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A Short Proof of the Schröder-Simpson Theorem

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We give a short and elementary proof of the Schröder-Simpson Theorem, viz., that the space of all continuous maps from a given space X, to the non-negative reals with their Scott topology, is the cone-theoretic dual of the probabilistic powerdomain on X.

1. The Schröder-Simpson Theorem

A *continuous valuation* on a topological space X is a map ν from the lattice $\mathcal{O}(X)$ of open subsets of X to $\overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{+\infty\}$ such that $\nu(\emptyset) = 0$, $\nu(U) \leq \nu(V)$ whenever $U \subseteq V$, $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$ for all U, V, and $\nu(\bigcup_{i \in I} U_i) = \sup_{i \in I} \nu(U_i)$ for every directed family of opens $(U_i)_{i \in I}$.

Let $\mathbf{V}_{wk}(X)$ denote the space of all continuous valuations on a topological space X, with the *weak topology*, whose subbasic open subsets are $[h > r] = \{\nu \mid \int_{x \in X} h(x) d\nu > r\}$, where h ranges over the continuous maps from X to $\overline{\mathbb{R}}^+_\sigma$, and $r \in \mathbb{R}^+$, $r > 0$; $\overline{\mathbb{R}}^+_\sigma$ is $\overline{\mathbb{R}}^+$ with the Scott topology of its natural ordering. We define integration by the Choquet formula $\int_{x \in X} h(x) d\nu = \int_0^{+\infty} \nu(h^{-1}(t, +\infty]) dt$; the right-hand side is a Riemann integral.

A Riesz-like representation theorem (Kirch, 1993; Tix, 1995) states that $\mathbf{V}_{wk}(X)$ is isomorphic, as a cone, to the space of all continuous linear maps from $[X \to \overline{\mathbb{R}}^+_\sigma]_\sigma$ to $\overline{\mathbb{R}}^+_\sigma$. The isomorphism maps ν to the continuous linear map $h \mapsto \int_{x \in X} h(x) d\nu$. Here, $[X \to \overline{\mathbb{R}}^+_\sigma]$ is the space of all continuous maps from X to $\overline{\mathbb{R}}^+_\sigma$, and $[X \to \overline{\mathbb{R}}^+_\sigma]_\sigma$ is this space with the Scott topology of the pointwise ordering. A map is *linear* iff it preserves sums and products by non-negative reals.

Schröder and Simpson (Schröder and Simpson, 2005) found another representation theorem: the continuous linear maps ψ from $\mathbf{V}_{wk}(X)$ to $\overline{\mathbb{R}}^+$ are exactly those of the form $\nu \in \mathbf{V}_{wk}(X) \mapsto \int_{x \in X} h(x) d\nu$, for some unique $h \in [X \to \overline{\mathbb{R}}^+_\sigma]$. This answered a question by (Heckmann, 1996). Their proof is very technical. Keimel's proof (Keimel, 2012) is more conceptual and leads to more general results. Our argument is elementary. To make a comparison, Schröder and Simpson start with the simple observation that if

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h exists, then h must be the map $x \in X \mapsto \psi(\delta_x)$, where δ_x is the Dirac mass at x. Checking that this map answers the question, however, proves to be difficult. Instead, we build h from the shape of the weak open $\psi^{-1}(1, +\infty]$, using two simple lemmas.

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2. The Proof

The key is Keimel's Lemma 5.5 (Keimel, 2012), which is perhaps (half of) the fundamental reason why the Schröder-Simpson Theorem holds. We state it in a slightly more general form; Keimel dealt with the case of only two maps h_i, and they were simple, i.e., of the form $\sum_{i=1}^n a_i \chi_{U_i}$ for some $a_i \in \mathbb{R}^+$ and opens U_i. (χ_U is the characteristic map of U.) Let $\mathbf{V}_b(X)$ denote the space of all *bounded* continuous valuations ν, i.e., such that $\nu(X) < +\infty$, with the subspace topology, which we again call the weak topology.

Lemma 2.1. Let ψ be a continuous linear map from $\mathbf{V}_b(X)$ to $\overline{\mathbb{R}}^+_\sigma$, and $(h_i)_{i \in I}$ be a family of continuous maps from X to $\overline{\mathbb{R}}^+_\sigma$. The following are equivalent:

1 $\psi(\nu) \geq \sup_{i \in I} \int_{x \in X} h_i(x) d\nu$ for every $\nu \in \mathbf{V}_b(X)$;
2 $\psi(\nu) \geq \int_{x \in X} \sup_{i \in I} h_i(x) d\nu$ for every $\nu \in \mathbf{V}_b(X)$.

Proof. The $2 \Rightarrow 1$ implication is clear. Conversely, assume 1. We start with Keimel's case, where $I = \{1, 2\}$, and h_1, h_2 are simple. Keimel argues as follows.

Write h_1 as $\sum_{i=1}^m a_i \chi_{U_i}$, h_2 as $\sum_{j=1}^n b_j \chi_{V_j}$, where $a_i, b_j \in \mathbb{R}^+$, U_i, V_j are open. Let C_k, $1 \leq k \leq p$, be an enumeration of the atoms of the Boolean algebra of subsets of X generated by the sets U_i and V_j, i.e., the non-empty subsets of the form $\bigcap_{i=1}^m \pm_i U_i \cap \bigcap_{j=1}^n \pm_{m+j} V_j$, where each sign $\pm_i$ is $+$ or $-$, and $+A = A$, $-A = X \smallsetminus A$. One can then write h_1 as $\sum_{k=1}^p a'_k \chi_{C_k}$ and h_2 as $\sum_{k=1}^p b'_k \chi_{C_k}$, with $a'_k, b'_k \in \mathbb{R}^+$.

Each C_k is a *crescent*, i.e., the difference $A \smallsetminus B$ of two opens A and B. The Smiley-Horn-Tarski (a.k.a., Pettis) Theorem states that each $\nu \in \mathbf{V}_b(X)$ extends to a unique additive function on the finite disjoint unions of crescents, defined on crescents by $\nu(A \smallsetminus B) = \nu(A \cup B) - \nu(B) = \nu(A) - \nu(A \cap B)$; additivity means that the ν value of a finite disjoint union of crescents is the sum of the ν values of each crescent.

Define the *restriction* $\nu_{|C}$ of ν to C by $\nu_{|C}(U) = \nu(C \cap U)$ for every $U \in \mathcal{O}(X)$. Writing $C = A \smallsetminus B$ with A, B open, $\nu_{|C}(U) = \nu((A \cap U) \cup B) - \nu(B)$, from which $\nu_{|C}$ is in $\mathbf{V}_b(X)$. Moreover, $\nu_{|C}(C) = \nu(C)$, and $\nu_{|C}(C') = 0$ for every crescent C' disjoint from C.

So $\int_{x \in X} h_1(x) d\nu_{|C_k} = \sum_{k'=1}^p a'_{k'} \nu_{|C_k}(C_{k'}) = a'_k$. Similarly $\int_{x \in X} h_2(x) d\nu_{|C_k} = b'_k$. Similarly again, $\int_{x \in X} \sup(h_1(x), h_2(x)) d\nu_{|C_k} = \sup(a'_k, b'_k)$. By assumption, $\psi(\nu_{|C_k}) \geq \sup(\int_{x \in X} h_1(x) d\nu_{|C_k}, \int_{x \in X} h_1(x) d\nu_{|C_k}) = \sup(a'_k, b'_k) = \int_{x \in X} \sup(h_1(x), h_2(x)) d\nu_{|C_k}$. As the disjoint union of C_k, $1 \leq k \leq n$, is the whole of X, $\nu = \sum_{k=1}^p \nu_{|C_k}$, so $\psi(\nu) = \sum_{k=1}^p \psi(\nu_{|C_k}) \geq \sum_{k=1}^p \int_{x \in X} \sup(h_1(x), h_2(x)) d\nu_{|C_k} = \int_{x \in X} \sup(h_1(x), h_2(x)) d\nu$.

We now prove $1 \Rightarrow 2$ in the case where $I = \{1, 2\}$, without assuming h_1 or h_2 simple. Every continuous map $h \colon X \to \overline{\mathbb{R}}^+_\sigma$ is the pointwise supremum of the countable chain

$(h_K)_{K\in\mathbb{N}}$ where $h_K = \frac{1}{2^K}\sum_{i=1}^{K2^K} \chi_{h^{-1}(i/2^K,+\infty]}$ is simple. Moreover, $\sup(h_{1K}, h_{2K}) = \sup(h_1, h_2)_K$. If $\psi(\nu) \geq \sup_{i\in\{1,2\}} \int_{x\in X} h_i(x)d\nu$ for every $\nu \in \mathbf{V}_b(X)$, then for every $K \in \mathbb{N}$, $\psi(\nu) \geq \sup_{i\in\{1,2\}} \int_{x\in X} h_{iK}(x)d\nu$ for every $\nu \in \mathbf{V}_b(X)$, and by Keimel's argument above, $\psi(\nu) \geq \int_{x\in X} \sup(h_{1K}(x), h_{2K}(x))d\nu = \int_{x\in X} \sup(h_1, h_2)_K(x)d\nu$. Taking sups over $K \in \mathbb{N}$, $\psi(\nu) \geq \int_{x\in X} \sup(h_1(x), h_2(x))d\nu$ for every $\nu \in \mathbf{V}_b(X)$.

The implication $1 \Rightarrow 2$ now follows easily when I is finite. For general I, assumption 1 implies $\psi(\nu) \geq \sup_{i\in J} \int_{x\in X} h_i(x)d\nu$ for every $\nu \in \mathbf{V}_b(X)$ and for every finite subset J of I. We have seen that this implied $\psi(\nu) \geq \int_{x\in X} \sup_{i\in J} h_i(x)d\nu$ for every $\nu \in \mathbf{V}_b(X)$. Taking (directed) sups over J, and since integration is Scott-continuous in the integrated function, $\psi(\nu) \geq \int_{x\in X} \sup_{i\in I} h_i(x)d\nu$ for every $\nu \in \mathbf{V}_b(X)$. □

Let us recall Keimel's Separation Theorem (Keimel, 2006, Theorem 9.1): let C be a topological cone, U be an open convex subset, and A be a non-empty convex subset of C disjoint from U; then there is a continuous linear map $\Lambda\colon C \to \overline{\mathbb{R}}^+_\sigma$ such that $\Lambda(a) \leq 1$ for every $a \in A$ and $\Lambda(u) > 1$ for every $u \in U$. A cone is a commutative monoid $(C, +)$ with a multiplicative action (scalar product) of $(\mathbb{R}^+, +, \times)$. A topological cone additionally has a topology such that both addition $+$ and scalar product are (jointly) continuous. A subset A of C is convex if and only if for all $\alpha \in [0, 1]$, $a, a' \in A$, $\alpha a + (1-\alpha)a'$ is in A. Keimel's Separation Theorem is a consequence of a sandwich theorem by Roth (Roth, 2000), which relies on the Axiom of Choice. The special case where we use it ($C = \overline{\mathbb{R}}^{+n}$) is very likely to be provable without the Axiom of Choice, though.

Lemma 2.2. Let ψ be a continuous linear map from $\mathbf{V}_b(X)$ to $\overline{\mathbb{R}}^+_\sigma$, and $h_1, \ldots, h_n$ be finitely many continuous maps from X to $\overline{\mathbb{R}}^+_\sigma$. If $\bigcap_{i=1}^n [h_i > 1] \subseteq \psi^{-1}(1, +\infty]$, then there is a continuous map $h\colon \to \overline{\mathbb{R}}^+_\sigma$ such that $\bigcap_{i=1}^n [h_i > 1] \subseteq [h > 1] \subseteq \psi^{-1}(1, +\infty]$.

Proof. Define $F\colon \mathbf{V}_b(X) \to \overline{\mathbb{R}}^{+n}_\sigma$ by $F(\nu) = \left(\int_{x\in X} h_i(x)d\nu\right)_{1\leq i\leq n}$. The image of $\bigcap_{i=1}^n [h_i > 1]$ by F is included in the open convex subset $U = (1, +\infty]^n$ of $\overline{\mathbb{R}}^{+n}_\sigma$. One easily checks that the image A of the complement of $\psi^{-1}(1, +\infty]$ by F is also convex, since ψ is linear. A is non-empty, since $F(0)$ is in A. We claim that A and U are disjoint: every element $(a_i)_{1\leq i\leq n}$ of A is of the form $F(\nu)$ with $\nu \notin \psi^{-1}(1, +\infty]$, hence $\nu \notin \bigcap_{i=1}^n [h_i > 1]$ by assumption; so $a_i = \int_{x\in X} h_i(x)d\nu \leq 1$ for some i, which implies $(a_i)_{1\leq i\leq n} \notin U$. By Keimel's Separation Theorem, there is a continuous linear map $\Lambda\colon C \to \overline{\mathbb{R}}^+_\sigma$ such that $\Lambda(a) \leq 1$ for every $a \in A$ and $\Lambda(u) > 1$ for every $u \in U$. Let $\gamma_i = \Lambda(\vec{e}_i)$ where $\vec{e}_i$ is a tuple of zeroes, except for a 1 at position i. If we agree that $(+\infty).0 = 0.(+\infty) = 0$, then $\Lambda(\vec{c}) = \sum_i \gamma_i c_i$ for every point $\vec{c} = (c_i)_{1\leq i\leq n}$ of $\overline{\mathbb{R}}^{+n}_\sigma$. (Continuity of Λ is essential here.) Letting $h = \sum_{i=1}^n \gamma_i h_i$, we obtain $(\Lambda \circ F)(\nu) = \sum_{i=1}^n \gamma_i \int_{x\in X} h_i(x)d\nu = \int_{x\in X} h(x)d\nu$ for every $\nu \in \mathbf{V}_b(X)$. Every element ν of $\bigcap_{i=1}^n [h_i > 1]$ is such that $F(\nu)$ is in U, so $(\Lambda \circ F)(\nu) > 1$, that is, ν is in $[h > 1]$. Every element ν outside $\psi^{-1}(1, +\infty]$ is such that $F(\nu)$ is in A, so $(\Lambda \circ F)(\nu) \leq 1$, i.e., ν is not in $[h > 1]$. This proves the other inclusion $[h > 1] \subseteq \psi^{-1}(1, +\infty]$. □

An equivalent statement, which we will not use but is more in the style of Lemma 2.1, is the following. Given ψ, $h_1, \ldots, h_n$ as above, if $\psi(\nu) \geq \min_{i=1}^n \int_{x\in X} h_i(x)d\nu$ for every

$\nu \in \mathbf{V}_b(X)$, then there is a continuous map $h\colon X \to \overline{\mathbb{R}}^+_\sigma$ such that $\psi(\nu) \geq \int_{x\in X} h(x)d\nu \geq \min_{i=1}^n \int_{x\in X} h_i(x)d\nu$ for every $\nu \in \mathbf{V}_b(X)$.

Theorem 2.3 (Schröder-Simpson). Let X be a topological space, and ψ be a continuous linear map from $\mathbf{V}_b(X)$ to $\overline{\mathbb{R}}^+_\sigma$. Then there is a unique continuous map $h\colon X \to \overline{\mathbb{R}}^+_\sigma$ such that $\psi(\nu) = \int_{x\in X} h(x)d\nu$ for every $\nu \in \mathbf{V}_b(X)$, and $h(x) = \psi(\delta_x)$ for every $x \in X$.

Proof. Since ψ is continuous, $\psi^{-1}(1, +\infty]$ is open, hence of the form $\bigcup_{i\in I} \bigcap_{j\in J_i} [h_{ij} > r_{ij}]$, where each J_i is finite, h_{ij} is continuous from X to $\overline{\mathbb{R}}^+_\sigma$, and $r_{ij} \in \mathbb{R}^+$, $r_{ij} > 0$. Without loss of generality, $r_{ij} = 1$, since $[h_{ij} > r_{ij}] = [1/r_{ij}.h_{ij} > 1]$.

For each $i \in I$, Lemma 2.2 gives us a continuous map $h_i\colon X \to \mathbb{R}^+$ such that $\bigcap_{j\in J_i}[h_{ij} > 1] \subseteq [h_i > 1] \subseteq \psi^{-1}(1, +\infty]$. So $\psi^{-1}(1, +\infty] = \bigcup_{i\in I} \bigcap_{j\in J_i}[h_{ij} > 1] \subseteq \bigcup_{i\in I}[h_i > 1] \subseteq \psi^{-1}(1, +\infty]$, whence $\psi^{-1}(1, +\infty] = \bigcup_{i\in I}[h_i > 1]$. Let $h = \sup_{i\in I} h_i$. By Lemma 2.1, $\psi(\nu) \geq \int_{x\in X} h(x)d\nu$ for every $\nu \in \mathbf{V}_b(X)$. If the inequality were strict for some ν, then $\psi(\nu) > t > \int_{x\in X} h(x)d\nu$ for some $\nu \in \mathbf{V}_b(X)$ and some $t > 0$. Replacing ν by $1/t.\nu$ if necessary, we may assume $t = 1$. Then $\nu \in \psi^{-1}(1, +\infty] = \bigcup_{i\in I}[h_i > 1]$, so $\nu \in [h_i > 1]$ for some $i \in I$. But then $t = 1 > \int_{x\in X} h(x)d\nu \geq \int_{x\in X} h_i(x)d\nu > 1$, contradiction.

□

Corollary 2.4. Let X be a topological space, and ψ be a continuous linear map from $\mathbf{V}_{wk}(X)$ to $\overline{\mathbb{R}}^+_\sigma$. Then there is a unique continuous map $h\colon X \to \overline{\mathbb{R}}^+_\sigma$ such that $\psi(\nu) = \int_{x\in X} h(x)d\nu$ for every $\nu \in \mathbf{V}_{wk}(X)$, and $h(x) = \psi(\delta_x)$ for every $x \in X$.

Proof. Every continuous valuation ν is the supremum of a directed family of bounded continuous valuations ν_i, $i \in I$ (Heckmann, 1996, Theorem 4.2). Let $h(x) = \psi(\delta_x)$. By Theorem 2.3, and since integration is Scott-continuous in the valuation, $\psi(\nu) = \sup_{i\in I} \psi(\nu_i) = \sup_{i\in I} \int_{x\in X} h(x)d\nu_i = \int_{x\in X} h(x)d\nu$. □

We also mention the following corollary to Theorem 2.3, also due to Schröder and Simpson (Schröder and Simpson, 2005). The argument below is theirs, too. Let $\mathbf{V}_{\leq 1\ wk}(X)$ denote the space of all subprobability valuations, i.e., of all continuous valuations ν on X such that $\nu(X) \leq 1$, with the subspace (a.k.a, *weak*) topology from $\mathbf{V}_b(X)$. Call $\psi\colon \mathbf{V}_{\leq 1\ wk}(X) \to \overline{\mathbb{R}}^+_\sigma$ *linear* in this case iff $\psi(a\nu + b\nu') = a\psi(\nu) + b\psi(\nu')$ for all $\nu, \nu' \in \mathbf{V}_{\leq 1\ wk}(X)$ and $a, b \in [0, 1]$.

Corollary 2.5. Let X be a topological space, and ψ be a continuous linear map from $\mathbf{V}_{\leq 1\ wk}(X)$ to $\overline{\mathbb{R}}^+_\sigma$. Then there is a unique continuous map $h\colon X \to \overline{\mathbb{R}}^+_\sigma$ such that $\psi(\nu) = \int_{x\in X} h(x)d\nu$ for every $\nu \in \mathbf{V}_{\leq 1\ wk}(X)$, and $h(x) = \psi(\delta_x)$ for every $x \in X$.

Proof. Define $\psi'\colon \mathbf{V}_b(X) \to \overline{\mathbb{R}}^+_\sigma$ by $\psi'(\nu) = a\psi(1/a.\nu)$, for any arbitrary $a > 0$ such that $\nu(X) \leq a$. This is well-defined by linearity, and is linear in the usual sense. Using the equivalence between $1/a.\nu \in [h > r]$ and $\nu \in [h > ar]$, one easily shows that ψ' is continuous. We finally apply Theorem 2.3 to ψ'. □

We finish with a remark on dual cones. Given a topological cone C, let its *dual cone* C^* be the set of all linear continuous maps from C to $\overline{\mathbb{R}}^+_\sigma$. This has an obvious structure of cone, and we topologize it with the topology induced by the inclusion of C^* into ${\overline{\mathbb{R}}^+_\sigma}^C$ with the product topology; this is variously known as the weak topology, or as the topology of pointwise convergence, and a subbasis of opens is given by the sets $[c > r]^*$ defined as $\{\psi \in C^* \mid \psi(c) > r\}$, $c \in C$, $r \in \mathbb{R}^+$. The Schröder-Simpson Theorem, as stated above, implies that $\mathbf{V}_{wk}(X)^*$ is isomorphic to $[X \to \overline{\mathbb{R}}^+_\sigma]$ as a cone, by the map $h \in [X \to \overline{\mathbb{R}}^+_\sigma] \mapsto (\nu \in \mathbf{V}_{wk}(X) \mapsto \int_{x \in X} h(x) d\nu)$, with inverse $\psi \in \mathbf{V}_{wk}(X)^* \mapsto (x \in X \mapsto \psi(\delta_x))$. This becomes an isomorphism of *topological* cones once we equip $C = [X \to \overline{\mathbb{R}}^+_\sigma]$ with the coarsest topology containing the sets $[\nu > r]^* = \{h \in C \mid \int_{x \in X} h(x) d\nu > r\}$ (the confusion of notation with $[\nu > r]^* = \{\psi \in C^* \mid \psi(\nu) > r\}$ is vindicated by the isomorphism); it is fair to call this the *weak** topology on $C = [X \to \overline{\mathbb{R}}^+_\sigma]$, and we write $[X \to \overline{\mathbb{R}}^+_\sigma]_{wk*}$ for the resulting topological cone.

By the Riesz-like representation theorem mentioned in the introduction, $[X \to \overline{\mathbb{R}}^+_\sigma]^*_\sigma$ is isomorphic to $\mathbf{V}_{wk}(X)$ as a cone. The very definition of the weak* topology (and of the isomorphism) shows that the isomorphism extends to an isomorphism of *topological* cones between $[X \to \overline{\mathbb{R}}^+_\sigma]^*_{wk*}$ and $\mathbf{V}_{wk}(X)$.

Call a topological cone *reflexive* if and only if the function that maps each $c \in C$ to $\widehat{c} = (h \in C^* \mapsto h(c)) \in C^{**}$ is an isomorphism of topological cones. We sum up this discussion as follows.

Theorem 2.6. For every topological space X, the topological cones $\mathbf{V}_{wk}(X)$ and $[X \to \overline{\mathbb{R}}^+_\sigma]_{wk*}$ are reflexive, and dual to each other (up to isomorphisms of topological cones).

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