Exponentiable Streams and Prestreams

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Abstract

Inspired by a construction of Escardó, Lawson, and Simpson, we give a general construction of $\mathcal{C}$-generated objects in a topological construct. When $\mathcal{C}$ consists of exponentiable objects, the resulting category is Cartesian-closed. This generalizes the familiar construction of compactly-generated spaces, and we apply this to Krishnan's categories of streams and prestreams, as well as to Haucourt streams. For that, we need to identify the exponentiable objects in these categories: for prestreams, we show that these are the preordered core-compact topological spaces, and for streams, these are the core-compact streams.

1 Introduction

Streams and prestreams were introduced by Krishnan [Kri09] as a foundation for *directed* algebraic topology, where topological spaces are equipped with a local notion of direction, typically of time. There are several competing proposals, see [Hau09a], and the references given therein. Streams are one of the most practical. Krishnan (op.cit.) shows that streams for a complete and cocomplete category. He also identifies a Cartesian-closed subcategory of so-called *compactly flowing* streams, modeled after the Cartesian-closed category of compactly generated weak Hausdorff spaces.

Our objective is to show that there are many Cartesian-closed subcategories, both of prestreams and of streams. The main construction is a more or less direct categorical generalization of a topological construction by Escardó, Lawson, and Simpson [ELS04], which we describe in Section 3, right after having recapitulated a few notions that we need in Section 2. Our initial motivation was to find Cartesian-closed subcategories of *prestreams*, instead of Krishnan's streams. Indeed, prestreams are simpler objects, with a clearer definition. As a general element of style, and in the name of clarity, one of our aims was to make all constructions as concrete as we could, and while this seems to be contradicted by the abstract style of Section 3, in the remaining sections we insist on giving explicit formulae for limits, colimits, and other notions. This is needed anyway later.

The construction of Section 3 is parameterized by a class of exponentiable objects (of prestreams, or of streams, in our case), and it is therefore interesting to characterize exponentiable objects in each of the relevant categories. This is what we do for prestreams in Section 5, and for streams in Section 6. We also study the restricted, intuitive class of streams introduced by Haucourt [Hau12], because of their intuitive appeal. In each case, we obtain Cartesian-closed categories: core-compactly generated streams, core-compactly generated Haucourt streams, core-compactly generated prestreams, compactly generated streams, compactly generated Haucourt streams, compactly generated prestreams, and orderly compactly generated prestreams all form Cartesian-closed categories. We conclude in Section 7.

2 Preliminaries

Exponentiable objects. In a category $\mathbf{C}$ with finite products $\times$, an *exponential* (from the object X to the object Y) is an object Y^X, together with a morphism $\mathrm{App}\colon Y^X \times X \to Y$ (application, evaluation), and a collection of morphisms $\Lambda(f)\colon Z \to Y^X$, one for each morphism $f\colon Z \times X \to Y$, where Z is an arbitrary object of $\mathbf{C}$, satisfying the equations:

$$\begin{array}{lrcll} (\beta) & \mathrm{App} \circ (\Lambda(f) \times \mathrm{id}_X) & = & f & \text{for every } f\colon Z \times X \to Y \\ (\eta) & \Lambda(\mathrm{App}) & = & \mathrm{id}_{Y^X} & \\ (\sigma) & \Lambda(f) \circ g & = & \Lambda(f \circ (g \times \mathrm{id}_X)) & \text{for all } f\colon Z \times X \to Y, g\colon Z' \to Z. \end{array}$$

This presentation is closest to the λ-calculus [Cur86]. An object X is *exponentiable* in $\mathbf{C}$ if and only if it has an exponential Y^X for every object Y of $\mathbf{C}$. Equivalently, X is exponentiable if and only if the functor $\_ \times X$ is left adjoint (to the functor $\_^X$). $\mathbf{C}$ is *Cartesian-closed* if and only if all its objects are exponentiable.

For example, the exponentiable objects in $\mathbf{Top}$, the category of topological spaces, are exactly the core-compact spaces. (See [ELS04], or [GL13, Section 5.4], for a comprehensive treatment.) These are defined as follows. Let $\mathcal{O}(X)$ denote the lattice of open subsets of a topological space X. For $U, V \in \mathcal{O}(X)$, write $U \Subset V$ if and only if every open cover of V contains a finite subcover of U. (This is the so-called way-below relation familiar to domain theorists.) X is *core-compact* if and only if for every open neighborhood V of any point $x \in X$, there is an open subset U such that $x \in U \Subset V$. Every locally compact space (i.e., every space in which every point has a neighborhood basis of compact subsets) is core-compact, where $U \Subset V$ if and only if $U \subseteq K \subseteq V$ for some compact subset K.

In Hausdorff spaces (and more generally, sober spaces), core-compactness coincides with local compactness. In particular, a Hausdorff space that is not locally compact cannot be exponentiable in $\mathbf{Top}$. Examples include $\mathbb{Q}$, the Sorgenfrey line, or Baire space $\mathbb{N}^{\mathbb{N}}$ [GL13, Exercises 4.8.4, 4.8.5, Example 4.8.12]: $\mathbf{Top}$ is not Cartesian-closed.

When X is core-compact, then Y^X can be taken to be the space of all continuous maps from X to Y, with the Isbell topology. In this case, the description of the latter simplifies to the following (see [EH02, Theorem 4.3], or [GL13, Theorem 5.4.4]), which we call the *core-open* topology: it is the coarsest topology that makes $[U \Subset$

$V]$ open for all opens $U \in \mathcal{O}(X)$, $V \in \mathcal{O}(Y)$, where $[U \Subset V]$ denotes the set of continuous maps f such that $U \Subset f^{-1}(V)$. When X is locally compact, the core-open topology coincides with the more familiar compact-open topology.

Topological functors. We need to recall the notion of a *topological functor* $|\_|$ [AHS09]. As an illustration in the sequel, think of $\mathbf{C}$ as $\mathbf{Top}$, $\mathbf{D}$ as $\mathbf{Set}$, $|\_|$ as the underlying set functor. We shall call this the *topological example*.

Let $|\_|$ be a faithful functor from a category $\mathbf{C}$ to a category $\mathbf{D}$. A morphism $g\colon |A| \to |B|$ in $\mathbf{D}$ *lifts* to a (necessarily unique) $f\colon A \to B$ if and only if $|f| = g$. In the topological example, the maps that have liftings are the continuous maps.

The *fiber* over an object D of $\mathbf{D}$ is the class of objects A of $\mathbf{C}$ such that $|A| = D$. In the topological example, one may think as objects in the fiber of D as topologies added on the set D. The fiber over D is preordered by $A \leq B$ if and only if the identity morphism on $|A| = |B| = D$ has a lifting from A to B: if so, we say that A is *finer* than B, and that B is *coarser* than A.

There are several equivalent definitions of a topological functor. One states that $|\_|$ is *topological* if and only if every $|\_|$-*source* $(g_i\colon D \to |A_i|)_{i \in I}$ (where I is any class of indices) has a unique $|\_|$-initial lift (see [AHS09, Definition 21.1]). Such a functor is automatically faithful, and *amnestic*, meaning that $\leq$ is an ordering, not just a preorder, on each fiber. It is also *uniquely transportable*: given an object A of $\mathbf{C}$, and an isomorphism $g\colon |A| \to D$ in $\mathbf{D}$, there is a unique element B of the fiber of D such that g lifts to an isomorphism between A and B. In the topological example, the latter means that we can transport topologies along any bijection.

An equivalent definition, which matches topological uses better, is as follows. A functor $|\_|$ is topological if and only if it is faithful, amnestic, and for every $|\_|$-source $(g_i\colon D \to |A_i|)_{i \in I}$, there is an object B in the fiber of D such that g_i lifts to a morphism $f_i\colon B \to A_i$ for every $i \in I$, and satisfying the following universal property: For every morphism $g\colon |C| \to |B|$ in $\mathbf{D}$, g lifts to a (unique) morphism from C to B in $\mathbf{C}$ if and only if $g_i \circ g$ lifts to a morphism from C to A_i for every $i \in I$. In this case, B is the coarsest object in the fiber of D such that g_i lifts to a morphism $f_i\colon B \to A_i$ for every $i \in I$.

This corresponds to the familiar construction in $\mathbf{Top}$ that there is a coarsest topology B on D that makes all the functions g_i continuous; and the universal property states that to show that a map g with codomain D (with topology B) is continuous, it is equivalent to show that $g_i \circ g$ is continuous for every $i \in I$.

When every g_i is an identity map, this also implies that every family of objects in the fiber has a greatest lower bound. Consequently, $\leq$ endows the fiber over each object D with the structure of a complete lattice. The largest (coarsest) element in the fiber is the *indiscrete* object over D, which we write D_1, and the smallest (finest) one is the *discrete* object D_0 over D. In general, we define an indiscrete object over D as a greatest lower bound D_1 of the empty $|\_|$-source; explicitly, D_1 is indiscrete over D iff $|D_1| = D$, and every morphism $g\colon |B| \to D$ lifts to one from B to D_1. Similarly for discrete objects.

In the definition of topological functors, we are not requiring the class I on which the $|\_|$-source is indexed to be a set. This does not make a difference for *fiber-small* topological functors, i.e., those where every fiber is a set, not a proper class [AHS09, Proposition 21.34]. In particular, a fiber-small functor is topological $|\_|$ if and only if it is faithful, amnestic, and every *small* $|\_|$-source $(g_i : D \to |A_i|)_{i \in I}$ defines a coarsest object B in the fiber of D such that every g_i lifts to a morphism from B to A_i, and satisfying the same universal property as above. This is the case in the topological example: the class of topologies on a set forms a set.

A dual statement exists, too. Let $|\_|$ be a topological functor from $\mathbf{C}$ to $\mathbf{D}$. For every $|\_|$-*sink* $(g_i : |A_i| \to D)_{i \in I}$ (where I is a class of indices), there is an object B in the fiber of D such that g_i lifts to a morphism $f_i : A_i \to B$ for every $i \in I$, and satisfying the following universal property: For every morphism $g : |B| \to |C|$ in $\mathbf{D}$, g lifts to a (unique) morphism from B to C in $\mathbf{C}$ if and only if $g \circ g_i$ lifts to a morphism from A_i to C for every $i \in I$. B is the finest object in the fiber of D such that g_i lifts to a morphism $f_i : A_i \to B$ for every $i \in I$.

A topological functor $|\_|$ is both left adjoint (to the indiscrete object functor) and right adjoint (to the discrete object functor), and as such preserves both limits and colimits. It also *lifts limits (uniquely)*, meaning that, given any functor $F : \mathbf{J} \to \mathbf{C}$ such that $|F| = |\_| \circ F$ has a limit D in $\mathbf{D}$, there is a (unique) limit A of F in $\mathbf{C}$ such that $|A| = D$. In fact, a functor $|\_|$ is topological if and only if it is faithful, lifts limits uniquely, and has indiscrete objects D_1 over each object D of $\mathbf{D}$ [AHS09, Theorem 21.18].

In the sequel, instead of saying that $|\_| : \mathbf{C} \to \mathbf{D}$ is topological, or amnestic, or has any other property, we shall say that $\mathbf{C}$ is topological, resp. amnestic, resp. has any other property, *over* $\mathbf{D}$, leaving the functor $|\_|$ implicit. For example, we say that $\mathbf{Top}$ is topological over $\mathbf{Set}$. We shall see that the categories of streams and prestreams are topological over $\mathbf{Top}$, hence also over $\mathbf{Set}$.

A *construct* is a pair $(\mathbf{C}, |\_|)$ where $\mathbf{C}$ is a category, and $|\_|$ is a faithful functor from $\mathbf{C}$ to $\mathbf{Set}$. The construct is topological if and only if the functor $|\_|$ is.

When $\mathbf{D}$ has a terminal object 1, and $|\_|$ is topological, then $\mathbf{C}$ also has a terminal object, which happens to be 1_1, the indiscrete object on 1. A topological functor $|\_|$ has *discrete terminal objects* if and only if 1_1 is discrete, iff $1_0 = 1_1$, iff 1_0 is terminal, iff the fiber over 1 contains only one element. A construct $(\mathbf{C}, |\_|)$ is *well-fibered* if and only if it is fiber-small and has discrete terminal objects [AHS09, Definition 27.20]. This is the case of the topological example.

3 The Escardó-Lawson-Simpson Construction

Escardó, Lawson and Simpson [ELS04] provide a useful construction of a Cartesian-closed category $\mathbf{Map}_{\mathcal{C}}$, from which one can easily derive construction of Cartesian-closed categories $\mathbf{Top}_{\mathcal{C}}$ of $\mathbf{Top}$. When $\mathcal{C}$ is the class of compact Hausdorff spaces, $\mathbf{Top}_{\mathcal{C}}$ is the familiar category of compactly generated spaces.

We observe that the same constructions work, almost without modification, in case we replace $\mathbf{Top}$ by a category $\mathbf{C}$ that is topological over $\mathbf{Set}$.

Barr also provided a general categorical construction for building monoidal closed categories, and in particular Cartesian-closed categories [Bar78]. The $\mathbf{Map}_{|\_|,\mathcal{C}}$ and

$\mathbf{C}_{|\_|,\mathcal{C}}$ constructions below rely on different assumptions, and are hopefully easier to apply.

3.1 The Category $\mathbf{Map}_{|\_|,\mathcal{C}}$

Definition 3.1 **($\mathbf{Map}_{|\_|,\mathcal{C}}$)** *Let $|\_|$ be a faithful functor from a category $\mathbf{C}$ to a category $\mathbf{D}$, and let $\mathcal{C}$ be a class of objects of $\mathbf{C}$. Call $\mathcal{C}$-*probe *(on X) any morphism $k: C \to X$ in $\mathbf{C}$, where $C \in \mathcal{C}$.*

*For any two objects X and Y of $\mathbf{C}$, a $\mathcal{C}$-*map *from X to Y is a morphism $g: |X| \to |Y|$ such that, for every $\mathcal{C}$-probe $k: C \to X$, $g \circ |k|$ has a (necessarily unique) lifting. We write $g \bullet k$ for this lifting, so that $|g \bullet k| = g \circ |k|$.*

The category $\mathbf{Map}_{|\_|,\mathcal{C}}$ has all objects of $\mathbf{C}$ as objects, and as morphisms from X to Y all $\mathcal{C}$-maps from X to Y. Identities and composition are given as in $\mathbf{D}$.

This is clearly a category. $\mathbf{Map}_{|\_|,\mathcal{C}}$ is like $\mathbf{C}$, except with possibly more morphisms:

Lemma 3.2 *Let $|\_|$ be a faithful functor from a category $\mathbf{C}$ to a category $\mathbf{D}$, and let $\mathcal{C}$ be a class of objects of $\mathbf{C}$. Every morphism $f: X \to Y$ of $\mathbf{C}$ defines a morphism $|f|$ from X to Y in $\mathbf{Map}_{|\_|,\mathcal{C}}$.*

Proof. For every $\mathcal{C}$-probe, $f \bullet k$ is just $f \circ k$. □

Lemma 3.3 *Let $|\_|$ be a topological functor from a category $\mathbf{C}$ to a category $\mathbf{D}$, and let $\mathcal{C}$ be a class of objects of $\mathbf{C}$. Assume that $\mathbf{D}$ has all finite products.*

The category $\mathbf{Map}_{|\_|,\mathcal{C}}$ has all finite products. For all objects $X_1, \ldots, X_n$, the product $X_1 \times \ldots \times X_n$ in $\mathbf{C}$ is a product in $\mathbf{Map}_{|\_|,\mathcal{C}}$, and projections and pairing maps are defined as in $\mathbf{D}$.

Proof. Since $|\_|$ lifts limits, $\mathbf{C}$ has all finite products, and $|\_|$ preserves them on the nose.

Let $X_1, \ldots, X_n$ be n objects in $\mathbf{Map}_{|\_|,\mathcal{C}}$ (equivalently, in $\mathbf{C}$). Write $X_1 \times \ldots \times X_n$ for their product in $\mathbf{C}$, π_i for ith projection, and $\langle g_1, \ldots, g_n \rangle: Z \to X_1 \times \ldots \times X_n$ for the pairing of $g_i: Z \to X_i$, $1 \leq i \leq n$. We also use similar notations in $\mathbf{D}$.

Let g_i be morphisms from Z to X_i in $\mathbf{Map}_{|\_|,\mathcal{C}}$ (i.e., from $|Z|$ to $|X_i|$ in $\mathbf{D}$), $1 \leq i \leq n$. By the definition of products in $\mathbf{D}$, $\langle g_1, \ldots, g_n \rangle$ is the unique morphism h in $\mathbf{Map}_{|\_|,\mathcal{C}}$ such that $\pi_i \circ h = g_i$ for every i, $1 \leq i \leq n$. We must show that it is a $\mathcal{C}$-map. Let $k: C \to X$ be any $\mathcal{C}$-probe. $\langle g_1 \bullet k, \ldots, g_n \bullet k \rangle$ is a lifting of $\langle g_1, \ldots, g_n \rangle \circ k$, since $|\langle g_1 \bullet k, \ldots, g_n \bullet k \rangle| = \langle |g_1 \bullet k|, \ldots, |g_n \bullet k| \rangle = \langle g_1 \circ |k|, \ldots, g_n \circ |k| \rangle = \langle g_1, \ldots, g_n \rangle \circ |k|$: the first equality is since $|\_|$ preserves products (hence pairings), the second equality is by definition of $\bullet$, the third one because pairings always distribute with composition on the right. □

In particular, there is no ambiguity in writing $\times$ for product, whether in $\mathbf{C}$, $\mathbf{D}$, or $\mathbf{Map}_{|\_|,\mathcal{C}}$.

Let us write Y^X for the exponential object from X to Y when it exists, whether in $\mathbf{C}$ or in $\mathbf{D}$, and let $\mathrm{App}: Y^X \times X \to Y$ be the application (or evaluation) morphism, $\Lambda(h): Z \to Y^X$ be the currification of $h: Z \times X \to Y$. Write $*$ for the unique element of the terminal object 1 in $\mathbf{Set}$. When $\mathbf{D} = \mathbf{Set}$, for every object Z of $\mathbf{C}$, for every element $z \in |Z|$, write z^1 for the unique morphism from 1_0 to Z such that $|z^1|$ maps $*$ to z. This exists and is unique because 1_0 is the discrete object on $\{*\}$. We characterize exponentials in a topological construct:

Lemma 3.4 *Let $(\mathbf{C}, |\_|)$ be a topological construct, and let C, X be two objects in $\mathbf{C}$ such that some exponential from C to X exists.*

There is a unique such exponential X^C with $|X^C| = \mathrm{Hom}_{\mathbf{C}}(1_0 \times C, X)$. Application App is such that $|\mathrm{App}|(h, c) = |h|(*, c)$ for all $h \in |X^C|$ and $c \in |C|$. Currification is such that for every morphism $f\colon Z \times C \to X$ in $\mathbf{C}$, $|\Lambda(f)|(z) = f \circ (z^1 \times \mathrm{id}_C)$.

Proof. We first fix an exponential A (we refrain from writing it X^C, so as to avoid any possible confusion), and build an isomorphism with some element in the fiber of $\mathrm{Hom}_{\mathbf{C}}(1_0 \times C, X)$. To do so, we build its image θ by $|\_|$.

Let $\mathrm{App}\colon A \times X \to C$ be application, $\Lambda(f)\colon Z \to A$ be the currification of $f\colon Z \times C \to X$. For every $h \in |A|$, let $\theta(h)$ be the morphism $\mathrm{App} \circ (h^1 \times \mathrm{id}_C)$, i.e., $1_0 \times C \xrightarrow{h^1 \times \mathrm{id}_C} A \times C \xrightarrow{\mathrm{App}} X$. We claim that its inverse is θ', defined by $\theta'(f) = |\Lambda(f)|(*)$, for every $f\colon 1_0 \times C \to X$. Note that $\theta'(f)^1 = \Lambda(h)$, by definition of z^1 as the unique object such that $|z^1|(*) = z$. We check the following:

$$\begin{aligned} \theta(\theta'(f)) &= \mathrm{App} \circ (\theta'(f)^1 \times \mathrm{id}_C) \\ &= \mathrm{App} \circ (\Lambda(h) \times \mathrm{id}_C) = h \end{aligned}$$

by (β). Conversely,

$$\begin{aligned} \theta'(\theta(h)) &= |\Lambda(\mathrm{App} \circ (h^1 \times \mathrm{id}_C))|(*) \\ &= |\Lambda(\mathrm{App}) \circ h^1|(*) \qquad \text{(by } (\sigma)\text{)} \\ &= |h^1|(*) \qquad \text{(by } (\eta)\text{)} \\ &= h. \end{aligned}$$

Therefore θ is a bijection. Since $|\_|$ is topological, it is uniquely transportable, and therefore there is a unique exponential object, call it X^C, in the fiber of $\mathrm{Hom}_{\mathbf{C}}(1_0 \times C, X)$. This is isomorphic to A through (the unique lifting of) θ.

Applying $|\_|$, $|\theta(h)| = |\mathrm{App}| \circ (|h^1| \times \mathrm{id}_{|C|})$, using the fact that $|\_|$ preserves products on the nose. It follows that, for every $c \in |C|$, $|\theta(h)|(*, c) = |\mathrm{App}|(h, c)$. Moreover, for every morphism $f\colon Z \times C \to X$ in $\mathbf{C}$, for every $z \in Z$, $\Lambda(f) \circ z^1 = \Lambda(f \circ (z^1 \times \mathrm{id}_C))$ by (σ), so $|\Lambda(f)|(z) = |\Lambda(f) \circ z^1|(*) = |\Lambda(f \circ (z^1 \times \mathrm{id}_C))|(*) = \theta^{-1}(f \circ (z^1 \times \mathrm{id}_C))$, by definition of $\theta^{-1} = \theta'$. When $A = X^C$ itself, θ is the identity, which allows us to conclude. ☐

In a topological construct with discrete terminal objects, 1_0 is terminal, and $1_0 \times X$ is naturally isomorphic to X, via $\pi_2\colon 1_0 \times X \to X$ in one direction, and $\langle !, \mathrm{id}_X \rangle\colon X \to 1_0 \times X$ in the other (we write $!\colon X \to 1_0$ for the unique morphism to the terminal object 1_0).

Corollary 3.5 *Let $(\mathbf{C}, |\_|)$ be a topological construct with discrete terminal objects, and let C, X be two objects in $\mathbf{C}$ such that some exponential from C to X exists.*

There is a unique such exponential X^C with $|X^C| = \mathrm{Hom}_{\mathbf{C}}(C, X)$. Application App *is such that* $|\mathrm{App}|(h, c) = |h|(c)$ *for all* $h \in |X^C|$ *and* $c \in |C|$. *Currification is such that for every morphism* $f\colon Z \times C \to X$ *in* $\mathbf{C}$, $|\Lambda(f)|(z) = f \circ \langle z^1 \circ !, \mathrm{id}_C\rangle$.

The construction $\mathbf{Map}_{|\_|,\mathcal{C}}$ is interesting when $\mathcal{C}$ is a strongly productive class, defined below. We shall see later that we can instead require $\mathcal{C}$ to satisfy a weaker requirement called productivity.

Definition 3.6 (Strongly Productive) *Let* $\mathbf{C}$ *be a category with finite products. A class* $\mathcal{C}$ *of objects of* $\mathbf{C}$ *is* strongly productive *if and only if every object of* $\mathcal{C}$ *is exponentiable in* $\mathbf{C}$, *and products of pairs of elements of* $\mathcal{C}$ *are in* $\mathcal{C}$.

Theorem 3.7 *Let* $(\mathbf{C}, |\_|)$ *be a well-fibered topological construct, and let* $\mathcal{C}$ *be a strongly productive class of objects of* $\mathbf{C}$. *The category* $\mathbf{Map}_{|\_|,\mathcal{C}}$ *is Cartesian-closed.*

More precisely, for any two objects X *and* Y *of* $\mathbf{C}$, *let* $\mathcal{C}[X, Y]$ *be the set of all* $\mathcal{C}$*-maps from* X *to* Y. *Given any* $\mathcal{C}$*-probe* $k\colon C \to X$, *let* $\_ \bullet k$ *be the map from* $\mathcal{C}[X, Y]$ *to* $\mathrm{Hom}_{\mathbf{C}}(C, Y)$ *that sends* f *to* $f \bullet k$. *An exponential object from* X *to* Y *is the coarsest object, written* $[Y^X]_{\mathcal{C}}$, *in the fiber of* $\mathcal{C}[X, Y]$ *such that the map* $\mathcal{C}[X, Y] \xrightarrow{\_\bullet k} \mathrm{Hom}_{\mathbf{C}}(C, Y)$ *lifts to a morphism from* $[Y^X]_{\mathcal{C}}$ *to* Y^C *in* $\mathbf{C}$, *for every* $\mathcal{C}$*-probe* $k\colon C \to X$.

Application $\mathrm{App}_{\mathcal{C}}\colon [Y^X]_{\mathcal{C}} \times X \to Y$ *(in* $\mathbf{Map}_{|\_|,\mathcal{C}}$*) is given by ordinary function application (in* $\mathbf{Set}$*), namely* $\mathrm{App}_{\mathcal{C}}(f, x) = f(x)$. *Currification* $\Lambda_{\mathcal{C}}(f)\colon Z \to [Y^X]_{\mathcal{C}}$ *of a morphism* $f\colon Z \times X \to Y$ *in* $\mathbf{Map}_{|\_|,\mathcal{C}}$ *(i.e., of a map* $f\colon |Z| \times |X| \to |Y|$*) is the map that sends each* $z \in |Z|$ *to* $f(z, \_)$.

Proof. Every object C of $\mathcal{C}$ is exponentiable, so Y^C exists, and can be chosen in the fiber of $\mathrm{Hom}_{\mathbf{C}}(C, Y)$ by Corollary 3.5. Also, among the objects E in the fiber of $\mathcal{C}[X, Y]$ such that $\mathcal{C}[X, Y] \xrightarrow{\_\bullet k} \mathrm{Hom}_{\mathbf{C}}(C, Y)$ lifts to a morphism $\widehat{k, E}$ from E to Y^C in $\mathbf{C}$, for every $\mathcal{C}$-probe $k\colon C \to X$, there is a coarsest one, because $|\_|$ is topological. So $[Y^X]_{\mathcal{C}}$ is well defined.

Let $\mathrm{App}_{\mathcal{C}}\colon \mathcal{C}[X, Y] \times |X| \to |Y|$ (in $\mathbf{Set}$) be defined by $\mathrm{App}_{\mathcal{C}}(f, x) = f(x)$. To show that $\mathrm{App}_{\mathcal{C}}$ is a morphism in $\mathbf{Map}_{|\_|,\mathcal{C}}$, we must check that $\mathrm{App}_{\mathcal{C}} \circ |k|$ lifts to a morphism $\mathrm{App}_{\mathcal{C}} \bullet k\colon C \to Y$ for every $\mathcal{C}$-probe $k\colon C \to [Y^X]_{\mathcal{C}} \times X$. Let $k_1 = \pi_1 \circ k\colon C \to [Y^X]_{\mathcal{C}}$, $k_2 = \pi_2 \circ k\colon C \to X$. Both are $\mathcal{C}$-probes. We claim that $\mathrm{App}_{\mathcal{C}} \bullet k$ is the composite:

$$C \xrightarrow{\langle k_1, \mathrm{id}_C\rangle} [Y^X]_{\mathcal{C}} \times C \xrightarrow{\hat{k}_2 \times \mathrm{id}_C} Y^C \times C \xrightarrow{\mathrm{App}} Y$$

where $\hat{k}_2$ is the unique lifting of $\_ \bullet k_2\colon \mathcal{C}[X, Y] \to \mathrm{Hom}_{\mathbf{C}}(C, Y)$. Temporary call this composite f. We must check that $|f| = \mathrm{App}_{\mathcal{C}} \circ |k|$. For every $c \in |C|$,

$$\begin{aligned} |f|(c) &= |\mathrm{App}|(|k_1|(c) \bullet k_2, c) \\ &= \big||k_1|(c) \bullet k_2\big|(c) \quad \text{(by Corollary 3.5 again)} \\ &= |k_1|(c)(|k_2|(c)) \end{aligned}$$

while, by definition,

$$(\mathrm{App}_{\mathcal{C}} \circ |k|)(c) = \mathrm{App}_{\mathcal{C}}(|k_1|(c), |k_2|(c)) = |k_1|(c)(|k_2|(c)).$$

Let us turn to currification. Fix an arbitrary $\mathcal{C}$-map f from $Z \times X$ to Y (i.e., f is a map from $|Z| \times |X|$ to $|Y|$ whose compositions with all relevant $\mathcal{C}$-probes have liftings).

For each $z \in |Z|$, there is a map $f(z, \_)$ from $|X|$ to $|Y|$. We first check that it is a $\mathcal{C}$-map. For every $\mathcal{C}$-probe $k \colon C \to X$, $\langle z^1 \circ !, k\rangle \colon C \to Z \times X$ is a $\mathcal{C}$-probe, and since f is a $\mathcal{C}$-map, $f \bullet \langle z^1 \circ !, k\rangle$ exists. For every $c \in |C|$,

$$\begin{aligned} |f \bullet \langle z^1 \circ !, k\rangle|(c) &= (f \circ \langle |z^1 \circ !|, |k|\rangle)(c) \quad \text{(since } |\_| \text{ preserves products)} \\ &= f(z, |k|(c)) = (f(z, \_) \circ |k|)(c). \end{aligned}$$

This means that $f \bullet \langle z^1 \circ !, k\rangle$ lifts $f(z, \_) \circ |k|$, showing that $f(z, \_)$ is a $\mathcal{C}$-map.

The map $z \mapsto f(z, \_)$ is therefore one from $|Z|$ to $\mathcal{C}[X, Y] = |[Y^X]_{\mathcal{C}}|$. Call this map $\Lambda_{\mathcal{C}}(f)$. We claim that this is a $\mathcal{C}$-map, too. Fixing a $\mathcal{C}$-probe $k_1 \colon C_1 \to Z$, we must show that $\Lambda_{\mathcal{C}}(f) \circ |k_1|$ lifts to a morphism from C_1 to $[Y^X]_{\mathcal{C}}$. By the universal property for $|\_|$-sources, applied to the definition of $[Y^X]_{\mathcal{C}}$, we only have to check that $(\_ \bullet k_2) \circ \Lambda_{\mathcal{C}}(f) \circ |k_1|$ lifts to a morphism from C_1 to Y^{C_2} for every $\mathcal{C}$-probe $k_2 \colon C_2 \to X$.

We claim that the required lifting is $\Lambda(f \bullet (k_1 \times k_2))$. Note that this makes sense: since $\mathcal{C}$ is strongly productive, $C_1 \times C_2$ is in $\mathcal{C}$, so that $k_1 \times k_2 \colon C_1 \times C_2 \to Z \times X$ is a $\mathcal{C}$-probe, whence $f \bullet (k_1 \times k_2)$ exists. Also, the latter can be currified, because C_2 is exponentiable.

Let us compute. For every $c_1 \in |C_1|$,

$$|\Lambda(f \bullet (k_1 \times k_2))|(c_1) \quad = \quad (f \bullet (k_1 \times k_2)) \circ \langle c_1^1 \circ !, \mathrm{id}_{C_2}\rangle \quad \text{(by Corollary 3.5)}$$

and we must show that this is equal to:

$$((\_ \bullet k_2) \circ \Lambda_{\mathcal{C}}(f) \circ |k_1|)\,(c_1) \quad = \quad f(|k_1|(c_1), \_) \bullet k_2.$$

Since both are elements of $\mathrm{Hom}_{\mathbf{C}}(C_2, Y)$, and $|\_|$ is faithful, it suffices to check that their images under $|\_|$ are the same. For every $c_2 \in |C_2|$,

$$\begin{aligned} \big||\Lambda(f \bullet (k_1 \times k_2))|(c_1)\big|(c_2) &= \big|(f \bullet (k_1 \times k_2)) \circ \langle c_1^1 \circ !, \mathrm{id}_{C_2}\rangle\big|(c_2) \\ &= (f \circ (|k_1| \times |k_2|))(c_1, c_2) \\ &= f(|k_1|(c_1), |k_2|(c_2)), \end{aligned}$$

while

$$\begin{aligned} \big|((\_ \bullet k_2) \circ \Lambda_{\mathcal{C}}(f) \circ |k_1|)\,(c_1)\big|(c_2) &= \big|f(|k_1|(c_1), \_) \bullet k_2\big|(c_2) \\ &= (f(|k_1|(c_1), \_) \circ |k_2|)(c_2) \\ &= f(|k_1|(c_1), |k_2|(c_2)) \end{aligned}$$

and we are done. We conclude that $\Lambda_{\mathcal{C}}(f)$ is a $\mathcal{C}$-map from Z to $[Y^X]_{\mathcal{C}}$.

The equations (β), (η), (σ) are obvious from the fact that they hold in $\mathbf{Set}$. E.g., for (β), we must show that $\mathrm{App}_{\mathcal{C}} \circ (\Lambda_{\mathcal{C}}(f) \times \mathrm{id}_X) = f$ for every $\mathcal{C}$-map f from $Z \times X$ to Y. For every pair (z, x) in $|Z| \times |X|$, the left-hand side applied to (z, x) yields $\mathrm{App}_{\mathcal{C}} \circ (f(z, \_), x) = f(z, x)$, hence equals the right-hand side (f) applied to (z, x). □

3.2 $\mathcal{C}$-Generated Objects

$\mathbf{Map}_{|\_|,\mathcal{C}}$ is not a subcategory of $\mathbf{C}$. We build a subcategory of $\mathbf{C}$ that is equivalent to $\mathbf{Map}_{|\_|,\mathcal{C}}$.

Definition 3.8 *Let $|\_|$ be a topological functor from a category $\mathbf{C}$ to a category $\mathbf{D}$, and let $\mathcal{C}$ be a class of objects of $\mathbf{C}$. For every object X of $\mathbf{C}$, let $\mathcal{C}X$ be the finest object in the fiber of $|X|$ such that, for every $\mathcal{C}$-probe $k\colon C \to X$, $|k|$ lifts to a morphism from C to $\mathcal{C}X$.*

An object X of $\mathbf{C}$ is $\mathcal{C}$-generated *if and only if $\mathcal{C}X = X$.*

The category $\mathbf{C}_{|\_|,\mathcal{C}}$ is the full subcategory of $\mathbf{C}$ whose objects are the $\mathcal{C}$-generated objects.

The following properties hold:

- $\mathcal{C}X \leq X$, i.e., there is a morphism i_X from $\mathcal{C}X$ to X that lifts the identity on $|X|$. Indeed, X itself is among the objects X' such that for every $\mathcal{C}$-probe k to X', $|k|$ lifts to a probe to X'; $\mathcal{C}X$ is the finest such object, hence is finer than X.

- Every $\mathcal{C}$-probe $k\colon C \to X$ factors through $i_X\colon \mathcal{C}X \to X$, i.e., there is a morphism $k'\colon C \to \mathcal{C}X$ such that $k = i_X \circ k'$, namely the lifting of $|k|$ to a morphism from C to $\mathcal{C}X$.

Lemma 3.9 *Under the assumptions of Definition 3.8, every object of $\mathcal{C}$ is $\mathcal{C}$-generated.*

Proof. Let $X \in \mathcal{C}$. The identity morphism on X is a $\mathcal{C}$-probe, so $|\mathrm{id}_X|$ lifts to a morphism from X to $\mathcal{C}X$, that is, $X \leq \mathcal{C}X$. We conclude since $\mathcal{C}X \leq X$, and $|\_|$ is amnestic. □

Lemma 3.10 *Under the assumptions of Definition 3.8, for every $C \in \mathcal{C}$, a morphism $g\colon |C| \to |X|$ in $\mathbf{D}$ lifts to one from C to X if and only if it lifts to one from C to $\mathcal{C}X$.*

Proof. If g lifts to $f\colon C \to \mathcal{C}X$, then $i_X \circ f\colon C \to X$ lifts g as well. Conversely, if g lifts to $f\colon C \to X$, then f is a $\mathcal{C}$-probe, hence factors through i_x, yielding a lifting of g to a morphism from C to $\mathcal{C}X$. □

Lemma 3.11 *Under the assumptions of Definition 3.8, every object of the form $\mathcal{C}X$ is $\mathcal{C}$-generated.*

Proof. For every $\mathcal{C}$-probe $k\colon C \to X$, $|k|$ lifts to a morphism from C to $\mathcal{C}X$, hence to one from C to $\mathcal{C}\mathcal{C}X$ by Lemma 3.10. That is, $\mathcal{C}\mathcal{C}X$ is an object Y in the fiber of $|X|$ such that for every $\mathcal{C}$-probe $k\colon C \to X$, $|k|$ lifts to a morphism from C to Y. $\mathcal{C}X$ is the finest, so $\mathcal{C}X \leq \mathcal{C}\mathcal{C}X$. Since $\mathcal{C}\mathcal{C}X \leq \mathcal{C}X$ and $|\_|$ is amnestic, $\mathcal{C}X = \mathcal{C}\mathcal{C}X$. □

Lemma 3.12 *Under the assumptions of Definition 3.8, the $\mathcal{C}$-maps from X to Y are exactly the morphisms in $\mathbf{D}$ that lift to a morphism from $\mathcal{C}X$ to Y in $\mathbf{C}$.*

Proof. Let g be a $\mathcal{C}$-map from X to Y (in particular, a morphism from $|X|$ to $|Y|$ in $\mathbf{D}$). For every $\mathcal{C}$-probe $k\colon C \to X$, by definition $g \circ |k|$ lifts to some morphism $g \bullet k\colon C \to Y$. By the universal property of $|\_|$-sinks, and Definition 3.8, g lifts to a morphism from $\mathcal{C}X$ to Y.

Conversely, assume g lifts to a morphism f from $\mathcal{C}X$ to Y. For every $\mathcal{C}$-probe $k\colon C \to X$, one can write k as $i_X \circ k'$ for some morphism $k'\colon C \to \mathcal{C}X$, and $|k'| = |k|$, so $|f \circ k'| = g \circ |k|$, showing that $f \circ k'$ lifts $g \circ |k|$: g is a $\mathcal{C}$-map. ☐

Proposition 3.13 *Let $|\_|$ be a topological functor from a category $\mathbf{C}$ to a category $\mathbf{D}$, and let $\mathcal{C}$ be a class of objects of $\mathbf{C}$. The categories $\mathbf{Map}_{|\_|,\mathcal{C}}$ and $\mathbf{C}_{|\_|,\mathcal{C}}$ are equivalent.*

The equivalence is given, in one direction, by the functor $\mathcal{C}\colon \mathbf{Map}_{|\_|,\mathcal{C}} \to \mathbf{C}_{|\_|,\mathcal{C}}$ that maps each object X to $\mathcal{C}X$ and each morphism g from X to Y in $\mathbf{Map}_{|\_|,\mathcal{C}}$ (i.e., from $|X|$ to $|Y|$ in $\mathbf{D}$) to $i_Y \circ f$ where f is the unique lifting of g as a morphism from $\mathcal{C}X$ to Y; in the other direction, by the functor that is the identity on objects and coincides with $|\_|$ on morphisms.

Proof. We must show that $\mathcal{C}$ is a well-defined functor. The existence and uniqueness of f follow from Lemma 3.12, and the fact that $|\_|$ is faithful. The fact that $\mathcal{C}$ preserves identities and composition is because $|\_|$ is faithful, again. In the converse direction, the functor I that is the identity on objects and coincides with $|\_|$ on morphisms is well-defined: we need to check that if $f\colon X \to Y$ is a morphism in $\mathbf{C}_{|\_|,\mathcal{C}}$, then $|f|$ is a $\mathcal{C}$-map from X to Y, and this is by Lemma 3.12, together with $\mathcal{C}X = X$.

Finally, for every object X of $\mathbf{Map}_{|\_|,\mathcal{C}}$, $\mathrm{id}_{|X|}$ is a (natural) isomorphism from X to $I\mathcal{C}X$ (i.e., from $|X|$ to $|I\mathcal{C}X| = |X|$ in $\mathbf{D}$): we need only check that $\mathrm{id}_{|X|}$ is both a $\mathcal{C}$-map from X to $I\mathcal{C}X = \mathcal{C}X$ and from $\mathcal{C}X$ to X. The first claim follows from Lemma 3.12, and the second one from $\mathcal{C}X \leq X$.

Finally, for every object X of $\mathbf{C}_{|\_|,\mathcal{C}}$, id_X is a (natural) isomorphism from X to $\mathcal{C}IX = \mathcal{C}X$, since X is $\mathcal{C}$-generated. ☐

Together with Theorem 3.7, we obtain:

Theorem 3.14 *Let $(\mathbf{C}, |\_|)$ be a well-fibered topological construct, and let $\mathcal{C}$ be a strongly productive class of objects of $\mathbf{C}$. The category $\mathbf{C}_{|\_|,\mathcal{C}}$ is Cartesian-closed.*

The terminal object of $\mathbf{C}_{|\_|,\mathcal{C}}$ is the terminal object of $\mathbf{C}$, products $X \times_{\mathcal{C}} Y$ are defined as $\mathcal{C}(X \times Y)$ where $\times$ is product in $\mathbf{C}$, and the exponential object from X to Y is $\mathcal{C}[Y^X]_{\mathcal{C}}$.

Proof. Only the second part remains to be checked. A terminal object of $\mathbf{C}_{|\_|,\mathcal{C}}$ is $\mathcal{C}1$ where 1 is terminal in $\mathbf{C}$. Observe that $\mathcal{C}1 = 1$, since there is only one object in the fiber of $|1|$, by well-fiberedness and amnesticity. For products, remember that product in $\mathbf{Map}_{|\_|,\mathcal{C}}$ coincides with product in $\mathbf{C}$. The rest is clear. ☐

The largest possible choice for $\mathcal{C}$ is the class of all exponentiable objects. We observe that this is indeed a strongly productive class: for any two exponentiable objects X and Y, $\_ \times (X \times Y)$ is left adjoint to $(\_^X)^Y$, or to $(\_^Y)^X$, so $X \times Y$ is exponentiable as well.

Lemma 3.15 *Let $|\_|$ be a topological functor from a category $\mathbf{C}$ to a category $\mathbf{D}$, and let $\mathcal{C}$ be a class of objects of $\mathbf{C}$.*

The category $\mathbf{C}_{|\_|,\mathcal{C}}$ is coreflective in $\mathbf{C}$. The right adjoint to the inclusion functor is the functor, again written $\mathcal{C}$, that maps each object X to $\mathcal{C}X$, and each morphism $f\colon X \to Y$ to the unique lifting of $|f|$ as a morphism from $\mathcal{C}X$ to $\mathcal{C}Y$.

Proof. We must first check that $\mathcal{C}X$ is indeed an object of $\mathbf{C}_{|\_|,\mathcal{C}}$: this is Lemma 3.11. Then, we must check that for every morphism $f\colon X \to Y$ in $\mathbf{C}$, $|f|$ lifts to a morphism from $\mathcal{C}X$ to $\mathcal{C}Y$. For every $\mathcal{C}$-probe $k\colon C \to X$, $i_Y \circ f \circ k$ is a lifting of $|f| \circ |k|$ to a morphism from C to $\mathcal{C}Y$, so $|f|$ is a $\mathcal{C}$-map from X to $\mathcal{C}Y$. Lemma 3.12 then implies the desired conclusion.

Write I for the inclusion functor. The unit of the coreflection is defined on each object X as the identity morphism from X to $\mathcal{C}IX = X$ (since X is $\mathcal{C}$-generated). The counit is defined as $i_X : I\mathcal{C}X \to X$ on each object X of $\mathbf{C}$. ☐

Since $\mathcal{C}$ is a left adjoint, it preserves all colimits, whence:

Lemma 3.16 *Let $(\mathbf{C}, |\_|)$ be a topological construct. Given any class $\mathcal{C}$ of topological spaces, $\mathbf{C}_{|\_|,\mathcal{C}}$ has all colimits, and they are computed as in $\mathbf{C}$. Every colimit of $\mathcal{C}$-generated objects in $\mathbf{C}$ is again $\mathcal{C}$-generated.*

3.3 $\mathcal{C}$-Generation as Colimits

It is well-known that the compactly generated spaces are exactly the quotients of compact Hausdorff spaces. More generally, given a (strongly) productive class $\mathcal{C}$ of topological spaces containing a non-empty space, the $\mathcal{C}$-generated spaces are exactly the colimits of spaces from $\mathcal{C}$ [ELS04]. A similar phenomenon occurs here.

We say that an object A in $\mathbf{C}$ is *non-empty*, where $(\mathbf{C}, |\_|)$ is a construct, if and only if $|A|$ is a non-empty set. In the sequel, we shall always consider that the elements of a set-theoretic coproduct $X = \coprod_{i \in I} X_i$ are pairs (i, x) where $x \in X_i$. This comes with canonical injections $\iota_i : X_i \to X$ defines by $\iota_i(x) = (i, x)$.

Proposition 3.17 *Let $(\mathbf{C}, |\_|)$ be a well-fibered topological construct, and let $\mathcal{C}$ be a class of objects of $\mathbf{C}$ containing a non-empty object A.*

The $\mathcal{C}$-generated objects of $\mathbf{C}$ are exactly the small colimits, taken in $\mathbf{C}$, of objects of $\mathcal{C}$.

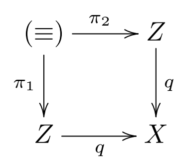
Proof. First, $\mathbf{C}$ is also cocomplete and finitely complete, since $\mathbf{Set}$ is. Every small colimit of objects in $\mathcal{C}$ is a colimit of $\mathcal{C}$-generated objects by Lemma 3.9, hence is itself $\mathcal{C}$-generated, by Lemma 3.16.

Conversely, let X be a $\mathcal{C}$-generated object of $\mathbf{C}$. Let I be the set of objects X' in the fiber of $|X|$ such that $X \not\leq X'$: this is a set, since $|\_|$ is fiber-small. For every object $X' \in I$, pick a $\mathcal{C}$-probe $k_{X'}\colon C_{X'} \to X$ such that $|k_{X'}|$ does *not* lift to a morphism from $C_{X'}$ to X'. This exists: an object X' in the fiber of $|X|$ for which such a $k_{X'}$ does not exist is by definition one such that for every $\mathcal{C}$-probe $k\colon C \to X$, $|k|$ lifts to a morphism from C to X', which implies $\mathcal{C}X \leq X'$ (Definition 3.8); since $\mathcal{C}X = X$, this would contradict the fact that $X' \in I$. Finally, we use the Axiom of Choice to collect one $\mathcal{C}$-probe $k_{X'}$ per object X' in I.

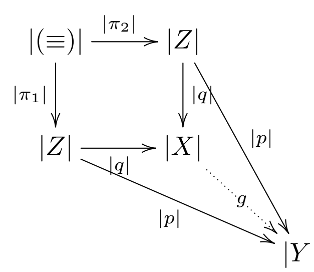
For every $x \in |X|$, $\{x\}$ is terminal in $\mathbf{Set}$, and since $|\_|$ lifts limits, there is a terminal object $1(x)$ in $\mathbf{C}$ such that $|1(x)| = \{x\}$. There is a unique morphism from A to $1(x)$, and since $|\_|$ is well-fibered, $1(x)$ is discrete, so there is a lifting of the inclusion map, from $\{x\}$ to $|X|$, to a morphism from $1(x)$ to X. Their composition is a morphism $c_x \colon A \to X$ such that $|c_x|$ is the constant x map from $|A|$ to $|X|$.

Let Z be the coproduct $\coprod_{X' \in I} C_{X'} \sqcup \coprod_{x \in |X|} A$, and q be the unique morphism from Z to X such that $q \circ \iota_{X'} = k_{X'}$ for every $X' \in I$ and $q \circ \iota_x = c_x$, where $\iota_{X'}$, resp. ι_x, is the canonical morphism from $C_{X'}$, resp. A, to the coproduct Z. We claim that q is a regular epi.

If it is, then it must be the coequalizer of its kernel pair. Form the latter; this is the following pullback $(\equiv)$ of two morphisms equal to q:



We must show that q is a coequalizer of the pair of parallel maps $\pi_1, \pi_2 \colon (\equiv) \to Z$. To this end, let $p \colon Z \to Y$ be any other morphism such that $p \circ \pi_1 = p \circ \pi_2$. Applying $|\_|$, we obtain the following diagram:



where the upper left square is again a pullback, since $|\_|$ preserves limits. The existence of the dotted g morphism is justified as follows. The elements of $|Z|$ are the pairs (X', z) with $z \in |C_{X'}|$, plus the pairs (x, a) where $x \in |X|$ and $a \in |A|$. Since $|q|$ maps (X', z) to z, and (x, a) to x, $|q|$ is surjective. In $\mathbf{Set}$, the surjective maps are the regular epis, which implies that g exists and is unique. (Concretely, we define $g(x)$, where $x \in |X|$, as $|p|(x, a)$, where a is any fixed element of $|A|$, which exists since A is non-empty.)

Consider the one-morphism $|\_|$-source (g). This lifts to a morphism $f \colon X' \to Y$, where X' is coarsest in the fiber of X with this property. The universal property of X' is: for every morphism $g' \colon |X''| \to |X'|$ in $\mathbf{Set}$, g' lifts to a morphism from X'' to X' iff $g \circ g'$ lifts to a morphism from X'' to Y.

If $X \not\leq X'$, by definition X' is in I, so $k_{X'} \colon C_{X'} \to X$ is such that $|k_{X'}|$ does not lift to a morphism from $C_{X'}$ to X'. Observing that $k_{X'} = q \circ \iota_{X'}$, and taking $X'' = C_{X'}$ and $g' = |k_{X'}|$ in the above universal property, we see that, since $g \circ g' = g \circ |q| \circ |\iota_{X'}| = |p| \circ |\iota_{X'}|$ lifts to $p \circ \iota_{X'} \colon C_{X'} \to Y$, $g' = |k_{X'}|$ must lift to a morphism from $C_{X'}$ to X'. This is a contradiction, so $X \leq X'$.

Since $X \leq X'$, there is a lifting $i\colon X \to X'$ of the identity. Note that $|(f \circ i) \circ q| = |p|$, which follows from $|i| = \mathrm{id}_{|X|}$, $|f| = g$, and $g \circ |q| = |p|$. Since $|\_|$ is faithful, $p = (f \circ i) \circ q$. The fact that $f \circ i$ is the unique morphism h such that $p = h \circ q$ follows from the fact that g is unique such that $|p| = g \circ |q|$ and that $|\_|$ is faithful. Therefore, q is regular epi.

This exhibits X as a coequalizer of $(\equiv) \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} Z$. Since Z is a small coproduct of objects in $\mathcal{C}$, X is a small colimit of objects of $\mathcal{C}$. ☐

It is time we relaxed the strong productivity condition.

Definition 3.18 (Productive) *Let $|\_|$ be a topological functor from a category $\mathbf{C}$ with finite products to a category $\mathbf{D}$. A class $\mathcal{C}$ of objects of $\mathbf{C}$ is* productive *if and only if every object of $\mathcal{C}$ is exponentiable in $\mathbf{C}$, and products of pairs of elements of $\mathcal{C}$ are $\mathcal{C}$-generated.*

As every object of $\mathcal{C}$ is $\mathcal{C}$-generated (Lemma 3.9), strongly productive classes are productive.

Proposition 3.19 *Let $(\mathbf{C}, |\_|)$ be a well-fibered topological construct, and $\mathcal{C}$ be a productive class of objects of $\mathbf{C}$ containing a non-empty object. The class $\overline{\mathcal{C}}$ of all exponentiable $\mathcal{C}$-generated objects is the largest class $\mathcal{C}'$ of exponentiable objects such that the $\mathcal{C}$-generated objects are exactly the $\mathcal{C}'$-generated objects.*

Proof. If X is $\mathcal{C}$-generated, then it is a colimit of objects in $\mathcal{C}$ by Proposition 3.17. Each such space is exponentiable by definition, and $\mathcal{C}$-generated by Lemma 3.9, hence in $\overline{\mathcal{C}}$. Using Lemma 3.16, we conclude that X is $\overline{\mathcal{C}}$-generated. Conversely, if X is $\overline{\mathcal{C}}$-generated, then it is a colimit of objects from $\overline{\mathcal{C}}$ by Proposition 3.17, hence a colimit of $\mathcal{C}$-generated objects, so X is itself $\mathcal{C}$-generated by Lemma 3.16.

If $\mathcal{C}'$ is another class of exponentiable objects such that the $\mathcal{C}'$-generated objects are $\mathcal{C}$-generated, then in particular every object of $\mathcal{C}'$ is both exponentiable and $\mathcal{C}$-generated (Lemma 3.9), hence in $\overline{\mathcal{C}}$, whence the claim of maximality. ☐

We need the following proposition to prove Lemma 3.21 below, but this is of independent interest. It describes a case where products $X \times_{\mathcal{C}} Y$ are ordinary products in $\mathbf{C}$. This is analogous to (and generalizes) the fact that the product of two compactly generated spaces X, Y in the category of compactly generated spaces is their ordinary topological product as soon as X or Y is locally compact.

Proposition 3.20 *Let $(\mathbf{C}, |\_|)$ be a well-fibered topological construct, and $\mathcal{C}$ be a productive class of objects of $\mathbf{C}$ containing a non-empty object. For any two $\mathcal{C}$-generated objects X and Y, the product $X \times Y$ in $\mathbf{C}$ is $\mathcal{C}$-generated whenever X or Y is exponentiable.*

Proof. We first claim that $X \times C$ is $\mathcal{C}$-generated for every $C \in \mathcal{C}$. Since X is $\mathcal{C}$-generated, it is a colimit of objects C_i, $i \in I$, of $\mathcal{C}$ by Proposition 3.17. Since C is exponentiable (as every object of $\mathcal{C}$ is), $\_ \times C$ is left adjoint, hence preserves colimits. It follows that $X \times C$ is a colimit of objects of the form $C_i \times C$. Since the latter are all $\mathcal{C}$-generated by productivity, $X \times C$ is $\mathcal{C}$-generated, by Lemma 3.16.

Without loss of generality, assume X is exponentiable. Since Y is $\mathcal{C}$-generated, Y is a colimit of objects C'_j, $j \in J$, of $\mathcal{C}$, by Proposition 3.17. Since X is exponentiable, $X \times \_$ is a left adjoint, hence preserves colimits. It follows that $X \times Y$ is a colimit of objects $X \times C_j$, $j \in J$. We have seen that each was $\mathcal{C}$-generated, so $X \times Y$ is, too, by Lemma 3.16. ☐

Lemma 3.21 *Let $(\mathbf{C}, |\_|)$ be a well-fibered topological construct, and $\mathcal{C}$ be a productive class of objects of $\mathbf{C}$ containing a non-empty object. The class $\overline{\mathcal{C}}$ is strongly productive.*

Proof. Every binary product $X \times Y$ of objects of $\overline{\mathcal{C}}$ is $\mathcal{C}$-generated by Proposition 3.20, and exponentiable (the right adjoint to $(X \times Y) \times \_$ being $(\_^Y)^X$, or equivalently $(\_^X)^Y$). ☐

This allows us to relax the conditions on Theorem 3.7.

Theorem 3.22 *Let $(\mathbf{C}, |\_|)$ be a well-fibered topological construct, and let $\mathcal{C}$ be a productive class of objects of $\mathbf{C}$ containing a non-empty object. The category $\mathbf{Map}_{|\_|,\mathcal{C}}$ coincides with $\mathbf{Map}_{|\_|,\overline{\mathcal{C}}}$, and is Cartesian-closed. The exponentials are as in Theorem 3.7.*

Proof. We must first show that the $\overline{\mathcal{C}}$-maps are exactly the $\mathcal{C}$-maps. Since $\mathcal{C} \subseteq \overline{\mathcal{C}}$, every $\overline{\mathcal{C}}$-map is a $\mathcal{C}$-map. Conversely, let g be a $\mathcal{C}$-map from X to Y, and $k \colon C \to X$ be a $\overline{\mathcal{C}}$-probe: we wish to show that $g \circ |k|$ lifts to a morphism from C to Y. Since C is $\mathcal{C}$-generated, by the universal property of $|\_|$-sinks, it suffices to show that $(g \circ |k|) \circ |k'|$ lifts to a morphism from C' to Y for every $\mathcal{C}$-probe $k' \colon C' \to C$. Since $k \circ k'$ is a $\mathcal{C}$-probe, and g is a $\mathcal{C}$-map, $g \bullet (k \circ k')$ is a lifting of $(g \circ |k|) \circ |k'|$.

We have established that $\mathbf{Map}_{|\_|,\mathcal{C}}$ is the same category as $\mathbf{Map}_{|\_|,\overline{\mathcal{C}}}$. It is, in particular, Cartesian-closed.

It also follows that $\mathcal{C}[X, Y] = \overline{\mathcal{C}}[X, Y]$, and that application and currification are given by the same formulas as in Theorem 3.7. Finally, we know that an exponential of X and Y is $[Y^X]_{\overline{\mathcal{C}}}$, but we wish to show that $[Y^X]_{\mathcal{C}}$ is one, too. It is enough to check that $[Y^X]_{\mathcal{C}}$ is isomorphic to $[Y^X]_{\overline{\mathcal{C}}}$. To this end, we first prove that $\mathcal{C}[Y^X]_{\mathcal{C}} \leq [Y^X]_{\overline{\mathcal{C}}} \leq [Y^X]_{\mathcal{C}}$.

$[Y^X]_{\overline{\mathcal{C}}} \leq [Y^X]_{\mathcal{C}}$. $[Y^X]_{\overline{\mathcal{C}}}$ is an object in the fiber of $\mathcal{C}[X, Y]$ such that $\_ \bullet k$ lifts to a morphism from $[Y^X]_{\overline{\mathcal{C}}}$ to Y^C for every $\overline{\mathcal{C}}$-probe $k \colon C \to X$, and in particular for every $\mathcal{C}$-probe $k \colon C \to X$. $[Y^X]_{\mathcal{C}}$ is the coarsest, whence the inequality.

$\mathcal{C}[Y^X]_{\mathcal{C}} \leq [Y^X]_{\overline{\mathcal{C}}}$. This is the complicated part. By Lemma 3.12, it is enough to show that the identity map $\mathrm{id}_{\mathcal{C}[X,Y]}$ on $\mathcal{C}[X, Y]$ is a $\mathcal{C}$-map from $[Y^X]_{\mathcal{C}}$ to $[Y^X]_{\overline{\mathcal{C}}}$. We have seen that it was equivalent to show that it was a $\overline{\mathcal{C}}$-map, i.e., a morphism in $\mathbf{Map}_{|\_|,\overline{\mathcal{C}}}$ from $[Y^X]_{\mathcal{C}}$ to $[Y^X]_{\overline{\mathcal{C}}}$. Since $\mathbf{Map}_{|\_|,\overline{\mathcal{C}}}$ is Cartesian-closed, and since $[Y^X]_{\overline{\mathcal{C}}}$ is the exponential from X to Y there (Theorem 3.7), we only have to exhibit $\mathrm{id}_{\mathcal{C}[X,Y]}$ as the currification of a morphism from $[Y^X]_{\mathcal{C}} \times X$ to Y in $\mathbf{Map}_{|\_|,\overline{\mathcal{C}}}$. The latter, as a morphism in $\mathbf{Set}$, must be the function *app* that maps $(f, x) \in \mathcal{C}[X, Y] \times |X|$ to $f(x)$. To show that this is indeed a morphism in $\mathbf{Map}_{|\_|,\overline{\mathcal{C}}}$, we must show that *app* is a $\overline{\mathcal{C}}$-map. Let $k \colon C \to [Y^X]_{\mathcal{C}} \times X$ be a $\overline{\mathcal{C}}$-probe, which we write as $\langle k_1, k_2 \rangle$. We must show that $app \circ |k|$, the map that sends $c \in |C|$ to $|k_1|(c)(|k_2|(c)) \in |Y|$, lifts to a morphism from

C to Y in $\mathbf{C}$. Using Proposition 3.17, write C as the colimit of a functor $F\colon \mathbf{J} \to \mathbf{C}$ such that $F(J) \in \mathcal{C}$ for every object J of $\mathbf{J}$; let also $\iota_J\colon F(J) \to C$ be the colimit maps.

Since $k_2 \circ \iota_J$ is a $\mathcal{C}$-probe, from $F(J)$ to X, $\_ \bullet (k_2 \circ \iota_J)$ is a morphism from $[Y^X]_{\mathcal{C}}$ to $Y^{F(J)}$ in $\mathbf{C}$. Let f_J be the morphism $\mathrm{App} \circ \langle (\_ \bullet (k_2 \circ \iota_J)) \circ k_1 \circ \iota_J, \mathrm{id}_{F(J)} \rangle$. We claim that $f_J\colon F(J) \to Y$ lifts $app \circ |k| \circ |\iota_J|$. To this end, we compute the following, for every $a \in |F(J)|$:

$$
\begin{aligned}
|f_J|(a) &= |\mathrm{App}|(|\_ \bullet (k_2 \circ \iota_J)|(|k_1|(|\iota_J|(a))), a) \\
&= |\mathrm{App}|(|k_1|(|\iota_J|(a)) \circ |k_2 \circ \iota_J|, a) \\
&= |k_1|(|\iota_J|(a))(|k_2|(|\iota_J|(a))) = (app \circ |k|)(|\iota_J|(a)).
\end{aligned}
$$

We check that $f_K \circ F(j) = f_J$ for every morphism $j\colon J \to K$ in $\mathbf{J}$. Since $|\_|$ is faithful, it is enough to check that $|f_K| \circ |F(j)| = |f_J|$, which follows from the fact that $|\iota_J| \circ |F(j)| = |\iota_J \circ F(j)| = |\iota_K|$. By the construction of C as a colimit of F, there is a unique morphism $f\colon C \to Y$ such that $f_J = f \circ \iota_J$ for every object J of $\mathbf{J}$. We claim that f is the desired lifting of $app \circ |k|$. It suffices to check that $|f|(c) = (app \circ |k|)(c)$ for every $c \in |C|$. Since $|\_|$ preserves colimits, it suffices to show that $|f|(|\iota_J|(a)) = (app \circ |k|)(|\iota_J|(a))$ for every object J of $\mathbf{J}$, and every $a \in |F(J)|$. We have seen that both sides of this equation were equal to $|f_J|(a)$. This concludes the argument that $\mathcal{C}[Y^X]_{\mathcal{C}} \leq [Y^X]_{\overline{\mathcal{C}}}$.

Now that we know that $\mathcal{C}[Y^X]_{\mathcal{C}} \leq [Y^X]_{\overline{\mathcal{C}}} \leq [Y^X]_{\mathcal{C}}$, we claim that the three objects are isomorphic in $\mathbf{Map}_{|\text{-}|,\mathcal{C}}$. There are liftings of $\mathrm{id}_{\mathcal{C}[X,Y]}$, from $\mathcal{C}[Y^X]_{\mathcal{C}}$ to $[Y^X]_{\overline{\mathcal{C}}}$, and from $[Y^X]_{\overline{\mathcal{C}}}$ to $[Y^X]_{\mathcal{C}}$. In particular, $\mathrm{id}_{\mathcal{C}[X,Y]}$ is a $\mathcal{C}$-map from $\mathcal{C}[Y^X]_{\mathcal{C}}$ to $[Y^X]_{\overline{\mathcal{C}}}$, and from $[Y^X]_{\overline{\mathcal{C}}}$ to $[Y^X]_{\mathcal{C}}$. It is also a $\mathcal{C}$-map from $[Y^X]_{\mathcal{C}}$ to $\mathcal{C}[Y^X]_{\mathcal{C}}$, by Lemma 3.12. These three identities therefore form a circle of three isomorphisms in $\mathbf{Map}_{|\text{-}|,\mathcal{C}}$ between $\mathcal{C}[Y^X]_{\mathcal{C}}$, $[Y^X]_{\overline{\mathcal{C}}}$, and $[Y^X]_{\mathcal{C}}$. Since the second one is an exponential of X and Y, so is the third one. $\square$

Theorem 3.23 *Let $(\mathbf{C}, |\_|)$ be a well-fibered topological construct, and let $\mathcal{C}$ be a productive class of objects of $\mathbf{C}$ containing a non-empty object. The category $\mathbf{C}_{|\text{-}|,\mathcal{C}}$ coincides with $\mathbf{C}_{|\text{-}|,\overline{\mathcal{C}}}$, and is Cartesian-closed. The finite products and exponentials are as in Theorem 3.14.*

Proof. We first show that $\mathbf{C}_{|\text{-}|,\mathcal{C}}$ is the same category as $\mathbf{C}_{|\text{-}|,\overline{\mathcal{C}}}$. This amounts to the fact that an object is $\mathcal{C}$-generated, i.e., a colimit of objects from $\mathcal{C}$ (Proposition 3.17), if and only if it is a colimit of objects from $\overline{\mathcal{C}}$, which is clear.

It follows that $\mathbf{C}_{|\text{-}|,\mathcal{C}}$ is Cartesian-closed, by Theorem 3.14.

To check that the finite products and exponentials are as in Theorem 3.14 (e.g., that $\mathcal{C}(X \times Y) = \overline{\mathcal{C}}(X \times Y)$), we show that $\mathcal{C}X = \overline{\mathcal{C}}X$ for every object X of $\mathbf{C}$. This is obvious, since the functor $\mathcal{C}\colon \mathbf{Map}_{|\text{-}|,\mathcal{C}} \to \mathbf{C}_{|\text{-}|,\mathcal{C}}$ is inverse to the functor I that is the identity on objects and coincides with $|\_|$ on morphisms (Proposition 3.13), and so is the functor $\overline{\mathcal{C}}\colon \mathbf{Map}_{|\text{-}|,\overline{\mathcal{C}}} \to \mathbf{C}_{|\text{-}|,\overline{\mathcal{C}}}$. Since $\mathbf{Map}_{|\text{-}|,\mathcal{C}} = \mathbf{Map}_{|\text{-}|,\overline{\mathcal{C}}}$ (Theorem 3.22) and $\mathbf{C}_{|\text{-}|,\mathcal{C}} = \mathbf{C}_{|\text{-}|,\overline{\mathcal{C}}}$, the two functors $\mathcal{C}$ and $\overline{\mathcal{C}}$ are the same. $\square$

When $\mathbf{C} = \mathbf{Top}$, we retrieve the results of [ELS04], and indeed our arguments are categorical versions of theirs (up to our introduction of strongly productive classes, in an attempt to make the presentation simpler). In particular, if $\mathcal{C}$ is taken to be the class of compact Hausdorff spaces, then $\mathbf{Top}_{|\_|,\mathcal{C}}$ is the Cartesian-closed category of compactly-generated spaces.

We shall now apply all this to streams and prestreams.

4 Streams, Prestreams

A prestream $\mathcal{X}$ is a topological space X with a *precirculation*, i.e., a collection $(\sqsubseteq_U)_{U \in \mathcal{O}(X)}$, indexed by open subsets U of X, where $\sqsubseteq_U$ is a preorder on U, and such that whenever $U \subseteq V$ then $x \sqsubseteq_U y$ implies $x \sqsubseteq_V y$. The space X itself is the *carrier* of the prestream.

Prestreams form a category $\mathbf{Prestr}$. The *prestream morphisms* f from $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ to $\mathcal{Y} = (Y, (\preceq_V)_{V \in \mathcal{O}(Y)})$ are those continuous maps $f \colon X \to Y$ that are *locally monotonic*, in the sense that, for every open subset V of Y, for all $x, y \in f^{-1}(V)$, if $x \sqsubseteq_{f^{-1}(V)} y$ then $f(x) \preceq_V f(y)$.

Example 4.1 *Call* preordered space *any topological space X with a partial ordering $\sqsubseteq$. Any preordered space defines a canonical prestream, where $\sqsubseteq_U$ is the restriction of $\sqsubseteq$ to U, for every open subset U of X.*

Example 4.2 ($\overrightarrow{\mathbb{R}}$) *Consider the real line $\mathbb{R}$, and let $\sqsubseteq_U^{\mathbb{R}}$ be defined by $t \sqsubseteq_U^{\mathbb{R}} t'$ if and only if the whole interval $[t, t']$ is included in U. This is a prestream (which we shall denote as $\overrightarrow{\mathbb{R}}$) that does not arise from a preordered space. E.g., for $U = (-3, -1) \cup (1, 3)$, the inequality $-2 \sqsubseteq_U^{\mathbb{R}} 2$ fails. We define similar prestreams $\overrightarrow{[a, b]}$ on any compact subinterval $[a, b]$ of $\mathbb{R}$: these prestreams were introduced by Haucourt [Hau09b, text before Proposition 3.8]. This is a fundamental example. We shall come back to it again later.*

Proposition 4.3 $\mathbf{Prestr}$ *is topological over* $\mathbf{Top}$*. More precisely, the forgetful functor that maps each prestream $(X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ to the underlying topological space, and each prestream morphism to the underlying continuous map, is topological.*

Proof. This forgetful functor, call it $|\_|$, is clearly faithful and amnestic. Given a $|\_|$-source $(g_i \colon X \to |\mathcal{A}_i|)_{i \in I}$, where X is a topological space (and each g_i is continuous), let $(\sqsubseteq_U)_{U \in \mathcal{O}(X)}$ be the coarsest precirculation on X that makes every g_i a prestream morphism, namely: $x \sqsubseteq_U y$ if and only if, for every $i \in I$ and every open subset V of $|\mathcal{A}_i|$ such that $U \subseteq g_i^{-1}(V)$, $g_i(x) \preceq_{iV} g_i(y)$, where $(\preceq_{iV})_{V \in \mathcal{O}(|\mathcal{A}_i|)}$ is the precirculation on $\mathcal{A}_i$. Let $\mathcal{Z} = (Z, (\leq_W)_{W \in \mathcal{O}(Z)})$ be a prestream. For every continuous map $g \colon Z \to X$, if $g_i \circ g$ is a prestream morphism (i.e., lifts to morphism in $\mathbf{Prestr}$) from $\mathcal{Z}$ to $\mathcal{A}_i$ for every $i \in I$, the definition of $\sqsubseteq_U$ makes it clear that g is a prestream morphism from $\mathcal{Z}$ to $(X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$. $\square$

Corollary 4.4 $\mathbf{Prestr}$ *is both complete and cocomplete.*

Moreover, the carrier of prestream limits (resp., colimits) are computed as the corresponding topological limit (resp., colimit) of the carriers. We give explicit definitions below.

Prestream limits. We start with products. For any element $\vec{x}$ of a product $X = \prod_{i \in I} X_i$ of sets, write x_i for its ith component, so that $\vec{x} = (x_i)_{i \in I}$. The ith projection $\pi_i \colon X \to X_i$ maps $\vec{x}$ to x_i. When every X_i is a topological space, and X is equipped with the product topology, π_i is both continuous and open. We write $f[A]$ for the image of the set A under the map f. In particular, for every open subset V of $\prod_{i \in I} X_i$, $\pi_i[V]$ is open.

The following is Example 3.25 of [Kri09].

Proposition 4.5 *Given a family of prestreams, $\mathcal{X}_i = (X_i, (\sqsubseteq_{iU})_{U \in \mathcal{O}(X_i)})$, $i \in I$, their product in $\mathbf{Prestr}$ is $\prod_{i \in I} \mathcal{X}_i$, defined as $(X, (\sqsubseteq_V)_{V \in \mathcal{O}(X)})$ where X is the topological product $\prod_{i \in I} X_i$, and $\vec{x} \sqsubseteq_V \vec{y}$ if and only if for every $i \in I$, $x_i \sqsubseteq_{iU_i} y_i$, where $U_i = \pi_i[V]$.*

Prestream products are fairly strange. For instance, in a binary product $\mathcal{X} \times \mathcal{Y}$, to decide whether $(x, y) \sqsubseteq_W (x', y')$, we need to find an *enclosing* open rectangle $U \times V \supseteq W$ ($U \in \mathcal{O}(X)$, $V \in \mathcal{O}(Y)$) such that x would be below x' relatively to U, and y below y' relatively to V. Although streams are more complex beasts, stream products will be more intuitive (see Lemma 4.24 below).

For every topological subspace A of a prestream $\mathcal{X} = (X, (\sqsubseteq_V)_{V \in \mathcal{O}(X)})$, define the preorder $\sqsubseteq_{|A,U}$ on the open subset U of A by $x \sqsubseteq_{|A,U} y$ if and only if $x \sqsubseteq_V y$ for every open subset V of X such that $U = V \cap A$. $(A, (\sqsubseteq_{|A,U})_{U \in \mathcal{O}(A)})$ is a prestream: this is the *subprestream* of X with carrier A.

The following is easy to see.

Proposition 4.6 *Let $\mathcal{X} = (X, (\sqsubseteq_V)_{V \in \mathcal{O}(X)})$ be a prestream, A be a subspace of X, and let $\mathcal{A}$ be the subprestream $(A, (\sqsubseteq_{|A,U})_{U \in \mathcal{O}(A)})$. The canonical injection $\iota \colon A \to X$ is a prestream morphism, and for every prestream morphism f from a prestream $\mathcal{Z} = (Z, (\preceq_W)_{W \in \mathcal{O}(Z)})$ to $\mathcal{X}$ such that $f[Z] \subseteq A$, there is a unique prestream morphism $f^{|A}$ from $\mathcal{Z}$ to $\mathcal{A}$ such that $\iota \circ f^{|A} = f$.*

Now we can construct any (small) limit in $\mathbf{Prestr}$ as a subprestream of a product prestream, just as in $\mathbf{Top}$. Given any functor $F \colon \mathbf{J} \to \mathbf{Prestr}$, where $\mathbf{J}$ is any small category, the limit of F is the subprestream of $\prod_{A \text{ object of } \mathbf{J}} F(A)$ consisting of all vectors $\vec{x}$ such that, for every morphism $f \colon A \to B$ in $\mathbf{J}$, $x_B = F(f)(x_A)$.

Prestream colimits. Coproducts are particularly elementary.

Proposition 4.7 *Given a family of prestreams $\mathcal{X}_i = (X_i, (\sqsubseteq_{iU})_{U \in \mathcal{O}(X_i)})$, $i \in I$, their coproduct $\coprod_{i \in I} \mathcal{X}_i$ is $(X, (\sqsubseteq_V)_{V \in \mathcal{O}(X)})$ where X is the topological coproduct $\coprod_{i \in I} X_i$, and $(i, x) \sqsubseteq_V (j, y)$ if and only if $i = j$ and $x \sqsubseteq_{i\, \iota_i^{-1}(V)} y$.*

(Recall that the elements of $\coprod_{i \in I} \mathcal{X}_i$ are the pairs (i, x) with $i \in I$, $x \in X_i$.)

Quotients are more interesting. Given an equivalence relation $\equiv$ on a space X, $X/{\equiv}$ is the space of equivalence classes $q_\equiv(x)$ of elements x of X under $\equiv$, and the topology of $X/{\equiv}$ is the finest that makes $q_\equiv$ continuous. In other words, a subset V of $X/{\equiv}$ is open if and only if $q_\equiv^{-1}(V)$ is open in X.

Given a binary relation R on some set, its reflexive-transitive closure R^* is the preorder with smallest graph containing the graph of R. Equivalently, $x \; R \; y$ if and only if there is a path $x_0 \; R \; x_1 \; R \; \ldots \; R \; x_n$, for some $n \in \mathbb{N}$, such that $x_0 = x$ and $x_n = y$. We shall build a relation of the form $(\sqsubseteq \cup \equiv)^*$, where $\sqsubseteq$ is a preorder and $\equiv$ is an equivalence relation. It is easy to check that $x \; (\sqsubseteq \cup \equiv)^* \; y$ if and only if there is a path $x_0 \equiv x_1 \sqsubseteq x_2 \equiv x_3 \sqsubseteq \ldots \sqsubseteq x_{2n} \equiv x_{2n+1}$ for some $n \in \mathbb{N}$, with $x = x_0$ and $y = x_{2n+1}$.

For every equivalence relation $\equiv$ on the carrier X of a prestream $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$, define the preorder $\sqsubseteq_V^{\equiv}$ on the open subset V of $X/\equiv$ by $q_{\equiv}(x) \sqsubseteq_V^{\equiv} q_{\equiv}(y)$ if and only if $x \; (\sqsubseteq_{q_{\equiv}^{-1}(V)} \cup \equiv_{|q_{\equiv}^{-1}(V)})^* \; y$. We call *quotient prestream* of $\mathcal{X}$ by $\equiv$ the prestream $\mathcal{X}/\equiv$ defined as $(X/\equiv, (\sqsubseteq_V^{\equiv})_{V \in \mathcal{O}(X/\equiv)})$.

Proposition 4.8 *The map $q_{\equiv}$ is a prestream morphism, and for every prestream morphism f from $\mathcal{X}$ to a prestream $\mathcal{Z} = (Z, (\preceq_W)_{W \in \mathcal{O}(Z)})$, there is a unique prestream morphism $f^{\equiv}$ from $\mathcal{X}/\equiv$ to $\mathcal{Z}$ such that $f^{\equiv} \circ q_{\equiv} = f$.*

Proof. It is easy to see that $\sqsubseteq_V^{\equiv}$ is well defined, i.e., that $q_{\equiv}(x) \sqsubseteq_V^{\equiv} q_{\equiv}(y)$ does not depend on the chosen representatives x and y in their equivalence classes.

To check that $q_{\equiv}$ is a prestream morphism, we must verify that it is locally monotonic. Let V be any open subset of $X/\equiv$, and check that whenever $x \sqsubseteq_{q_{\equiv}^{-1}(V)} y$, then $q_{\equiv}(x) \sqsubseteq_V q_{\equiv}(y)$. This is clear since $x \sqsubseteq_{q_{\equiv}^{-1}(V)} y$ implies $x \; (\sqsubseteq_{q_{\equiv}^{-1}(V)} \cup \equiv_{|q_{\equiv}^{-1}(V)})^*$ y.

Let us check the universal property. The map $f^{\equiv}$ is uniquely determined by $f^{\equiv}(q_{\equiv}(x)) = f(x)$, is continuous, and we must check that it is locally monotonic. Let W be an open subset of Z, and assume that $q_{\equiv}(x) \sqsubseteq_{f^{\equiv -1}(W)} q_{\equiv}(y)$. There is a path $x_0 \equiv_{|q_{\equiv}^{-1}(f^{\equiv -1}(W))} x_1 \sqsubseteq_{q_{\equiv}^{-1}(f^{\equiv -1}(W))} x_2 \equiv_{|q_{\equiv}^{-1}(f^{\equiv -1}(W))} x_3 \sqsubseteq_{q_{\equiv}^{-1}(f^{\equiv -1}(W))} \ldots \sqsubseteq_{q_{\equiv}^{-1}(f^{\equiv -1}(W))} x_{2n} \equiv_{|q_{\equiv}^{-1}(f^{\equiv -1}(W))} x_{2n+1}$ with $x_0 = x$ and $x_{2n+1} = y$. Since $q_{\equiv}^{-1}(f^{\equiv -1}(W)) = (f^{\equiv} \circ q_{\equiv})^{-1}(W) = f^{-1}(W)$, we have $x_{2i} \equiv_{|f^{-1}(W)} x_{2i+1}$ for each i, $0 \leq i \leq n$, so $f(x_{2i}) = f^{\equiv}(q_{\equiv}(x_{2i})) = f^{\equiv}(q_{\equiv}(x_{2i+1})) = f(x_{2i+1})$; and $x_{2i+1} \sqsubseteq_{f^{-1}(W)} x_{2i+2}$ for each i, $0 \leq i \leq n-1$, so $f(x_{2i+1}) \sqsubseteq_W f(x_{2i+2})$. As a consequence, $f(x) \sqsubseteq_W f(y)$, which is what we wanted to prove. □

It follows that given any functor $F \colon \mathbf{J} \to \mathbf{Prestr}$, where $\mathbf{J}$ is a small category, the colimit of F is the quotient prestream of $\coprod_{A \text{ object of } \mathbf{J}} F(A)$ by the equivalence relation $\equiv$ defined as the smallest such that $(A, x) \equiv (B, y)$ whenever $f \colon A \to B$ is a morphism in $\mathbf{J}$ and $y = F(f)(x)$.

Example 4.9 *Define $\equiv_{\mathbb{Z}}$ on $\mathbb{R}$ by $t \equiv_{\mathbb{Z}} t'$ iff $t - t' \in \mathbb{Z}$. The prestream quotient $\overrightarrow{\mathbb{R}}/\equiv_{\mathbb{Z}}$ (see Example 4.2 for $\overrightarrow{\mathbb{R}}$) is a good candidate for a* directed circle*, i.e., the one-dimensional circle with a preferred direction of rotation, say counterclockwise. Equating points of the circle with complex numbers e^{it}, $t \in \mathbb{R}$, one can check that e^{it} is less than or equal to $e^{it'}$ relatively to an open subset U if and only if one can pick t and t' modulo 2π so that $t \leq t'$ and U contains the arc $\{e^{i(rt'+(1-r)t)} \mid r \in [0, 1]\}$. See Figure 1 (left), where U is shown as a union of fat arcs. One can go from one point e^{it} to another one $e^{it'}$ provided we can reach the latter, turning counterclockwise while remaining in the current fat arc.*

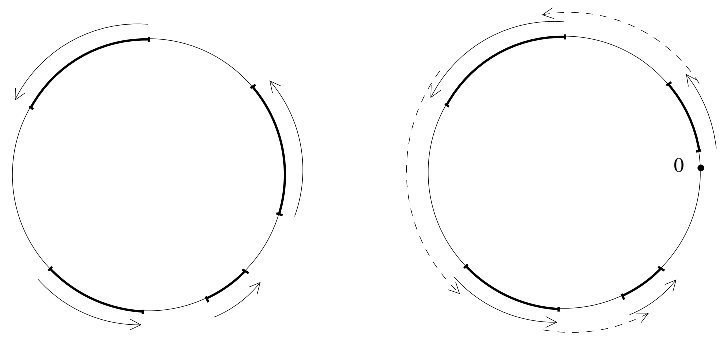


Figure 1: Two versions of the directed circle

Example 4.10 *One might instead try and build a directed circle as* $(\mathbb{R}, \leq)/{\equiv_{\mathbb{Z}}}$*, i.e., as a quotient of* $\mathbb{R}$ *as a preordered space, but this is uninteresting. Its precirculation is trivial: given any open subset* U *of* $\mathbb{R}/{\equiv}$*,* $x \sqsubseteq_U y$ *for* all *points* x *and* y*.*

Example 4.11 *Another directed circle candidate is the quotient* $([0,1], \leq)/{\equiv_{\mathbb{Z}}}$*. (This is due to Krishnan, using streams instead of prestreams. This difference is inconsequential: we shall see that quotients are the same in streams and in prestreams.) This one is not trivial, but produces a prestream with a distinguished base point. Indeed, equate the carrier of* $([0,1], \leq)/{\equiv_{\mathbb{Z}}}$ *with* $[0,1)$*. The quotient precirculation is then given as* $(\sqsubseteq_U)_{U \in \mathcal{O}([0,1))}$*, where: if* $0 \notin U$*, then* $x \sqsubseteq_U y$ *iff* $x \leq y$ *(one can jump from one fat arc to the next one, see Figure 1, right); if* $0 \in U$*, then* $\sqsubseteq_U$ *is trivial (*$x \sqsubseteq_U y$ *for all* $x, y \in U$*).*

Streams. A *stream* [Kri09] is a prestream $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ whose precirculation $(\sqsubseteq_U)_{U \in \mathcal{O}(X)}$ is a *circulation*, i.e., satisfies $\sqsubseteq_{\bigcup_{i \in I} U_i} = (\bigcup_{i \in I} \sqsubseteq_{U_i})^*$, for every family $(U_i)_{i \in I}$ of open subsets of X. Here is a definition that is operationally slightly simpler.

Lemma 4.12 *A precirculation* $(\sqsubseteq_U)_{U \in \mathcal{O}(X)}$ *on a topological space* X *is a circulation if and only if, for every open subset* U *of* X*, for all points* x, y *in* U *such that* $x \sqsubseteq_U y$*, for every open cover* $(U_i)_{i \in I}$ *of* U*,* $x \ (\bigcup_{i \in I} \sqsubseteq_{U \cap U_i})^* \ y$*.*

In other words, to compare two points x and y in $\sqsubseteq_U$, one can divide U infinitely, by taking an open cover $(U_i)_{i \in I}$ of U (with $U = \bigcup_{i \in I} U_i$), and finding a path from x to y such that any two consecutive points will be related by some $\sqsubseteq_{U_i}$.

Example 4.13 *A preordered space (Example 4.1) is almost never a stream. For example,* $\mathbb{R}$ *with its canonical ordering is not a stream. Consider* $U = (-3, -1) \cup (1, 3)$*, with its obvious two open covering. If* $\mathbb{R}$ *were a stream, since* $-2 \leq 2$ *in* U*, then there would exist a point* x *such that* $-2 \leq x$ *in* $(-3, -1)$ *and* $x \leq 2$ *in* $(1, 3)$*. This is impossible since* $(-3, -1) \cap (1, 3)$ *is empty.*

One can show that the prestream $\overrightarrow{\mathbb{R}} = (\mathbb{R}, (\sqsubseteq^{\mathbb{R}}_U)_{U \in \mathcal{O}(\mathbb{R})})$ of Example 4.2 is a stream. It is more than that:

Definition 4.14 *A* Haucourt circulation *on X is a precirculation $(\sqsubseteq_U)_{U\in\mathcal{O}(X)}$ such that, for every open subset U, for all $x, y \in U$, $x \sqsubseteq_U y$ if and only if there is a prestream morphism $\gamma\colon ([0,1], \leq) \to (X, (\sqsubseteq_U)_{U\in\mathcal{O}(X)})$ such that $\gamma(0) = x$, $\gamma(1) = y$, and the image of γ lies entirely inside U. (Such a map γ is called a* dipath from x to y in U.) *A* Haucourt stream *is a prestream whose precirculation is a Haucourt circulation.*

Haucourt streams arise from work by Haucourt [Hau09b] on the comparison between streams and Grandis' d-spaces [Gra03, Gra09]: see Appendix A. The following justifies the names 'Haucourt stream' and 'Haucourt circulation', rather than 'Haucourt prestream' and 'Haucourt precirculation'.

Lemma 4.15 *Every Haucourt stream is a stream.*

Proof. Let $(U_i)_{i\in I}$ be an open cover of an open subset U of X, and let $x, y \in U$ be such that $x \sqsubseteq_U y$. Since $\mathcal{X}$ is a Haucourt stream, there is a dipath γ from x to y in U. The open subset $\gamma^{-1}(U \cap U_i)$ can be written as the intersection of $[0, 1]$ with some union of open intervals of $\mathbb{R}$, say as $[0, 1] \cap \bigcup_{j\in J_i}(a_{ij}, b_{ij})$. Since $[0, 1]$ is compact, there is a finite set E of pairs (i, j), $i \in I$, $j \in J_i$ such that $[0, 1] \subseteq \bigcup_{i,j}(a_{ij}, b_{ij})$. It is easy to check that there is a finite, non-decreasing sequence of elements t_k in $[0, 1]$, $0 \leq k \leq N$, with $t_0 = 0$, $t_N = 1$, and such that for every k, $1 \leq k \leq N$, t_{k-1} and t_k both lie in some interval (a_{ij}, b_{ij}), $(i, j) \in E$. In particular, the whole interval $[t_{k-1}, t_k]$ is included in $(a_{ij}, b_{ij}) \subseteq \gamma^{-1}(U \cap U_i)$, so $\gamma(t_{k-1}) \sqsubseteq_{U\cap U_i} \gamma(t_k)$. This implies that $x\ (\bigcup_{i\in I} \sqsubseteq_{U\cap U_i})^*\ y$. We conclude by Lemma 4.12. □

Example 4.16 *The prestream $\overrightarrow{\mathbb{R}} = (\mathbb{R}, (\sqsubseteq^{\mathbb{R}}_U)_{U\in\mathcal{O}(\mathbb{R})})$ of Example 4.2 is a Haucourt stream, hence a stream. Indeed, if $t \sqsubseteq^{\mathbb{R}}_U t'$, then $[t, t'] \in U$, and it suffices to consider the dipath γ defined by $\gamma(r) = rt' + (1 - r)t$, $r \in [0, 1]$.*

Together with stream morphisms (which are just prestream morphisms between two streams), streams form a category $\mathbf{Str}$. Krishnan shows that $\mathbf{Str}$ is complete and cocomplete, as a consequence of the fact that the forgetful functor from $\mathbf{Str}$ to $\mathbf{Top}$ is topological [Kri09, Lemma 3.22].

While this is more complicated than in $\mathbf{Prestr}$, it is still instructive to obtain as concrete a description of limits and colimits in $\mathbf{Str}$. We start by examining the so-called cosheafification functor, from $\mathbf{Prestr}$ to $\mathbf{Str}$.

Definition 4.17 *Given any precirculation $(\sqsubseteq_U)_{U\in\mathcal{O}(X)}$ on a space X, its* one-step cosheafification $(\widehat{\sqsubseteq}_U)_{U\in\mathcal{O}(X)}$ *is defined by: for all $x, y \in U$, $x \mathrel{\widehat{\sqsubseteq}_U} y$ if and only if, for every open cover $(U_i)_{i\in I}$ of U, $x\ \left(\bigcup_{i\in I} \sqsubseteq_{U\cap U_i}\right)^*\ y$; equivalently, for every such open cover, there is a path $x = x_0 \sqsubseteq_{U_{i_1}} x_1 \sqsubseteq_{U_{i_2}} \ldots \sqsubseteq_{U_{i_n}} x_n = y$, for some $n \in \mathbb{N}$, $i_1, \ldots, i_n \in I$, and $x_j \in U_{i_j} \cap U_{i_{j+1}}$ whenever $1 \leq j \leq n - 1$.*

We write $Sh^1(X, (\sqsubseteq_U)_{U\in\mathcal{O}(X)})$ for $(X, (\widehat{\sqsubseteq}_U)_{U\in\mathcal{O}(X)})$, all also call it the one-step cosheafification of the prestream $(X, (\sqsubseteq_U)_{U\in\mathcal{O}(X)})$.

The one-step cosheafification map is a monotonic map on the complete lattice that is the fiber of X. Moreover, if $|\mathcal{X}| = X$, then $Sh^1(\mathcal{X}) \leq \mathcal{X}$. By Tarski's fixed point theorem, Sh^1 has a largest fixed point $Sh^\infty(\mathcal{X})$ below $\mathcal{X}$, which one can obtain by iterating Sh^1 transfinitely, starting from $\mathcal{X}$. Alternatively, $Sh^\infty(\mathcal{X})$ is the largest post-fixed point of Sh^1 below $\mathcal{X}$, namely, the coarsest object $\mathcal{Y}$ in the fiber of $\mathcal{X}$ such that $\mathcal{Y} \leq \mathcal{X}$ and $\mathcal{Y} \leq Sh^1(\mathcal{Y})$.

By definition $Sh^\infty(\mathcal{X})$ is a stream, and its (pre)circulation is the coarsest circulation finer than the precirculation of $\mathcal{X}$. Therefore $Sh^\infty(\mathcal{X})$ is exactly Krishnan's *cosheafification* $\mathcal{X}^!$ of $\mathcal{X}$.

Lemma 4.18 *Sh^1 defines a functor on* $\mathbf{Prestr}$*, whose action on morphisms is the identity.*

Proof. Let f be a prestream morphism from $(X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ to $(Y, (\preceq_V)_{V \in \mathcal{O}(Y)})$. We claim that whenever $x \mathrel{\widehat{\sqsubseteq}_{f^{-1}(V)}} y$, $f(x) \mathrel{\widehat{\preceq}_V} f(y)$. For every open cover $(V_i)_{i \in I}$ of V, $(f^{-1}(V_i))_{i \in I}$ is an open cover of $f^{-1}(V)$, so, if $x \mathrel{\widehat{\sqsubseteq}_{f^{-1}(V)}} y$, then there is a path $x = x_0 \sqsubseteq_{f^{-1}(V) \cap f^{-1}(V_{i_1})} x_1 \sqsubseteq_{f^{-1}(V) \cap f^{-1}(V_{i_2})} \cdots \sqsubseteq_{f^{-1}(V) \cap f^{-1}(V_{i_n})} x_n = y$. Clearly, $f(x) = f(x_0) \preceq_{V \cap V_{i_1}} f(x_1) \preceq_{V \cap V_{i_2}} \cdots \prec_{V \cap V_{i_n}} f(x_n) = f(y)$. ☐

This is, of course, a categorical construction on any fiber-small topological functor $|\_| \colon \mathbf{C} \to \mathbf{D}$. Let $|\_| \colon \mathbf{C} \to \mathbf{D}$ be a fiber-small topological functor. Call a functor S from $\mathbf{C}$ to $\mathbf{C}$ *deflationary* if and only if $S(X) \leq X$ for every object X of $\mathbf{C}$ (in particular, $|S(X)| = |X|$), and $|S(f)| = |f|$ for every morphism f. Let $Fix(S)$ be the full subcategory of $\mathbf{C}$ whose objects are the fixed points of S.

Given any deflationary functor S^1 on $\mathbf{C}$, S^1 is monotonic on the fibers, that is, if $X \leq Y$, then $S^1(X) \leq S^1(Y)$. Indeed, letting i be the lifting of the identity, from X to Y, $|S^1(i)| = |i|$ is the identity. Since $S^1(X) \leq X$, S^1 restricts to a monotonic map on the complete lattice of objects Y such that $Y \leq X$. In particular, S^1 has a largest fixed point $S^\infty(X)$ below X; $S^\infty(X)$ is the coarsest fixed point of S^1 finer than X in the fiber of $|X|$.

Lemma 4.19 *Let $|\_| \colon \mathbf{C} \to \mathbf{D}$ be a fiber-small topological functor, and S^1 be a deflationary functor on $\mathbf{C}$. For every object $\mathcal{X}$ of $Fix(S^1)$, for every object $\mathcal{Y}$ of $\mathbf{C}$, and for every morphism $g \colon \mathcal{X} \to \mathcal{Y}$, $|g|$ lifts to a morphism from $\mathcal{X}$ to $S^\infty(\mathcal{Y})$.*

Proof. Let $\mathcal{A}$ be the set of objects $\mathcal{Z}$ in the fiber of $|\mathcal{Y}|$ such that $|g|$ lifts to a morphism from $\mathcal{X}$ to $\mathcal{Z}$. In particular, $\mathcal{Y}$ is in $\mathcal{A}$. Also, for every $\mathcal{Z} \in \mathcal{A}$, $S^1(\mathcal{Z})$ is in $\mathcal{A}$, because $|g|$ lifts to $S^1(g) \colon S^1(\mathcal{X}) = \mathcal{X} \to S^1(\mathcal{Y})$. The family $(\mathrm{id}_{|\mathcal{Y}|} \colon |\mathcal{Y}| \to |\mathcal{Z}|)_{\mathcal{Z} \in \mathcal{Z}}$ is a $|\_|$-source, so we can build the coarsest object $\mathcal{Z}_0$ in the fiber of $|\mathcal{Y}|$ such that the identity lifts from $\mathcal{Z}_0$ to every object of $\mathcal{A}$: this is the greatest lower bound of $\mathcal{A}$. By the universal property for $|\_|$-sources, $|g|$ lifts to a morphism g_0 from $\mathcal{X}$ to $\mathcal{Z}_0$. This shows that $\mathcal{Z}_0$ is in $\mathcal{A}$, and is therefore the smallest element of $\mathcal{A}$.

Since $\mathcal{Z}_0$ is in $\mathcal{A}$, $S^1(\mathcal{Z}_0)$ is also in $\mathcal{A}$, so $\mathcal{Z}_0 \leq S^1(\mathcal{Z}_0)$. In other words, $\mathcal{Z}_0$ is a post fixed point of S^1. Since $S^\infty(Y)$ is the largest, $\mathcal{Z}_0 \leq S^\infty(Y)$. Write $i \colon \mathcal{Z}_0 \to S^\infty(\mathcal{Y})$ for the lifting of the identity: then $i \circ g_0 \colon \mathcal{X} \to S^\infty(\mathcal{Y})$ lifts $|g|$. ☐

Proposition 4.20 *Let $|\_| \colon \mathbf{C} \to \mathbf{D}$ be a fiber-small topological functor, and S^1 be a deflationary functor on $\mathbf{C}$.*

S^∞ defines a functor from $\mathbf{C}$ to $Fix(S)$ such that $|S^\infty(g)| = |g|$ for every morphism g.

S^∞ is right adjoint to the inclusion functor. The unit at the object $\mathcal{X}$ of $Fix(S^1)$ is the identity map from $\mathcal{X}$ to $S^\infty(\mathcal{X}) = \mathcal{X}$, the counit at the object $\mathcal{X}$ of $\mathbf{C}$ is the unique lifting of the identity as a morphism from $S^\infty(\mathcal{X})$ to $\mathcal{X}$.

In particular, $Fix(S^1)$ is a coreflective subcategory of $\mathbf{C}$.

Proof. The action of the functor S^∞ on morphisms is as follows. For every morphism $g\colon \mathcal{X} \to \mathcal{Y}$ in $\mathbf{C}$, $g \circ j$ is a morphism from $S^\infty(\mathcal{X}) \to \mathcal{Y}$, where $j\colon S^\infty(\mathcal{X}) \to \mathcal{X}$ lifts the identity, and then $|g \circ j| = |j|$ lifts to a unique morphism from $S^\infty(\mathcal{X})$ to $S^\infty(\mathcal{Y})$ by Lemma 4.19. This is $S^\infty(g)$. Checking that this defines an adjunction is straightforward: as usual, we check identity of morphisms $f = f'$ by checking $|f| = |f'|$. $\square$

Corollary 4.21 *Cosheafification is right adjoint to inclusion.* $\mathbf{Str}$ *is a coreflective subcategory of* $\mathbf{Prestr}$.

It follows that $\mathbf{Str}$ is complete, as shown by Krishnan [Kri09]. Another way of arriving at this result is to show that $\mathbf{Str}$ is also topological over $\mathbf{Top}$. This was Krishnan's way of proving completeness, and follows from a more general categorical result:

Lemma 4.22 *Let $|\_|\colon \mathbf{C} \to \mathbf{D}$ be a fiber-small topological functor, and S^1 be a deflationary functor on $\mathbf{C}$. The functor $|\_|\colon Fix(S^1) \to \mathbf{D}$ is topological.*

Proof. To make things clearer, write $|\_|'$ for the restriction of $|\_|$ to $Fix(S^1)$. The functor $|\_|'$ is clearly faithful and amnestic. Consider an arbitrary $|\_|'$-source $(g_i\colon D \to |A_i|')_{i\in I}$. This means that A_i is an object of $Fix(S^1)$ for each $i \in I$, and $(g_i\colon D \to |A_i|)_{i\in I}$ is a $|\_|$-source. There is a coarsest object B in the $|\_|$-fiber of D such that g_i $|\_|$-lifts to a morphism $f_i\colon B \to A_i$ for each $i \in I$. (We prefix "fiber" and "lift" with the intended functor, for readability.) Let $j\colon S^\infty(B) \to B$ be the unique $|\_|$-lifting of the identity, then g_i $|\_|'$-lifts to $f_i \circ j$ for each $i \in I$.

Let us show the universal property for $|\_|'$-sources. Let $g\colon |C|' \to D$ (where C is an object of $Fix(S^1)$), and assume that $g_i \circ g$ $|\_|'$-lifts to a morphism h_i from C to A_i in $Fix(S^1)$ for every $i \in I$. The morphism $g_i \circ g$ trivially $|\_|$-lifts to h_i in $\mathbf{C}$, and by the universal property for $|\_|$-sources, g $|\_|$-lifts to a morphism from C to B in $\mathbf{C}$. By Lemma 4.19, g also $|\_|$-lifts to a morphism h from C to $S^\infty(B)$ in $\mathbf{C}$. It follows that h is a morphism in $Fix(S^1)$, and $|\_|'$-lifts g. $\square$

We therefore retrieve:

Proposition 4.23 (Krishnan) $\mathbf{Str}$ *is topological over* $\mathbf{Top}$*, and therefore complete and cocomplete.*

Corollary 4.21 tells us a bit more: Sh^∞ preserves limits. Since $Sh^\infty(\mathcal{X}) = \mathcal{X}$ for every stream $\mathcal{X}$, one deduces immediately that limits of streams are computed as the cosheafification of the corresponding prestream limits. This is how Krishnan builds them [Kri09, Section 3.2]. For example, the stream product of the streams $\mathcal{X}$ and $\mathcal{Y}$ is $Sh^\infty(\mathcal{X} \times \mathcal{Y})$, where $\times$ denotes prestream product. In this case, we only need *one* iteration of Sh^1 to reach the fixpoint defining Sh^∞, as we shall see.

Recall that the open subsets W of a product $X \times Y$ are the unions of *open rectangles* $U \times V, U \in \mathcal{O}(X), V \in \mathcal{O}(Y)$.

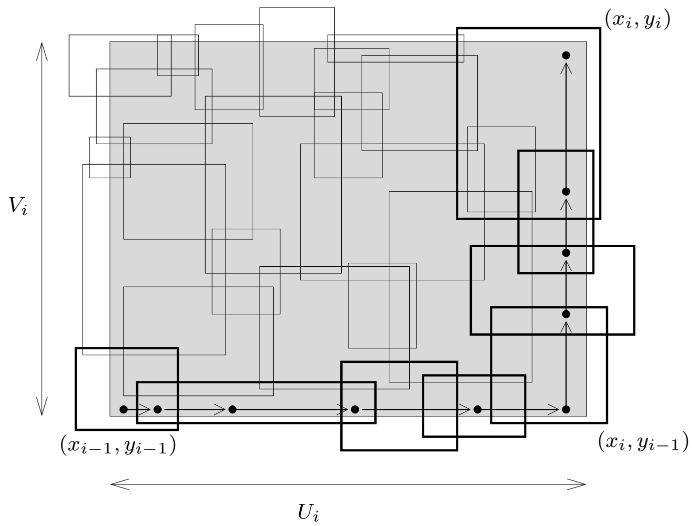


Figure 2: The construction of Lemma 4.24

Lemma 4.24 *Let $\mathcal{X} = (X, (\sqsubseteq_U)_{U\in\mathcal{O}(X)})$ and $\mathcal{Y} = (Y, (\preceq_V)_{V\in\mathcal{O}(Y)})$ be two streams. Define $(\sqsubseteq \otimes \preceq)_W$, for every $W \in \mathcal{O}(X \times Y)$, as the following preorder on W: $(x, y)\ (\sqsubseteq \otimes \preceq)_W\ (x', y')$ if and only if there are finitely many open rectangles $U_1 \times V_1$, $\ldots$, $U_n \times V_n$ included in W, and paths $x = x_0 \sqsubseteq_{U_1} x_1 \sqsubseteq_{U_2} \ldots \sqsubseteq_{U_n} x_n = x'$, $y = y_0 \sqsubseteq_{V_1} y_1 \sqsubseteq_{V_2} \ldots \sqsubseteq_{V_n} y_n = y'$, of the same length.*

The family $((\sqsubseteq \otimes \preceq)_W)_{W\in\mathcal{O}(X\times Y)}$ is a circulation.

Proof. That it is a precirculation is obvious. Let $(W_j)_{j\in J}$ be an open covering of W. Each W_j is a union of open rectangles, so one can refine this covering by another one, $(U'_k \times V'_k)_{k\in K}$, consisting of open rectangles. (By refining, we mean that every $U'_k \times V'_k$ is included in some W_j.) Assume $(x, y)\ (\sqsubseteq \otimes \preceq)_W\ (x', y')$, and consider paths $x = x_0 \sqsubseteq_{U_1} x_1 \sqsubseteq_{U_2} \ldots \sqsubseteq_{U_n} x_n = x'$, $y = y_0 \sqsubseteq_{V_1} y_1 \sqsubseteq_{V_2} \ldots \sqsubseteq_{V_n} y_n = y'$, where $U_1 \times V_1, \ldots, U_n \times V_n$ are open rectangles included in W.

The idea of the construction below is given in Figure 2, where $U_i \times V_i$ is the gray rectangle, and some of the rectangles $U'_k \times V'_k$ are shown as smaller rectangles covering $U_i \times V_i$. For each i, $1 \leq i \leq n$, we have $x_{i-1} \sqsubseteq_{U_i} x_i$. Now $U_i \times V_i \subseteq W \subseteq \bigcup_{k\in K}(U'_k \times V'_k)$, and y_{i-1} is in V_i, so $U_i \subseteq \bigcup_{k\in K_{i-1}} U'_k$, where $K_{i-1} = \{k \in K \mid y_{i-1} \in V'_k\}$. (The rectangles $U'_k \times V'_k$ with $k \in K_{i-1}$ are shown with fat borders, at the bottom of the figure.) Since $\mathcal{X}$ is a stream, there is a path $x_{i-1}\ (\bigcup_{k\in K_{j-1}} \sqsubseteq_{U_i\cap U'_k})^*\ x_i$. By pairing all the points on this path with y_{i-1}, we obtain a path $(x_{i-1}, y_{i-1})\ (\bigcup_{k\in K_{j-1}}(\sqsubseteq_{U_i\cap U'_k} \times \preceq_{V_i\cap V'_k}))^*\ (x_i, y_{i-1})$. In particular, $(x_{i-1}, y_{i-1})\ (\bigcup_{k\in K}(\sqsubseteq \otimes \preceq)_{W\cap(U'_k\times V'_k)})^*\ (x_i, y_{i-1})$, and since $(U'_k \times V'_k)_{k\in K}$ refines $(W_j)_{j\in J}$, $(x_{i-1}, y_{i-1})\ (\bigcup_{j\in J}(\sqsubseteq \otimes \preceq)_{W\cap W_j})^*\ (x_i, y_{i-1})$.

Symmetrically, $(x_i, y_{i-1})\ (\bigcup_{j\in J}(\sqsubseteq \otimes \preceq)_{W\cap W_j})^*\ (x_i, y_i)$. By concatenating these paths, we obtain $(x, y)\ (\bigcup_{j\in J}(\sqsubseteq \otimes \preceq)_{W\cap W_j})^*\ (x', y')$. □

Proposition 4.25 *Given any two streams $\mathcal{X}$ and $\mathcal{Y}$, their product in* $\mathbf{Str}$ *is given by the construction* $\sqsubseteq \otimes \preceq$ *of Lemma 4.24, and coincides with* $Sh^1(\mathcal{X} \times \mathcal{Y})$.

Proof. Let X be the carrier of $\mathcal{X}$, Y be that of $\mathcal{Y}$. Let $\mathcal{Z}$ denote $X \times Y$ with the circulation given in Lemma 4.24. Clearly, $\mathcal{Z} \leq \mathcal{X} \times \mathcal{Y}$, and since $\mathcal{Z}$ is a circulation, it follows that $\mathcal{Z} \leq Sh^\infty(\mathcal{X} \times \mathcal{Y})$.

Conversely, let $\mathcal{Z}'$ be any stream such that $\mathcal{Z}' \leq \mathcal{X} \times \mathcal{Y}$. Write $(\lhd_W)_{W \in \mathcal{O}(X \times Y)}$ for the circulation on $\mathcal{Z}'$. If $x \lhd_W y$, then $x \left(\bigcup_{i \in I} \lhd_{(U_i \times V_i)}\right)^* y$, where $(U_i \times V_i)_{i \in I}$ is the collection of open rectangles included in W, since $\mathcal{Z}'$ is a stream. Any two pairs related by $\lhd_{(U_i \times V_i)}$ are related by $\sqsubseteq_{U_i} \times \preceq_{V_i}$, since $\mathcal{Z}' \leq \mathcal{X} \times \mathcal{Y}$. It follows that $x \ (\sqsubseteq \otimes \preceq)_W \ y$, where we take the notation from Lemma 4.24. We have proved that $\mathcal{Z}' \leq \mathcal{Z}$. As this holds for every stream $\mathcal{Z}' \leq \mathcal{X} \times \mathcal{Y}$, it holds for the largest one, $Sh^\infty(\mathcal{X} \times \mathcal{Y})$. We conclude that $\mathcal{Z} = Sh^\infty(\mathcal{X} \times \mathcal{Y})$.

Finally, $Sh^\infty(\mathcal{X} \times \mathcal{Y}) \leq Sh^1(\mathcal{X} \times \mathcal{Y}) \leq \mathcal{Z}$, so all three are equal. ☐

While we are considering explicit constructions, we note that the situation is extremely simple for colimits.

Proposition 4.26 *Colimits in* $\mathbf{Str}$ *are computed as in* $\mathbf{Prestr}$*: any colimit in of streams taken in* $\mathbf{Prestr}$ *is a stream.*

Proof. It suffices to show this for coproducts and for quotients. This is clear for coproducts, in the light of Proposition 4.7. For quotients, let $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ be a stream, and recall that the precirculation on $X/\equiv$ is defined by $q_\equiv(x) \sqsubseteq^{\equiv}_V q_\equiv(y)$ if and only if $x \left(\sqsubseteq_{q_\equiv^{-1}(V)} \cup \equiv_{|q_\equiv^{-1}(V)}\right)^* y$. If so, then there is a path $x = x_0 \equiv_{|q_\equiv^{-1}(V)} x_1 \sqsubseteq_{q_\equiv^{-1}(V)} x_2 \equiv_{|q_\equiv^{-1}(V)} x_3 \sqsubseteq_{q_\equiv^{-1}(V)} \cdots \sqsubseteq_{q_\equiv^{-1}(V)} x_{2n} \equiv_{|q_\equiv^{-1}(V)} x_{2n+1} = y$ in V. For every open cover $(V_i)_{i \in I}$ of V, we observe that: for j odd, $x_j \sqsubseteq_{q_\equiv^{-1}(V)} x_{j+1}$ implies that $x_j \left(\bigcup_{i \in I} \sqsubseteq_{q_\equiv^{-1}(V \cap V_i)}\right)^* x_{j+1}$, so $q_\equiv(x_j) \left(\bigcup_{i \in I} \sqsubseteq^{\equiv}_{V \cap V_i}\right)^* q_\equiv(x_{j+1})$; for j even, $x_j \equiv_{|q_\equiv^{-1}(V)} x_{j+1}$ implies that $q_\equiv(x_j) = q_\equiv(x_{j+1})$ is in V, hence in some V_i, $i \in I$, whence $q_\equiv(x_j) \sqsubseteq^{\equiv}_{V \cap V_i} q_\equiv(x_{j+1})$. Combining these results, $q_\equiv(x) \left(\bigcup_{i \in I} \sqsubseteq^{\equiv}_{V \cap V_i}\right)^* q_\equiv(y)$. ☐

One can play exactly the same game with Haucourt streams, and the situation is even simpler. One possibility is to define $S^1(X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ now as X together with the precirculation $(\sqsubseteq^1_U)_{U \in \mathcal{O}(X)}$ defined by: for all $x, y \in U$, $x \sqsubseteq^1_U y$ if and only if there is a dipath from x to y inside U, and replay the arguments above.

We can do better using Haucourt's notion of *directed path from x to y* in $\mathcal{X}$ (resp., in an open subset U). By definition, this is a prestream morphism γ from $\overrightarrow{[0,1]}$ to $\mathcal{X}$ such that $\gamma(0) = x$ and $\gamma(1) = y$ (resp., and the image of γ lies entirely inside U); we have introduced the stream $\overrightarrow{[0,1]}$ in Example 4.2. Our *dipaths* are prestream morphisms from $([0,1], \leq)$ to $\mathcal{X}$ instead. Since the identity map is a prestream morphism from $\overrightarrow{[0,1]}$ to $([0,1], \leq)$, every dipath is a directed path, but the converse may fail. However, it does not make a difference whether we define Haucourt streams with dipaths or with directed paths:

Lemma 4.27 *Let $\mathcal{X}$ be a prestream. For every open subset U, for all points $x, y \in U$, the dipaths from x to y* in U *coincide with the directed paths from x to y* in U.

Proof. Let $(\sqsubseteq_U)_{U\in\mathcal{O}(X)}$ be the precirculation on $\mathcal{X}$. Every dipath from x to y in U is a directed path from x to y in U. Conversely, if γ is a directed path from x to y in U, then by definition for all $t, t' \in [0,1]$ such that $t \sqsubseteq^{\mathbb{R}}_{\gamma^{-1}(U)} t'$, $\gamma(t) \leq_U \gamma(t')$. Since $[0,1] \subseteq \gamma^{-1}(U)$, and $\sqsubseteq^{\mathbb{R}}_{[0,1]}$ is the usual ordering $\leq$, we obtain that $t \leq t'$ implies $\gamma(t) \leq_U \gamma(t')$. It follows that γ is a dipath from x to y in U. ☐

Definition 4.28 *Given any precirculation $(\sqsubseteq_U)_{U\in\mathcal{O}(X)}$ on a space X, its* haucourtification *$(\overrightarrow{\sqsubseteq}_U)_{U\in\mathcal{O}(X)}$ is defined by: for all $x, y \in U$, $x \overrightarrow{\sqsubseteq}_U y$ if and only if there is a directed path from x to y inside U.*

We write $\mathcal{H}(X, (\sqsubseteq_U)_{U\in\mathcal{O}(X)})$ for $(X, (\overrightarrow{\sqsubseteq}_U)_{U\in\mathcal{O}(X)})$, and also call it the haucourtification of the prestream $(X, (\sqsubseteq_U)_{U\in\mathcal{O}(X)})$.

Example 4.29 *The haucourtification of $([0,1], \leq)$ is $\overrightarrow{[0,1]}$. Similarly, $\mathcal{H}(\mathbb{R}, \leq) = \overrightarrow{\mathbb{R}}$. Indeed, we have seen in Example 4.16 that $\overrightarrow{[0,1]}$ and $\overrightarrow{\mathbb{R}}$ are Haucourt streams, and one easily checks that any Haucourt stream finer that $([0,1], \leq)$, resp., $(\mathbb{R}, \leq)$, must be finer than $\overrightarrow{[0,1]}$, resp., $\overrightarrow{\mathbb{R}}$.*

The following says in particular that any iteration of $\mathcal{H}$ starting from $\mathcal{X}$ will stop after the first step, i.e., $\mathcal{H}(\mathcal{H}(\mathcal{X})) = \mathcal{H}(\mathcal{X})$. We are in the nice case where, by taking $S^1 = \mathcal{H}$, S^∞ is equal to S^1 itself.

Lemma 4.30 *The haucourtification of a prestream $\mathcal{X}$ is a Haucourt stream.*

Proof. Assume $x \overrightarrow{\sqsubseteq}_U y$. By definition there is a directed path $\gamma\colon \overrightarrow{[0,1]} \to \mathcal{X}$ from x to y in U. We must show that it is a prestream morphism from $\overrightarrow{[0,1]}$ to $\mathcal{H}(\mathcal{X})$. Let V be any open subset of the carrier of $\mathcal{X}$, and write $(\sqsubseteq_U)_{U\in\mathcal{O}(X)}$ for the precirculation on $\mathcal{X}$. We must show that for all $t, t' \in [0,1]$ such that $t \leq t'$ and $[t, t'] \subseteq \gamma^{-1}(V)$, $\gamma(t) \overrightarrow{\sqsubseteq}_V \gamma(t')$, i.e., we must find a directed path $\gamma'\colon \overrightarrow{[0,1]} \to \mathcal{X}$ from $\gamma(t)$ to $\gamma(t')$ in $\gamma^{-1}(V)$. We just take the reparameterization $\gamma'(r) = \gamma(rt' + (1-r)t)$. ☐

It is clear that $\mathcal{H}$ defines a deflationary functor on $\mathbf{Prestr}$. The fact that it is a functor such that $|\mathcal{H}(f)| = |f|$ follows from the fact that if there is a directed path γ from x to y in $f^{-1}(V)$, where V is open in $\mathcal{Y}$, and f is a prestream morphism from $\mathcal{X}$ to $\mathcal{Y}$, then $f \circ \gamma$ is a directed path from $f(x)$ to $f(y)$ in V. Lemma 4.22 then applies, so:

Proposition 4.31 $\mathbf{HStr}$ *is topological over* $\mathbf{Top}$*, and therefore complete and cocomplete.*

Proposition 4.20 also applies:

Proposition 4.32 *Haucourtification, as a functor from* $\mathbf{Prestr}$ *to the category* $\mathbf{HStr}$ *of Haucourt streams, is right adjoint to inclusion.* $\mathbf{HStr}$ *is a coreflective subcategory of* $\mathbf{Prestr}$.

In particular, $\mathbf{HStr}$ is complete, and limits are computed as the haucourtifications of the corresponding prestream limits. This time, $\mathcal{H}(\mathcal{X} \times \mathcal{Y})$ is directly the product in $\mathbf{HStr}$ of the two Haucourt streams $\mathcal{X}$ and $\mathcal{Y}$, and we do not need an intermediate result such as Lemma 4.24. The latter, nonetheless, has an analog here.

Lemma 4.33 *Let $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ and $\mathcal{Y} = (Y, (\preceq_V)_{V \in \mathcal{O}(Y)})$ be two Haucourt streams. The circulation $((\sqsubseteq \otimes \preceq)_W)_{W \in \mathcal{O}(X \times Y)}$ defined in Lemma 4.24 is a Haucourt circulation, and coincides with the circulation on $\mathcal{H}(\mathcal{X} \times \mathcal{Y})$.*

Proof. It suffices to prove the second claim. Write $(\triangleleft_W)_{W \in \mathcal{O}(X \times Y)}$ for the circulation on $\mathcal{H}(\mathcal{X} \times \mathcal{Y})$. Let $W \in \mathcal{O}(X \times Y)$, and assume $(x, y) \triangleleft_W (x', y')$. Since $\mathcal{H}(\mathcal{X} \times \mathcal{Y})$ is a Haucourt stream (Lemma 4.30), hence a stream (Lemma 4.15), there is a path $(x, y) = (x_0, y_0) \triangleleft_{U_1 \times V_1} (x_1, y_1) \triangleleft_{U_2 \times V_2} \ldots \triangleleft_{U_n \times V_n} (x_n, y_n) = (x', y')$. It follows easily that $(x, y)\ (\sqsubseteq \otimes \preceq)_W\ (x', y')$.

Conversely, if $(x, y)\ (\sqsubseteq \otimes \preceq)_W\ (x', y')$, then by definition there are finitely many open rectangles $U_1 \times V_1, \ldots, U_n \times V_n$ included in W, and two paths $x = x_0 \sqsubseteq_{U_1} x_1 \sqsubseteq_{U_2} \ldots \sqsubseteq_{U_n} x_n = x'$, $y = y_0 \sqsubseteq_{V_1} y_1 \sqsubseteq_{V_2} \ldots \sqsubseteq_{V_n} y_n = y'$. (Without loss of generality, assume $n \geq 1$.) Since $\mathcal{X}$ and $\mathcal{Y}$ are Haucourt streams, for each i, $1 \leq i \leq n$, there is a directed path γ_i from x_{i-1} to x_i in U_i, and a directed path δ_i from y_{i-1} to y_i in V_i. The pairing $\langle \gamma_i, \delta_i \rangle \colon \overrightarrow{[0,1]} \to \mathcal{X} \times \mathcal{Y}$ is then a directed path from (x_{i-1}, x_i) to (y_{i-1}, y_i) in $U_i \times V_i$, hence in W. Build the concatenation of these directed paths. By definition, this is the map $\kappa \colon [0, 1] \to X \times Y$ such that $\kappa(t) = \langle \gamma_i, \delta_i \rangle (nt - i + 1)$ for all $t \in [(i-1)/n, i/n]$, $1 \leq i \leq n$. It is easy to see that κ is a directed path from (x, y) to (x', y') in W, so $(x, y) \triangleleft_W (x', y')$. ☐

We finish with colimits.

Proposition 4.34 *Colimits in* $\mathbf{HStr}$ *are computed as in* $\mathbf{Prestr}$*: any colimit in of Haucourt streams taken in* $\mathbf{Prestr}$ *is a Haucourt stream.*

Proof. This is clear for coproducts. For quotients, let $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ be a stream, and recall that the precirculation on $X/{\equiv}$ is defined by $q_\equiv(x) \sqsubseteq^{\equiv}_V q_\equiv(y)$ if and only if $x\ (\sqsubseteq_{q_\equiv^{-1}(V)} \cup \equiv_{|q_\equiv^{-1}(V)})^*\ y$. If so, then there is a path $x = x_0 \equiv_{|q_\equiv^{-1}(V)} x_1 \sqsubseteq_{q_\equiv^{-1}(V)} x_2 \equiv_{|q_\equiv^{-1}(V)} x_3 \sqsubseteq_{q_\equiv^{-1}(V)} \cdots \sqsubseteq_{q_\equiv^{-1}(V)} x_{2n} \equiv_{|q_\equiv^{-1}(V)} x_{2n+1} = y$ in V. Since $\mathcal{X}$ is a Haucourt stream, there are dipaths γ_j from x_j to x_{j+1} in $q_\equiv^{-1}(V)$ for each odd j. They induce dipaths $q_\equiv \circ \gamma_j$ from $q_\equiv(x_j)$ to $q_\equiv(x_{j+1})$ in V. For j even, $q_\equiv(x_j) = q_\equiv(x_{j+1})$. By concatenating the dipaths $q_\equiv \circ \gamma_1, q_\equiv \circ \gamma_3, \ldots, q_\equiv \circ \gamma_{2n-1}$, we obtain a dipath from $q_\equiv(x)$ to $q_\equiv(y)$ in V. ☐

5 Exponentiable Prestreams

Since $\mathbf{Prestr}$ is topological over $\mathbf{Top}$, hence over $\mathbf{Set}$, and there is only one prestream in the fiber over the terminal object $\{*\}$, Corollary 3.5 applies: for all prestreams $\mathcal{X}$, $\mathcal{Y}$ such that the exponential $\mathcal{Y}^{\mathcal{X}}$ exists in $\mathbf{Prestr}$, up to isomorphism this exponential must be the set $[\mathcal{X} \to \mathcal{Y}]$ of all prestream morphisms from $\mathcal{X}$ to $\mathcal{Y}$, with some topology, and some precirculation. Moreover, and omitting any explicit mention of the functor $|\_|$, App must be ordinary function application, and $\Lambda(f)$ must be defined by $\Lambda(f)(z) = f(z, \_)$.

Let $\mathbb{S}$ be *Sierpiński space*, i.e., $\{0, 1\}$ with opens all subsets except $\{0\}$. This has the important property that a subset U of a space X is open if and only if the characteristic map $\chi_U \colon X \to \mathbb{S}$ is continuous.

Let the *Sierpiński stream* be $\mathbb{S}$ with the trivial precirculation at all open subsets. Since any open cover of a non-empty open subset V of $\mathbb{S}$ must contain V, $\mathbb{S}$ is trivially a stream, not just a prestream. The following lemma therefore applies to both $\mathbf{Prestr}$ and $\mathbf{Str}$.

Lemma 5.1 *Let* $\mathbf{C}$ *be any full subcategory of* $\mathbf{Prestr}$ *with finite products. Assume that* $1 = \{*\}$*, with the obvious topology and precirculation, is an object of* $\mathbf{C}$*, and that the Sierpiński stream is an object of* $\mathbf{C}$*.*

The carrier X *of any exponentiable object in* $\mathbf{C}$ *is core-compact.*

Proof. Let $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ be exponentiable in $\mathbf{C}$. The continuous maps from X to $\mathbb{S}$ are the characteristic maps χ_U of open subsets of X, and they all define prestream morphisms. Up to isomorphism, the exponential object, which is $[\mathcal{X} \to \mathcal{Y}]$, must be the set $\mathcal{O}(X)$, with some topology and some precirculation.

The application map $\mathrm{App}\colon [X \to Y] \times X \to Y$ is then continuous, and $\Lambda(f)$ is continuous from Z to $[X \to Y] \cong \mathcal{O}(X)$ for every continuous map $f\colon Z \times X \to Y$. In this case, X must be core-compact [GL13, Proposition 5.3.3]; see also [GHK+03, Theorem II.4.12] (for T_0 spaces) or [EH02, Theorem 4.3]. ☐

Recall that X is exponentiable in $\mathbf{Top}$ if and only if it is core-compact. The unique exponential object Y^X is $[X \to Y]^{\mathrm{o}}$, the space of all continuous maps from X to Y with the core-open topology. For a prestream $\mathcal{X}$ with core-compact carrier X and a prestream $\mathcal{Y}$ with carrier Y, write $[\mathcal{X} \to \mathcal{Y}]^{\mathrm{o}}$ for $[\mathcal{X} \to \mathcal{Y}]$ with the subspace topology induced from $[X \to Y]^{\mathrm{o}}$. This is generated by subbasic opens sets that we again write $[U \Subset V]$ ($U \in \mathcal{O}(X)$, $V \in \mathcal{O}(Y)$), now denoting the set of prestream morphisms f such that $U \Subset f^{-1}(V)$.

Theorem 5.2 *The exponentiable objects in* $\mathbf{Prestr}$ *are the preordered core-compact spaces, i.e., the prestreams* $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ *such that* X *is core-compact and* $\sqsubseteq_U$ *is the restriction to* U *of the preorder* $\sqsubseteq_X$*, for every open subset* U *of* X*.*

For every prestream $\mathcal{Y} = (Y, (\preceq_V)_{V \in \mathcal{O}(Y)})$*, for every open subset* V *of* Y*, let* $\Diamond V$ *be the set of prestream morphisms* f *from* $\mathcal{X}$ *to* $\mathcal{Y}$ *whose image* $f[X]$ *intersects* V*. The exponential object* $\mathcal{Y}^{\mathcal{X}}$ *in* $\mathbf{Prestr}$ *is (up to isomorphism)* $([\mathcal{X} \to \mathcal{Y}]^o, (\leq^o_W)_{W \in \mathcal{O}([\mathcal{X} \to \mathcal{Y}]^o)})$*, where* $\leq^o_W$ *is defined by:*

for all $f, g \in W$*,* $f \leq^o_W g$ *iff for every open subset* V *of* Y *such that* $W \subseteq \Diamond V$*, for all* $x, x' \in X$ *such that* $f(x)$ *and* $g(x')$ *are in* V *and* $x \sqsubseteq_X x'$*,* $f(x) \preceq_V g(x')$*.*

Proof. Let $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ be an exponentiable object in $\mathbf{Prestr}$, and $\mathcal{Y} = (Y, (\preceq_V)_{V \in \mathcal{O}(Y)})$ be a prestream. Let $\mathcal{Y}^{\mathcal{X}}$ be the exponential object. We can assume that its carrier is $[\mathcal{X} \to \mathcal{Y}]$, with some topology, and we decide to write $(\leq_W)_{W \in \mathcal{O}([\mathcal{X} \to \mathcal{Y}])}$ for its precirculation.

Since the application map $\mathrm{App}\colon \mathcal{Y}^{\mathcal{X}} \times \mathcal{X} \to \mathcal{Y}$ is a prestream morphism, and recalling the definition of products in $\mathbf{Prestr}$ (Proposition 4.5), we have that for all opens V of Y, for all $f, g \in [\mathcal{X} \to \mathcal{Y}]$, if f and g are in $\pi_1[\mathrm{App}^{-1}(V)]$ then $f \leq_{\pi_1[\mathrm{App}^{-1}(V)]} g$ implies that for all $x, x' \in \pi_2[\mathrm{App}^{-1}(V)]$ such that (f, x) and (g, x') are in $\mathrm{App}^{-1}(V)$ and $x \sqsubseteq_{\pi_2[\mathrm{App}^{-1}(V)]} x'$, $f(x) \preceq_V g(x')$.

This definition is simplified slightly once we realize that, whenever V is non-empty, $\pi_2[\mathrm{App}^{-1}(V)]$ is the whole of X. Indeed, pick $y \in V$, and note that for every $x \in X$, there is a prestream morphism h such that $\mathrm{App}(h, x) = h(x)$ is in V, namely, the constant map with value y. Also, $\pi_1[\mathrm{App}^{-1}(V)]$ is the set of all $h \in [\mathcal{X} \to \mathcal{Y}]$ whose image $h[X]$ intersects V. We have decided to write $\Diamond V$ for this set. It follows that for every open subset W of $[\mathcal{X} \to \mathcal{Y}]$, if $f \leq_W g$ then $f \leq^{\mathrm{o}}_W g$.

We now examine currifications. Given any third prestream $\mathcal{Z} = (Z, (\lhd_O)_{O \in \mathcal{O}(Z)})$, and any prestream morphism $h \colon \mathcal{Z} \times \mathcal{X} \to \mathcal{Y}$, $\Lambda(h)$ should be a prestream morphism from $\mathcal{Z}$ to $\mathcal{Y}^{\mathcal{X}}$. Recall that $\Lambda(h) = h(z, \_)$. If this is a prestream morphism, then for every open subset W of $[\mathcal{X} \to \mathcal{Y}]$, we must have:

$$\text{for all } z \lhd_{\Lambda(h)^{-1}(W)} z',\ h(z, \_) \leq_W h(z', \_),$$

in particular,

$$\text{for all } z \lhd_{\Lambda(h)^{-1}(W)} z',\ h(z, \_) \leq^{\mathrm{o}}_W h(z', \_).$$

The latter expands to the following formula, where z and z' are arbitrary points of Z, V is an arbitrary open subset of Y, x and x' are arbitrary points of X:

$$\begin{array}{cr} \text{if } z \lhd_{\Lambda(h)^{-1}(W)} z', & (a) \\ \text{if } W \subseteq \Diamond V, & (b) \\ \text{if } x \sqsubseteq_X x', & (c) \\ \text{and if } h(z, x) \in V \text{ and } h(z', x') \in V, & (d) \\ \text{then } h(z, x) \preceq_V h(z', x'). & (e) \end{array}$$

We claim that this implies that $\mathcal{X}$ is a preordered space, i.e., that $\sqsubseteq_U$ is the restriction to U of the preorder $\sqsubseteq_X$, for every open subset U of X. Assume the contrary. Let $\mathcal{Z}$ be the terminal object 1, $\mathcal{Y} = \mathcal{X}$, and h be the map that sends $(*, x)$ to x. This is just second projection π_2, hence is a prestream morphism. Since $\mathcal{X}$ is not a preordered space, there is an open subset U of X and there are two points $x, x' \in U$ such that $x \sqsubseteq_X x'$ but $x \not\sqsubseteq_U x'$. Take $z = z' = *$, so that condition (a) is trivially satisfied. Take $V = U$, and $W = \Diamond U$, so that (b) holds. By assumption, (c) holds. Also, $h(z, x) = x$, $h(z', x') = x'$ are in $V = U$, so (d) holds. Since $x \not\sqsubseteq_U x'$, $h(z, x) \sqsubseteq_U h(z', x')$ fails.

It follows that $\mathcal{X}$ is a preordered space. X is core-compact by Lemma 5.1.

In the other direction, assume that $\mathcal{X}$ is a preordered core-compact space. We claim that $([\mathcal{X} \to \mathcal{Y}]^{\mathrm{o}}, (\leq^{\mathrm{o}}_W)_{W \in \mathcal{O}([\mathcal{X} \to \mathcal{Y}]^{\mathrm{o}})})$ is an exponential from $\mathcal{X}$ to $\mathcal{Y}$. This reduces to showing that the usual formulas for application and for currification define prestream morphisms.

In the case of application, we must first show that ordinary function application App is continuous from $[\mathcal{X} \to \mathcal{Y}]^{\mathrm{o}} \times X$ to Y. This is because App is continuous from the larger space $[X \to Y]^{\mathrm{o}} \times X$ (X is core-compact), and $[\mathcal{X} \to \mathcal{Y}]^{\mathrm{o}}$ has the subspace topology from $[X \to Y]^{\mathrm{o}}$. We proceed to local monotonicity, and show that for every open subset V of Y, for all $(f, x), (g, x') \in \mathrm{App}^{-1}(V)$ (i.e., if $f(x)$ and $g(x')$ are in V), if $f \leq^{\mathrm{o}}_{\pi_1[\mathrm{App}^{-1}(V)]} g$ and $x \sqsubseteq_{\pi_2[\mathrm{App}^{-1}(V)]} x'$, then $f(x) \preceq_V g(x')$. Recall that $\pi_1[\mathrm{App}^{-1}(V)] = \Diamond V$, and that if V is non-empty, then $\pi_2[\mathrm{App}^{-1}(V)] = X$. Since $f \leq^{\mathrm{o}}_{\pi_1[\mathrm{App}^{-1}(V)]} g$, $f \leq^{\mathrm{o}}_{\Diamond V} g$. In particular, $\Diamond V$ is non-empty, so V is non-empty, so

$\pi_2[\mathrm{App}^{-1}(V)] = X$. Since $f \leq^{\mathrm{o}}_{\Diamond V} g$, $x \sqsubseteq_X x'$, and $f(x)$ and $g(x')$ are in V, the definition of $\leq^{\mathrm{o}}_{\Diamond V}$ yields $f(x) \preceq_V g(x')$ immediately.

In the case of currification, for every prestream morphism $h\colon \mathcal{Z} \times \mathcal{X} \to \mathcal{Y}$, the map $\Lambda(h)\colon z \mapsto h(z, \_)$ is continuous, again because X is core-compact and $[\mathcal{X} \to \mathcal{Y}]^{\mathrm{o}}$ has the subspace topology from $[X \to Y]^{\mathrm{o}}$. It remains to show that $\Lambda(h)$ is locally monotonic, i.e., that for all $z \lhd_{\Lambda(h)^{-1}(W)} z'$, $h(z, \_) \leq^{\mathrm{o}}_W h(z', \_)$. We have seen that this amounts to showing (e) from assumptions (a)–(d). From (a) and (b), $z \lhd_{\Lambda(h)^{-1}(\Diamond V)} z'$. Note that $\Lambda(h)^{-1}(\Diamond V) = \{z \in Z \mid \exists x \in X \cdot h(z,x) \in V\} = \pi_1[h^{-1}(V)]$. By (d), x and x' are in $\pi_2[h^{-1}(V)]$, and by (c), $x \sqsubseteq_X x'$. Since $\mathcal{X}$ is a preordered space, $\sqsubseteq_{\pi_2[h^{-1}(V)]}$ is the restriction to $\pi_2[h^{-1}(V)]$ of the preorder $\sqsubseteq_X$, so $x \sqsubseteq_{\pi_2[h^{-1}(V)]} x'$. Together with $z \lhd_{\pi_1[h^{-1}(V)]} z'$ and the fact that h is a prestream morphism, we obtain (e). ☐

We now return to the construction of Section 3. Theorem 3.23, together with Theorem 5.2, immediately implies the following. (We write $|\_|$ for the forgetful functor from $\mathbf{Prestr}$ to $\mathbf{Set}$ here. The part about small colimits is by Proposition 3.17.)

Theorem 5.3 *Let $\mathcal{C}$ be any class of preordered core-compact spaces, and such that any binary prestream product of objects in $\mathcal{C}$ is $\mathcal{C}$-generated (equivalently, when $\mathcal{C}$ contains at least one non-empty space, such that every binary product of objects of $\mathcal{C}$ is a small colimit in $\mathbf{Prestr}$ of objects of $\mathcal{C}$).*

The category $\mathbf{Prestr}_{|\_|,\mathcal{C}}$ is Cartesian-closed.

In this category, product of $\mathcal{X}$ and $\mathcal{Y}$ is given by $\mathcal{C}(\mathcal{X} \times \mathcal{Y})$. When $\mathcal{X}$ or $\mathcal{Y}$ is core-compact (e.g., locally compact), this is just ordinary prestream product $\mathcal{X} \times \mathcal{Y}$ (Proposition 3.20).

We have the following examples, inter alia. All are in fact strongly productive: the product of any two objects of $\mathcal{C}$ will always be in $\mathcal{C}$.

Example 5.4 *The largest example of this construction is obtained by taking $\mathcal{C}$ to be the class of* all *preordered core-compact spaces. It is fair to call these spaces, a.k.a., the prestream quotients of coproducts of preordered core-compact spaces, the* core-compactly generated prestreams.

Example 5.5 *Instead, take $\mathcal{C}$ to be the class of all preordered compact Hausdorff spaces. Call* compactly generated prestreams *those prestreams that are $\mathcal{C}$-generated. This is again a Cartesian-closed category. Every coproduct of compact Hausdorff spaces is locally compact Hausdorff. Since the carriers of coproducts of prestreams are the coproducts of the carriers, and since coproducts in $\mathbf{Prestr}$ of preordered spaces are preordered spaces, every compactly generated prestream is the quotient in $\mathbf{Prestr}$ of a preordered locally compact Hausdorff space. An argument similar to Proposition 5.7 below shows that the compactly generated prestreams are exactly the quotients in $\mathbf{Prestr}$ of preordered locally compact Hausdorff spaces.*

Example 5.6 *We can restrict $\mathcal{C}$ further, and take it to be the class of all* compact pospaces. *A pospace is a preordered topological space whose preorder is an ordering, and also one whose graph is closed in the product. (They are all Hausdorff.) Compact pospaces have a rich theory, see [Nac65], or [GL13, Chapter 9]. We call the* *objects of* $\mathbf{Prestr}_{|\text{-}|,\mathcal{C}}$ *the* orderly compactly generated prestreams *in this case. These again form a Cartesian-closed category, and one whose objects are easy to describe, as the following Proposition shows. Moreover, despite the fact that it is smaller than all the previous categories, it is large enough to build geometric realizations of cubical sets, since all the standard directed cubes are compact pospaces [Hau12].*

Proposition 5.7 *The orderly compactly generated prestreams are exactly the quotients, taken in* $\mathbf{Prestr}$*, of the locally compact pospaces.*

Proof. Let $\mathcal{X}$ be an orderly compactly generated prestream, i.e., a quotient of some coproduct $\coprod_{i\in I}\mathcal{X}_i$ in $\mathbf{Prestr}$, where each $\mathcal{X}_i = (X_i, \leq_i)$ is a compact pospace. By Proposition 4.7, $\coprod_{i\in I}\mathcal{X}_i$ is obtained as the topological coproduct $\coprod_{i\in I} X_i$, with the coproduct ordering $\coprod_{i\in I} \leq_i$, defined by $(i,x) \coprod_{i\in I} \leq_i (j,y)$ iff $i=j$ and $x \leq_i j$. We show that the graph of $\coprod_{i\in I} \leq_i$ is closed in $(\coprod_{i\in I} \leq_i)^2$ by showing that its complement W is open. Let $((i,x),(j,y))$ be a pair of points in W. Either $i \neq j$, in which case $(\{i\}\times X_i)\times(\{j\}\times X_j)$ is an open neighborhood of the pair that is included in W, or $i=j$, in which case the set of all pairs of points $((i,x'),(i,y'))$ such that $x' \not\leq_i y'$ is an open neighborhood of $((i,x),(j,y))$ included in W.

Conversely, we claim that every prestream quotient of a locally compact pospace is orderly compactly generated. By Proposition 3.17, it suffices to show that every locally compact pospace $(X,\leq)$ is orderly compactly generated, or equivalently, is a quotient of a coproduct of compact pospaces. Since X is locally compact Hausdorff, pick a compact neighborhood K_x of x, for each point $x \in X$, and consider the equivalence relation $\equiv$ on $\coprod_{x\in X} K_x$ given by $(x,y) \equiv (x',z)$ iff $y=z$. We equip each K_x with the restriction of $\leq$ to K_x, and claim that the map $f^{\equiv}\colon (\coprod_{x\in X} K_x, \leq_{|K_x})/{\equiv} \to X$ defined by $f^{\equiv}(q_{\equiv}(x,y)) = y$ is a prestream isomorphism. (The notation $f^{\equiv}$ is from Proposition 4.8, with $f\colon (\coprod_{x\in X} K_x, \leq_{|K_X}) \to X$ defined by $f(x,y)=y$.) Clearly, $f^{\equiv}$ is bijective, and its inverse is given by $f^{\equiv\,-1}(x) = q_{\equiv}(x,x)$.

To check that $f^{\equiv}$ is a prestream morphism, we only need to check that f is, and by the universal property of coproducts, this boils down to checking that, for every $x\in X$, the function that maps (x,y) to y is a prestream morphism from $(K_x, \leq_{|K_x})$ to $(X,\leq)$, i.e., a continuous monotonic map. This is obvious.

To check that $f^{\equiv\,-1}$ is a prestream morphism, we first check that it is continuous. Let V be any open subset of $\coprod_{x\in X} K_x/{\equiv}$, and x be a point in the inverse image of V, i.e., such that $q_{\equiv}(x,x)\in V$. Note that $\iota_x^{-1}(q_{\equiv}^{-1}(V))$ is an open subset of K_x, meaning that one can write it as $K_x \cap U$ for some open subset U of X. The intersection W of the interior of K_x with U is then an open neighborhood of x in X (since K_x is a neighborhood of x). W is included in the inverse image of V, i.e., for every point $y\in W$, $f^{\equiv\,-1}(y)$ is in V: indeed, $f^{\equiv\,-1}(y) = q_{\equiv}(y,y) = q_{\equiv}(x,y)$ (since $x\equiv y$, and (x,y) makes sense since $y\in K_x$) $= q_{\equiv}(\iota_x(y))$ is in V since $y \in K_x \cap U = \iota_x^{-1}(q_{\equiv}^{-1}(V))$. We conclude that the inverse image of V by $f^{\equiv\,-1}$ contains open neighborhoods of each of its points, and is therefore open.

Finally, we check that for every open subset V of $\coprod_{x\in X} K_x/{\equiv}$, for all x,y in the inverse image U of V by $f^{\equiv\,-1}$, if $x\leq y$ then $f^{\equiv\,-1}(x) \sqsubseteq_V^{\equiv} f^{\equiv\,-1}(y)$, where $(\sqsubseteq_V^{\equiv})_{V\in\mathcal{O}((\coprod_{x\in X} K_x, \leq_{|K_x})/\equiv)}$ is the quotient precirculation. Using Proposition 4.8, $f^{\equiv\,-1}(x) \sqsubseteq_V^{\equiv} f^{\equiv\,-1}(y)$ is equivalent to $(x,x)\ (\sqsubseteq_{|q_{\equiv}^{-1}(V)} \cup \equiv_{|q_{\equiv}^{-1}(V)})^*\ (y,y)$, where

$\sqsubseteq$ is the coproduct ordering ($(x_1, x_2) \sqsubseteq (x_3, x_4)$ iff $x_1 = x_3$ and $x_2 \leq x_4$). If $x \leq y$, then $(x, x) \sqsubseteq_{|q_{\equiv}^{-1}(V)} (x, y) \equiv_{|q_{\equiv}^{-1}(V)} (y, y)$ (where the only subtlety to check is $(x, y) \in q_{\equiv}^{-1}(V)$, which follows from $q_{\equiv}(x, y) = q_{\equiv}(y, y) = f^{\equiv\,-1}(y) \in V$), hence $(x, x)\ (\sqsubseteq_{|q_{\equiv}^{-1}(V)} \cup \equiv_{|q_{\equiv}^{-1}(V)})^*\ (y, y)$. ☐

6 Exponentiable Streams

As for prestreams, Corollary 3.5 implies that for all streams $\mathcal{X}$, $\mathcal{Y}$ such that the exponential $\mathcal{Y}^{\mathcal{X}}$ exists in $\mathbf{Str}$, $\mathcal{Y}^{\mathcal{X}}$ must be the set $[\mathcal{X} \to \mathcal{Y}]$ of all stream morphisms from $\mathcal{X}$ to $\mathcal{Y}$, with some topology, and some circulation (up to isomorphism). By Lemma 5.1, if $\mathcal{Y}^{\mathcal{X}}$ exists then the carrier of $\mathcal{X}$ must be core-compact.

We must keep in mind that binary products in $\mathbf{Str}$ are very different from binary products in $\mathbf{Prestr}$ (compare Proposition 4.25 with Proposition 4.5). As a result, the exponentiable objects are very different, too. Remarkably, no condition at all on the circulation is required for exponentiability.

Theorem 6.1 *The exponentiable objects in* $\mathbf{Str}$ *are exactly the core-compact streams, i.e., the streams* $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ *whose carrier* X *is core-compact.*

For every stream $\mathcal{Y} = (Y, (\preceq_V)_{V \in \mathcal{O}(Y)})$*, the exponential object* $\mathcal{Y}^{\mathcal{X}}$ *in* $\mathbf{Str}$ *is (up to isomorphism) the cosheafification of* $([\mathcal{X} \to \mathcal{Y}]^o, (\leq^s_W)_{W \in \mathcal{O}([\mathcal{X} \to \mathcal{Y}]^o)})$*, where* $\leq^s_W$ *is defined by:*

for all $f, g \in W$, $f \leq^s_W g$ *iff for all open subsets* U *of* X *and* V *of* Y *such that* $W \times U \subseteq \mathrm{App}^{-1}(V)$*, for all* $x, x' \in U$ *such that* $x \sqsubseteq_U x'$, $f(x) \preceq_V g(x')$.

Proof. The only thing left to prove is that App is a stream morphism, and that $\Lambda(f)$ is a stream morphism for every stream morphism $f\colon Sh^\infty(\mathcal{Z} \times \mathcal{X}) \to \mathcal{Y}$, using the above definition, and assuming X core-compact. (Recall that we write $\times$ for product in $\mathbf{Prestr}$, so the product of $\mathcal{Z}$ and $\mathcal{X}$ in $\mathbf{Str}$ is $Sh^\infty(\mathcal{Z} \times \mathcal{X})$.) Continuity follows from general topology (and the fact that cosheafification does not change the topology), and we only need to check local monotonicity.

For short, write $[\mathcal{X} \to \mathcal{Y}]'$ for $([\mathcal{X} \to \mathcal{Y}]^{\mathrm{o}}, (\leq^{\mathrm{s}}_W)_{W \in \mathcal{O}([\mathcal{X} \to \mathcal{Y}]^{\mathrm{o}})})$, so that $\mathcal{Y}^{\mathcal{X}}$ is $Sh^\infty([\mathcal{X} \to \mathcal{Y}]')$.

Application. We first show that App is locally monotonic from $Sh^1([\mathcal{X} \to \mathcal{Y}]' \times \mathcal{X})$ to $\mathcal{Y}$. Let $V \in \mathcal{O}(Y)$ and assume that (f, x) is less than or equal to (g, x') relatively to $\mathrm{App}^{-1}(V)$ in $Sh^1([\mathcal{X} \to \mathcal{Y}]' \times \mathcal{X})$. There is a path $(f, x) = (f_0, x_0)\ (\leq^{\mathrm{s}}_{W_1} \times \sqsubseteq_{U_1})\ (f_1, x_1)\ (\leq^{\mathrm{s}}_{W_2} \times \sqsubseteq_{U_2})\ \ldots\ (\leq^{\mathrm{s}}_{W_n} \times \sqsubseteq_{U_n})\ (f_n, x_n) = (g, x')$, for some $n \in \mathbb{N}$, and some open rectangles $W_i \times U_i$ included in $\mathrm{App}^{-1}(V)$, $1 \leq i \leq n$. By definition of $\leq^{\mathrm{s}}_{W_i}$, for each i, $f_{i-1}(x_{i-1}) \preceq_V f_i(x_i)$, so $f(x) \preceq_V g(x')$.

Since $\mathcal{Y}^{\mathcal{X}} = Sh^\infty([\mathcal{X} \to \mathcal{Y}]') \leq [\mathcal{X} \to \mathcal{Y}]'$, App is also locally monotonic from $Sh^1(\mathcal{Y}^{\mathcal{X}} \times \mathcal{X})$ to $\mathcal{Y}$. We conclude since $Sh^1(\mathcal{Y}^{\mathcal{X}} \times \mathcal{X})$ is the product of $\mathcal{Y}^{\mathcal{X}}$ and $\mathcal{X}$ in $\mathbf{Str}$, by Proposition 4.25.

Currification. Let $\mathcal{Z} = (Z, (\triangleleft_O)_{O \in \mathcal{O}(Z)})$ be a third stream, and assume a stream morphism $h\colon \mathcal{Z} \times \mathcal{X} \to \mathcal{Y}$. We must show that $\Lambda(h)$ is locally monotonic from $\mathcal{Z}$ to $\mathcal{Y}^{\mathcal{X}} = Sh^\infty([\mathcal{X} \to \mathcal{Y}]')$. By Lemma 4.19 applied to $S^1 = Sh^1$, it suffices to show that

$\Lambda(h)$ is locally monotonic from $\mathcal{Z}$ to $[\mathcal{X} \to \mathcal{Y}]'$. Let W be an open subset of $[\mathcal{X} \to \mathcal{Y}]'$. Assume $z \triangleleft_{\Lambda(h)^{-1}(W)} z'$. We wish to show that for all opens $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(Y)$ such that $W \times U \subseteq \mathrm{App}^{-1}(V)$, for all $x \sqsubseteq_U x'$, $h(z, x) \preceq_V h(z', x')$. Fix U and V as above, i.e., $W \times U \subseteq \mathrm{App}^{-1}(V)$. It is easy to check that $\Lambda(h)^{-1}(W) \times U \subseteq h^{-1}(V)$. Since h is a stream morphism, and $z \triangleleft_{\Lambda(h)^{-1}(W)} z'$, for all $x \sqsubseteq_U x'$ we obtain $h(z, x) \preceq_V h(z', x')$. ☐

We deal with Haucourt streams right away. Lemma 5.1 does not apply in this setting, since the Sierpiński stream is not a Haucourt stream. Indeed, because $[0, 1]$ is connected, there is no directed path from 0 to 1, or from 1 to 0, in $\mathbb{S}$. As a result, it may well be that there are exponentiable Haucourt streams with non-core-compact carriers. We only have the following partial result.

Theorem 6.2 *The core-compact Haucourt streams are exponentiable in* $\mathbf{HStr}$.

For every core-compact Haucourt stream $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$, and every Haucourt stream $\mathcal{Y} = (Y, (\preceq_V)_{V \in \mathcal{O}(Y)})$, the exponential object $\mathcal{Y}^{\mathcal{X}}$ in $\mathbf{Str}$ is (up to isomorphism) the haucourtification of $([\mathcal{X} \to \mathcal{Y}]^o, (\leq^s_W)_{W \in \mathcal{O}([\mathcal{X} \to \mathcal{Y}]^o)})$.

Proof. The proof is the same as for Theorem 6.1. We only give indications of changes. As before, App is locally monotonic from $Sh^1([\mathcal{X} \to \mathcal{Y}]' \times \mathcal{X})$ to $\mathcal{Y}$, so it is also locally monotonic from $Sh^1(\mathcal{Y}^{\mathcal{X}} \times \mathcal{X})$ to $\mathcal{Y}$, where $\mathcal{Y}^{\mathcal{X}}$ is now the exponential in $\mathbf{HStr}$. This is because $\mathcal{Y}^{\mathcal{X}} = \mathcal{H}([\mathcal{X} \to \mathcal{Y}]') \leq [\mathcal{X} \to \mathcal{Y}]'$. Since $\mathcal{H}(\mathcal{Z})$ is a stream and $\mathcal{H}(\mathcal{Z}) \leq \mathcal{Z}$ for every Haucourt stream $\mathcal{Z}$, $\mathcal{H}(\mathcal{Z}) \leq Sh^\infty(\mathcal{Z}) \leq Sh^1(\mathcal{Z})$, so App is locally monotonic from $\mathcal{H}(\mathcal{Y}^{\mathcal{X}} \times \mathcal{X})$ to $\mathcal{X}$. We show that currification maps stream morphisms to stream morphisms as in Theorem 6.1, replacing Sh^1 by $\mathcal{H}$. ☐

As for Theorem 6.3, we apply the constructions of Section 3 and obtain the following. Recall that the colimits in $\mathbf{Str}$ and in $\mathbf{HStr}$ are computed exactly as in $\mathbf{Prestr}$ (Proposition 4.26, Proposition 4.34).

Theorem 6.3 *Let $\mathcal{C}$ be any class of core-compact streams (resp., core-compact Haucourt streams), and such that any binary product (in $\mathbf{Str}$, resp. $\mathbf{HStr}$) of objects in $\mathcal{C}$ is $\mathcal{C}$-generated (equivalently, when $\mathcal{C}$ contains at least one non-empty space, such that every binary product in $\mathbf{Str}$, resp. $\mathbf{HStr}$, of objects of $\mathcal{C}$ is a small colimit of objects of $\mathcal{C}$).*

The category $\mathbf{Str}_{|\text{-}|,\mathcal{C}}$ (resp., $\mathbf{HStr}_{|\text{-}|,\mathcal{C}}$) is Cartesian-closed.

In this category, product of $\mathcal{X}$ and $\mathcal{Y}$ is given by $\mathcal{C}(Sh^1(\mathcal{X} \times \mathcal{Y}))$, resp., $\mathcal{C}(\mathcal{H}(\mathcal{X} \times \mathcal{Y}))$. When $\mathcal{X}$ or $\mathcal{Y}$ is core-compact (e.g., locally compact), this is just ordinary stream product $Sh^1(\mathcal{X} \times \mathcal{Y})$ (Proposition 3.20, Proposition 4.25, Lemma 4.33).

Example 6.4 *The largest example of this construction is obtained by taking $\mathcal{C}$ to be the class of* all *core-compact streams. It is fair to call these spaces, namely the quotients of coproducts of core-compact streams (in $\mathbf{Str}$, equivalently, in $\mathbf{Prestr}$), the* core-compactly generated streams. *There is no a priori relationship with the core-compactly generated prestreams of Example 5.4, in particular a core-compactly generated stream need not be a core-compactly generated prestream.*

Similarly, the core-compactly generated Haucourt streams *are the quotients of coproducts of core-compact Haucourt streams, and form again a Cartesian-closed category. Every core-compactly generated Haucourt stream is a core-compactly generated stream.*

Example 6.5 *When $\mathcal{C}$ is the class of all compact Hausdorff streams, we call the $\mathcal{C}$-generated streams* compactly generated streams. *This is again a Cartesian-closed category, and again is unrelated to the compactly generated prestreams of Example 5.5. Reasoning as in that example, we see that the compactly generated streams are the quotients (in $\mathbf{Str}$, equivalently in $\mathbf{Prestr}$) of locally compact Hausdorff streams.*

Similarly, the compactly generated Haucourt streams *are the quotients of locally compact Hausdorff Haucourt streams. They are generated by the compact Hausdorff Haucourt streams, and form a Cartesian-closed category. Every compactly generated Haucourt stream is a compactly generated stream.*

Example 6.6 *Krishnan defines* compactly flowing streams *[Kri09, Section 5] as the streams whose carrier X is weak Hausdorff (i.e., such that continuous images of compact Hausdorff spaces in X are closed) and whose circulation $(\sqsubseteq_U)_{U \in \mathcal{O}(X)}$ is such that $\sqsubseteq_U = (\bigcup_K \sqsubseteq_{\upharpoonright K})^*$, where K ranges over the compact Hausdorff subspaces of U. The notation $\sqsubseteq_{\upharpoonright K}$ denotes the preorder $\leq_K$, where $(\leq_V)_{V \in \mathcal{O}(K)}$ is the circulation of the cosheafification of K, seen as a subprestream of X (see Proposition 4.6). Compactly flowing streams form a Cartesian-closed category [Kri09]. Using weak Hausdorffness, it is fairly easy to see that a weak Hausdorff stream is compactly flowing if and only if it is $\mathcal{C}$-generated, where $\mathcal{C}$ is the class of all its compact Hausdorff substreams. (A substream is the cosheafification of a subprestream.) It follows that Krishnan's compactly flowing streams are exactly those compactly generated streams (Example 6.5) that are weak Hausdorff.*

7 Conclusion

We have given a categorical reformulation of a construction due to Escardó, Lawson, and Simpson [ELS04]. This construction allows us to build (many) Cartesian-closed subcategories of certain topological constructs. The original Escardó-Lawson-Simpson construction allows one to build Cartesian-closed categories of $\mathbf{Top}$, including the prominent category of compactly-generated spaces, but also many more.

We have applied this to categories of prestreams, Krishnan and Haucourt streams, providing several examples of Cartesian-closed subcategories, of which Krishnan's construction of compactly flowing streams is one instance. Rather fortunately, all these categories share the same notion of colimit, so that taking a geometric realization, say of cubical sets, in any of these categories, always yields the same prestream.

The main import of our work, and that which made the above results possible, is our characterization of those prestreams, and of those streams, that are exponentiable: the exponentiable prestreams are the preordered core-compact spaces, and the exponentiable streams are the core-compact streams.

We leave the following problems open:

1. While all core-compact Haucourt streams are exponentiable in $\mathbf{HStr}$, must an exponentiable Haucourt stream be core-compact? This is unlikely, but finding an explicit counterexample seems hard.

2. The cosheafification $Sh^{\infty}(\mathcal{X})$ of a prestream $\mathcal{X}$ was built as a fixed point of Sh^1, therefore a priori requiring transfinitely many iterations of Sh^1. All our examples of cosheafifications were obtained in one step, as $Sh^1(\mathcal{X})$. Is there any example of a prestream such that $Sh^1(\mathcal{X})$ is not a stream? Where no finite iteration of Sh^1 on $\mathcal{X}$ ever yields a stream? What is the least ordinal number α at which iterating Sh^1 α-times yields the cosheafification? Recall that haucourtification always produces a Haucourt stream in one step.

3. The notions of (core-compactly, resp. compactly) generated streams and prestreams appear unrelated. What inclusions do hold between these classes of objects?

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A Haucourt Streams and d-Spaces

The notion of Haucourt streams arises from Haucourt's work [Hau09b] on comparing two models of directed algebraic topology, Krishnan's streams [Kri09] on the one hand, and Grandis' d-spaces [Gra03, Gra09] on the other hand.

A *d-space* is a pair (X, dX) of a topological space X and a family dX of continuous maps γ from $[0, 1]$ to X containing all the constant maps, and stable under reparameterization and concatenation. (A *reparameterization* of γ is any map $\gamma \circ \delta$, where $\delta\colon [0, 1] \to [0, 1]$ is continuous and increasing.) The elements of dX are called the *d-paths*. A d-space morphism from (X, dX) to (Y, dY) is a continuous map $f\colon X \to Y$ such that $f \circ \gamma \in dY$ for every $\gamma \in dX$. The d-spaces and their morphisms form a category $d\mathbf{Top}$.

For every prestream $\mathcal{X} = (X, (\sqsubseteq_U)_{U\in\mathcal{O}(X)})$, Haucourt defines the *directed paths* in $\mathcal{X}$ as the prestream morphisms from $\overrightarrow{[0, 1]}$ to $\mathcal{X}$ (not from $([0, 1], \leq)$ to $\mathcal{X}$, which would define dipaths instead). Taking dX to consist exactly of the directed paths in $\mathcal{X}$, we obtain a d-space $D(\mathcal{X})$. This defines the object part of a functor $D\colon \mathbf{Prestr} \to d\mathbf{Top}$. Conversely, given a d-space (X, dX), one defines a precirculation $(\sqsubseteq_U)_{U\in\mathcal{O}(X)}$ on X by $x \sqsubseteq_U y$ if and only if there is a d-path $\gamma \in dX$ such that $\gamma(0) = x$, $\gamma(1) = y$, and the image of γ lies entirely in U. $(X, (\sqsubseteq_U)_{U\in\mathcal{O}(X)})$ is a stream

[Hau09b, Lemma 5.1], which one writes $S(X, dX)$. This defines the object part of a functor $S\colon d\mathbf{Top} \to \mathbf{Str}$.

Haucourt shows that S is left adjoint to D [Hau09b, Lemma 5.9]., whether these are considered as functors between $d\mathbf{Top}$ and $\mathbf{Prestr}$, or as functors between $d\mathbf{Top}$ and $\mathbf{Str}$.

Moreover, $DSD = D$ and $SDS = S$ [Hau09b, Proposition 5.10]. This implies that the full subcategory $\mathbf{Str}^*$ of $\mathbf{Str}$ whose objects are those of the form $S(X, dX)$ for some d-space (X, dX), is isomorphic to the full subcategory $d\mathbf{Top}^*$ of $d\mathbf{Top}$ whose objects are those of the form $D(\mathcal{X})$ for some stream $\mathcal{X}$ [Hau09b, Theorem 5.13].

By definition, the objects of $\mathbf{Str}^*$ are the prestreams $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ such that for every open subset U, for all $x, y \in U$, $x \sqsubseteq_U y$ if and only if there is a directed path $\gamma\colon \overrightarrow{[0,1]} \to \mathcal{X}$ such that $\gamma(0) = x$, $\gamma(1) = y$, and the image of γ lies entirely inside U. This differs from Definition 4.14 only in the fact that γ is a directed path from x to y in U, while Definition 4.14 uses *dipaths* from x to y in U. However, the two notions agree, as we show in Lemma 4.27. So the objects of $\mathbf{Str}^*$ are exactly what we have called the Haucourt streams.

Since $\mathbf{Str}^* = \mathbf{HStr}$ is a subcategory of $\mathbf{Str}$, this also provides another proof that every Haucourt stream is a stream.