacts are measurable, and we would retrieve the usual definition of Radon measure, where $\mu(A) = \sup_{Q \text{ compact}} \mu(Q)$ for all measurable sets A. To show that any weakly Radon measure restricts to a continuous valuation on open sets, we reason as follows. Let $U = \bigcup_{i \in I} U_i$, where $(U_i)_{i \in I}$ is a directed family of opens, and let $\epsilon > 0$; we need to prove that $\nu(U_i) \geq \nu(U) - \epsilon$ for some $i \in I$. Using the fact that μ is weakly Radon, let $B \subseteq Q \subseteq U$ as above. Since X is locally compact, there is a compact saturated subset Q_1 such that $Q \subseteq int(Q_1) \subseteq Q_1 \subseteq U$. Then $Q_1 \subseteq U_i$ for some $i \in I$, so $\nu(U_i) \geq \nu(int(Q_1)) = \mu(int(Q_1)) \geq \mu(B) \geq \mu(U) - \epsilon$.

Conversely, if X is locally compact and well-filtered, every continuous valuation ν extends to a (unique, bounded) measure μ on all Borel measurable subsets of X. This is Theorem 5.3 of (Keimel and Lawson, 2005), once restricted to bounded measures. Moreover, one can observe that the measure μ thus constructed is weakly Radon: for every open U, since X is locally compact, hence $\mathcal{O}(X)$ is a continuous dcpo, $\nu(U) = \sup_{V \Subset U} \nu(V)$; so for every $\epsilon > 0$, there is an open V and a compact saturated subset Q such that $V \subseteq Q \subseteq U$ and $\nu(V) \geq \nu(U) - \epsilon$; then take $B = V$.

One can therefore conclude that measures and continuous valuations are one and the same thing, under mild assumptions. (See (Keimel and Lawson, 2005) for other extension results from continuous valuations to measures.) Precisely, weakly Radon measures are in one-to-one correspondence with continuous valuations, on any locally compact and well-filtered space.

This is one example of a measure extension theorem. There are others: later we shall cite Norberg's Theorem 3.9 (Norberg, 1989), see also (Alvarez-Manilla et al., 1997, Corollary 5.2), which applies to second countable continuous dcpos. Keimel and Lawson (Keimel and Lawson, 2005, Theorem 8.3) show that when X is stably locally compact, i.e., T_0, well-filtered, locally compact and coherent, every locally finite (not necessarily bounded) continuous valuation on X extends to a Radon measure on the Borel σ-algebra of X^{patch}, where X^{patch} is X with its patch topology—a σ-algebra that is in general larger than the Borel σ-algebra of X.

2.5. *Credibilities, plausibilities.*

Let us step back to games, i.e., monotone capacities. A game ν is *totally convex* iff:

$$\nu\left(\bigcup_{i=1}^{n} U_i\right) \geq \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} U_i\right) \tag{1}$$

for every finite family $(U_i)_{i=1}^n$, $n \geq 1$, of opens of X. A *credibility* is a totally convex game. We called credibilities *belief functions* in (Goubault-Larrecq, 2007a), following common usage. Choquet used the term "monotonic of infinite order" (Choquet, 54), and one often sees the term "totally monotonic" for this notion, see, e.g., (Gilboa and Schmeidler, 1992), sometimes with a slightly different definition (Molchanov, 2005, Definition 1.8). Similarly, the standard name for "totally concave" below is "totally alternating". However these standard names fail to reveal a fundamental duality, which the first author called convex-concave duality (Goubault-Larrecq, 2010).

Dually, a game ν is *totally concave* iff (1) holds with $\geq$ replaced by $\leq$, and the roles of unions and intersections are swapped, i.e.:

$$\nu\left(\bigcap_{i=1}^{n} U_i\right) \leq \sum_{I\subseteq\{1,\ldots,n\},I\neq\emptyset} (-1)^{|I|+1}\nu\left(\bigcup_{i\in I} U_i\right) \tag{2}$$

A totally concave game is a *plausibility* (Goubault-Larrecq, 2007a; Molchanov, 2005). Note that, if ν is a valuation, then (1) holds with $=$ instead of $\geq$. This equation is the well-known *inclusion-exclusion principle* of probability theory. Dually, we call (2) with $=$ instead of $\leq$ the *exclusion-inclusion principle*; this again holds whenever ν is a valuation.

Two of the three theorems that form the topic of this paper relate continuous credibilities, resp. plausibilities, on X with continuous valuations on the Smyth, resp. Hoare powerdomain on X. These will be established in Section 4 and Section 5. We will deal with the Plotkin powerdomain in Section 6. However we postpone the definition of the notion corresponding to valuations on this powerdomain, *estimates*, to Section 6.

Given any capacity ν on a space X, and any continuous map $f : X \to Y$, the *image capacity* $f[\nu]$ on Y is defined by $f[\nu](V) = \nu(f^{-1}(V))$. It is easy to see that $f[\nu]$ is a game, a continuous game, a credibility, a plausibility, or a valuation, as soon as ν is.

3. Generating valuations: A general setting

In this section we deal with the following question: Given a collection $\mathcal{S}$ of subsets of a set L and a real valued function ν defined on $\mathcal{S}$, under which conditions is it possible to extend ν to a uniquely determined valuation on the lattice of subsets of L generated by $\mathcal{S}$. We will always suppose that $\mathcal{S}$ is a $\cap$-semilattice, that is, finite intersections of members of $\mathcal{S}$ also belong to $\mathcal{S}$. Under this hypothesis Groemer (Groemer, 1978, Theorem 1) has given a complete solution to the problem. We refer to the book by Klain and Rota for an elegant presentation of Groemer's *Integral Theorem* (Klain and Rota, 1997, Theorem 2.2.1). Note that strict modular real valued functions on lattices are called valuations by Klain and Rota. If one requires valuations to be monotone as we do, one has to add a hypothesis on the set function ν that will turn out to be exactly Choquet's monotonicity of infinite order, namely, total convexity.

For our purposes we will only use a very special case of Groemer's Integral Theorem. As in this special case the proof is short and elementary, we will present it below. We need some preparations.

For every subset A of a set L, we denote by $\mathcal{X}_A$ the characteristic function of A, that is, $\mathcal{X}_A(x) = 1$ if $x \in A$, else $= 0$. For subsets A_1 and A_2, one has

$$\mathcal{X}_{A_1 \cap A_2} = \mathcal{X}_{A_1} \cdot \mathcal{X}_{A_2} \quad (3)$$

$$\mathcal{X}_{A_1 \cup A_2} = \mathcal{X}_{A_1} + \mathcal{X}_{A_2} - \mathcal{X}_{A_1 \cap A_2} \quad (4)$$

$$\mathcal{X}_{A_1 \setminus A_2} = \mathcal{X}_{A_1} - \mathcal{X}_{A_1 \cap A_2} \quad (5)$$

$$\mathcal{X}_{L \setminus A_1} = 1 - \mathcal{X}_{A_1} \quad (6)$$

For a non-empty finite family $A_1, \ldots, A_n$ of subsets, we have $A_1 \cup \cdots \cup A_n = L \setminus \big((L \setminus A_1) \cap \cdots \cap (L \setminus A_n)\big)$. From Equations (3) and (6), we deduce that $\mathcal{X}_{A_1 \cup \cdots \cup A_n} = 1 - \prod_{i=1}^n (1 - \mathcal{X}_{A_n}) = 1 - \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \prod_{i \in I} \mathcal{X}_{A_i} = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \prod_{i \in I} \mathcal{X}_{A_i} = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \mathcal{X}_{(\bigcap_{i \in I} A_i)}$, so we have the the following generalization of (4) that will be called the *Inclusion-Exclusion Formula*:

$$\mathcal{X}_{A_1 \cup \cdots \cup A_n} = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \mathcal{X}_{(\bigcap_{i \in I} A_i)} \quad (7)$$

We adopt the usual convention that the union of an empty family of subsets of a set L is the empty set and that the intersection is the whole set L. Equation (7) then also holds for the empty family of subsets, as the sum over an empty index set is 0.

For the characteristic function of the relative complement $S = A \setminus (A_1 \cup \ldots \cup A_n)$, $\mathcal{X}_S = \mathcal{X}_A - \mathcal{X}_{A \cap (A_1 \cup \ldots \cup A_n)} = \mathcal{X}_A - \mathcal{X}_{(A \cap A_1) \cup \ldots \cup (A \cap A_n)} = \mathcal{X}_A - \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \mathcal{X}_{\bigcap_{i \in I} (A \cap A_i)}$. Thus:

$$\mathcal{X}_{A \setminus (A_1 \cup \ldots \cup A_n)} = \mathcal{X}_A - \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \mathcal{X}_{(A \cap \bigcap_{i \in I} A_i)} \quad (8)$$

or equivalently,

$$\mathcal{X}_{A \setminus (A_1 \cup \ldots \cup A_n)} = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \mathcal{X}_{(A \cap \bigcap_{i \in I} A_i)} \quad (9)$$

For a collection $\mathcal{S}$ of subsets of L denote by $V(\mathcal{S})$ the vector space of real valued functions on L generated by the characteristic functions $\mathcal{X}_A$, $A \in \mathcal{S}$, that is:

$$V(\mathcal{S}) = \{\sum_{i=1}^n r_i \mathcal{X}_{A_i} \mid A_i \in \mathcal{S}, r_i \in \mathbb{R} \text{ for } i = 1, \ldots, n\}.$$

We shall say that $\mathcal{S}$ is a $\cap$-*semilattice* of subsets of L, if $A_1 \cap A_2 \in \mathcal{S}$ whenever $A_1 \in \mathcal{S}$ and $A_2 \in \mathcal{S}$. In this paper we also require that $L \in \mathcal{S}$. If in addition $A_1 \cup A_2 \in \mathcal{S}$ and $\emptyset \in \mathcal{S}$, then we say that $\mathcal{S}$ is a *lattice* of subsets. If moreover $L \setminus A_2 \in \mathcal{S}$, then $\mathcal{S}$ is called an *algebra* of subsets.

Remark 3.1. Let $\mathcal{S}$ be a $\cap$-semilattice of subsets of L.

(a) The finite unions $B = A_1 \cup \cdots \cup A_n$ of non-empty families $A_i \in \mathcal{S}$ form a lattice $\lambda\mathcal{S}$ of subsets of L, the lattice generated by $\mathcal{S}$. The characteristic functions of these finite unions belong to the vector space $V(\mathcal{S})$ by Equation (7).

(b) For $A \in \mathcal{S}$ and for any finite (possibly empty) family $A_1, \ldots, A_n$ in $\mathcal{S}$, the characteristic function of the relative complement

$$S = A \setminus (A_1 \cup \cdots \cup A_n)$$

also belongs to the vector space $V(\mathcal{S})$ by Equation (5).

(c) Every $B \in \lambda\mathcal{S}$ and every relative complement $B \backslash B'$ of members of the lattice $\lambda\mathcal{S}$ can be represented as a finite union of disjoint sets of the form S. (Indeed let $B = A_1 \cup \cdots \cup A_n$ with A_i in $\mathcal{S}$, let $S_1 = A_1 \setminus B'$ and $S_i = A_i \setminus (A_1 \cup \cdots \cup A_{i-1} \cup B')$ for $i = 2, \ldots, n$. Then the S_i are pairwise disjoint and their union is $B \setminus B'$.)

(d) The finite unions of sets of the form S form an algebra $\rho\mathcal{S}$, the algebra of subsets generated by $\mathcal{S}$. The characteristic functions of sets $C \in \rho\mathcal{S}$ all belong to the vector space $V(\mathcal{S})$. In fact, the characteristic function $\mathcal{X}_C$ of a set C belongs to $V(\mathcal{S})$ if and only if $C \in \rho\mathcal{S}$.

Let us consider now a real valued function ν defined on a collection $\mathcal{S}$ of subsets of L. Slightly relaxing our previous definition, we will say that ν is *strict* if $\nu(\emptyset) = 0$ as soon as $\emptyset \in \mathcal{S}$, and that ν is *monotone* if $\nu(A_1) \le \nu(A_2)$ whenever $A_1 \subseteq A_2$ in $\mathcal{S}$. If $\mathcal{S}$ is a lattice, we again say that ν is *modular* if the equation

$$\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$$

holds for all $U, V \in \mathcal{S}$. If $\mathcal{S}$ is an algebra, then ν is said to be *additive*, if $\nu\big(\bigcup_{i=1}^n A_i\big) = \sum_{i=1}^n \nu(A_i)$ for all finite families of pairwise disjoint members $A_i \in \mathcal{S}$. An additive map on an algebra is easily seen to be modular and strict, thus it induces a strict modular map on each sublattice.

Remark 3.2. There are canonical bijections between
(i) the [monotone] linear functionals $\mu\colon V(\mathcal{S}) \to \mathbb{R}$,
(ii) the [monotone] additive maps $\mu\colon \rho\mathcal{S} \to \mathbb{R}$,
(iii) the strict [monotone] modular maps $\mu\colon \lambda\mathcal{S} \to \mathbb{R}$.

Indeed, if μ is a [monotone] linear functional on the vector space $V(\mathcal{S})$, we define its 'restriction' to the algebra $\rho\mathcal{S}$ by considering the values on the characteristic functions, that is, we set $\mu(C) = \mu(\mathcal{X}_C)$ for $C \in \rho\mathcal{S}$; the linearity of μ on $V(\mathcal{S})$ implies the additivity of its 'restriction' to $\rho\mathcal{S}$ and, if μ is monotone on $V(\mathcal{S})$, it is also monotone on $\rho\mathcal{S}$. Restricting further yields a strict [monotone] modular function on the lattice $\lambda\mathcal{S}$. Conversely, every strict [monotone] modular map on $\lambda\mathcal{S}$ extends uniquely to a [monotone] additive map on the algebra $\rho\mathcal{S}$ according to the well-known Smiley-Horn-Tarski Theorem (see, e.g., (König, 1997, Theorem 3.4)). And any [monotone] additive map μ on $\rho\mathcal{S}$ defines a [monotone] linear functional by considering the integral $\int f d\mu = \sum_{i=1}^n r_i \mu(A_i)$ for $f = \sum_{i=1}^n r_i \mathcal{X}_{A_i} \in V(\mathcal{S})$ with respect to μ (Klain and Rota, 1997, Theorem 2.2.1).

In the following we will always identify additive maps on $\rho\mathcal{S}$ and modular maps on $\lambda\mathcal{S}$ according to the previous remark.

We now note that an extension of a function $\nu\colon \mathcal{S} \to \mathbb{R}$ to a linear functional on $V(\mathcal{S})$ (to an additive function on $\rho\mathcal{S}$, a modular function on $\lambda\mathcal{S}$, respectively) is unique, if it exists.

Remark 3.3. (a) As the characteristic functions $\mathcal{X}_A$, $A \in \mathcal{S}$, generate the vector space $V(\mathcal{S})$, a linear functional μ on $V(\mathcal{S})$ is uniquely determined by its values on these characteristic functions. Thus, for a function $\nu\colon \mathcal{S} \to \mathbb{R}$ there is at most one linear functional $\nu^*\colon V(\mathcal{S}) \to \mathbb{R}$ such that $\nu^*(\mathcal{X}_A) = \nu(A)$ for all $A \in \mathcal{S}$.

(b) As the linear functionals on $V(\mathcal{S})$ are in bijective correspondence with the strict modular maps on the lattice $\lambda\mathcal{S}$ (with the additive maps on the Boolean ring $\rho\mathcal{S}$, respectively), a function $\nu\colon \mathcal{S} \to \mathbb{R}$ has at most one strict modular extension $\nu^*\colon \lambda\mathcal{S} \to \mathbb{R}$ (at most one additive extension $\nu^*\colon \rho\mathcal{S} \to \mathbb{R}$, respectively). If such an extension exists, it is given by:

$$\nu^*(A_1 \cup \ldots \cup A_n) = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \nu\big(\bigcap_{i \in I} A_i\big) \tag{10}$$

$$\nu^*(A \setminus (A_1 \cup \ldots \cup A_n)) = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \nu\big(A \cap \bigcap_{i \in I} A_i\big) \tag{11}$$

This is seen by applying the functional ν^* to Equations (7) and (8).

For the existence of an extension in the general case we refer to Groemer's Integral Theorem (Klain and Rota, 1997, Theorem 2.2.1). We will only use it in a special case for which we will include a simple proof: see Theorem 3.3 below.

Definition 3.1. A member A of a $\cap$-semilattice $\mathcal{S}$ of subsets will be called $\cup$-*irreducible* if it cannot be represented as a finite union of strictly smaller members of $\mathcal{S}$. We will say that $\mathcal{S}$ is $\cup$-*irredundant*, if all of its members are $\cup$-irreducible.

Lemma 3.2. Suppose $\mathcal{S}$ is a $\cup$-irredundant $\cap$-semilattice. Then the characteristic functions $\mathcal{X}_A$, $\emptyset \neq A \in \mathcal{S}$, are linearly independent, hence, form a basis for the vector space $V(\mathcal{S})$.

Proof. Suppose that $\sum_{i=1}^n r_i \mathcal{X}_{A_i} = 0$ for pairwise distinct non-empty $A_i \in \mathcal{S}$ and $0 \neq r_i \in \mathbb{R}$. Among the sets A_i choose a maximal one, say A_1. As A_1 is supposed to be $\cup$-irreducible, it contains an element x not contained in $A_2 \cup \cdots \cup A_n$. Thus $\mathcal{X}_{A_1}(x) = 1$ and $\mathcal{X}_{A_i}(x) = 0$ for $i = 2, \ldots, n$, whence $0 = \sum_{i=1}^n r_i \mathcal{X}_{A_i}(x) = r_1$, a contradiction. □

Theorem 3.3. Suppose that $\mathcal{S}$ is a $\cup$-irredundant $\cap$-semilattice of subsets of a set L and $\nu\colon \mathcal{S} \to \mathbb{R}$ a strict real valued function. Then:
(i) There is a unique linear functional $\nu^*\colon V(\mathcal{S}) \to \mathbb{R}$ such that $\nu^*(\mathcal{X}_A) = \nu(A)$ for all $A \in \mathcal{S}$.
(ii) There is a unique additive function $\nu^*\colon \rho\mathcal{S} \to \mathbb{R}$ extending ν.
(iii) There is a unique strict modular function $\nu^*\colon \lambda\mathcal{S} \to \mathbb{R}$ extending ν.
The extension ν^* is given by the formulas (10) and (11).

Property (i) follows from the preceding Lemma since any real valued function defined on a basis of a vector space has a unique extension to a linear map. Properties (ii) and (iii) follow by restricting the linear extension ν^* from (i) to the characteristic functions of members of $\lambda\mathcal{S}$ and $\rho\mathcal{S}$, respectively, that is, $\nu^*(B) = \nu^*(\mathcal{X}_B)$ for B in $\lambda\mathcal{S}$ or $\rho\mathcal{S}$.

Groemer's Integral Theorem has the same conclusion as Theorem 3.3, but is more general. Instead of $\lambda\mathcal{S}$ it assumes a general lattice of sets, and $\mathcal{S}$ is only assumed to be a $\cap$-semilattice that is also a generating subset for this lattice, in the sense that every element of the lattice should be a finite union of elements of $\mathcal{S}$. On the other hand, Groemer's Theorem requires one to check that the inclusion-exclusion formula $\nu\left(\bigcup_{i=1}^n A_i\right) = \sum_{I\subseteq\{1,\ldots,n\},I\neq\emptyset}(-1)^{|I|+1}\nu\left(\bigcap_{i\in I} A_i\right)$ is satisfied for all $A_1, \ldots, A_n$ in $\mathcal{S}$ such that $\bigcup_{i=1}^n A_i$ is also in $\mathcal{S}$. Theorem 3.3 is the special case where $\mathcal{S}$ is $\cup$-irredundant, in which case the latter condition is vacuously true.

The modular extension ν^* of ν from $\mathcal{S}$ to $\lambda\mathcal{S}$ need not be monotone, even when ν is monotone on $\mathcal{S}$. As a simple example one may consider the three element set a, b, c with the $\cap$-semilattice $\mathcal{S} = \{\emptyset, \{a\}, \{b\}, \{a, b, c\}\}$ and the function $\nu(\emptyset) = 0, \nu(\{a\} = \nu(\{b\}) = \nu(\{a, b, c\} = 1$ which is monotone on $\mathcal{S}$, but its extension is not monotone as it satisfies $\nu^*(\{a, b\}) = 2 > \nu(\{a, b, c\} = 1$.

We want to characterize those ν for which the extension ν^* is also monotone. For this we consider the pointwise order of functions in the vector space $V(\mathcal{S})$ and the inclusion order on $\lambda\mathcal{S}$ and $\rho\mathcal{S}$. Note that an additive function on a ring of sets is monotone if and only if all its values are non-negative and, similarly, a linear functional on the vector space $V(\mathcal{S})$ is monotone if and only if non-negative elements have non-negative values. Such functionals are often called *positive*.

Lemma 3.4. Let $\mathcal{S}$ be a $\cap$-semilattice of subsets of a set L and $\lambda\mathcal{S}$ the lattice generated by $\mathcal{S}$. If a modular function $\mu\colon \lambda\mathcal{S} \to \mathbb{R}$ is monotone its restriction ν to $\mathcal{S}$ satisfies the following property: For every $A \in \mathcal{S}$ and every finite family $A_1, \ldots, A_n$ in $\mathcal{S}$ such that $A_1 \cup \ldots \cup A_n \subseteq A$, the following inequality holds:

$$\nu(A) \quad \geq \sum_{\substack{I\subseteq\{1,\ldots,n\}\\ I\neq\emptyset}} (-1)^{|I|+1}\nu\Big(\bigcap_{i\in I} A_i\Big). \tag{12}$$

Proof. If $A_1 \cup \ldots \cup A_n \subseteq A$ in $\mathcal{S}$, the monotonicity of μ implies $\mu(A_1 \cup \ldots \cup A_n) \leq \mu(A) = \nu(A)$. Calculating $\mu(A_1 \cup \ldots \cup A_n)$ according to Equation (10) yields the desired inequality. □

The previous lemma gives rise to the following definition:

Definition 3.5. A real valued function ν defined on a $\cap$-semilattice $\mathcal{S}$ of subsets of a set L is called *totally convex*, iff it satisfies the inequality (12) for all $A, A_1, \ldots A_n$ in $\mathcal{S}$ such that $A_1 \cup \ldots \cup A_n \subseteq A$. This is tantamount to saying that, for every $A \in \mathcal{S}$ and every finite family $A_1, \ldots, A_n$ in $\mathcal{S}$ the following inequality holds:

$$\sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|}\nu\big(A \cap \bigcap_{i\in I} A_i\big) \geq 0. \tag{13}$$

In the inequalities (12) and (13) the case $n = 0$ is admitted which implies that a totally convex function has only non-negative values. The case $n = 1$ shows that totally convex functions are monotone.

The following is the main result of this section:

Theorem 3.6. For a $\cup$-irredundant $\cap$-semilattice $\mathcal{S}$ of subsets of a set L there is a

canonical bijection between:
(i) monotone additive real valued functions μ on the algebra $\rho\mathcal{S}$ generated by $\mathcal{S}$,
(ii) valuations on the lattice $\lambda\mathcal{S}$ generated by $\mathcal{S}$,
(iii) totally convex strict real valued functions ν defined on $\mathcal{S}$.

Proof. Every monotone additive real valued function μ on $\rho\mathcal{S}$ restricts to a monotone strict modular function on $\lambda\mathcal{S}$ and every monotone strict modular function on $\lambda\mathcal{S}$ restricts to a strict totally convex function on $\mathcal{S}$ by the preceding lemma. It remains to show that every strict totally convex real valued function ν on $\mathcal{S}$ extends uniquely to a monotone additive real valued function ν^* on $\rho\mathcal{S}$.

By Theorem 3.3, ν has a unique extension ν^* to an additive real valued function on $\rho\mathcal{S}$. The value of every set of the form $S = A \setminus (A_1 \cup \ldots \cup A_n)$ with $A, A_1 \ldots, A_n$ in $\mathcal{S}$ is given by Equation (11). If ν is totally convex, then $\nu^*(S) \geq 0$. As every set C belonging to the algebra $\rho\mathcal{S}$ is a disjoint union of sets of the form S, the additivity of ν^* implies that $\nu^*(C) \geq 0$. But additive functions with non-negative values on an algebra are monotone. □

We can dualize the developments of this section in the following way: Complementation $A \mapsto L \setminus A$ is an anti-isomorphism of the algebra of subsets of a set L. It interchanges $\cap$ and $\cup$. It allows to transfer every statement about a collection $\mathcal{S}$ of subsets of L to a dual statement about the complementary collection $\mathcal{S}^c = \{L \setminus A \mid A \in \mathcal{S}\}$.

For a real valued map ν on a collection $\mathcal{S}$ of subsets the *conjugate* map ν^c is defined on the complementary collection $\mathcal{S}^c$ by $\nu^c(B) = \nu(L) - \nu(L \setminus B)$ provided that $L \in \mathcal{S}$. The following properties are straightforward:

Remark 3.4. Let $\mathcal{S}$ be a collection of subsets of L containing L itself, and ν be a real valued function defined on L.

(1) $\mathcal{S}$ is a $\cap$-semilattice if and only if $\mathcal{S}^c$ is a $\cup$-semilattice, ν^c is always strict, and ν is monotone if and only if ν^c is.

(2) If $\mathcal{S}$ is a $\cap$-semilattice, then ν is totally convex on $\mathcal{S}$ if and only if ν^c is *totally concave* on $\mathcal{S}^c$, which means that, for all $B \in \mathcal{S}^c$ and for each finite non-empty family $B_1, \ldots, B_n$ in $\mathcal{S}^c$, with $n \geq 1$, the following inequality holds:

$$\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \nu^c\big(B \cup \bigcup_{i \in I} B_i\big) \quad \leq \quad 0. \tag{14}$$

(3) $\mathcal{S}$ is a lattice if and only if $\mathcal{S}^c$ is a lattice, too, and then ν is modular if and only if ν^c is.

(4) If $\mathcal{S}$ is an algebra, then $\mathcal{S}^c = \mathcal{S}$ and, if ν is additive on $\mathcal{S}$, then $\nu = \nu^c$.

Only the second statement needs a proof. We first note that ν^c is always strict and that (14) with $n = 1$ implies that ν^c is monotone, hence that $\nu^c(B) \geq 0$ by definition. Note also that, contrarily to (13), we do not allow n to be 0 in (14): this would require $\nu^c(B) \leq 0$; since already $\nu^c(B) \geq 0$, this would force ν^c to be identically zero, hence ν to be constant.

Let $A = L \setminus B$ and $A_i = L \setminus B_i$ for each i. We compute:

$$\begin{aligned}\sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|}\nu^c\big(B \cup \bigcup_{i\in I} B_i\big) &= \sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|}\left(\nu(L) - \nu\big(A \cap \bigcap_{i\in I} A_i\big)\right) \\ &= \left(\sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|}\right)\nu(L) - \sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|}\nu\big(A \cap \bigcap_{i\in I} A_i\big).\end{aligned}$$

When $n \geq 1$, $\sum_{I\subseteq\{1,\ldots,n\}}(-1)^{|I|} = 0$, as the development of $(1-1)^n$ shows. So (14) holds in this case if and only if $\sum_{I\subseteq\{1,\ldots,n\}}(-1)^{|I|}\nu\big(A \cap \bigcap_{i\in I} A_i\big) \geq 0$, which is just (13) in the case $n \geq 1$. In particular, if ν is totally convex, then (14) is satisfied whenever $n \geq 1$. Conversely, if ν^c is totally concave, then (13) is satisfied whenever $n \geq 1$. We must show that it is also satisfied when $n = 0$, i.e., that $\nu(A) \geq 0$. (14) with $n = 1$ yields $\nu^c(B) - \nu^c(B \cup B_1) \leq 0$ for any B_1. For $B_1 = L$, we obtain $\nu^c(B) - \nu(L) \leq 0$, which is the desired inequality.

Dualizing our theorems 3.3 and 3.6, and in particular using the obvious dual notions of $\cup$-semilattice, $\cap$-irreducibility, and $\cap$-irredundancy, we obtain:

Theorem 3.7. Let $\mathcal{S}$ be a $\cap$-irredundant $\cup$-semilattice of subsets of L. Then every strict real valued function ν defined on $\mathcal{S}$ extends uniquely to a modular function ν_* on the lattice $\lambda\mathcal{S}$ generated by $\mathcal{S}$. The extension is a valuation if and only if ν is totally concave.

The members of the lattice $\lambda\mathcal{S}$ are the finite intersections $B_1 \cap \ldots \cap B_n$ of sets $B_1, \ldots, B_n \in \mathcal{S}$, and the extension ν_* on such sets is given by

$$\nu_*(B_1 \cap \ldots \cap B_n) = \sum_{\substack{I\subseteq\{1,\ldots,n\}\\ I\neq\emptyset}} (-1)^{|I|+1}\nu\Big(\bigcup_{i\in I} B_i\Big). \tag{15}$$

4. The Demonic Case: Continuous Credibilities

Throughout this section we consider a topological space X. Recall that we denote by $\mathcal{O}(X)$ the lattice of all open subsets of X and by $\mathcal{Q}_{\mathcal{V}}(X)$ the set of all non-empty compact saturated subsets $Q \subseteq X$ with the upper Vietoris topology which is given by the *basic open* sets

$$\Box U = \{Q \in \mathcal{Q}(X) \mid Q \in U\},\ U \in \mathcal{O}(X).$$

Let us note in passing that $\mathcal{Q}_{\mathcal{V}}(X)$ is sober provided that X is, see (Schalk, 1993, Proposition 7.20). We start with a useful lemma on the basic open sets $\Box U$.

Lemma 4.1. (1) $\Box\emptyset = \emptyset$ and $\Box X = \mathcal{Q}_{\mathcal{V}}(X)$; (2) $\Box U \cap \Box V = \Box(U \cap V)$; (3) $\Box\big(\bigcup_{i\in I} U_i\big) = \bigcup_{i\in I} \Box U_i$ for every directed family of opens $(U_i)_{i\in I}$.

Proof. For (1) we use that the empty set is excluded from $\mathcal{Q}_{\mathcal{V}}(X)$. (2) and the $\supseteq$ direction of (3) are obvious. For any $Q \in \Box\big(\bigcup_{i\in I} U_i\big)$, i.e., $Q \subseteq \bigcup_{i\in I} U_i$, there is an $i \in I$ such that $Q \subseteq U_i$ since Q is compact; so $Q \in \Box U_i$, hence $Q \in \bigcup_{i\in I} \Box U_i$. $\Box$

It is item (2) that allowed us to claim that the sets of the form $\Box U$ formed not just a subbase, but a base of the topology of $\mathcal{Q}_{\mathcal{V}}(X)$.

Lemma 4.2. The basic open sets $\Box U$ form a $\cup$-irredundant $\cap$-semilattice $\mathcal{S}$ of subsets of $\mathcal{Q}(X)$ and the map $U \mapsto \Box U : \mathcal{O}(X) \to \mathcal{S}$ is an isomorphism of $\cap$-semilattices.

Proof. Property (2) in Lemma 4.1 tells us that $U \mapsto \Box U$ is a $\cap$-semilattice homomorphism. If U, V are different open sets, for example $U \not\subseteq V$, then there is an element $x \in U \setminus V$; hence $\uparrow x$ is a compact saturated set contained in $\Box U \setminus \Box V$, which implies that $\Box U \neq \Box V$. Thus $U \mapsto \Box U : \mathcal{O}(X) \to \mathcal{S}$ is injective and, hence, a $\cap$-semilattice isomorphism.

In order to show that $\mathcal{S}$ is $\cup$-irredundant, we take an open set U and we suppose that $U_1, \ldots, U_n$ are open sets such that $\Box U_i$ is a proper subset of $\Box U$ for every i. By the previous paragraph we can find an $x_i \in U \setminus U_i$ for every i. The saturation $\uparrow\{x_1, \ldots, x_n\}$ is a compact saturated subset of U not contained in any U_i. Thus the union $\Box U_1 \cup \ldots \cup \Box U_n$ is properly contained in $\Box U$ which shows that $\Box U$ is $\cup$-irreducible. $\Box$

The preceding lemma allows us to apply Theorem 3.3 and Theorem 3.6. The finite unions $\Box U_1 \cup \ldots \cup \Box U_n$ of basic open sets with $U_i \in \mathcal{O}(X)$ will be called *elementary open* sets; they form the lattice $\lambda\mathcal{S}$ generated by the basic open sets. And we obtain:

Corollary 4.3. For every strict map $\nu : \mathcal{O}(X) \to \mathbb{R}$ there is a unique strict modular function $\nu^* : \lambda\mathcal{S} \to \mathbb{R}$ such that $\nu^*(\Box U) = \nu(U)$ for every open U in X. The extension is given by

$$\nu^*(\Box U_1 \cup \ldots \cup \Box U_n) = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \nu\Big(\bigcap_{i \in i} U_i\Big).$$

The extension ν^* is a valuation if and only if ν is a credibility on X.

We would like to extend the valuation ν^* on the elementary open sets to a continuous valuation on the lattice of all open subsets of $\mathcal{Q}_{\mathcal{V}}(X)$. We first observe that the continuity of the credibility ν is a necessary condition.

Lemma 4.4. For any continuous valuation P on $\mathcal{Q}_{\mathcal{V}}(X)$, the game $P_{\rceil X}$ defined by $P_{\rceil X}(U) = P(\Box U)$, for all opens U of X, is a continuous credibility on X.

Proof. It is clear that $\nu = P_{\rceil X}$ is a game, and ν is continuous, as it is the composition of the map $U \mapsto \Box U$, which is continuous by Lemma 4.1 (3), followed by the continuous map P. Clearly, ν is totally convex by Corollary 4.3, as the restriction of P to the lattice $\lambda\mathcal{S}$ is a valuation. $\Box$

The notation $P_{\rceil X}$ is meant to be a reminder that it is obtained by *restriction*. If we equate X with the subspace of $\mathcal{Q}_{\mathcal{V}}(X)$ of all elements of the form $\uparrow x$, $x \in X$, the opens U of X are exactly the opens of the form $\Box U \cap X$, so that we have actually defined $P_{\rceil X}$ by the restriction formula $P_{\rceil X}(\Box U \cap X) = P(\Box U)$.

Our goal in this section is to show that, under mild conditions on X, the converse holds: any continuous credibility on X can be *extended* to a continuous valuation on the whole of $\mathcal{Q}_{\mathcal{V}}(X)$. Moreover, the extension is unique.

This can be interpreted in many ways. In economics, this theorem is a refinement of the "completion of a misspecified model" result of (Gilboa and Schmeidler, 1992), where games really are games between players, namely the elements of the space X. When X is finite at least, valuations are naturally identified with games (in the usual sense) where each player plays independently of (or against) each other, expecting some payoff for herself (utility). Capacities model situations where players can form coalitions, where payoff will be allotted to coalitions as a whole: this is the so-called cooperative game with transferable utility model. (Utility is transferable because we don't care how the players in a given coalition will share their earnings, and don't prefer any way of sharing among the coalition over any other.) Coalitions are non-empty sets of players. When X is topologized, our constructions show that coalitions should be refined to mean non-empty compact saturated sets of players, although this may now lack some of the economic intuition.

In mathematics, this is a representation theorem for random sets. This states that it is equivalent to give oneself a distribution over subsets of X (here, non-empty compact saturated subsets), or to give oneself a continuous credibility on X directly (Molchanov, 2005).

In a computer science context, the first author has argued elsewhere (Goubault-Larrecq, 2007a) that continuous credibilities provided a semantic model for mixed probabilistic choice and demonic non-determinism. Remember that $\mathcal{Q}_{\mathcal{V}}(X)$ is the standard powerdomain for demonic non-determinism. Our theorem then states that continuous credibilities are in one-to-one correspondence with continuous valuations on $\mathcal{Q}_{\mathcal{V}}(X)$. This model is one where probabilistic choice is resolved first, then non-determinism, i.e., the opposite of some other models (Mislove, 2000; Tix et al., 2005; Goubault-Larrecq, 2007b). I.e., you first draw some non-empty compact saturated subset $Q \in \mathcal{Q}_{\mathcal{V}}(X)$ at random, then pick non-deterministically some element from Q.

The following lemma probably illustrates the point more concretely. Let δ_x be the *Dirac mass* at x: $\delta_x(U)$ is 1 if $x \in U$, 0 otherwise; this is a continuous valuation. Let $\mathfrak{u}_Q$ be the *unanimity game* on Q: $\mathfrak{u}_Q(U)$ is 1 if $Q \subseteq U$, 0 otherwise. Unanimity games generalize Dirac masses in that $\delta_x = \mathfrak{u}_{\uparrow x} = \mathfrak{u}_{\eta_{\mathcal{Q}}(x)}$. Call a valuation *simple* iff it is of the form $\sum_{i=1}^n a_i \delta_{x_i}$, $a_1, \ldots, a_n \in \mathbb{R}^+$; every simple valuation is a continuous valuation. Call *simple credibility* any game of the form $\sum_{i=1}^n a_i \mathfrak{u}_{Q_i}$, $a_1, \ldots, a_n \in \mathbb{R}^+$, $Q_1, \ldots, Q_n \in \mathcal{Q}_{\mathcal{V}}(X)$. Then we have:

Lemma 4.5. Any simple credibility is a continuous credibility. Moreover, every restriction of a simple valuation on $\mathcal{Q}_{\mathcal{V}}(X)$ is a simple credibility on X; namely, for any $a_1, \ldots, a_n \in \mathbb{R}^+$, $Q_1, \ldots, Q_n \in \mathcal{Q}_{\mathcal{V}}(X)$,

$$\left(\sum_{i=1}^n a_i \delta_{Q_i} \right)_{\downarrow X} = \sum_{i=1}^n a_i \mathfrak{u}_{Q_i}.$$

Proof. The first claim follows from the second one and Lemma 4.4. The second one is immediate: on the open U, the two sides of the equality simplify to $\sum_{\substack{1 \leq i \leq n \\ Q_i \subseteq U}} a_i$. □

Unanimity games $\mathfrak{u}_Q$ are in bijection with elements of $\mathcal{Q}(X)$, and are accordingly models of pure demonic non-deterministic choice, among all continuous credibilities. Pure non-deterministic choice can be characterized by the fact that the game takes the values 0 or 1 only:

Proposition 4.6. Let X be a sober space. Any continuous credibility ν with $\nu(X) = 1$ and such that ν only takes values 0 or 1 is of the form $\mathfrak{u}_Q$, $Q \in \mathcal{Q}_\mathcal{V}(X)$. In fact, this already holds of all convex games that take values 0 or 1 only and such that $\nu(X) = 1$, where ν is *convex* iff $\nu(U \cup V) + \nu(U \cap V) \geq \nu(U) + \nu(V)$ for all opens U, V.

Proof. We first check that every continuous credibility is convex: this is the case $n = 2$ in (1). Next, assume ν is a convex game, $\nu(X) = 1$, and ν only takes values 0 or 1. Let $\mathcal{F}$ be the collection of all opens U such that $\nu(U) = 1$. This is a filter of opens, i.e., it is non-empty (since $\nu(X) = 1$), $U \subseteq V$ and $U \in \mathcal{F}$ imply $V \in \mathcal{F}$ and $U, V \in \mathcal{F}$ imply $U \cap V \in \mathcal{F}$. The latter is because $\nu(U \cap V) \geq \nu(U) + \nu(V) - \nu(U \cup V) \geq 1 + 1 - 1 = 1$, and $\nu(U \cap V)$ is either 0 or 1. $\mathcal{F}$ is Scott-continuous, i.e., for any directed family of opens $(U_i)_{i \in I}$, if $\bigcup_{i \in I} U_i \in \mathcal{F}$ then $U_i \in \mathcal{F}$ already for some $i \in I$. And $\mathcal{F}$ is non-trivial, i.e., not the whole of $\mathcal{O}(X)$, since $\emptyset \notin \mathcal{F}$. The Hofmann-Mislove Theorem, see (Abramsky and Jung, 1994, Theorem 7.2.9) or (Gierz et al., 2003, Theorem II-1.20), implies that the intersection Q of all elements of $\mathcal{F}$ is a non-empty compact saturated subset of X, and that $Q \subseteq U$ iff $U \in \mathcal{F}$, whence $\nu = \mathfrak{u}_Q$. □

We come finally to our demonic extension of the Choquet-Kendall-Matheron theorem. We need some facts about the upper Vietoris topology. It will be convenient to use the following notation for compact saturated sets Q in X: $\blacksquare Q = \{Q' \in \mathcal{Q}_\mathcal{V}(X) \mid Q' \subseteq Q\}$.

Lemma 4.7. The set $\blacksquare Q$ is compact and saturated in $\mathcal{Q}_\mathcal{V}(X)$ for every compact saturated set Q in X,

Proof. Consider a family of basic open sets $(\Box U_i)_i$, $U_i \in \mathcal{O}(X)$, such that $\blacksquare Q \subseteq \bigcup_i \Box U_i$. Then $Q \in \bigcup_i \Box U_i$. Thus, there is an i such that $Q \in \Box U_i$ which means that $Q \subseteq U_i$ that is, $\blacksquare Q \subseteq \Box U_i$. □

Among the elementary open sets we consider those of the form $\Box int(Q_1) \cup \ldots \cup \Box int(Q_n)$ where $Q_1, \ldots, Q_n$ are compact saturated sets. We call them *special* elementary open sets.

Lemma 4.8. If X is locally compact, then every elementary open set $\Box U_1 \cup \ldots \cup \Box U_n$ is the union of the directed family of special elementary open sets $\Box int(Q_1) \cup \ldots \cup \Box int(Q_n)$ with $Q_1 \subseteq U_1$, $\ldots$, $Q_n \subseteq U_n$. In particular, the sets $\Box int(Q)$, where Q ranges over all compact saturated sets of X form a basis of the upper Vietoris topology.

Proof. It suffices to show that $\Box U = \bigcup\{\Box int(Q) \mid Q \in \Box U\}$. As X is locally compact, an open subset U of X is the union of the $int(Q)$ where Q ranges over the compact saturated sets contained in U, i.e., $U = \bigcup\{int(Q) \mid Q \in \Box U\}$. As the union of finitely many compact saturated sets contained in U is again a compact saturated set contained in U, $\Box U$ is directed and Lemma 4.1 (3) implies that $\Box U = \bigcup\{\Box int(Q) \mid Q \in \Box U\}$. □

Lemma 4.8 shows that if X is locally compact, then $\mathcal{Q}_{\mathcal{V}}(X)$ is a C-space in the sense of Erné (Erné, 1991), that is, a space Z in which for every $z \in Z$, for every open neighborhood V of z, there is a further element $z' \in V$ such that z is in the interior of the upward-closure of z'. Specifically, for every $Q \in \mathcal{Q}_{\mathcal{V}}(X)$, for every open neighborhood $\mathcal{U}$ of Q in $\mathcal{Q}_{\mathcal{V}}(X)$, there is a basic open $\Box int(Q')$ containing Q and included in $\mathcal{U}$. But $\Box int(Q')$ is included in $\blacksquare Q'$, which is precisely the upward-closure of Q' in the specialization ordering of $\mathcal{Q}_{\mathcal{V}}(X)$, which is reversed inclusion.

Every C-space is, by definition, locally compact. Moreover, the lattice of open subsets of any C-space is completely distributive, hence in particular continuous.

Lemma 4.9. Let X be a locally compact space. Then $\mathcal{Q}_{\mathcal{V}}(X)$ is locally compact, too. The lattice of open subsets of $\mathcal{Q}_{\mathcal{V}}(X)$ is continuous; for two open subsets $\mathcal{U}$ and $\mathcal{V}$ we have $\mathcal{U} \Subset \mathcal{V}$ if and only if there are finitely many $Q_1, \dots, Q_n \in \mathcal{V}$ such that $\mathcal{U} \subseteq \Box int(Q_1) \cup \dots \cup \Box int(Q_n)$.

Proof. Only the last statement remains to be proved. Let $\mathcal{U}$ and $\mathcal{V}$ be open subsets for the upper Vietoris topology. If there are $Q_1, \dots, Q_n \in \mathcal{V}$ such that $\mathcal{U} \subseteq \Box int(Q_1) \cup \dots \cup \Box int(Q_n)$, then $\mathcal{U} \subseteq \blacksquare Q_1 \cup \dots \cup \blacksquare Q_n \subseteq \mathcal{V}$. Being a finite union of compact sets $\blacksquare Q_1 \cup \dots \cup \blacksquare Q_n$ is compact. It follows that $\mathcal{U} \Subset \mathcal{V}$. Suppose conversely that $\mathcal{U} \Subset \mathcal{V}$. The open set $\mathcal{V}$ is a union of basic open sets $\Box U$, and each of those is a union of basic open sets $\Box int(Q)$ with $Q \subseteq U$. Hence, $\mathcal{V}$ a union of sets of the form $\Box int(Q)$ with $Q \in \mathcal{V}$. Thus, if $\mathcal{U} \Subset \mathcal{V}$, there are finitely many $Q_1, \dots, Q_n \in \mathcal{V}$ such that $\mathcal{U} \subseteq \Box int(Q_1) \cup \dots \cup \Box int(Q_n)$ as desired. □

As an immediate consequence of the previous lemma we have:

Corollary 4.10. Suppose that X is locally compact. The special elementary opens $\Box int(Q_1) \cup \dots \cup \Box int(Q_n)$, where $Q_1, \dots, Q_n$ range over finite families of compact saturated subsets of X, form a basis of the continuous poset $\mathcal{O}(\mathcal{Q}_{\mathcal{V}}(X))$.

Proposition 4.11. For every continuous credibility ν on X, there is at most one continuous valuation ν^* on $\mathcal{Q}_{\mathcal{V}}(X)$ such that $\nu(U) = \nu^*(\Box U)$ for every open U of X; and such a ν^* exists whenever X is locally compact.

Proof. Assume ν is a continuous credibility on X. By Corollary 4.3, there a unique valuation ν^* defined on the lattice $\lambda\mathcal{S}$ of elementary opens such that $\nu^*(\Box U) = \nu(U)$ for every open U of X.

As every open subset $\mathcal{U}$ of $\mathcal{Q}_{\mathcal{V}}(X)$ is the union of elementary opens contained in $\mathcal{U}$ and as this family is directed, a continuous extension of ν^* to $\mathcal{O}(\mathcal{Q}_{\mathcal{V}}(X))$ must satisfy

$$\nu^*(\mathcal{U}) = \sup\{\nu^*(\mathcal{V}) \mid \mathcal{V} \in \lambda\mathcal{S}, \mathcal{V} \subseteq \mathcal{U}\}$$

and hence is uniquely determined by the values of ν^* on the elementary opens.

In order to show the existence, we suppose that X is locally compact. By Lemma 4.9, the lattice of open subsets of $\mathcal{Q}_{\mathcal{V}}(X)$ is continuous, the elementary opens form a basis and we have identified the way-below relation there. According to the Extension Lemma recorded in the Preliminaries, we obtain a continuous map $\nu^{**} \colon \mathcal{O}(\mathcal{Q}_{\mathcal{V}}(X)) \to \mathbb{R}$ by defining

$$\nu^{**}(\mathcal{U}) = \sup_{\mathcal{J} \text{ finite } \subseteq \mathcal{U}} \nu^*\big(\bigcup_{Q \in \mathcal{J}} \Box int(Q)\big).$$

Let us show that ν^{**} is indeed an extension of ν^*, that is, $\nu^{**}(\mathcal{U}) = \nu^*(\mathcal{U})$ holds for elementary opens $\mathcal{U} = \Box U_1 \cup \ldots \cup \Box U_n$ where $U_1, \ldots, U_n$ are open in X. Without loss of generality, assume $U_1, \ldots, U_n$ to be non-empty.

By Lemma 4.8 and Lemma 4.9, the definition of $\nu^{**}(\mathcal{U})$ can be simplified for elementary opens:

$$\nu^{**}(\Box U_1 \cup \ldots \cup \Box U_n) = \sup_{Q_1 \subseteq U_1, \ldots, Q_n \subseteq U_n} \nu^*\big(\Box int(Q_1) \cup \ldots \cup \Box int(Q_n)\big).$$

Using the definition of ν^* (see Corollary 4.3),

$$\nu^*\big(\Box int(Q_1) \cup \ldots \cup \Box int(Q_n)\big) + \sum_{\substack{I \subseteq \{1,\ldots,n\}, I \neq \emptyset \\ |I| \text{ even}}} \nu\Big(\bigcap_{i \in I} int(Q_i)\Big) = \sum_{\substack{I \subseteq \{1,\ldots,n\}, I \neq \emptyset \\ |I| \text{ odd}}} \nu\Big(\bigcap_{i \in I} int(Q_i)\Big).$$

The family of all tuples $(Q_1, \ldots, Q_n)$ of non-empty compact saturated subsets such that $Q_1 \subseteq U_n, \ldots, Q_n \subseteq U_n$, ordered by pointwise inclusion, is directed. Take sups in the above equality, using the fact that $+$ is Scott-continuous, and that ν is continuous:

$$\nu^{**}(\Box U_1 \cup \ldots \cup \Box U_n) + \sum_{\substack{I \subseteq \{1,\ldots,n\}, I \neq \emptyset \\ |I| \text{ even}}} \nu(V_I) = \sum_{\substack{I \subseteq \{1,\ldots,n\}, I \neq \emptyset \\ |I| \text{ odd}}} \nu(V_I),$$

where V_I stands for $\bigcup_{Q_1 \subseteq U_1, \ldots, Q_n \subseteq U_n} \bigcap_{i \in I} int(Q_i)$. Now, using Lemma 4.8, $V_I = \bigcap_{i \in I} U_i$: in the $\supseteq$ direction, let $x \in \bigcap_{i \in I} U_i$, pick a compact saturated subset Q such that $x \in int(Q) \subseteq Q \subseteq \bigcap_{i \in I} U_i$, let $Q_i = Q$ for all $i \in I$, and $Q_i \subseteq U_i$ arbitrary otherwise (remember that each U_i is non-empty). So

$$\nu^{**}(\Box U_1 \cup \ldots \cup \Box U_n) + \sum_{\substack{I \subseteq \{1,\ldots,n\}, I \neq \emptyset \\ |I| \text{ even}}} \nu\Big(\bigcap_{i \in I} U_i\Big) = \sum_{\substack{I \subseteq \{1,\ldots,n\}, I \neq \emptyset \\ |I| \text{ odd}}} \nu\Big(\bigcap_{i \in I} U_i\Big).$$

By Corollary 4.3, a similar equation holds with ν^* in place of ν^{**}, so indeed $\nu^{**}(\mathcal{U}) = \nu^*(\mathcal{U})$ where $\mathcal{U} = \Box U_1 \cup \ldots \cup \Box U_n$.

As ν^{**} is an extension of ν^* we may use the notation ν^* instead of ν^{**} also for the extended map.

We finally claim that ν^* is modular on $\mathcal{O}(\mathcal{Q}_\mathcal{V}(X))$. Let $\mathcal{U}$ and $\mathcal{U}'$ be two arbitrary opens of $\mathcal{Q}_\mathcal{V}(X)$, and write them as unions of directed families of elementary opens $(\mathcal{U}_i)_{i \in I}$ and $(\mathcal{U}'_j)_{j \in J}$ respectively. Then $\mathcal{U} \cup \mathcal{U}'$ is the union of the directed family of elementary opens $(\mathcal{U}_i \cup \mathcal{U}'_j)_{i \in I, j \in J}$, and similarly for $\mathcal{U} \cap \mathcal{U}'$. Since ν^* is continuous and since $+$ is Scott-continuous, $\nu^*(\mathcal{U} \cup \mathcal{U}') + \nu^*(\mathcal{U} \cap \mathcal{U}')$ is the least upper bound of all $\nu^*(\mathcal{U}_i \cup \mathcal{U}'_j) + \nu^*(\mathcal{U}_i \cap \mathcal{U}'_j)$, where $i \in I$, $j \in J$, which equals $\nu^*(\mathcal{U}_i) + \nu^*(\mathcal{U}'_j)$ as ν^* is a valuation. It follows that $\nu^*(\mathcal{U} \cup \mathcal{U}') + \nu^*(\mathcal{U} \cap \mathcal{U}') = \nu^*(\mathcal{U}) + \nu^*(\mathcal{U}')$. ☐

Putting Lemma 4.4 and Proposition 4.11 together, we get:

Theorem 4.12 (Representation, Demonic Case). Let X be locally compact. The function that maps any continuous valuation P on $\mathcal{Q}_\mathcal{V}(X)$ to the continuous credibility

$P_{|X}$ on X is one-to-one. For every continuous credibility ν on X, there is a unique continuous valuation ν^* on $\mathcal{Q}_{\mathcal{V}}(X)$ such that $\nu^*_{|X} = \nu$, i.e., such that $\nu^*(\Box U) = \nu(U)$ for all open subsets U of X.

To give an illustration of what ν^* is, it follows immediately from Lemma 4.5 and the uniqueness of ν^* that $(\sum_{i=1}^n a_i \mathfrak{u}_{Q_i})^*$ is just the simple valuation $\sum_{i=1}^n a_i \delta_{Q_i}$ on $\mathcal{Q}_{\mathcal{V}}(X)$.

For those who have a preference for measures as compared to continuous valuations, we obtain:

Corollary 4.13. Let X be locally compact and well-filtered. Any weakly Radon measure μ on $\mathcal{Q}_{\mathcal{V}}(X)$ restricts to a continuous credibility $\mu_{|X}$ on X. Conversely, for every continuous credibility ν on X, there is a unique weakly Radon measure μ on $\mathcal{Q}_{\mathcal{V}}(X)$ such that $\mu(\Box U) = \nu(U)$ for all opens U of X.

Proof. This follows from the above using the Keimel-Lawson result that continuous valuations and weakly Radon measures are in one-to-one correspondence on any locally compact and well-filtered space, and from the fact that $\mathcal{Q}_{\mathcal{V}}(X)$ is indeed locally compact and well-filtered, since $\mathcal{Q}_{\mathcal{V}}(X) = \mathcal{Q}(X)$ is a continuous dcpo under our assumptions. □

We shall later argue for a representation theorem for the angelic case, which only assumes X core-compact, not locally compact (Theorem 5.12). A similar generalization can be obtained here by replacing $\mathcal{Q}_{\mathcal{V}}(X)$ by the space $\mathcal{F}_V(X)$ of all non-trivial (i.e., not containing the empty open) Scott-open filters of open subsets of X. When X is sober, the Hofmann-Mislove Theorem states that these two spaces are order-isomorphic through the isomorphism that sends each $Q \in \mathcal{Q}_{\mathcal{V}}(X)$ to its filter of open neighborhoods, where we agree to order $\mathcal{Q}_{\mathcal{V}}(X)$ by its specialization ordering $\supseteq$ and $\mathcal{F}_V(X)$ by inclusion. This easily extends to a homeomorphism provided we equip $\mathcal{F}_V(X)$ with the topology having as basic open subsets those of the form $\{\mathcal{F} \in \mathcal{F}_V(X) \mid U \in \mathcal{F}\}$ when U ranges over the open subsets of X. We decide to write the latter, again, $\Box U$.

For any space X, let the *sobrification* X^s of X be the space of all irreducible closed subsets C of X, with the so-called hull-kernel topology. This is none other than the lower Vietoris topology on X^s, whose subbasic open sets are $\diamond U = \{C \in X^s \mid C \cap U \neq \emptyset\}$, U open in X. Irreducibility entails that the latter are the *only* open subsets of X^s. In particular, the topologies $\mathcal{O}(X)$ and $\mathcal{O}(X^s)$ are order-isomorphic, through the map $U \mapsto \diamond U$.

It is remarkable that X^s is locally compact as soon as X is core-compact. This is an instance of the Hofmann-Lawson duality between continuous distributive lattices and locally compact sober spaces, see (Abramsky and Jung, 1994, Theorem 7.2.16) or (Gierz et al., 2003, Theorem V-5.5). So, if X is core-compact, then X^s is locally compact, and we can apply Theorem 4.12 on X^s instead of X. Since X^s is also sober, $\mathcal{Q}_{\mathcal{V}}(X^s)$ is homeomorphic to $\mathcal{F}_V(X^s)$. However, there is an order-isomorphism between the opens of X^s and those of X, which implies that $\mathcal{F}_V(X^s)$ is homeomorphic to $\mathcal{F}_V(X)$. Putting all this together, we obtain the following result.

Corollary 4.14. Let X be core-compact. The function that maps any continuous valuation P on $\mathcal{F}_V(X)$ to the continuous credibility $P_{|X}$ on X, defined by $P_{|X}(U) = P(\Box U)$, is one-to-one. For every continuous credibility ν on X, there is a unique continuous valuation ν^* on $\mathcal{F}_V(X)$ such that $\nu^*_{\downarrow X} = \nu$, i.e., such that $\nu^*(\Box U) = \nu(U)$ for all open subsets U of X.

We finish this section with an application to integration. We use *Choquet integration* (Choquet, 54) of any continuous map $f : X \to \mathbb{R}^+_\sigma$ (i.e., any lower semi-continuous map from X to $\mathbb{R}^+$) along any game ν, which we write $\oint_{x\in X} f(x)d\nu$:

$$\oint_{x\in X} f(x)d\nu \quad = \quad \int_0^{+\infty} \nu(f^{-1}(t,+\infty))dt \tag{16}$$

Denneberg (Denneberg, 1994, Chapters 5, 6) gives an in-depth treatment of the Choquet integral, in a more standard, measure-theoretic fashion, where f above does not need to be continuous, but the resulting integral only commutes with limits of increasing sequences, not of directed families. Tix (Tix, 1995) also used the same formula (16), in the restricted case where ν is a continuous valuation. The integral on the right is an ordinary improper Riemann integral, and is well-defined, since $f^{-1}(t,+\infty)$ is open for every $t \in \mathbb{R}$ by assumption, and ν measures opens. Also, since f is bounded, the improper integral above really is an ordinary Riemann integral over some closed interval. The function $t \mapsto \nu(f^{-1}(t,+\infty))$ is non-increasing, and every non-increasing function is Riemann-integrable.

It is immediate from (16) that Choquet integration satisfies the change of variables formula: $\oint_{x\in X} f(g(x))d\nu = \oint_{y\in Y} f(y)dg[\nu]$, whenever g is continuous from X to Y, f is continuous from Y to $\mathbb{R}^+_\sigma$, and ν is any game on X. Recall that the image game $g[\nu]$ is defined by $g[\nu](V) = \nu(g^{-1}(V))$.

It is easy to see that the Choquet integral is monotonic in both the f and the ν argument. It is Scott-continuous in ν, because Riemann integrals of non-increasing functions are Scott-continuous in the integrated function, see for example (Tix, 1995, Lemma 4.2), or use elementary reasoning on Darboux sums; we leave this as an exercise. Similarly, the Choquet integral is Scott-continuous in f as soon as ν is a continuous game, meaning that if f is continuous bounded and the sup of the directed family $(f_i)_{i\in I}$ of continuous bounded maps, then $\oint_{x\in X} f(x)d\nu = \sup_{i\in I} \oint_{x\in X} f_i(x)d\nu$: again, reason on the formula (16).

The Choquet integral is also linear in the game, in the sense that $\oint_{x\in X} f(x)d\sum_{i=1}^n a_i\nu_i = \sum_{i=1}^n a_i \oint_{x\in X} f(x)d\nu_i$. This is because the Riemann integral is linear in the integrated function. The Choquet integral is not in general linear in f (integration along unanimity games provides easy counterexamples, see Lemma 4.17 below), although it is certainly *positively homogeneous*: whenever $a \in \mathbb{R}^+$, $\oint_{x\in X} af(x)d\nu = a\oint_{x\in X} f(x)d\nu$ (do the change of variables $t \mapsto t/a$ in (16), at least when $a \neq 0$).

An alternate definition consists in observing that any *step function* $\sum_{i=1}^n a_i\chi_{U_i}$, where $a_1, \ldots, a_n \in \mathbb{R}^+$, $X \supseteq U_1 \supseteq \ldots \supseteq U_n$ is a decreasing sequence of opens, and χ_U is the indicator function of U ($\chi_U(x) = 1$ if $x \in X$, $\chi_U(x) = 0$ otherwise) is continuous from X to $\mathbb{R}^+_\sigma$, and of integral along ν equal to $\sum_{i=0}^n a_i\nu(U_i)$—for *any* game ν:

Lemma 4.15. Let ν be any game on X. For any step function $\sum_{i=1}^{n} a_i \chi_{U_i}$, where $a_1, \ldots, a_n \in \mathbb{R}^+$, $X \supseteq U_1 \supseteq \ldots \supseteq U_n$,

$$\oint_{x \in X} f(x) d\nu \quad = \quad \sum_{i=1}^{n} a_i \nu(U_i).$$

Proof. Without loss of generality, assume $a_1, \ldots, a_n > 0$. For every $t \in \mathbb{R}^+$, $\nu(f^{-1}(t, +\infty))$ equals $\nu(U_1)$ for all $t \in [0, a_1)$, $\nu(U_2)$ for all $t \in [a_1, a_1 + a_2)$, $\ldots$, $\nu(U_n)$ for all $t \in [a_1 + \ldots + a_{n-1}, a_1 + \ldots + a_{n-1} + a_n)$, and 0 for all $t \in [a_1 + \ldots + a_n, +\infty)$. So, splitting the Riemann integral of (16) at the corresponding boundaries, $\oint_{x \in X} f(x) d\nu = a_1 \nu(U_1) + a_2 \nu(U_2) + \ldots + a_n \nu(U_n)$. ☐

It is well-known that every bounded continuous function $f : X \to \mathbb{R}^+_\sigma$ can be written as the least upper bound of a sequence of step functions $f_K = \frac{1}{2^K} \sum_{k=1}^{\lfloor b2^K \rfloor} \chi_{f^{-1}(\frac{k}{2^K}, +\infty)}(x)$, $K \in \mathbb{N}$, where $b = \sup_{x \in X} f(x)$. Then the integral of f along ν is the least upper bound of the increasing sequence of the integrals of f_K along ν (even if ν is not continuous). The resulting formula is known as Jones' integral when ν is a continuous valuation:

Lemma 4.16. Let ν be any game on X. For every bounded continuous function $f : X \to \mathbb{R}^+_\sigma$, the sequence $(f_K)_{K \in \mathbb{N}}$ is non-decreasing, f_K tends uniformly to f in the sense that $f(x) - \frac{1}{2^K} \leq f_K(x) \leq f(x)$ for all $x \in X$, and:

$$\oint_{x \in X} f(x) d\nu \quad = \quad \sup_{K \in \mathbb{N}} \oint_{x \in X} f_K(x) d\nu$$

Proof. If $b = \sup_{x \in X} f(x)$ is zero, then this is obvious. So assume $b > 0$. Let more generally $f_{(\epsilon)}(x) = \epsilon \sum_{k=1}^{\lfloor b/\epsilon \rfloor} \chi_{f^{-1}]k\epsilon, +\infty[}(x)$ for any $\epsilon > 0$; $f_{(\epsilon)}$ is a step function, and $f_K = f_{(1/2^K)}$. It is standard that $f_{(\epsilon)}$ is non-increasing in ϵ, and below f; moreover, it is easy to see that $f(x) - f_{(\epsilon)}(x) \leq \epsilon$ for all $x \in X$.

Moreover, by Lemma 4.15, $\oint_{x \in X} f_{(\epsilon)}(x) d\nu = \sum_{k=1}^{\lfloor b/\epsilon \rfloor} \epsilon \nu(f^{-1}(k\epsilon, +\infty))$. Since Choquet integration is monotonic in the integrated function, this is less than or equal to $\oint_{x \in X} f(x) d\nu$. Since $f^{-1}(t, +\infty)$ is non-increasing in t, for every $k \geq 1$, $\epsilon \nu(f^{-1}(k\epsilon, +\infty)) \geq \int_{k\epsilon}^{(k+1)\epsilon} \nu(f^{-1}(t, +\infty)) dt$. So:

$$\begin{aligned}
\oint_{x \in X} f_{(\epsilon)}(x) d\nu &= \sum_{k=1}^{\lfloor b/\epsilon \rfloor} \epsilon \nu(f^{-1}(k\epsilon, +\infty)) \\
&\geq \sum_{k=1}^{\lfloor b/\epsilon \rfloor} \int_{k\epsilon}^{(k+1)\epsilon} \nu(f^{-1}(t, +\infty)) dt \\
&= \int_{\epsilon}^{(\lfloor b/\epsilon \rfloor + 1)\epsilon} \nu(f^{-1}(t, +\infty)) dt \\
&= \int_{\epsilon}^{+\infty} \nu(f^{-1}(t, +\infty)) dt \quad \text{since } f^{-1}(t, +\infty) = \emptyset \text{ whenever } t \geq b \\
&= \oint_{x \in X} f(x) d\nu - \int_0^{\epsilon} \nu(f^{-1}(t, +\infty)) dt \geq \oint_{x \in X} f(x) d\nu - \epsilon \nu(X)
\end{aligned}$$

since $f^{-1}(t, +\infty) \subseteq X$ and ν is monotone. $\Box$

Choquet integration has other properties, notably that the integral of $f+g$ does indeed coincide with the sums of the integrals of f and g when f and g are comonotonic, i.e., when there is no pair $x, x' \in X$ such that $f(x) < f(x')$ and $g(x) > g(x')$. See (Gilboa and Schmeidler, 1992) for the finite case, (Goubault-Larrecq, 2007, chapitre 4) for the topological case, or (Molchanov, 2005, Theorem 5.5).

We are more interested here in integrating along continuous credibilities. We first note the following, which is probably the most intuitive reason why picking from a compact saturated set Q should be called *demonic* non-determinism: taking the average along $\mathfrak{u}_Q$ means *minimizing* your earnings $f(x)$ over all possible choices $x \in Q$.

Lemma 4.17. For every non-empty compact saturated set Q of X, for every bounded continuous map $f : X \to \mathbb{R}^+_\sigma$, let $f_*(Q) = \min_{x\in Q} f(x)$. Then $\oint_{x\in X} f(x) d\mathfrak{u}_Q = f_*(Q)$.

Proof. First, the minimum $\min_{x\in Q} f(x)$ is indeed attained: since Q is compact (and non-empty), the image $f(Q)$ is compact in $\mathbb{R}^+_\sigma$ (and non-empty), so $\uparrow f(Q)$ is non-empty compact saturated, hence of the form $[t, +\infty)$ for some real t: then $t = \min_{x\in Q} f(x)$. Now observe that $\mathfrak{u}_Q(f^{-1}(t, +\infty)) = 1$ iff $Q \subseteq f^{-1}(t, +\infty)$ iff $\min_{x\in Q} f(x) > t$ iff $t < f_*(Q)$, and $\mathfrak{u}_Q(f^{-1}(t, +\infty)) = 0$ otherwise. By (16), $\oint_{x\in X} f(x) d\mathfrak{u}_Q = \int_0^{+\infty} \mathfrak{u}_Q(f^{-1}(t, +\infty)) dt = \int_0^{f_*(Q)} 1 dt = f_*(Q)$. $\Box$

It follows that integrating along simple credibilities means taking "means of mins" (Gilboa and Schmeidler, 1992): $\oint_{x\in X} f(x) d \sum_{i=1}^n a_i d\mathfrak{u}_{Q_i} = \sum_{i=1}^n a_i \min_{x\in Q_i} f(x)$. This is another way to see that picking x at random along a credibility means picking a non-empty compact saturated subset Q_i at random with probability a_i, then picking some element from Q_i in a demonic way. This generalizes to all continuous, not just simple credibilities:

Proposition 4.18. For every continuous valuation P on $\mathcal{Q}_\mathcal{V}(X)$, for every bounded continuous map $f : X \to \mathbb{R}^+_\sigma$, $\oint_{x\in X} f(x) dP_{\downarrow X} = \oint_{Q\in\mathcal{Q}_\mathcal{V}(X)} f_*(Q) dP$.

Let X be locally compact. For every continuous credibility ν on X, for every bounded continuous map $f : X \to \mathbb{R}^+_\sigma$, $\oint_{x\in X} f(x) d\nu = \oint_{Q\in\mathcal{Q}_\mathcal{V}(X)} f_*(Q) d\nu^*$.

Proof. Note that $f_*^{-1}(t, +\infty) = \{Q \mid \min_{x\in Q} f(x) > t\} = \{Q \mid Q \subseteq f^{-1}(t, +\infty)\} = \Box f^{-1}(t, +\infty)$. (That the minimum is attained is important.) Then $\oint_{Q\in\mathcal{Q}_\mathcal{V}(X)} f_*(Q) dP = \int_0^{+\infty} P(f_*^{-1}(t, +\infty)) dt = \int_0^{+\infty} P(\Box f^{-1}(t, +\infty)) dt = \int_0^{+\infty} P_{\downarrow X}(f^{-1}(t, +\infty)) dt$. The latter is just $\oint_{x\in X} f(x) dP_{\downarrow X}$. The second part then follows from Theorem 4.12. $\Box$

Using Lemma 4.17, one can also see this as a form of disintegration, viz. $\oint_{x\in X} f(x) d\nu = \oint_{Q\in\mathcal{Q}_\mathcal{V}(X)} \left(\oint_{x\in Q} f(x) d\mathfrak{u}_Q\right) d\nu^*$.

5. The Angelic Case: Continuous Plausibilities

There is an easy way in which we can reduce the case of continuous plausibilities to that of continuous credibilities, at least when X is stably compact. In this case indeed, there is a form of duality between angelic and demonic non-determinism that immediately entails that there is a one-to-one correspondence between continuous (normalized) plausibilities on X and continuous (normalized) valuations on the Hoare powerdomain $\mathcal{H}_{\mathcal{V}}(X)$ of all non-empty closed subsets of X. This is the subject of (Goubault-Larrecq, 2010, Section 6).

However, we can also prove the same result under weaker assumptions on X. The procedure is parallel to that in Section 4.

The upshot is that, dually to continuous credibilities, continuous plausibilities provide a semantic model for mixed probabilistic choice and *angelic* non-determinism, where one draws some non-empty closed subset $F \in \mathcal{H}_{\mathcal{V}}(X)$ at random, then picks non-deterministically some element from F.

The topology of $\mathcal{H}_{\mathcal{V}}(X)$ is the lower Vietoris topology the opens of which are unions of finite intersections of *subbasic open* sets

$$\Diamond U = \{F \in \mathcal{H}_{\mathcal{V}}(X) \mid F \cap U \neq \emptyset\}.$$

Contrarily to the case of $\mathcal{Q}_{\mathcal{V}}(X)$, they only form a subbase, not a base. Let us notice in passing that $\mathcal{H}_{\mathcal{V}}(X)$ is always sober as remarked by (Schalk, 1993, Proposition 1.7). Some obvious properties of the subbasic opens are:

Lemma 5.1. (1) $\Diamond\emptyset = \emptyset$ and $\Diamond X = \mathcal{H}_{\mathcal{V}}(X)$; (2) $\Diamond U \cup \Diamond V = \Diamond(U \cup V)$; (3) $\Diamond\big(\bigcup_{i\in I} U_i\big) = \bigcup_{i\in I} \Diamond U_i$ for every directed family of opens $(U_i)_{i\in I}$.

Note (2): $\Diamond$ commutes with finite unions, while $\Box$ commutes with finite intersections.

Lemma 5.2. The subbasic open sets $\Diamond U$ for U open in X form a $\cap$-irredundant $\cup$-semilattice $\mathcal{S}$ of subsets of $\mathcal{H}_{\mathcal{V}}(X)$ and $U \mapsto \Diamond U$ is a $\cup$-semilattice isomorphism from the set $\mathcal{O}(X)$ of all open subsets of X onto the semilattice $\mathcal{S}$ of subbasic open sets in $\mathcal{H}_{\mathcal{V}}(X)$.

Proof. For the second claim it suffices to show that $U \mapsto \Diamond U$ is injective. Indeed, if U and U' are two different open sets, say $U \not\subseteq U'$, then there is an element $x \in U \setminus U'$. The closure of the singleton $\{x\}$ meets U but not U' and, hence, is then a member of $\Diamond U$ but not of $\Diamond U'$.

It remains to show that the members of $\mathcal{S}$ are $\cap$-irreducible. This is the case in a strong sense: Consider collection of open subsets U_i of X such that $\Diamond U_i$ properly contains $\Diamond U$ for each i. Then for each i there is a closed set F_i meeting U_i but disjoint from U The closure F of the union of all the sets F_i meets every U_i but is still disjoint from the open set U, which implies that $F \in \bigcap_i \Diamond U_i \setminus \Diamond U$. □

The finite intersections $\Diamond U_1 \cap \ldots \cap \Diamond U_n$ of subbasic opens are the *basic open* sets of the lower Vietoris topology and they form a lattice of subsets of $\mathcal{H}_{\mathcal{V}}(X)$. The preceding lemma allows us to apply Theorem 3.7 and we obtain:

Corollary 5.3. For every strict map $\nu\colon \mathcal{O}(X) \to \mathbb{R}$ there is a unique strict modular function ν_* defined on the lattice of basic opens of $\mathcal{H}_{\mathcal{V}}(X)$ such that $\nu^*(\Diamond U) = \nu(U)$ for every open set U in X. The extension is given by

$$\nu_*(\Diamond U_1 \cap \ldots \cap \Diamond U_n) = \sum_{\substack{I\subseteq\{1,\ldots,n\}\\ I\neq\emptyset}} (-1)^{|I|+1} \nu\big(\bigcup_{i\in i} U_i\big).$$

The extension ν_* is a valuation if and only if ν is a plausibility on X.

We would like to extend ν to a continuous valuation on $\mathcal{H}_\mathcal{V}(X)$. We observe:

Lemma 5.4. For any continuous valuation P on $\mathcal{H}_\mathcal{V}(X)$, the game $P_{\upharpoonright X}$ defined by $P_{\upharpoonright X}(U) = P(\Diamond U)$, for all opens U of X, is a continuous plausibility on X.

Proof. Continuity is by Lemma 5.1 (3). As the restriction of P to the lattice of elementary opens is a valuation, ν is totally concave by Corollary 5.3 □

Again, $P_{\upharpoonright X}$ is obtained by restriction, considering X as a subspace of $\mathcal{H}_\mathcal{V}(X)$. The canonical embedding justifying this is $\eta_\mathcal{H} : X \to \mathcal{H}_\mathcal{V}(X), x \mapsto \downarrow x$, assuming X is T_0.

Dually to unanimity games $\mathfrak{u}_Q$, let $\mathfrak{e}_F$ be the *example game* on the non-empty closed set F: $\mathfrak{e}_F(U)$ is 1 if $F \cap U \neq \emptyset$, 0 otherwise. Example games also generalize Dirac masses, since $\delta_x = \mathfrak{e}_{\downarrow x}$.

Call *simple plausibility* any game of the form $\sum_{i=1}^n a_i \mathfrak{u}_{F_i}$, $a_1, \ldots, a_n \in \mathbb{R}^+$, $F_1, \ldots, F_n \in \mathcal{H}_\mathcal{V}(X)$. Then we have:

Lemma 5.5. Any simple plausibility is a continuous plausibility. Moreover, every restriction of a simple valuation on $\mathcal{H}_\mathcal{V}(X)$ is a simple plausibility on X; namely, for any $a_1, \ldots, a_n \in \mathbb{R}^+$, $F_1, \ldots, F_n \in \mathcal{H}_\mathcal{V}(X)$,

$$\left(\sum_{i=1}^n a_i \delta_{F_i}\right)_{\upharpoonright X} = \sum_{i=1}^n a_i \mathfrak{e}_{F_i}.$$

Proof. The first claim follows from the second one and Lemma 5.4. The second one is immediate: on the open U, the two sides of the equality simplify to $\sum_{\substack{1 \leq i \leq n \\ F_i \cap U \neq \emptyset}} a_i$. □

Similarly to Proposition 4.6, example games can be justified as models of pure angelic non-determinism, where again pure non-determinism is characterized by games taking values 0 or 1 only:

Proposition 5.6. Let X be a topological space. Any continuous plausibility ν with $\nu(X) = 1$ and such that ν only takes values 0 or 1 is of the form $\mathfrak{e}_F$, $F \in \mathcal{H}_\mathcal{V}(X)$. In fact, this already holds of all concave games that take values 0 or 1 only and such that $\nu(X) = 1$, where ν is *concave* iff $\nu(U \cup V) + \nu(U \cap V) \leq \nu(U) + \nu(V)$ for all opens U, V.

Proof. We first check that every continuous plausibility is convex: this is the case $n = 2$ in (2). Next, consider the union U_∞ of all opens U such that $\nu(U) = 0$. Note that this is a directed union: if $\nu(U) = 0$ and $\nu(V) = 0$ then $\nu(U \cup V) \leq \nu(U) + \nu(V) - \nu(U \cap V) = 0$. Since ν is continuous, $\nu(U_\infty) = 0$. Let F be the complement of U_∞. F is non-empty, otherwise $\nu(X) = \nu(U_\infty) = 0$. For every open set U, if U does not intersect F then $U \subseteq U_\infty$, so $\nu(U) = 0$; if U does, then U is not contained in U_∞, so $\nu(U) \neq 0$, whence $\nu(U) = 1$. So $\nu = \mathfrak{e}_F$. □

To establish a converse to Lemma 5.4, we will need to study the topology of $\mathcal{H}_\mathcal{V}(X)$. It will be convenient to use the following notation for every compact subset Q of X: $\blacklozenge Q = \{F \in \mathcal{H}_\mathcal{V}(X) \mid F \cap Q \neq \emptyset\}$

Lemma 5.7. $\blacklozenge Q_1 \cap \ldots \cap \blacklozenge Q_n$ is compact and saturated in $\mathcal{H}_{\mathcal{V}}(X)$ for any non-empty finite family $Q_1, \ldots, Q_n$ of non-empty compact subsets of X.

Proof. It is certainly saturated. By Alexander's Subbase Lemma, it is enough to show that for every family $(U_j)_{j \in J}$ of opens of X, such that $\bigcap_{i=1}^n \blacklozenge Q_i \subseteq \bigcup_{j \in J} \Diamond U_j$, there is a finite subset J' of J such that $\bigcap_{i=1}^n \blacklozenge Q_i \subseteq \bigcup_{j \in J'} \Diamond U_j$. Since $\bigcap_{i=1}^n \blacklozenge Q_i \subseteq \bigcup_{j \in J} \Diamond U_j$, consider the closed set F, complement of $\bigcup_{j \in J} U_j$. F is certainly not in $\bigcup_{j \in J} \Diamond U_j$, so it is not in $\bigcap_{i=1}^n \blacklozenge Q_i$ either. It follows that F does not intersect some Q_i, $1 \leq i \leq n$. Equivalently, Q_i is contained in the complement of F, namely $\bigcup_{j \in J} U_j$. Since Q_i is compact, there is a finite subset J' of J such that $Q_i \subseteq \bigcup_{j \in J'} U_j$. We claim that $\bigcap_{i=1}^n \blacklozenge Q_i \subseteq \bigcup_{j \in J'} \Diamond U_j$: every closed set F in the left hand side must meet Q_i, hence also $\bigcup_{j \in J'} U_j$. So F meets U_j for some $j \in J'$, whence $F \in \bigcup_{j \in J'} \Diamond U_j$. □

Among the basic open sets of $\mathcal{H}_{\mathcal{V}}(X)$ we consider those of the form $\Diamond int(Q_1) \cap \ldots \cap \Diamond int(Q_n)$ where $Q_1, \ldots, Q_n$ are compact saturated sets in X. We call them *special* basic open sets.

Lemma 5.8. If X is locally compact, then every basic open set $\Diamond U_1 \cap \ldots \cap \Diamond U_n$ is the union of the directed family of special basic open sets $\Diamond int(Q_1) \cap \ldots \cap \Diamond int(Q_n)$ with $Q_1 \subseteq U_1, \ldots, Q_n \subseteq U_n$. In particular, the sets $\Diamond int(Q)$ form a subbasis of the lower Vietoris topology, when Q ranges over the compact saturated sets in X.

Proof. As X is locally compact, an open subset U_i of X is the union of the $int(Q_i)$ where Q_i ranges over the compact saturated sets in X contained in U_i, i.e., $U_i = \bigcup\{int(Q_i) \mid Q_i \subseteq U_i\}$. As the union of finitely many compact saturated set contained in U_i is again a compact saturated sets contained in U_i, the family of $Q_i \subseteq U_i$ is directed. Lemma 5.1 (3) implies that $\Diamond U_i = \bigcup\{\Diamond int(Q)_i \mid Q_i \subseteq U_i\}$. We deduce $\Diamond U_1 \cap \ldots \cap \Diamond U_n = \bigcup_{Q_1 \subseteq U_1} \Diamond int(Q_1) \cap \ldots \cap \bigcup_{Q_n \subseteq U_n} \Diamond int(Q_n) = \bigcup_{Q_1 \subseteq U_1, \ldots, Q_n \subseteq U_n} \Diamond int(Q_1) \cap \ldots \cap \Diamond int(Q_n)$, where we have used that directed unions commute with finite intersections. □

It is known (Schalk, 1993, Proposition 6.11) that for a locally compact space X, the Hoare powerdomain is locally compact and sober. We reprove local compactness and give some additional information:

Lemma 5.9. Let X be a locally compact space. Then $\mathcal{H}_{\mathcal{V}}(X)$ is locally compact. The lattice of open subsets of $\mathcal{H}_{\mathcal{V}}(X)$ is continuous; for two open subsets $\mathcal{U}$ and $\mathcal{V}$ we have $\mathcal{U} \Subset \mathcal{V}$ if and only if there are finitely many compact saturated sets $Q_1, \ldots, Q_n$ such that $\mathcal{U} \subseteq \Diamond int(Q_1) \cap \ldots \cap \Diamond int(Q_n)$ and $\blacklozenge Q_1 \cap \ldots \cap \blacklozenge Q_n \subseteq \mathcal{V}$

Proof. We first show that $\mathcal{H}_{\mathcal{V}}(X)$ is locally compact. For this, consider any $F \in \mathcal{H}_{\mathcal{V}}(X)$ and any basic open neighborhood $\Diamond U_1 \cap \ldots \cap \Diamond U_n$ of F. Then $F \cap U_i \neq \emptyset$ whence there are points $x_i \in F \cap U_i$ for $i = 1, \ldots, n$. As X is locally compact, we may find open sets V_i and compact saturated sets Q_i such that $x_i \in V_i \subseteq Q_i \subseteq U_i$ for each i. Then $F \in \Diamond V_1 \cap \ldots \cap \Diamond V_n$ as $x_i \in F \cap V_i$ for each i and $\Diamond V_1 \cap \ldots \cap \Diamond V_n \subseteq \blacklozenge Q_1 \cap \ldots \cap \blacklozenge Q_n \subseteq \Diamond U_1 \cap \ldots \cap \Diamond U_n$. As $\blacklozenge Q_1 \cap \ldots \cap \blacklozenge Q_n$ is compact by Lemma 5.7, we have found a compact neighborhood of F contained in the given basic neighborhood.

For any locally compact space—hence also for $\mathcal{H}_{\mathcal{V}}(X)$—the lattice of open subsets is continuous and, for open subsets $\mathcal{U}$ and $\mathcal{V}$, one has $\mathcal{U} \Subset \mathcal{V}$ if and only of there is a compact set $\mathcal{K}$ such that $\mathcal{U} \subseteq \mathcal{K} \subseteq \mathcal{V}$. Thus, if there are compact saturated sets $Q_1, \ldots, Q_n$ such that $\mathcal{U} \subseteq \Diamond int(Q_1) \cap \ldots \cap \Diamond int(Q_n)$ and $\blacklozenge Q_1 \cap \ldots \cap \blacklozenge Q_n \subseteq \mathcal{V}$, then $\mathcal{U} \Subset \mathcal{V}$, as by Lemma 5.7 $\blacklozenge Q_1 \cap \ldots \cap \blacklozenge Q_n$ is compact. Suppose conversely that $\mathcal{U} \Subset \mathcal{V}$. The open set $\mathcal{V}$ is a union of basic open sets $\Diamond U_1 \cap \cdots \cap \Diamond U_n$, and each of those is a union of special basic open sets $\Diamond int(Q_1) \cap \ldots \cap \Diamond int(Q_n)$ with $Q_i \subseteq U_i$ for $i = 1, \ldots, n$. Hence, $\mathcal{V}$ a union of special basic opens $\Diamond int(Q_1) \cap \ldots \cap \Diamond int(Q_n)$ with $\blacklozenge Q_1 \cap \ldots \cap \blacklozenge Q_n \subseteq \mathcal{V}$. Thus, if $\mathcal{U} \Subset \mathcal{V}$, there are finitely many compact saturated sets $Q_1, \ldots, Q_n$ such that $\mathcal{U} \subseteq \Diamond int(Q_1) \cap \ldots \cap \Diamond int(Q_n)$ and $\blacklozenge Q_1 \cap \ldots \cap \blacklozenge Q_n \subseteq \mathcal{V}$. as desired. □

As an immediate consequence of the previous lemma we have;

Corollary 5.10. Suppose that X is locally compact. The special basic opens $\Diamond int(Q_1) \cap \ldots \cap \Diamond int(Q_n)$, where $Q_1, \ldots, Q_n$ range over finite families of compact saturated subsets of X, form a basis of the continuous lattice $\mathcal{O}(\mathcal{H}_\mathcal{V}(X))$.

Proposition 5.11. For every continuous plausibility ν on X, there is at most one continuous valuation ν_* on $\mathcal{H}_\mathcal{V}(X)$ such that $\nu(U) = \nu_*(\Diamond U)$ for every open U of X; and such a ν_* exists whenever X is locally compact.

Proof. Assume ν is a continuous plausibility on X. By Corollary 5.3, there is a unique valuation ν_* defined on the lattice of basic opens such that $\nu_*(\Diamond U) = \nu(U)$ for every open U of X.

As every open subset $\mathcal{U}$ of $\mathcal{H}_\mathcal{V}(X)$ is the union of its basic open subsets and as this family is directed, a continuous extension of ν_* to $\mathcal{O}(\mathcal{H}_\mathcal{V}(X))$ must satisfy

$$\nu_*(\mathcal{U}) = \sup\{\nu_*(\mathcal{V}) \mid \mathcal{V} \text{ basic open}, \mathcal{V} \subseteq \mathcal{U}\}$$

and hence is uniquely determined by the values of ν_* on the basic opens.

In order to show the existence, we suppose that X is locally compact. By Lemma 5.9 and Corollary 5.10, the lattice of open subsets of $\mathcal{H}_\mathcal{V}(X)$ is continuous, the basic opens form a basis of this continuous lattice and we have identified the way-below relation there. According to the Extension Lemma recorded in the Preliminaries, we define $\nu_{**} \colon \mathcal{O}(\mathcal{H}_\mathcal{V}(X)) \to \mathbb{R}$ by

$$\nu_{**}(\mathcal{U}) = \sup_{\mathcal{J}} \nu_*\Big(\bigcap_{Q \in \mathcal{J}} \Diamond int(Q)\Big)$$

where $\mathcal{J}$ ranges over the finite sets of compact saturated sets Q such that $\bigcap_{Q \in \mathcal{J}} \Diamond int(Q) \Subset \mathcal{U}$. The map ν_{**} is always continuous. It is an extension of ν_* if and only if $\nu_{**}(\mathcal{U}) = \nu_*(\mathcal{U})$ for all basic opens $\mathcal{U}$ and, in this case, ν_{**} is the unique continuous extension of ν_*.

Thus, let us show that $\nu_{**}(\mathcal{U}) = \nu_*(\mathcal{U})$ holds for $\mathcal{U} = \Diamond U_1 \cap \ldots \cap \Diamond U_n$ where $U_1, \ldots, U_n$ are open in X. Without loss of generality, assume $U_1, \ldots, U_n$ to be non-empty. By Lemma 5.8 and Lemma 5.9, the definition of $\nu_{**}(\mathcal{U})$ can be simplified for basic opens:

$$\nu_{**}(\Diamond U_1 \cap \ldots \cap \Diamond U_n) = \sup_{Q_1 \subseteq U_1, \ldots, Q_n \subseteq U_n} \nu_*\big(\Diamond int(Q_1) \cap \ldots \cap \Diamond int(Q_n)\big).$$

Using the definition of ν_* (see Corollary 5.3),

$$\nu_*\big(\Diamond int(Q_1)\cap\ldots\cap\Diamond int(Q_n)\big)+\sum_{\substack{I\subseteq\{1,\ldots,n\},I\neq\emptyset\\|I|\text{ even}}}\nu\Big(\bigcup_{i\in I}int(Q_i)\Big)=\sum_{\substack{I\subseteq\{1,\ldots,n\},I\neq\emptyset\\|I|\text{ odd}}}\nu\Big(\bigcup_{i\in I}int(Q_i)\Big).$$

The family of all tuples $(Q_1,\ldots,Q_n)$ of non-empty compact saturated subsets such that $Q_1\subseteq U_n$, $\ldots$, $Q_n\subseteq U_n$, ordered by pointwise inclusion, is directed. Take sups in the above equality, using the fact that $+$ is Scott-continuous, and that ν is continuous:

$$\nu_{**}(\Diamond U_1\cap\ldots\cap\Diamond U_n)+\sum_{\substack{I\subseteq\{1,\ldots,n\},I\neq\emptyset\\|I|\text{ even}}}\nu(V_I)=\sum_{\substack{I\subseteq\{1,\ldots,n\},I\neq\emptyset\\|I|\text{ odd}}}\nu(V_I),$$

where V_I stands for $\bigcup_{Q_1\subseteq U_1,\ldots,Q_n\subseteq U_n}\bigcup_{i\in I}int(Q_i)$. Now $V_I=\bigcup_{i\in I}U_i$, as in a locally compact space every element in an open set U_i is contained in the interior of some compact saturated set $Q_i\subseteq U_i$. So

$$\nu_{**}(\Diamond U_1\cap\ldots\cap\Diamond U_n)+\sum_{\substack{I\subseteq\{1,\ldots,n\},I\neq\emptyset\\|I|\text{ even}}}\nu\Big(\bigcup_{i\in I}U_i\Big)=\sum_{\substack{I\subseteq\{1,\ldots,n\},I\neq\emptyset\\|I|\text{ odd}}}\nu\Big(\bigcup_{i\in I}U_i\Big).$$

By Corollary 5.3, a similar equation holds with ν_* in place of ν_{**}, so indeed $\nu_{**}(\mathcal{U})=\nu_*(\mathcal{U})$ where $\mathcal{U}=\Diamond U_1\cap\ldots\cap\Diamond U_n$.

As ν_{**} is an extension of ν_* we may use the notation ν_* instead of ν_{**} also for the extended map. To conclude, we show that ν_* is modular exactly as for ν^* in Proposition 4.11, using the facts that ν_* is continuous and $+$ is Scott-continuous. □

Proposition 5.11 can be extended to the case of core-compact spaces. Remember that the topologies $\mathcal{O}(X)$ of X and $\mathcal{O}(X^s)$ of the sobrification X^s of X are order-isomorphic, through the map $U\mapsto\diamond U$.

It follows that $\mathcal{H}_\mathcal{V}(X)$ is isomorphic to $\mathcal{H}_\mathcal{V}(X^s)$. The isomorphism is given, in one direction by the map $box: F\in\mathcal{H}_\mathcal{V}(X)\mapsto\square F\in\mathcal{H}_\mathcal{V}(X^s)$, where we define $\square F$ as the complement of $\diamond U$ where U is the complement of F (this is continuous since the inverse image of $\Diamond\diamond U$ is $\Diamond U$); in the other direction by the map *unbox* sending $\square F$ (all closed sets of X^s are of this form) to F, which is continuous since the inverse image of $\Diamond U$ is $\Diamond\diamond U$.

If X is core-compact, then use Proposition 5.11 on the locally compact space X^s. For every continuous plausibility ν on X, $\nu'(\diamond U)=\nu(U)$ defines a continuous plausibility on X^s: this follows from the fact that not only $\diamond$ commutes with finite unions, but also with finite intersections, a consequence of irreducibility. By Proposition 5.11, ν'_* exists, in such a way that $\nu'_*(\Diamond\diamond U)=\nu'(\diamond U)$ for all opens U of X. Then define ν_* as the image game $unbox[\nu'_*]$, where *unbox* maps $\square F$ to F. As the image game of a continuous valuation by a continuous map, ν_* is a continuous valuation, and $\nu_*(\Diamond U)=\nu'_*(unbox^{-1}(\Diamond U))=\nu'_*(\Diamond\diamond U)=\nu'(\diamond U)=\nu(U)$. Putting this together with Lemma 5.4, we get:

Theorem 5.12 (Representation, Angelic Case). Let X be core-compact. The function that maps any continuous valuation P on $\mathcal{H}_\mathcal{V}(X)$ to the continuous plausibility $P_{\upharpoonright X}$ on X is one-to-one. For every continuous plausibility ν on X, there is a unique continuous valuation ν_* on $\mathcal{H}_\mathcal{V}(X)$ such that $\nu_{*\upharpoonright X} = \nu$, i.e., such that $\nu_*(\Diamond U) = \nu(U)$ for all open subsets U of X.

Note that, from Lemma 5.5 and the uniqueness of ν_*, $(\sum_{i=1}^n a_i \mathfrak{e}_{F_i})_*$ is just the simple valuation $\sum_{i=1}^n a_i \delta_{F_i}$ on $\mathcal{H}_\mathcal{V}(X)$.

Again, this has the following measure-theoretic consequence.

Corollary 5.13. Let X be core-compact. Any weakly Radon measure μ on $\mathcal{H}_\mathcal{V}(X)$ restricts to a continuous plausibility $\mu_{\upharpoonright X}$ on X. Conversely, for every continuous plausibility ν on X, there is a unique weakly Radon measure μ on $\mathcal{H}_\mathcal{V}(X)$ such that $\mu(\Diamond U) = \nu(U)$ for all opens U of X.

Proof. As $\mathcal{H}_\mathcal{V}(X)$ is locally compact (by Lemma 5.9) and well-filtered (because sober, using the Hofmann-Mislove Theorem), we can apply the Keimel-Lawson result that continuous valuations and weakly Radon measures are in one-to-one correspondence on any locally compact and well-filtered space. □

The latter result is closest to Norberg's Theorem (Norberg, 1989, Theorem 6.1). He requires X to be core-compact, second countable and sober, while we only need X to be core-compact, and not even T_0. Sobriety seems to be the price to pay for deducing topological facts from their localic counterparts. Second countability is also required by Norberg, however this can be accounted for by the fact that he deals with measures instead of valuations. The connection can be made explicit by using a measure extension theorem due to Norberg again (Norberg, 1989, Theorem 3.9), see also (Alvarez-Manilla et al., 1997, Corollary 5.2), instead of the Keimel-Lawson result we have been using so far: on a continuous dcpo L with a second countable Scott topology, for any locally finite map $\mu : \mathcal{O}(L) \to \mathbb{R}^+ \cup \{+\infty\}$ (i.e., $\mu(U) < +\infty$ whenever $U \Subset L$), μ has a unique extension to a Borel measure on the Borel σ-algebra of L iff its is an ω-continuous valuation (where ω-continuous means that $\mu(\bigcup_{n\in\mathbb{N}} U_n) = \sup_{n\in\mathbb{N}} \mu(U_n)$ for all non-decreasing sequences of opens $(U_n)_{n\in\mathbb{N}}$).

Other small differences occur in the definition of the topology of $\mathcal{H}_\mathcal{V}(X)$, which Norberg takes as generated by the sets $\blacklozenge Q$, Q compact saturated in X in his Theorem 6.1.

As in the demonic case, we finish with a look at Choquet integration. The next lemma is an intuitive explanation why picking from a closed set F should be called *angelic* non-determinism: taking the average along $\mathfrak{e}_F$ means getting as high an earning $f(x)$ as you can over all possible choices $x \in F$ (and even a bit more, as this is a sup, not a max).

Lemma 5.14. For every non-empty closed set F of X, for every bounded continuous map $f : X \to \mathbb{R}^+_\sigma$, let $f^*(F) = \sup_{x\in F} f(x)$. Then $\oint_{x\in X} f(x) d\mathfrak{e}_F = f^*(F)$.

Proof. Observe that $\mathfrak{e}_F(f^{-1}(t, +\infty)) = 1$ iff $F \cap f^{-1}(t, +\infty) \neq \emptyset$ iff $\sup_{x\in F} f(x) > t$ iff $t < f^*(F)$, and $\mathfrak{e}_F(f^{-1}(t, +\infty)) = 0$ otherwise. Using the definition (16), $\oint_{x\in X} f(x) d\mathfrak{e}_F = \int_0^{+\infty} \mathfrak{e}_F(f^{-1}(t, +\infty)) dt = \int_0^{f^*(F)} 1 dt = f^*(F)$. □

So integrating along simple plausibilities means taking means of sups: we obtain the equality $\oint_{x\in X} f(x) d \sum_{i=1}^n a_i d\mathfrak{e}_{F_i} = \sum_{i=1}^n a_i \sup_{x\in F_i} f(x)$. This is another way to see that picking x at random along a credibility means picking a non-empty closed subset F_i at random with probability a_i, then picking some element from F_i in an angelic way. This generalizes to all continuous, not just simple plausibilities:

Proposition 5.15. For every continuous valuation P on $\mathcal{H}_\mathcal{V}(X)$, for every bounded continuous map $f : X \to \mathbb{R}^+_\sigma$, $\oint_{x\in X} f(x)dP_{\upharpoonright X} = \oint_{F\in\mathcal{H}_\mathcal{V}(X)} f^*(x)dP$.

Let X be core-compact. For every continuous plausibility ν on X, for every bounded continuous map $f : X \to \mathbb{R}^+_\sigma$, $\oint_{x\in X} f(x)d\nu = \oint_{F\in\mathcal{H}_\mathcal{V}(X)} f^*(x)d\nu_*$.

Proof. Note that ${f^*}^{-1}(t,+\infty) = \{F \mid \sup_{x\in F} f(x) > t\} = \{F \mid F \cap f^{-1}(t,+\infty) \neq \emptyset\} = \Diamond f^{-1}(t,+\infty)$. Then $\oint_{F\in\mathcal{H}_\mathcal{V}(X)} f^*(x)dP = \int_0^{+\infty} P({f^*}^{-1}(t,+\infty))dt$ is equal to $\int_0^{+\infty} P(\Diamond f^{-1}(t,+\infty))dt = \int_0^{+\infty} P_{\upharpoonright X}(f^{-1}(t,+\infty))dt$. The latter is just $\oint_{x\in X} f(x)dP_{\upharpoonright X}$. The second part then follows from Theorem 5.12. □

Using Lemma 5.14, one can also see this as a form of disintegration, viz. $\oint_{x\in X} f(x)d\nu = \oint_{F\in\mathcal{H}_\mathcal{V}(X)} \left(\oint_{x\in Q} f(x)d\mathfrak{e}_F\right) d\nu_*$.

6. The Erratic Case: Sesqui-Continuous Estimates

We now turn to the third powerdomain, the Plotkin powerdomain $\mathcal{P}\ell_\mathcal{V}(X)$, which combines the angelic with the demonic one. If not specified otherwise, X will be an arbitrary topological space and $\mathcal{O}(X)$ the lattice of open subsets. Our goal in this section is to elucidate what kind of game on X would be in a similar correspondence to continuous valuations on $\mathcal{P}\ell_\mathcal{V}(X)$. These will be the *estimates*, see below. However, be warned that estimates won't actually be games, since they won't measure opens, rather *crescents*. A crescent C is the difference $U \setminus V$ of two opens U and V, equivalently, the intersection $C = U \cap F$ of an open set U with a closed subset F; crescents are sometimes called locally closed subsets. In the literature the presentation of crescents as differences of open sets prevails. For our purposes it turns out that the presentation as an intersection of an open with a closed set is preferable for technical reasons. Note that open as well as closed sets are crescents. The intersection of finitely many crescents is again a crescent so that the crescents of X form a $\cap$-semilattice $\mathsf{C}(X)$ of subsets of X.

Crescents are usually mentioned to describe the smallest algebra of sets $\mathcal{A}(X)$ containing the topology of X: the elements of this algebra are the finite, disjoint unions of crescents. The Smiley-Horn-Tarski Theorem, a.k.a., Pettis' Theorem, states that every real valued strict modular function μ on $\mathcal{O}(X)$ extends to a unique additive set function $\mu^\#$ on $\mathcal{A}(X)$. (It is traditional to write simply μ instead of $\mu^\#$, however we prefer to make it clear which is the valuation, and which is the extension.)

We draw the attention of the reader to the fact that, contrarily to $\mu^\#$, estimates will be defined not on finite disjoint unions of crescents, but only on crescents.

Recall that $\mathcal{P}\ell_\mathcal{V}(X)$ is the set of all lenses, where a lens is the intersection $L = Q \cap F$ of a compact saturated set Q with a closed set F, provided the intersection is non-empty.

$$\Box U = \{L \in \mathcal{P}\ell_\mathcal{V}(X) \mid L \subseteq U\} \text{ and } \Diamond V = \{L \in \mathcal{P}\ell_\mathcal{V}(X) \mid L \cap V \neq \emptyset\}$$

are the subbasic open sets of the topology on $\mathcal{P}\ell_\mathcal{V}(X)$, where U and V range over the open subsets of X. We extend these notations to crescents C of X: $\Box C = \{L \in \mathcal{P}\ell_{\mathcal{V}}(X) \mid L \subseteq C\}$ and $\Diamond C = \{L \in \mathcal{P}\ell_{\mathcal{V}}(X) \mid L \cap C \neq \emptyset\}$. As closed sets F are crescents, $\Box F$ and $\Diamond F$ are defined for closed sets, too.

The operators $\Box$ and $\Diamond$ have the following properties:

Lemma 6.1. (1) For any closed or open subset G of X one has

$$\Box G = \mathcal{P}\ell_{\mathcal{V}}(X) \setminus \Diamond(X \setminus G), \quad \Diamond G = \mathcal{P}\ell_{\mathcal{V}}(X) \setminus \Box(X \setminus G).$$

In particular, for closed sets F in X, the sets $\Box F$ and $\Diamond F$ are (subbasic) closed sets in $\mathcal{P}\ell_{\mathcal{V}}(X)$.

(2) For crescents C, C' of X, one has

$$\Box C \cap \Box C' = \Box(C \cap C').$$

In particular, for every crescent $C = U \cap F$ with U open and F closed in X, $\Box C = \Box U \cap \Box F$; hence, $\Box C$ is a crescent in $\mathcal{P}\ell(X)$.

(3) The crescents in $\mathcal{P}\ell_{\mathcal{V}}(X)$ of the form $\Box C$, $C \in \mathsf{C}(X)$, generate the same algebra $\mathcal{A}_0$ of subsets as the subbasic open sets $\Box U$ and $\Diamond V$, $U, V \in \mathcal{O}(X)$.

Proof. (1) For a lens L one has $L \in \Box G$ iff $L \subseteq G$ iff $L \cap (X \setminus G) = \emptyset$ iff $L \notin \Diamond(X \setminus G)$ iff $L \in \mathcal{P}\ell_{\mathcal{V}}(X) \setminus \Diamond(X \setminus G)$. The second claim follows in the same way. (2) is straightforward. (3) The algebra generated by the $\Box U$ and $\Diamond V$ contains all differences $\Box U \setminus \Diamond V = \Box(U \setminus V)$. Conversely, the algebra generated by the $\Box C$, where C ranges over the crescents of X, contains $\Box U$ and $\Box F$ for all all open sets U and all closed sets F in X, as open and closed sets are crescents, hence it also contains $\Diamond V = \mathcal{P}\ell_{\mathcal{V}}(X) \setminus \Box(X \setminus V)$ for open V. ☐

Although crescents C of X are not $\cup$-irreducible in the $\cap$-semilattice $\mathsf{C}(X)$, it is remarkable that the crescents $\Box C$ are $\cup$-irreducible in $\mathcal{P}\ell_{\mathcal{V}}(X)$:

Lemma 6.2. The set $\Box\mathsf{C}(X) = \{\Box C \mid C \in \mathsf{C}(X)\}$ of crescents of $\mathcal{P}\ell_{\mathcal{V}}(X)$ is a $\cup$-irredundant $\cap$-semilattice and $C \mapsto \Box C$ is a $\cap$-semilattice isomorphism from the set $\mathsf{C}(X)$ of all crescents in X to the semilattice $\Box\mathsf{C}(X)$.

Proof. As singletons are lenses, every crescent C of X is the union of the lenses $L \in \Box C$. Thus, if we have two crescents $C \neq C'$, then also $\Box C \neq \Box C'$. Together with Lemma 6.1 (2) this implies that $C \mapsto \Box C$ is a $\cap$-semilattice isomorphism.

We now show that every $\Box C$ is $\cup$-irreducible. Suppose that $C_1, \ldots, C_n$ are crescents in X such that $\Box C_i$ is a proper subsets of $\Box C$ for $i = 1, \ldots, n$. Then C_i is a proper subset of C. Choose $x_i \in C \setminus C_i$. Then the lens $L = \uparrow\{x_1, \ldots, x_n\} \cap cl\{x_1, \ldots, x_n\}$ belongs to $\Box C$, but not to $\Box C_i$ for $i = 1 \ldots, n$, that is, $\Box C_1 \cup \ldots \cup \Box C_n \neq \Box C$. ☐

Lemma 6.2 allows us to apply Theorem 3.3 and Theorem 3.6 and we obtain:

Corollary 6.3. Let X be a topological space. Let $\mathcal{A}_0$ be the algebra of subsets of $\mathcal{P}\ell_{\mathcal{V}}(X)$ generated by the crescents $\Box C$, where C is a crescent of X. Then, for every function $\nu \colon \mathsf{C}(X) \to \mathbb{R}$, there is a unique additive map $\nu^*_* \colon \mathcal{A}_0 \to \mathbb{R}$ with the property that $\nu^*_*(\Box C) = \nu(C)$ for each crescent C of X. The map ν^*_* is monotone if and only if ν is totally convex, that is, if for all crescents $C, C_1, \ldots, C_n$ of X the following inequality holds:

$$\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \nu(C \cap \bigcap_{i \in I} C_i) \quad \geq \quad 0. \tag{17}$$

Thus, a map $\nu\colon \mathsf{C}(X) \to \mathbb{R}$ can be extended to a monotone additive map on the algebra generated by the crescents $\Box C$, $C \in \mathsf{C}(X)$ if and only if it is an estimate according to the following definition;

Definition 6.4 (Estimate). Let X be a topological space. An *estimate* on X is a strict totally convex map ν from the $\cap$-semilattice $\mathsf{C}(X)$ of all crescents of X to $\mathbb{R}^+$.

Let us comment on the relation between estimates on the one side and credibilities and plausibilities on the other side:

Remark 6.1. For every estimate ν on X, define the maps:

$$\begin{aligned} \nu^{\uparrow}(U) &= \nu(U) \\ \nu^{\downarrow}(V) &= \nu(X) - \nu(X \setminus V) \end{aligned}$$

for all opens U, V. Then $\nu^{\uparrow}$ is a credibility on X and $\nu^{\downarrow}$ is a plausibility on X.

Indeed, all open sets U are crescents. Thus, the restriction $\nu^{\uparrow}$ of an estimate ν to the open sets remains totally convex, i.e., a credibility on X. Also the closed sets F are crescents. So the restriction of an estimate to the closed sets remains totally convex. It follows that its conjugate $\nu^{\downarrow}$ is totally concave on the open sets, i.e., a plausibility on X.

It is tempting to guess that an estimate ν should be determined in a unique way by giving just the credibility $\nu^{\uparrow}$ and the plausibility $\nu^{\downarrow}$. However, estimates carry more information than just a credibility and a plausibility together. We shall see this in Proposition 6.19.

In order to prepare the continuous version of Corollary 6.3 we define:

Definition 6.5. We say that a function $\nu\colon \mathsf{C}(X) \to \mathbb{R}$ is *sesqui-continuous* if it is monotone and satisfies

— for every directed family of opens $(U_i)_{i \in I}$ and for every closed F in X,

$$\nu\Big(\bigcup_{i \in I} U_i \cap F\Big) \quad = \quad \sup_{i \in I} \nu(U_i \cap F) \tag{18}$$

— for every open U and for every filtered family of closed $(F_j)_{j \in J}$ in X,

$$\nu\Big(U \cap \bigcap_{j \in J} F_j\Big) \quad = \quad \inf_{j \in J} \nu(U \cap F_j) \tag{19}$$

This definition is justified by the following lemmas and the subsequent proposition:

Lemma 6.6. If P is a continuous valuation on a space X, then its Smiley-Horn-Tarski extension $P^{\#}$ is sesqui-continuous on the crescents of X.

Proof. Let U be open and let $(F_i)_{i\in I}$ be a filtered family of closed sets. Then $(X \setminus F_i)_{i\in I}$ is a directed family of open sets and, by the continuity of P on the opens and the additivity of $P^\#$, we have $P^\#(U \cap \bigcap_{i\in I} F_i) = P(U) - P(U \setminus \bigcap_{i\in I} F_i) = P(U) - P(\bigcup_{i\in I}(U \setminus F_i) = P(U) - \sup_{i\in I} P(U \setminus F_i) = \inf_{i\in I}(P(U) - P(U \setminus F_i)) = \inf_{i\in I} P^\#(U \cap F_i)$. In the particular case $U = X$ we obtain $P^\#(\bigcap_i F_i) = \inf_{i\in I} P^\#(F_i)$.

Now let F be closed and let $(U_i)_{i\in I}$ be a directed family of opens sets. Using the additivity of $P^\#$ again we obtain: $P^\#(\bigcup_{i\in I} U_i \cap F) = P^\#(F) - P^\#(F \setminus \bigcup_{i\in I} U_i) = P^\#(F) - P^\#(\bigcap_{i\in I}(F \setminus U_i))$. As $(F \setminus U_i)_{i\in I}$ is a filtered family of closed sets, the result of the previous paragraph allows to conclude that $P^\#(\bigcap_{i\in I} U_i \cap F) = P^\#(F) - \inf_{i\in I} P^\#(F \setminus U_i) = \sup_{i\in I}(P^\#(F) - P^\#(F \setminus U_i)) = \sup_{i\in I} P^\#(U_i \cap F)$, as desired. □

Lemma 6.7. (1) For every directed family of opens $(U_i)_{i\in I}$ and for every closed set F in X,

$$\Box\Big(\bigcup_{i\in I} U_i \cap F\Big) = \bigcup_{i\in I} \Box(U_i \cap F).$$

(2) For every open set U and every family of closed sets $(F_j)_{j\in J}$ in X,

$$\Box\Big(U \cap \bigcap_{j\in J} F_j\Big) = \bigcap_{j\in J} \Box(U \cap F_j).$$

Proof. (1) $L \in \Box\big(\bigcup_{i\in I} U_i \cap F\big)$ iff $L \subseteq \bigcup_{i\in I} U_i \cap F$, iff $L \subseteq \bigcup_{i\in I} U_i$, and $L \subseteq F$. Since L is compact and the family (U_i) directed, if $L \subseteq \bigcup_{i\in I} U_i$, then $L \subseteq U_i$ for some $i \in I$, so $L \subseteq U_i \cap F$, hence $L \in \Box(U_i \cap F)$. The converse is clear.

(2) $L \in \Box\big(U \cap \bigcap_{i\in J} V_j\big)$ iff $L \subseteq U \cap \bigcap_{i\in J} F_j = \bigcap_{j\in J}(U \cap F_j)$, and this is equivalent to $L \in \bigcap_{j\in J} \Box(U \cap F_j)$. □

Every valuation P on $\mathcal{P}\ell_{\mathcal{V}}(X)$ extends to a monotone additive map $P^\#$ on the algebra $\mathcal{A}(\mathcal{P}\ell_{\mathcal{V}}(X))$ generated by all the open subsets of $\mathcal{P}\ell_{\mathcal{V}}(X)$. Thus $P^\#$ is also monotone and additive on the algebra $\mathcal{A}_0$ generated by the subbasic open sets which is equally generated by the $\Box C$ for each $C \in \mathsf{C}(X)$ by Lemma 6.1 (3). Then the restriction of $P^\#$ to the $\cap$-semilattice $\Box\mathsf{C}(X)$ is strict and totally convex by Corollary 6.3. Thus, for every valuation on $\mathcal{P}\ell_{\mathcal{V}}(X)$ its restriction $P_{\downarrow\uparrow X}(C) = P^\#(\Box C)$ is an estimate on X.

Proposition 6.8. Let X be a topological space, P a valuation on $\mathcal{P}\ell_{\mathcal{V}}(X)$. Then $P_{\downarrow\uparrow X}$ is an estimate. If moreover P is continuous, then $P_{\downarrow\uparrow X}$ is sesqui-continuous.

Proof. Let us show that $P_{\downarrow\uparrow X}$ is sesqui-continuous, assuming P continuous. For every crescent $C = U \cap F$, $\Box C = \Box U \cap \Box F$ by Lemma 6.1 (2), so $P_{\downarrow\uparrow X}(C) = P^\#(\Box C) = P^\#(\Box U \cap \Box F)$. Using that $P^\#$ is sesqui-continuous on the crescents of $\mathcal{P}\ell_{\mathcal{V}}(X)$ by Lemma 6.6 and that the map $C \mapsto \Box C$ from the crescents of X to the crescents of $\mathcal{P}\ell(X)$ has the continuity properties established in Lemma 6.7, we infer that $P_{\downarrow\uparrow X}$ is sesqui-continuous on the crescents of X, too. □

We shall see that, under some conditions on the space X, all sesqui-continuous estimates are in fact obtained this way. Before we do so, we give an example of estimates, the simple estimates.

Definition 6.9 (Unanimity Estimate). Let X be a topological space. For every subset A of X, the *unanimity estimate* on A is the map from $\mathsf{C}(X)$ to $\mathbb{R}^+$ defined by: for every crescent C, $\mathfrak{ue}_A(C) = 1$ if $A \subseteq C$, $\mathfrak{ue}_A(C) = 0$ otherwise.

A *simple estimate* is any map of the form $\sum_{i=1}^n a_i \mathfrak{ue}_{L_i}$, $a_1, \ldots, a_n \in \mathbb{R}^+$, $L_1, \ldots, L_n \in \mathcal{P}\ell_{\mathcal{V}}(X)$.

Lemma 6.10. Every simple estimate is a sesqui-continuous estimate; in fact $\sum_{i=1}^n a_i \mathfrak{ue}_{L_i} = (\sum_{i=1}^n a_i \delta_{L_i})_{\downarrow\uparrow X}$.

Proof. For any crescent C, $(\sum_{i=1}^n a_i \mathfrak{ue}_{L_i})(C) = \sum_{\substack{1 \leq i \leq n \\ L_i \subseteq C}} a_i = \sum_{\substack{1 \leq i \leq n \\ L_i \in \Box C}} a_i$, and this is just $(\sum_{i=1}^n a_i \delta_{L_i})(\Box C)$. The first part of the Lemma now follows from Proposition 6.8. □

Simple estimates are models of mixed probabilistic and erratic non-deterministic choice: pick L_i at random first, then choose erratically, from L_i.

Note that, when $\nu = \sum_{i=1}^n a_i \mathfrak{ue}_{L_i}$ is a simple estimate, $\nu^\uparrow$ is the simple credibility $\sum_{i=1}^n a_i \mathfrak{u}_{(\uparrow L_i)}$, while $\nu^\downarrow$ is the simple plausibility $\sum_{i=1}^n a_i \mathfrak{e}_{cl(L_i)}$. The first claim is obvious, because $L_i \subseteq U$ iff $\uparrow L_i \subseteq U_i$. The second claim is proved as follows: $\nu^\downarrow(V) = \sum_{i=1}^n a_i - \sum_{\substack{1 \leq i \leq n \\ L_i \subseteq X \setminus V}} a_i = \sum_{\substack{1 \leq i \leq n \\ L_i \cap V \neq \emptyset}} a_i$; then since V is open, $L_i \cap V \neq \emptyset$ iff $cl(L_i) \cap V \neq \emptyset$, so that $\nu^\downarrow(V) = (\sum_{i=1}^n a_i \mathfrak{e}_{cl(L_i)})(V)$.

As for Proposition 4.6 and Proposition 5.6, unanimity estimates are models of pure erratic non-determinism.

Proposition 6.11. Let X be either T_2 or stably compact. Any sesqui-continuous estimate ν such that ν only takes values 0 or 1 and with $\nu(X) = 1$ is of the form $\mathfrak{ue}_L$, $L \in \mathcal{P}\ell_{\mathcal{V}}(X)$. In fact, this already holds if $\nu^\uparrow$ is a continuous convex game, $\nu^\downarrow$ is a continuous concave game, $\nu(U \setminus V)$ is monotone in U and antitone in V, and $\nu^\uparrow(U) \leq \nu(U \setminus V) + \nu^\downarrow(V)$ for all opens U, V.

Proof. First, if ν is a sesqui-continuous estimate, then $\nu^\uparrow$ is a continuous credibility, and $\nu^\downarrow$ is a continuous plausibility. In particular, $\nu^\uparrow$ is convex and $\nu^\downarrow$ is concave. Taking $n = 1$ in (17) where $C_1 \subseteq C$ yields $\nu(C) - \nu(C_1) \geq 0$, i.e., ν is monotone in its argument. In particular, $\nu(U \setminus V)$ is monotone in U and antitone in V. Taking $n = 2$, $C = X$, $C_1 = X \setminus V$, $C_2 = U$ in (17) yields $\nu^\downarrow(V) - \nu^\uparrow(U) + \nu(U \setminus V) = \nu(X) - \nu(X \setminus V) - \nu(U) + \nu(U \setminus V) = \nu(C) - \nu(C \cap C_1) - \nu(C \cap C_2) + \nu(C \cap C_1 \cap C_2) \geq 0$.

We now assume only the latter inequality, namely that $\nu^\uparrow(U) \leq \nu(U \setminus V) + \nu^\downarrow(V)$ for all opens U, V, plus the facts that $\nu^\uparrow$ is a continuous convex game, $\nu^\downarrow$ is a continuous concave game, and that $\nu(U \setminus V)$ is monotone in U and antitone in V.

To prove the claim, it is instructive to go through Heckmann's **A**-*valuations* (Heckmann, 1997). Let **A** the dcpo with three elements 0, M, and 1, with ordering $\sqsubseteq$ such that $0 \sqsubseteq \mathsf{M} \sqsubseteq 1$. An **A**-valuation $\alpha : \mathcal{O}(X) \to \mathbf{A}$ is a strict, monotone, continuous map such $\alpha(X) = 1$, whenever $\alpha(U) = 0$ then $\alpha(U \cup V) = \alpha(V)$ for all opens V, and whenever $\alpha(U) = 1$ then $\alpha(U \cap V) = \alpha(V)$ for all opens V. When X is stably compact, $\mathcal{P}\ell_{\mathcal{V}}(X)$ is isomorphic to the space of **A**-valuations with a form of Vietoris topology (Goubault-Larrecq, 2010, Proposition 5.3); in any case for any **A**-valuation α there is a unique lens

L such that $\alpha(U) = 1$ if $L \subseteq U$, $\alpha(U) = 0$ if $L \cap U = \emptyset$, and $\alpha(U) = \mathsf{M}$ otherwise. The same holds when X is T_2 (Heckmann, 1997, Theorem 5.1).

We shall show that the map defined by $\alpha(U) = 1$ if $\nu^\uparrow(U) = 1$, $\alpha(U) = 0$ if $\nu^\downarrow(U) = 0$, and $\alpha(U) = \mathsf{M}$ otherwise, is an **A**-valuation (without any assumption at all on X).

This is well-defined, in particular we cannot have both $\nu^\uparrow(U) = 1$ and $\nu^\downarrow(U) = 0$, otherwise $1 = \nu^\uparrow(U) \leq \nu(U \setminus U) + \nu^\downarrow(U) = 0$. Then, α is monotonic and continuous because $\nu^\uparrow$ and $\nu^\downarrow$ are.

We must show that, if $\alpha(U) = 0$ then $\alpha(U \cup V) = \alpha(V)$ for all opens V. Since $\alpha(U) = 0$, $\nu^\downarrow(U) = 0$. Then $\nu^\uparrow(U \cup V) \leq \nu((U \cup V) \setminus U) + \nu^\downarrow(U) \leq \nu(V \setminus U) + 0 \leq \nu(V \setminus \emptyset) = \nu^\uparrow(V)$. Since $\nu^\uparrow(V) \leq \nu^\uparrow(U \cup V)$ by monotonicity, $\nu^\uparrow(U \cup V) = \nu^\uparrow(V)$. Also, $\nu^\downarrow(U \cup V) \leq \nu^\downarrow(U) + \nu^\downarrow(V) - \nu^\downarrow(U \cap V) \leq \nu^\downarrow(V)$, so $\nu^\downarrow(U \cup V) = \nu^\downarrow(V)$. Since $\nu^\uparrow$ and $\nu^\downarrow$ both agree on $U \cup V$ and V, so does α, i.e., $\alpha(U \cup V) = \alpha(V)$.

We must then show that, if $\alpha(U) = 1$, i.e., if $\nu^\uparrow(U) = 1$, then $\alpha(U \cap V) = \alpha(V)$. Again, it is enough to show $\nu^\uparrow(U \cap V) = \nu^\uparrow(V)$ and $\nu^\downarrow(U \cap V) = \nu^\downarrow(V)$. The first equality is because $\nu^\uparrow(U \cap V) \geq \nu^\uparrow(U) + \nu^\uparrow(V) - \nu^\uparrow(U \cup V) \geq 1 + \nu^\uparrow(V) - 1 = \nu^\uparrow(V)$, the converse inequality being clear. The second equality is because $1 = \nu^\uparrow(U) \leq \nu(U \setminus (U \cap V)) + \nu^\downarrow(U \cap V) = \nu(U \setminus V) + \nu^\downarrow(U \cap V) \leq \nu(X \setminus V) + \nu^\downarrow(U \cap V) = 1 - \nu^\downarrow(V) + \nu^\downarrow(U \cap V)$. □

In order to achieve our goal to show that every sesqui-continuous estimate ν extends to a continuous valuation on $\mathcal{P}\ell_{\mathcal{V}}(X)$, we will require the space X not only to be locally compact as in the demonic and angelic cases, but also to be coherent in the sense that the intersection of any two compact saturated subsets is compact, too. This will allow us to derive the relevant topological properties of $\mathcal{P}\ell_{\mathcal{V}}(X)$.

For compact saturated subsets Q of X we use the notations

$$\blacksquare Q = \{L \in \mathcal{P}\ell_{\mathcal{V}}(X) \mid L \subseteq Q\} \text{ and } \blacklozenge Q = \{L \in \mathcal{P}\ell_{\mathcal{V}}(X) \mid L \cap Q \neq \emptyset\}$$

and we show:

Lemma 6.12. Let X be a coherent space. Then $\blacksquare Q \cap \blacklozenge Q_1 \cap \cdots \cap \blacklozenge Q_n$ is compact in $\mathcal{P}\ell_{\mathcal{V}}(X)$ for all compact saturated sets $Q, Q_1, \ldots, Q_n$ in X.

Proof. Let $\mathcal{P} = \blacksquare Q \cup \bigcup_{i=1}^n \blacklozenge Q_i$. Use Alexander's Subbase Lemma, and show that one can extract a finite subcover from a cover of $\mathcal{P}$ by subbasic opens $\Box U_i$, $i \in I$, and $\Diamond V_j$, $j \in J$. Since this is a cover, $\blacksquare Q \cap \bigcap_{i=1}^n \blacklozenge Q_i \subseteq \bigcup_{i \in I} \Box U_i \cup \bigcup_{j \in J} \Diamond V_j$.

Let $L_0 = Q \setminus \bigcup_{j \in J} V_j$.

If L_0 is empty, then $Q \subseteq \bigcup_{j \in J} V_j$, so $Q \subseteq \bigcup_{j \in J_0} V_j$ for some finite subset J_0 of J. Then $(\Diamond V_j)_{j \in J_0}$ is the desired finite subcover.

Otherwise, L_0 is a lens, $L_0 \in \blacksquare Q$, and $L_0 \notin \bigcup_{j \in J} \Diamond V_j$. So either $L_0 \notin \bigcap_{i=1}^n \blacklozenge Q_i$ or $L_0 \subseteq U_i$ for some $i \in I$.

In the latter case, $Q \subseteq U_i \cup \bigcup_{j \in J} V_j$, so $Q \subseteq U_i \cup \bigcup_{j \in J_0} V_j$ for some finite subset J_0 of J. Then $\Box U_i$ and $(\Diamond V_j)_{j \in J_0}$ form the desired finite subcover.

In the former case, where $L_0 \notin \bigcap_{i=1}^n \blacklozenge Q_i$, there is an i, $1 \leq i \leq n$, such that $L_0 \cap Q_i$ is empty, i.e., $Q \cap Q_i \subseteq \bigcup_{j \in J} V_i$. We now use the fact that X is coherent to conclude that $Q \cap Q_i \subseteq \bigcup_{j \in J_0} V_i$ for some finite subset J_0 of J. Then $(\Diamond V_j)_{j \in J_0}$ is the desired finite subcover. □

Remark 6.2. If X is compact and coherent, then we may choose $Q = X$ in the above lemma and we obtain that $\bigcap_{i=1}^{n} \blacklozenge Q_i$ is compact saturated for any compact saturated subsets $Q_1, \ldots, Q_n$ of X. For this conclusion, the hypothesis of compactness for X cannot be omitted. If we choose $X = \mathbb{R}^-$, the negative reals including 0 with the Scott topology, then $\mathbb{R}^-$ is coherent and even stably locally compact. But the sets $\blacklozenge Q$ are not compact.

Remark 6.3. We do not claim that $\blacksquare Q \cap \blacklozenge Q_1 \cap \cdots \cap \blacklozenge Q_n$ is saturated, and we won't need it. This is compact saturated when X is not just coherent, but stably compact. Indeed, while $\blacksquare Q$ is always saturated, to show that $\blacklozenge Q$ is, we need to show that for any two lenses L, L' such that $L \subseteq cl(L')$, if $L \cap Q \neq \emptyset$ then $L' \cap Q \neq \emptyset$. Indeed, when X is stably compact, $\downarrow L'$ is closed (see e.g., (Goubault-Larrecq, 2010, Fact 4.1)). So $\downarrow L' = cl(L')$. If $L \cap Q \neq \emptyset$, then there is an element x in $L \cap Q$. Since $L \subseteq \downarrow L'$, there is an element x' in L' such that $x \leq x'$; then $x' \in Q$.

A finite union of compact sets of the form $P = \blacksquare Q \cap \blacklozenge Q_1 \cap \cdots \cap \blacklozenge Q_n$, where $Q, Q_1, \ldots Q_n$ are compact saturated in X, is again compact in $\mathcal{P}\ell_{\mathcal{V}}(X)$; we call such finite unions *elementary compacts* and we denote by $\mathcal{Q}_{\text{el}}(\mathcal{P}\ell(X))$ the set of these elementary compacts.

Lemma 6.13. Over a coherent space X, the intersection of two elementary compact sets in $\mathcal{P}\ell_{\mathcal{V}}(X)$ is again an elementary compact set.

Proof. Using distributivity, the intersection of two elementary compact sets in $\mathcal{P}\ell_{\mathcal{V}}(X)$ is a finite union of intersections of two sets of the form $P = \blacksquare Q \cap \blacklozenge Q_1 \cap \cdots \cap \blacklozenge Q_n$ and $P' = \blacksquare Q' \cap \blacklozenge Q'_1 \cap \cdots \cap \blacklozenge Q'_m$. As X is supposed to be coherent, $Q \cap Q'$ is compact and saturated. As $\blacksquare(Q \cap Q') = \blacksquare Q \cap \blacksquare Q'$, the intersection of P and P' is again a compact set according to Lemma 6.12. ☐

For the rest of this section we adopt the following terminology for purely technical reasons. A finite union of basic open sets $\Box U \cap \Diamond U_1 \cap \cdots \cap \Diamond U_n$ will be called an *elementary open*, and a finite union of special basic sets $\Box int(Q) \cap \Diamond int(Q_1) \cap \ldots \Diamond int(Q_n)$, where $Q, Q_1, \ldots, Q_n$ are compact and saturated in X, will be called a *special elementary open.* We denote by $\mathcal{O}_{\text{el}}(\mathcal{P}\ell_{\mathcal{V}}(X))$ the collection of all elementary open sets.

Lemma 6.14. Let X be a locally compact coherent space. Then $\mathcal{P}\ell_{\mathcal{V}}(X)$ is locally compact, too. The lattice $\mathcal{O}(\mathcal{P}\ell_{\mathcal{V}}(X))$ of open subsets is continuous; for two open subsets $\mathcal{U}$ and $\mathcal{V}$ one has $\mathcal{U} \Subset \mathcal{V}$ if and only if there is an elementary compact set $\mathcal{P}$ such that $\mathcal{U} \subseteq \mathcal{P} \subseteq \mathcal{V}$. The elementary open sets form a basis of the continuous lattice $\mathcal{O}(\mathcal{P}\ell_{\mathcal{V}}(X)$.

Proof. We first show that $\mathcal{P}\ell_{\mathcal{V}}(X)$ is locally compact. For this, consider any lens L and any of its basic open neighborhood $\Box U \cap \Diamond U_1 \cap \ldots \cap \Diamond U_n$. Then $L \subseteq U$ and $L \cap U_i \neq \emptyset$ so that there are points $x_i \in L \cap U_i$ for $i = 1, \ldots, n$. As X is locally compact, we may find open sets $V, V_1, \ldots, V_n$ and compact saturated sets $Q, Q_1, \ldots, Q_n$ such that $L \subseteq V \subseteq Q \subseteq U$ and $x_i \in V_i \subseteq Q_i \subseteq U_i$ for each i. Then $L \in \Box U \cap \Diamond V_1 \cap \ldots \cap \Diamond V_n$ and $\Box V \cap \Diamond V_1 \cap \ldots \cap \Diamond V_n \subseteq \blacksquare Q \cap \blacklozenge Q_1 \cap \ldots \cap \blacklozenge Q_n \subseteq \Box U \cap \Diamond U_1 \cap \ldots \cap \Diamond U_n$. As $\blacksquare Q \cap \blacklozenge Q_1 \cap \ldots \cap \blacklozenge Q_n$ is compact by Lemma 6.12, we have found a compact neighborhood of L contained in the given basic neighborhood.

For any locally compact space—hence also for $\mathcal{P}\ell_{\mathcal{V}}(X)$—the lattice of open subsets is continuous and, for open subsets $\mathcal{U}$ and $\mathcal{V}$, one has $\mathcal{U} \Subset \mathcal{V}$ if and only of there is a compact set $\mathcal{K}$ such that $\mathcal{U} \subseteq \mathcal{K} \subseteq \mathcal{V}$. Thus, if there is an elementary compact $\mathcal{P}$ such that $\mathcal{U} \subseteq \mathcal{P} \subseteq \mathcal{V}$, then $\mathcal{U} \Subset \mathcal{V}$. Suppose conversely that $\mathcal{U} \Subset \mathcal{V}$. The open set $\mathcal{V}$ is a union of basic open sets $\Box U \cap \Diamond U_1 \cap \cdots \cap \Diamond U_n$, and each of those is a union of special basic open sets

$$(\dagger) \qquad \Box int(Q) \cap \Diamond int(Q_1) \cap \ldots \cap \Diamond int(Q_n) \text{ with } Q \subseteq U,\ Q_i \subseteq U_i \qquad (i = 1, \ldots, n).$$

Hence, $\mathcal{V}$ is also a union of special basic opens of the form (†). Thus, if $\mathcal{U} \Subset \mathcal{V}$, there are finitely many basic open sets of the form (†) such that $\mathcal{U}$ is contained in their union. This yields a corresponding elementary compact set $\mathcal{P}$ with $\mathcal{U} \subseteq \mathcal{P} \subseteq \mathcal{V}$. At the same time this shows that finite unions of sets of the form (†) yield a basis of the the continuous lattice $\mathcal{O}(\mathcal{P}\ell_{\mathcal{V}}(X))$. □

According to Corollary 6.3 an estimate ν on X has a unique extension to a monotone additive map ν_*^* on the algebra $\mathcal{A}_0$ generated by the crescents $\Box C$, $C \in \mathsf{C}(X)$. As the elementary open sets are contained in this algebra, the restriction of ν_*^* to $\mathcal{O}_{\mathrm{el}}(\mathcal{P}\ell_{\mathcal{V}}(X))$ is a valuation. In order to find a continuous valuation on $\mathcal{P}\ell_{\mathcal{V}}(X)$ extending the valuation ν_*^* we proceed according to the Extension Lemma recorded in the preliminaries using that $\mathcal{O}_{\mathrm{el}}(\mathcal{P}\ell(X))$ is a basis of the continuous lattice $\mathcal{O}(\mathcal{P}\ell_{\mathcal{V}}(X))$:

Lemma 6.15. Let X be locally compact and coherent, and let ν be an estimate on X. For every open subset $\mathcal{V}$ of $\mathcal{P}\ell_{\mathcal{V}}(X)$, let:

$$(\!|\nu|\!)(\mathcal{V}) \quad = \sup_{\substack{\mathcal{U} \in \mathcal{O}_{\mathrm{el}}(\mathcal{P}\ell_{\mathcal{V}}(X)) \\ \mathcal{U} \Subset \mathcal{V}}} \nu_*^*(\mathcal{U})$$

Then $(\!|\nu|\!)$ is a continuous valuation on $\mathcal{P}\ell_{\mathcal{V}}(X)$.

Proof. That $(\!|\nu|\!)$ is continuous, is a consequence of the Extension Lemma. Then, note the following, abbreviating $\mathcal{O}_{\mathrm{el}}(\mathcal{P}\ell_{\mathcal{V}}(X))$ as B. These are well-known as well, but we still provide a proof.

1 For any opens $\mathcal{V}_1$, $\mathcal{V}_2$ of $\mathcal{P}\ell_{\mathcal{V}}(X)$, for all $\mathcal{U} \in B$, $\mathcal{U} \Subset \mathcal{V}_1 \cup \mathcal{V}_2$ iff there are $\mathcal{U}_1, \mathcal{U}_2 \in B$ such that $\mathcal{U} \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$, $\mathcal{U}_1 \Subset \mathcal{V}_1$, and $\mathcal{U}_2 \Subset \mathcal{V}_2$.
Indeed, as $\mathcal{V}_i$ is the union of the directed family of elementary opens $\mathcal{U}_i$ that are relatively compact in $\mathcal{V}_i$ for $i = 1, 2$, the union $\mathcal{V}_1 \cup \mathcal{V}_2$ is the union of the directed family of the $\mathcal{U}_1 \cup \mathcal{U}_2$ where $\mathcal{U}_i$ are elementary opens relatively compact in $\mathcal{V}_i$ $(i = 1, 2)$. If $\mathcal{U} \Subset \mathcal{V}_1 \cup \mathcal{V}_2$ it follows that there are elementary basic sets $\mathcal{U}_1, \mathcal{U}_2$ which are relatively compact in $\mathcal{V}_1, \mathcal{V}_2$, respectively, such that $\mathcal{U} \Subset \mathcal{U}_1 \cap \mathcal{U}_2$. The converse direction is obvious.

2 For every opens $\mathcal{V}_1$, $\mathcal{V}_2$ of $\mathcal{P}\ell_{\mathcal{V}}(X)$, for every $\mathcal{U} \in B$, $\mathcal{U} \Subset \mathcal{V}_1 \cap \mathcal{V}_2$ iff there are $\mathcal{U}_1, \mathcal{U}_2 \in B$ such that $\mathcal{U} \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$, $\mathcal{U}_1 \Subset \mathcal{V}_1$, and $\mathcal{U}_2 \Subset \mathcal{V}_2$.
Indeed, if $\mathcal{U} \Subset \mathcal{V}_1 \cap \mathcal{V}_2$, then take $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U}$. Conversely, if $\mathcal{U} \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$, $\mathcal{U}_1 \Subset \mathcal{V}_1$, and $\mathcal{U}_2 \Subset \mathcal{V}_2$ then, by Lemma 6.14, there are two elementary compact sets $\mathcal{Q}_1$, $\mathcal{Q}_2$ such that $\mathcal{U}_1 \subseteq \mathcal{Q}_1 \subseteq \mathcal{V}_1$ and $\mathcal{U}_2 \subseteq \mathcal{Q}_2 \subseteq \mathcal{V}_2$, whence $\mathcal{U} \subseteq \mathcal{Q}_1 \cap \mathcal{Q}_2 \subseteq \mathcal{V}_1 \cap \mathcal{V}_2$. The intersection $\mathcal{Q}_1 \cap \mathcal{Q}_2$ of two elementary compact sets is elementary compact by Lemma 6.13. Hence $\mathcal{U} \Subset \mathcal{V}_1 \cap \mathcal{V}_2$.

We then compute:

$$\begin{aligned}
(\!|\nu|\!)(\mathcal{V}_1 \cup \mathcal{V}_2) + (\!|\nu|\!)(\mathcal{V}_1 \cap \mathcal{V}_2) &= \sup_{\substack{\mathcal{U} \in B \\ \mathcal{U} \Subset \mathcal{V}_1 \cup \mathcal{V}_2}} \nu_*^*(\mathcal{U}) + \sup_{\substack{\mathcal{U} \in B \\ \mathcal{U} \Subset \mathcal{V}_1 \cap \mathcal{V}_2}} \nu_*^*(\mathcal{U}) \\
&= \sup_{\substack{\mathcal{U}_1, \mathcal{U}_2 \in B \\ \mathcal{U}_1 \Subset \mathcal{V}_1 \\ \mathcal{U}_2 \Subset \mathcal{V}_2}} \nu_*^*(\mathcal{U}_1 \cup \mathcal{U}_2) + \sup_{\substack{\mathcal{U}_1, \mathcal{U}_2 \in B \\ \mathcal{U}_1 \Subset \mathcal{V}_1 \\ \mathcal{U}_2 \Subset \mathcal{V}_2}} \nu_*^*(\mathcal{U}_1 \cap \mathcal{U}_2) \\
&\qquad \text{by Items 1 and 2 above} \\
&= \sup_{\substack{\mathcal{U}_1, \mathcal{U}_2 \in B \\ \mathcal{U}_1 \Subset \mathcal{V}_1 \\ \mathcal{U}_2 \Subset \mathcal{V}_2}} \left(\nu_*^*(\mathcal{U}_1 \cup \mathcal{U}_2) + \nu_*^*(\mathcal{U}_1 \cap \mathcal{U}_2)\right) \\
&= \sup_{\substack{\mathcal{U}_1, \mathcal{U}_2 \in B \\ \mathcal{U}_1 \Subset \mathcal{V}_1 \\ \mathcal{U}_2 \Subset \mathcal{V}_2}} \nu^*(\mathcal{U}_1) + \sup_{\substack{\mathcal{U}_1, \mathcal{U}_2 \in B \\ \mathcal{U}_1 \Subset \mathcal{V}_1 \\ \mathcal{U}_2 \Subset \mathcal{V}_2}} \nu_*^*(\mathcal{U}_2) \\
&= \sup_{\substack{\mathcal{U}_1 \in B \\ \mathcal{U}_1 \Subset \mathcal{V}_1}} \nu_*^*(\mathcal{U}_1) + \sup_{\substack{\mathcal{U}_2 \in B \\ \mathcal{U}_2 \Subset \mathcal{V}_2}} \nu_*^*(\mathcal{U}_2) = (\!|\nu|\!)(\mathcal{V}_1) + (\!|\nu|\!)(\mathcal{V}_2)
\end{aligned}$$

where we use that ν_*^* itself is modular. □

The continuous valuation $(\!|\nu|\!)$ according to Lemma 6.15 can be extended to an additive map $(\!|\nu|\!)^\#$ on the algebra $\mathcal{A}(\mathcal{P}\ell_{\mathcal{V}}(X))$ generated by the open sets. It remains to show:

Lemma 6.16. Let X be locally compact and coherent, and ν be a sesqui-continuous estimate on X. Then, for every crescent C on X, $\nu(C) = (\!|\nu|\!)^\#(\Box C)$.

Proof. Clearly, $(\!|\nu|\!)(\mathcal{U}) \leq \nu_*^*(\mathcal{U})$ for every $\mathcal{U} \in \mathcal{O}_{\text{el}}(\mathcal{P}\ell_{\mathcal{V}}(X))$, since ν_*^* is monotone; this applies in particular when $\mathcal{U}$ is of the form $\Box U$ or $\Box U \cap \Diamond V$, so (a) $(\!|\nu|\!)(\Box U) \leq \nu_*^*(\Box U) = \nu(U)$ (using Corollary 6.3), and (b) $(\!|\nu|\!)(\Box U \cap \Diamond V) \leq \nu_*^*(\Box U \cap \Diamond V) = \nu_*^*(\Box U) - \nu_*^*(\Box(U \setminus V)) = \nu(U) - \nu(U \setminus V)$, using Corollary 6.3 again.

When $\mathcal{U} = \Box U$, where U is open in X, since X is locally compact, U is the sup of the directed family of all opens $U' \Subset U$. However, if $U' \Subset U$ then there is a compact saturated subset Q such that $U' \subseteq Q \subseteq U$, whence $\Box U' \subseteq \blacksquare Q \subseteq \Box U$. Using Lemma 6.12, $\Box U' \Subset \Box U$. Since $\nu_*^*(\Box U') = \nu(U')$ by Corollary 6.3, $(\!|\nu|\!)(\Box U) \geq \nu(U')$ for all $U' \Subset U$. Since ν is sesqui-continuous, and taking sups, $(\!|\nu|\!)(\Box U) \geq \nu(U)$. By (a), $(\!|\nu|\!)(\Box U) = \nu(U)$.

When $\mathcal{U} = \Box U \cap \Diamond V$, U is the sup of the directed family of all opens $U' \Subset U$, and V is the sup of the directed family of all opens $V' \Subset V$. By similar reasoning using Lemma 6.12, $\Box U' \cap \Diamond V' \Subset \Box U \cap \Diamond V$, so $(\!|\nu|\!)(\Box U \cap \Diamond V) \geq \nu_*^*(\Box U' \cap \Diamond V') = \nu(U') - \nu(U' \setminus V')$ (reasoning as in (b)) $\geq \nu(U') - \nu(U \setminus V')$ since ν is monotone). By sesqui-continuity, the sup of the latter quantity over $U' \Subset U$ and $V' \Subset V$ is $\nu(U) - \nu(U \setminus V)$. So, using (b), $(\!|\nu|\!)(\Box U \cap \Diamond V) = \nu(U) - \nu(U \setminus V)$.

Combining this with $(\!|\nu|\!)(\Box U) = \nu(U)$, we obtain $(\!|\nu|\!)(\Box U) - (\!|\nu|\!)(\Box U \cap \Diamond V) = \nu(U \setminus V)$, which is the desired equality. □

Lemma 6.17. Let X be locally compact and coherent, and ν be a sesqui-continuous estimate on X. Let $(\!|\nu|\!)$ be defined as in Lemma 6.15. Then $(\!|\nu|\!)^\#$ coincides with ν_*^* on the algebra generated by the elementary opens of $\mathcal{P}\ell_{\mathcal{V}}(X)$; and $(\!|\nu|\!)$ coincides with ν_*^* on $\mathcal{O}_{\text{el}}(\mathcal{P}\ell_{\mathcal{V}}(X))$.

Proof. $P = (\!|\nu|\!)$ is a continuous valuation on $\mathcal{P}\ell_{\mathcal{V}}(X)$ such that $\nu(C) = P^{\#}(\Box C)$ for every crescent C, by Lemma 6.15 and Lemma 6.16. By Corollary 6.3, $P^{\#}$ coincides with ν_*^* on the smallest algebra containing the elementary opens of $\mathcal{P}\ell_{\mathcal{V}}(X)$. The second part follows immediately from the first. □

Theorem 6.18. Let X be locally compact and coherent. For every sesqui-continuous estimate ν on X, $(\!|\nu|\!)$ is the unique continuous valuation P on $\mathcal{P}\ell_{\mathcal{V}}(X)$ such that $P_{|\uparrow X} = \nu$, i.e., such that, for every crescent C on X, $\nu(C) = P^{\#}(\Box C)$.

This is obtained by:

$$(\!|\nu|\!)(\mathcal{V}) \quad = \quad \sup_{\substack{\mathcal{U} \in \mathcal{O}_{\mathrm{el}}(\mathcal{P}\ell_{\mathcal{V}}(X)) \\ \mathcal{U} \Subset \mathcal{V}}} \nu_*^*(\mathcal{U})$$

where ν_*^* is defined over elementary opens by:

$$\begin{aligned} & \nu_*^*\left(\bigcup_{i=1}^m (\Box U_i \cap \Diamond U_{i1} \cap \ldots \cap \Diamond U_{in_i})\right) \qquad (20) \\ = & \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ I \neq \emptyset}} \sum_{\substack{J_i \subseteq \{1,\ldots,n_i\} \\ \text{for each } i \in I}} (-1)^{|I| + \sum_{i\in I} |J_i| + 1} \nu(\bigcap_{i \in I} U_i \setminus \bigcup_{\substack{i \in I \\ j \in J_i}} U_{ij}) \end{aligned}$$

The map $\nu \mapsto (\!|\nu|\!)$ is one-to-one from the set of sesqui-continuous estimates on X onto the set of continuous valuations on $\mathcal{P}\ell_{\mathcal{V}}(X)$, with inverse the map $P \mapsto P_{|\uparrow X}$.

Proof. The first part of the theorem follows from Lemma 6.17 and the last from Proposition 6.8. Let us check (20). Necessarily,

$$\begin{aligned} \nu_*^*(\Box U \cap \Diamond U_1 \cap \ldots \cap \Diamond U_n) &= \nu_*^*(\Box U \setminus (\Box(X \setminus U_1) \cup \ldots \cup \Box(X \setminus U_n))) \\ &= \nu_*^*(\Box U) - \nu_*^*(\Box(U \setminus U_1) \cup \ldots \cup \Box(U \setminus U_n)) \\ &= \sum_{J \subseteq \{1,\ldots,n\}} (-1)^{|J|} \nu_*^*(\Box(U \setminus \bigcup_{j \in J} U_j)) \\ &= \sum_{J \subseteq \{1,\ldots,n\}} (-1)^{|J|} \nu(U \setminus \bigcup_{j \in J} U_j) \end{aligned}$$

Equation (20) follows by a further application of the inclusion-exclusion principle. □

We may apply the previous theorem to unanimity estimates. Together with Lemma 6.10 it implies that:

$$\sum_{i=1}^n a_i \delta_{L_i} = (\!|\sum_{i=1}^n a_i \mathfrak{u}_{L_i}|\!).$$

We have claimed that estimates ν carried more information than just that given by the credibility $\nu^{\uparrow}$ and the plausibility $\nu^{\downarrow}$. Indeed:

Proposition 6.19. The estimates ν are not determined uniquely from $\nu^{\uparrow}$ and $\nu^{\downarrow}$. More precisely, there is a space X, and two estimates ν and ν' such that $\nu^{\uparrow} = \nu'^{\uparrow}$ and $\nu^{\downarrow} = \nu'^{\downarrow}$, but $\nu \neq \nu'$.

One can even take X finite, and require one to find uncountably many sesqui-continuous estimates ν with given $\nu^{\uparrow}$ and $\nu^{\downarrow}$.

Proof. Let X be the space $\{1 < 2 < \ldots < n\}$. With the Scott topology this is trivially a locally compact coherent space, so any sesqui-continuous estimate ν on X extends to a continuous valuation $(\!|\nu|\!)$ on $\mathcal{P}\ell_{\mathcal{V}}(X)$ by Theorem 6.18. Since $\mathcal{P}\ell_{\mathcal{V}}(X)$ is finite, $(\!|\nu|\!)$ must be a simple valuation, hence ν must be a simple estimate.

X has $n+1$ opens: the intervals $[i,n]$ with $1 \leq i \leq n$, and the empty set. X has $n(n+1)/2$ lenses: the intervals $[i,j]$ with $1 \leq i \leq j \leq n$. So ν can be written as $\sum_{1 \leq i \leq j \leq n} a_{ij} \mathfrak{u}_{[i,j]}$. Let us consider each a_{ij} as an unknown. To give oneself $\nu^{\uparrow}$ means giving oneself the n sums $\sum_{\substack{1 \leq i \leq j \leq n \\ [i,j] \subseteq U}} a_{ij}$, one for each non-empty open U of X. (The case of the empty open is trivial, since $\nu(\emptyset) = 0$.) Similarly, giving oneself $\nu^{\downarrow}$ means giving oneself the values of the n sums $\sum_{\substack{1 \leq i \leq j \leq n \\ [i,j] \cap V \neq \emptyset}} a_{ij}$, one for each non-empty open V of X. This is only $2n$ equations for $n(n+1)/2$ unknowns. When $n \geq 4$, there are more unknowns than equations, from which we shall see that the value of a_{ij} will not be unique.

Here is an explicit counter-example, found by solving the above constraints with $n = 4$. Fix an estimate $\nu_0 = \sum_{1 \leq i \leq j \leq 4} \frac{1}{10} \mathfrak{u}_{[i,j]}$. For all reals a, b such that $0 \leq a, b \leq 2/10$ and $-1/10 \leq b - a \leq 1/10$, let:

$$
\begin{array}{rcllll}
\nu_{a,b} & = & \frac{1}{10}\mathfrak{u}_{[1,1]} & + \frac{1}{10}\mathfrak{u}_{[1,2]} & + a\mathfrak{u}_{[1,3]} & + (\frac{2}{10} - a)\mathfrak{u}_{[1,4]} \\
 & & & + \frac{1}{10}\mathfrak{u}_{[2,2]} & + (\frac{2}{10} - b)\mathfrak{u}_{[2,3]} & + b\,\mathfrak{u}_{[2,4]} \\
 & & & & + (\frac{1}{10} + b - a)\mathfrak{u}_{[3,3]} & + (\frac{1}{10} - b + a)\mathfrak{u}_{[3,4]} \\
 & & & & & + \frac{1}{10}\mathfrak{u}_{[3,4]}
\end{array}
$$

One checks that:

$$
\begin{array}{rcll}
\nu^{\uparrow}_{a,b}([1,4]) & = & 1 & \\
\nu^{\uparrow}_{a,b}([2,4]) & = & \frac{6}{10} & \text{(sum of all coefficients on all rows except the first)} \\
\nu^{\uparrow}_{a,b}([3,4]) & = & \frac{3}{10} & \text{(all rows except the first and the second one)} \\
\nu^{\uparrow}_{a,b}([4,4]) & = & \frac{1}{10} & \\
\nu^{\uparrow}_{a,b}(\emptyset) & = & 0 &
\end{array}
$$

and also that:

$$
\begin{array}{rcll}
\nu^{\downarrow}_{a,b}([1,4]) & = & 1 & \\
\nu^{\downarrow}_{a,b}([2,4]) & = & \frac{9}{10} & \text{(sum of all coefficients on all columns except the first)} \\
\nu^{\downarrow}_{a,b}([3,4]) & = & \frac{7}{10} & \text{(all columns except the first and the second one)} \\
\nu^{\downarrow}_{a,b}([4,4]) & = & \frac{4}{10} & \\
\nu^{\downarrow}_{a,b}(\emptyset) & = & 0 &
\end{array}
$$

So $\nu^{\uparrow}_{a,b}$ are $\nu^{\downarrow}_{a,b}$ are independent of a and b, provided $0 \leq a, b \leq 2/10$ and $-1/10 \leq b - a \leq 1/10$. □

7. Conclusion

One aspect of this work is naturally that we have extended Choquet-Kendall-Matheron theorems to yet another level of generality. Dealing with non-Hausdorff spaces is needed if we ever want to apply these results to situations like those encountered in domain theory, where no dcpo of interest is Hausdorff. However we believe that the most important aspect of this work is the precise connection between credibilities, resp. plausibilities, and models of mixed probabilistic choice and non-deterministic choice, where the role of the Smyth, resp. the Hoare powerdomains, appear clearly. A nice intuition, obtained using Choquet integration along credibilities, resp. plausibilities, is that demonic choice minimizes your earnings while angelic choice maximizes them.

The erratic case is stranger, and we had to invent the new notion of sesqui-continuous estimate. Our last result shows that we cannot reduce the description of an estimate ν to the description of a pair of a credibility $\nu^{\uparrow}$ and a plausibility $\nu^{\downarrow}$, contrarily to what happens with forks (Goubault-Larrecq, 2007b), another model for mixed probabilistic choice and erratic non-determinism, which are just pairs of a continuous lower prevision and a continuous upper prevision satisfying some conditions. (The latter two are models of mixed probabilistic choice and demonic, resp. angelic choice. In both kinds of previsions, and in forks, as well as in (Mislove, 2000; Tix et al., 2005), one can see the non-deterministic choices to be done first, then the probabilistic choices. This is opposite to the models considered here.)

Another question we would like to answer is: is there a meaningful notion of integration along estimates ν that would require the extra information in ν that one cannot get from $\nu^{\uparrow}$ and $\nu^{\downarrow}$ alone? We shall explore this in another paper.

Acknowledgments

The main result of Section 4 was hidden in an appendix of an initial version of an earlier paper by the first author (Goubault-Larrecq, 2010), with longer and more complex proofs. It was suggested that it should be given more visibility, and deserved a paper of its own. We thank Achim Jung for giving us the opportunity to do so. We thank the anonymous referee for suggesting the actual simplified version of the proof of Proposition 4.11.

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