The Ideal Theory for WSTS [⋆]

Alain Finkel

LSV, ENS Cachan & CNRS, Université Paris-Saclay
finkel@lsv.ens-cachan.fr

Abstract. We begin with a survey on well structured transition systems and, in particular, we present the ideal framework [FG09a,BFM14] which was recently used to obtain new deep results on Petri nets and extensions. We argue that the theory of ideals prompts a renewal of the theory of WSTS by providing a way to define a new class of monotonic systems, the so-called Well Behaved Transition Systems, which properly contains WSTS, and for which coverability is still decidable by a forward algorithm. We then recall the completion of WSTS which leads to defining a conceptual Karp-Miller procedure that terminates in more cases than the generalized Karp-Miller procedure on extensions of Petri nets.

1 Introduction

Context. "The concept of a *well-structured transition system* (WSTS) arose thirty years ago, in 1987 precisely [Fin87,Fin90], where such systems were initially called *structured transition systems* and shown to have decidable termination and boundedness problems. WSTS were developed for the purpose of capturing properties common to a wide range of formal models (generating infinite-state systems) used in model-checking, system verification and concurrent programming. The coverability for such systems —given states s, t, decide whether $s \to^* t_1 \geq t$ for some t_1– was shown decidable in 1996 [AČJYK96,AČJT00], thus generalizing the decidability of coverability for lossy channel systems [AJ93] but also generalizing a much older result by Arnold and Latteux [AL78, Theorem 5, p. 391], published in French and thus less accessible, stating that coverability for vector addition systems with resets is decidable. It is interesting to note that the algorithm used by Arnold and Latteux in 1979 is an instance of the backward algorithm presented in [AČJYK96] and applied to $\mathbb{N}^n$." [1]

Ideals everywhere ? We believe that we have only now begun to understand that all (?) existing forward coverability algorithms were based on the use of ideals, i.e., directed downward closed sets, and on the fact that the *cover*, $\downarrow Post^*(s)$, i.e., the downward closure of the reachability set from s, is equal to a finite

[⋆] This paper contains results and parts of texts of the following published papers [FG09a,FG09b,FG12,BFM14,BFM16b] and also some results from a paper "Well Behaved Transition Systems" [BFM16a], in preparation with Michael Blondin and Pierre McKenzie.

[1] This citation is drawn from our paper [BFM16a].

union of ideals. Indeed, we may say now that the algorithm of Karp and Miller [KM69], for coverability in Petri nets, computes a finite set of ideals whose union is equal to the *cover*. Finkel introduced the framework of WSTS [Fin87,Fin90] and generalized the Karp-Miller procedure to a class of complete WSTS by building a non-effective completion of the set of states (the completion is done by quotienting equivalent increasing sequences of states; this construction is equivalent to the ideals completion), and replacing ω-accelerations of strictly increasing sequences of states (in Petri nets) by least upper bounds.

Emerson and Namjoshi [EN98] take into account the labeling of WSTS and consequently adapt the generalized Karp-Miller algorithm to model-checking. They assume the existence of a compatible dcpo (a dcpo is a directed complete partial ordering), and generalize the Karp-Miller procedure to the case of broadcast protocols. However, termination is then not guaranteed [EFM99], and in fact neither is the existence of a finite representation of the cover. This problem was solved latter in [FG09a].

Abdulla, Collomb-Annichini, Bouajjani and Jonsson proposed a forward procedure for lossy channel systems [ACABJ04a] using downward-closed regular languages as symbolic representations. We realize now that these symbolic representations were the ideals ! In [GRvB06b,GRvB06a], Ganty, Geeraerts, Raskin and Van Begin proposed the first forward procedure for solving the coverability problem for general WSTS equipped with an effective adequate domain of limits, or equipped with a finite set D used as a parameter to tune the precision of an abstract domain. Both solutions ensure that every downward-closed set has a finite representation and still ideals were implicit but they were not seen as the crucial mathematical object. Abdulla, Deneux, Mahata and Nylén also proposed a symbolic framework for dealing with downward-closed sets for Timed Petri nets [ADMN04] and this was still a story of ideals.

The starting point of the series of papers entitled *Forward analysis for WSTS, part I: Completions* [FG09a], and *Forward analysis for WSTS, part II: Complete WSTS* [FG09b,FG12], both written with Jean Goubault-Larrecq, came from our desire to derive similar general algorithms working *forwards*, namely algorithms computing the *cover* of any WSTS (and not for a particular class of WSTS). Our initial completion (of the set of states) was originally based on topology (the completion by sobrification), orderings (the completion by ideals) and the strong connection between both; after some years, we may now only work with the ideals completion [BFM16b] which is quite simple. While computing the cover allows one to decide coverability, by testing whether $t \in \downarrow Post^*(s)$, it also allows to decide whether the reachability set, $Post^*(s)$, is finite (the boundedness problem). No backward algorithm can decide this. In fact, boundedness is undecidable in general, e.g., on reset Petri nets [DFS98]. So computing the cover is not possible for general WSTS. Despite this, the known forward algorithms are felt to be more efficient than backward procedures in general: e.g., for lossy channel systems, although the backward procedure always terminates, only a (necessarily non-terminating) forward procedure is implemented in the TREX tool [ABJ98]. Another argument in favor of forward procedures is the following:

for depth-bounded processes, a fragment of the π-calculus, the backward algorithm of [AČJT00] is not applicable when the maximal depth of configurations is not known in advance because, in this case, the predecessor configurations are not effectively computable [WZH10]. But the *forward* Expand, Enlarge and Check algorithm of [GRvB07], which operates on complete WSTS, solves coverability even though the depth of the process is not known a priori [WZH10].

Our Contribution. Most of the material in Sections 2,5,6 of this paper is not original and appeared in previous papers [FG09a,FG09b,FG12,BFM14,BFM16b]. Section 3 is a survey on WSTS. Section 4 presents the ideals framework and some recent and deep results using ideals. Section 4 also recalls the Erdös and Tarsky Theorem that says that a quasi-ordered set X is without infinite antichain if and only if every downward closed subset of X is equal to a finite union of ideals. This Theorem paves the way to the new definition of *Well Behaved Transition System (WBTS)*, more general than WSTS, with its decidability of coverability [BFM16a] by a forward coverability algorithm.

In Section 5, we introduce the completion of a WSTS and building on our own theory of completions [FG09a,BFM16b], we recall that ω^2*-WSTS* are the right class of WSTS to consider: the completion $\widehat{S}$ of a WSTS S is a WSTS if and only if S is an ω^2-WSTS. All naturally occurring WSTS are in fact ω^2-WSTS. Despite the fact that $\mathbf{Clover}_{\mathfrak{S}}$ cannot terminate on all inputs, that $\mathfrak{S}$ is an ω^2-WSTS will ensure *progress*, i.e., will ensure that every opportunity of accelerating a loop will eventually be taken by $\mathbf{Clover}_{\mathfrak{S}}$.

In Section 6, we recall *complete WSTS* which are functional WSTS $\mathfrak{S} = (S, \xrightarrow{F}, \leq)$ where $(S, \leq)$ is a wqo and a continuous dcpo and every function in F is partial ω-continuous. This allows us to design a conceptual procedure $\mathbf{Clover}_{\mathfrak{S}}$ that looks for a finite representation (we say now, a finite set of ideals) of the cover. Our procedure also terminates in more cases than the well-known (generalized) Karp-Miller procedure [EN98,Fin90].

2 Preliminaries

2.1 Orderings

We borrow from theories of order, as used in model-checking [EN98,FS01], and also from domain theory [AJ94,GHK$^+$03].

Let X be a set and let $\leq \subseteq X \times X$. The relation $\leq$ is a *quasi-ordering* if it is reflexive and transitive. If $\leq$ is additionally antisymmetric, then $\leq$ is a *partial order*. We write $\geq$ for the converse quasi-ordering, $<$ for the associated strict ordering $(\leq \setminus \geq)$. There is also an associated equivalence relation $\equiv$, defined as $\leq \cap \geq$. A set X with a partial ordering $\leq$ is a *poset* $(X, \leq)$, or just X when $\leq$ is clear. If X is merely quasi-ordered by $\leq$, then the quotient $X/\!\equiv$ is ordered by the relation induced by $\leq$ on equivalence classes. So there is not much difference in dealing with quasi-orderings or partial orderings, and we shall essentially be concerned with the latter.

The set X is *well-founded (under $\leq$)* if there is no infinite strictly decreasing sequence $x_0 > x_1 > \dots$ of elements of X. An *antichain (under $\leq$)* is a subset $A \subseteq X$ of pairwise incomparable elements, i.e. for every $a, b \in A$, $a \not\leq b$ and $b \not\leq a$. We say that a quasi-ordering $\leq$ is a *well-quasi-ordering* for X if X is well-founded and contains no infinite antichain under $\leq$.

Let $A \subseteq X$, we define the *downward closure* and *upward closure* of A respectively as $\uparrow A \stackrel{\text{def}}{=} \{x \in X : x \geq a \text{ for some } a \in A\}$ and $\downarrow A \stackrel{\text{def}}{=} \{x \in X : x \leq a \text{ for some } a \in A\}$. A subset $A \subseteq X$ is said to be *downward closed* if $A = \downarrow A$ and *upward closed* if $A = \uparrow A$. An *ideal* is a downward closed subset $I \subseteq X$ that is also *directed*, i.e. it is nonempty and for every $a, b \in I$, there exists $c \in I$ such that $a \leq c$ and $b \leq c$. Chains, i.e., totally ordered subsets, and one-element sets are examples of directed subsets. The set of ideals of X is denoted $\mathsf{Ideals}(X) \stackrel{\text{def}}{=} \{I \subseteq X : I = \downarrow I \text{ and } I \text{ is directed}\}$.

An *upper bound* $x \in X$ of $E \subseteq X$ is such that $y \leq x$ for every $y \in E$. The *least upper bound (lub)* of a set E, if it exists, is written $\text{lub}(E)$. An element x of E is *maximal* (resp. minimal) iff $\uparrow x \cap E = \{x\}$ (resp. $\downarrow x \cap E = \{x\}$). Write $\text{Max}\, E$ (resp. $\text{Min}\, E$) for the set of maximal (resp. minimal) elements of E.

A *dcpo* is a poset in which every directed subset has a least upper bound. For any subset E of a dcpo X, let $\text{Lub}(E) = \{\text{lub}(D) \mid D \text{ directed subset of } E\}$. Clearly, $E \subseteq \text{Lub}(E)$; $\text{Lub}(E)$ can be thought of E plus all limits from elements of E. When $\leq$ is a well partial ordering that also turns X into a dcpo, we say that X is a *directed complete well order*, or *dcwo*.

3 A Survey on Well-Structured Transition Systems

The theory of WSTS has now been used for 30 years as a foundation for verification in various models, such as (monotonic extensions of) Petri nets, broadcast protocols, fragments of the pi-calculus, rewriting systems, lossy systems, timed Petri nets, etc. Two journal papers synthesise the known results and show the possible applications [AČJT00,FS01].

3.1 Monotonic Transition Systems

A *transition system* is a pair $\mathfrak{S} = (S, \to)$ of a set S, whose elements are called *states*, and a *transition relation* $\to \subseteq S \times S$. We write $s \to s'$ for $(s, s') \in \to$. Let $\xrightarrow{*}$ be the transitive and reflexive closure of the relation $\to$. We write $Post_{\mathfrak{S}}(s) = \{s' \in S \mid s \to s'\}$ for the set of immediate successors of the state s. The *reachability set* of a transition system $\mathfrak{S} = (S, \to)$ from an initial state s_0 is $Post^*_{\mathfrak{S}}(s_0) = \{s \in S \mid s_0 \xrightarrow{*} s\}$. The *reachability tree* $RT(S, \to, s_0)$ of a transition system $(S, \to)$ with an initial state s_0 is defined as follows: the root is labeled by s_0 and there is an arc between two nodes n, n' labeled by the states s, s' iff $s \to s'$.

We shall be interested in effective transition systems. Intuitively, a transition system $(S, \to)$ is *effective* iff one can compute the set of successors $Post_{\mathfrak{S}}(s)$ of any state s. We shall take this to imply that $Post_{\mathfrak{S}}(s)$ is finite (for simplicity, transition systems are supposed to be finitely banching), and each of its elements is computable. Formally, one would need to find a representation of the states $s \in S$. For reasons of readability, we shall make an abuse of language, and say that the pair $(S, \rightarrow)$ is itself an effective transition system in this case, leaving the representation of states and the *post* function implicit (see [FG12] for more precise definitions).

We say that an ordered transition system $\mathfrak{S} = (S, \rightarrow, \leq)$, where $\leq$ is a quasi ordering, is *monotonic* (resp. *strictly monotonic*) iff for all $s, s', s_1 \in S$ such that $s \rightarrow s'$ and $s_1 \geq s$ (resp. $s_1 > s$), there exists an $s_1' \in S$ such that $s_1 \xrightarrow{*} s_1'$ and $s_1' \geq s'$ (resp. $s_1' > s'$). $\mathfrak{S}$ is *transitive monotonic* iff for all $s, s', s_1 \in S$ such that $s \rightarrow s'$ and $s_1 \geq s$, there exists an $s_1' \in S$ such that $s_1 \xrightarrow{+} s_1'$ and $s_1' \geq s'$. $\mathfrak{S}$ is *strongly monotonic* iff for all $s, s', s_1 \in S$ such that $s \rightarrow s'$ and $s_1 \geq s$, there exists an $s_1' \in S$ such that $s_1 \rightarrow s_1'$ and $s_1' \geq s'$. These variations on monotonicity were studied in [Fin87,FS01]. Originally, three different definitions of monotonicity (hence six definitions with the strict variant) were given in [Fin87] and four with the stuttering variant (resp. eight) were studied in [FS01].

3.2 The Properties

Finite representations of $Post^*_{\mathfrak{S}}(s)$, e.g., as Presburger formulae or finite automata, usually don't exist even for monotonic transition systems (not even speaking of being computable). However, the *cover* set $Cover_{\mathfrak{S}}(s) = \downarrow Post^*_{\mathfrak{S}}(\downarrow s)$ ($= \downarrow Post^*_{\mathfrak{S}}(s)$ when $\mathfrak{S}$ is monotonic) will be much better behaved. Note that being able to compute the cover allows one to decide *coverability* ($t \in Cover_{\mathfrak{S}}(s)$?), and *boundedness* (is $Post^*_{\mathfrak{S}}(s)$ finite?). Let us recall that the *control-state reachability problem* (when the set S of states is $S = Q \times X$ with Q a finite set of control states) can be reduced to coverability. However, the *repeated control state reachability problem* (does there exist an infinite computation that visits infinitely often a control state q?) cannot be reduced to coverability.

The *eventuality* property for a given upward closed set I, is the following property: $\mathrm{EG}\, I$ is true in a state s_0 iff there is a computation from s_0 in which all states are in I. Given two labeled transition systems $\mathfrak{S}_1 = (S_1, \rightarrow_1)$ and $\mathfrak{S}_2 = (S_2, \rightarrow_2)$, on the same alphabet Σ, the relation $R \subseteq S_1 \times S_2$ is a *simulation* of $\mathfrak{S}_1$ by $\mathfrak{S}_2$ if for each $(s_1, s_2) \in R$, $s_1' \in S_1$ and $a \in \Sigma$, if $s_1 \xrightarrow{a} s_1'$ then there exists $s_2' \in S_2$ such that $s_2 \xrightarrow{a} s_2'$ and $(s_1', s_2') \in R$. We say that $s_1 \in S_1$ is *simulated* by $s_2 \in S_2$ if there is a simulation R of $\mathfrak{S}_1$ by $\mathfrak{S}_2$ such that $(s_1, s_2) \in R$.

3.3 Well-Structured Transition Systems

WSTS were originally thought of as generalizations of Petri nets (and classes of FIFO nets) in which the set of states (called markings) of a Petri net with n places, $\mathbb{N}^n$, is abstracted into a set X equipped with a wqo $\leq$; the Petri net transitions (which are particular affine translations from $\mathbb{N}^n$ into $\mathbb{N}^n$) are abstracted to general recursive monotonic relations in X. WSTS were defined and studied in the author's PhD thesis in 1986, the results were presented at ICALP'87 [Fin87] and published in the journal "information and computation" [Fin90].

Definition 1 ([Fin87,Fin90]). *A* Well Structured Transition System *(*WSTS*)* $\mathfrak{S} = (S, \rightarrow, \leq)$ *is a monotonic transition system such that* $(S, \leq)$ *is wqo.*

We will need *effective WSTS* $\mathfrak{S} = (S, \rightarrow, \leq)$, i.e., $(S, \rightarrow)$ is effective and $\leq$ is decidable. Generally WSTS are finitely banching. Some of the decidability results [BFM14] do not require this but, for simplicity, we will make this assumption. A WSTS (or more generally, an ordered transition system) $\mathfrak{S} = (S, \rightarrow, \leq)$ has the *effective PredBasis* property if there exists an algorithm which computes $\uparrow Pre(\uparrow s)$ for each $s \in S$; $\mathfrak{S}$ is *intersection effective* if there is an algorithm which computes a finite basis of $\uparrow s \cap \uparrow s'$, for all states $s, s' \in S$.

We now summarize the main decidability results on WSTS till the year 2000.

Theorem 1. *The following are* decidable*:*

- *Termination, for effective transitive monotonic WSTS [Fin87,FS01].*
- *Boundedness, for effective strictly monotonic transitive WSTS [Fin87,FS01].*
- *Coverability (hence control-state reachability), for effective WSTS with effective PredBasis ([AČJYK96], extended in [FS01]).*
- *Eventuality, for effective strongly monotonic finitely branching WSTS (see [KS96,AČJT00], extended in [FS01]).*
- *Simulation of a labeled WSTS by a finite automaton, for intersection effective and effective strongly monotonic WSTS with effective PredBasis [AČJYK96].*
- *Simulation of a finite automaton by a labeled WSTS, for effective strongly monotonic WSTS [AČJYK96].*

The following are undecidable*:*

- *Reachability, for effective strongly strictly monotonic WSTS (Transfer Petri nets, [DFS98]).*
- *Repeated control-state reachability (hence LTL), for effective strongly strictly monotonic WSTS (Transfer Petri nets, [DFS98]).* □

To prove these decidability results we alternatively use forward and backward algorithms. Termination, boundedness, eventuality and one part of simulation can be proved by using a forward algorithm that builds the so-called Finite Reachability Tree (FRT) [Fin87]: we develop the reachability tree until a state larger than or equal to one of its ancestors is encountered, in which case the current branch is definitely closed. The place-boundedness problem (to decide whether a place can contain an unbounded number of tokens) is undecidable for transfer Petri nets [DFS98], although they are strongly and strictly monotonic WSTS. It is decidable for Petri nets. This requires a richer structure than the FRT, the Karp-Miller tree. The set of labels of the Karp-Miller tree is a finite representation of the cover.

Almost all the assumptions used above are necessary:

Theorem 2. *The following are* undecidable*:*

- *Termination, for transitive monotonic WSTS.*
- *Boundedness, for effective strongly monotonic WSTS.*
- *Coverability, for effective strongly strictly monotonic WSTS.* □

For termination, Turing machines are transitive WSTS for which the termination ordering $\leq_{termination}$ is undecidable, [FS01]. For the second claim, Reset Petri nets have an undecidable bounded problem, and are effective strongly monotonic WSTS; but they are not strictly monotonic [DFS98]. For the last claim, there are WSTS composed of two recursive strictly monotonic functions from $\mathbb{N}^2$ into $\mathbb{N}^2$ that are not recursive on $\mathbb{N}_\omega^2$ hence there are no algorithm computing a PredBasis, [FMP04].

The status of eventuality and simulation is open: for each of these properties, we know of no natural class of WSTS for which this property would be undecidable.

3.4 WSTS Everywhere [2]

Here are *some* (this is not an exhaustive list) of the papers that introduced new points of view, in our opinion:

Forward coverability algorithm and forward analysis for WSTS

Ganty, Geeraerts, Raskin and Van Begin proposed a new forward procedure for deciding the coverability problem [GRB04,GRvB06a,GRvB06b]. This was the first forward procedure for this problem in the general framework of WSTS (to which they explicitely added, to the set of states, an Adequate Domain of Limits). Their procedure computes a sufficient part (to decide coverability) of a finite representation of the cover.

Goubault-Larrecq and I began in 2009 a series entitled *"Forward analysis for WSTS, Part I: Completions"* [FG09a] and *"Forward Analysis for WSTS, Part II: Complete WSTS"* [FG09b] in which we provide the missing theoretical fundations of finite representations of downward closed sets. Most of used ordering in WSTS are ω^2-ordering and in fact also better quasi ordering. This allows to extend the wqo to the completion of a WSTS and the completed system is still a WSTS. An ω^2-ordering that is extended on downward closed sets is also a wqo [FG09a,FG09b,AN00]. This work, based on both order and topology, allowed us to design a conceptual coverability set procedure for *all* WSTS. Bounded WSTS [CFS11] are a particular recursive class of WSTS for which our coverability set procedure terminates.

Expressive power of WSTS

In [ADB07,GRB07], Abdulla, Delzanno, Geeraerts, Raskin and Van Begin studied the expressive power of WSTS by means of the set of coverability languages which are well-adapted to WSTS. Bonnet, Finkel, Haddad and Rosa-Velardo proposed in [BFHR11] to use a new tool, the order type of

[2] "WSTS Everywhere" was the title of our survey with Philippe Schnoebelen [FS01].

posets, to prove, for example, that the class of all WSTS with set of states of type $\mathbb{N}^n$ are less expressive than WSTS with set of states of type $\mathbb{N}^{n+1}$. This strategy unifies the previous proofs and allows to compare models of different natures, such as lossy channel systems and timed Petri nets.

Petri net extensions and complexity of WSTS

Affine Petri nets extensions were studied a long time ago by Valk [Val78] under the name *self modified nets*; more recently, many Petri nets extensions were studied like recursive Petri nets [HP07], PRS [May00], Reset/Transfer Petri nets [DFS98,DJS99] and affine well-structured nets [FMP04]. More recently, since the first paper on Petri nets with data (which extend affine nets) by Lazić, Newcomb, Ouaknine, Roscoe and Worrell [LNO$^+$07], many authors like Rosa-Velardo, Frutos-Escrig [RdF07,RMdF11], Lazić, Haddad, Schmitz and Schnoebelen have began to study the complexity for many classes of Petri net extensions where tokens carry data: data nets, Petri data nets, ν-Petri nets, ordered and unordered data Petri nets. D. Figueira, S. Figueira, Schmitz and Schnoebelen began the study of the ordinal-recusive complexity of general WSTS. They characterized the ordinal length of bad sequences of vectors of integers [FFSS11] (using the Dickson lemma) and of words [SS11] (using the Higman lemma). Haddad, Schmitz and Schnoebelen showed *"how to reliably compute fast-growing functions with timed-arc Petri nets and data nets. They provided ordinal-recursive lower bounds on the complexity of the main decidable properties (safety, termination, regular simulation, etc.) of these models. Since these new lower bounds match the upper bounds that one can derive from wqo theory, they precisely characterise the computational power of these so-called "enriched" nets"* in[HSS12].

In [BHM15], Badouel, Hélouët and Morvan addressed a WSTS extension of Petri Nets whose transitions manipulate structured data via patterns and queries. Very recently, Hofman, Lasota, Lazić, Leroux, Schmitz and Totzke extended the construction of coverability trees to Petri Nets with Unordered Data [HLL$^+$16] and Lazić and Schmitz proved that coverability for ν-Petri nets is complete for "double Ackermann" time [LS16a].

Pushdown VASS and Well-Structured Pushdown Systems

Mixing pushdown and counters is possible even if one reaches undecidability or high complexity. Cai, Ogawa, Lazić, Leroux, Sutre, Totzke studied reachability and coverability for VASS with a stack and subclasses of Pushdown WSTS. Coverability is decidable for one dimensional Pushdown VASS but it is Tower-hard (while Boundedness is in exponential time) and its decidability is an open problem for general Pushdown VASS [Laz13,LST15b,LST15a].

We could also quote other applications and use of the WSTS theory to: *Well-Structured Graph Transformation Systems* [BDK$^+$12,KS14] ; to decide properties in the *pi-Calculus* [Mey08,ZWH12,HMM14]; and we could also mention the recent paper from Lasota [Las16] who proposes an interesting *"WQO Dichotomy Conjecture: under a mild assumption, either a data domain exhibits a well quasi-order (in which case one can apply the general setting of well-structured transi-*

tion systems to solve problems like coverability or boundedness), or essentially all the decision problems are undecidable for Petri nets over that data domain.".

4 The Ideal Framework of Ideals

Recall that an *ideal* is a downward closed subset $I \subseteq X$ that is also *directed*, i.e. it is nonempty and for every $a, b \in I$, there exists $c \in I$ such that $a \le c$ and $b \le c$. The set of ideals of X is denoted $\mathsf{Ideals}(X) \overset{\text{def}}{=} \{I \subseteq X : I = \downarrow I \text{ and } I \text{ is directed}\}$.

The two following examples come from [BFM16b].

Example 1. Let us consider the ideals of $\mathbb{N}^d$. It can be shown that

$$\mathsf{Ideals}(\mathbb{N}^d) = \underbrace{\mathsf{Ideals}(\mathbb{N}) \times \mathsf{Ideals}(\mathbb{N}) \times \cdots \times \mathsf{Ideals}(\mathbb{N})}_{d \text{ times}}$$

and that $I \in \mathsf{Ideals}(\mathbb{N})$ is either $\mathbb{N}$ or of the form $\downarrow x$ for some $x \in \mathbb{N}$. Therefore, any ideal $I \in \mathsf{Ideals}(\mathbb{N}^d)$ may be represented by some $x \in \mathbb{N}_\omega^d$ where $x_i = \omega$ represents $\mathbb{N}$ and $x_i = y$ represents $\downarrow y$. Consider the following downward closed set

$$X = \{(x_1, x_2) \in \mathbb{N}^2 : (x_1 \le 4) \lor (x_1 \le 8 \land x_2 \le 10) \lor (x_2 \le 5)\}.$$

As illustrated in Fig. 1, it is possible to write X as the following finite union of ideals:

$$\downarrow 4 \times \mathbb{N} \cup \downarrow 8 \times \downarrow 10 \ \cup \ \mathbb{N} \times \downarrow 5$$

which can be represented by $\{(4, \omega), (8, 10), (\omega, 5)\}$.

Example 2. It has been recently shown that downward closed languages (under the subword ordering) coincide with the class of *strictly piecewise-testable languages* [RHB+10]. Previously, downward closed languages were studied and used in [ACABJ04b] for representing infinite reachability subsets of lossy channel systems; it is proved that every downward closed language on Σ^*, where Σ is a finite alphabet, is a finite union of *products* $P_1 P_2 \cdots P_m$ where each P_i is either $\{\varepsilon, \sigma\}$ for some $\sigma \in \Sigma$, or A^* for some $A \subseteq \Sigma$. It has been remarked in [FG09a] that every ideal $I \in \mathsf{Ideals}(\Sigma^*)$, is exactly a product $I = P_1 P_2 \cdots P_m$ like in [ACABJ04b]. Following [FG09a], the previous result on downward closed languages is then a particular instance of a more general result: every downward closed set (here a downward closed language on Σ^*), in a wqo, is a finite union of ideals.

For example, consider the language of words over $\Sigma = \{a, b, c\}$ where the first letter does not reappear, *i.e.*, let

$$L = \{w \in \Sigma^+ : w_i \ne w_1 \text{ for } 1 < i \le |w|\}$$
$$= a\{b, c\}^* \cup b\{a, c\}^* \cup c\{a, b\}^* \ .$$

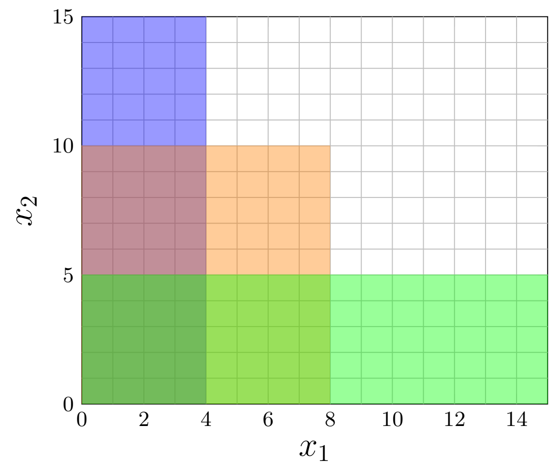

Fig. 1. Decomposition of $X = \{(x_1, x_2) \in \mathbb{N}^2 : (x_1 \leq 4) \vee (x_1 \leq 8 \wedge x_2 \leq 10) \vee (x_2 \leq 5)\}$ into finitely many ideals. The three ideals $\downarrow 4 \times \mathbb{N}, \downarrow 8 \times \downarrow 10$ and $\mathbb{N} \times \downarrow 5$ appear respectively in blue, orange and green.

It can be shown that

$$\begin{aligned}\downarrow L &= L \cup \{w \in \Sigma^* : |w|_\sigma = 0 \text{ for some } \sigma \in \Sigma\} \\ &= L \cup \{a, b\}^* \cup \{a, c\}^* \cup \{b, c\}^* \\ &= \{a, \varepsilon\}\{b, c\}^* \cup \{b, \varepsilon\}\{a, c\}^* \cup \{c, \varepsilon\}\{a, b\}^* \ .\end{aligned}$$

Hence, $\downarrow L$ decomposes into finitely many ideals.

It was observed in [FG09a,BFM14] that any downward closed subset of a well-quasi-ordered set is equal to a finite union of ideals, which led to further applications in the study of WSTS.

4.1 Recent Use of Ideals

- Leroux et Schmitz used in *Demystifying Reachability in Vector Addition Systems* [LS15b] and in *Ideal Decompositions for Vector Addition Systems* [LS16b] the decomposition of downward closed sets into finite many ideals on runs (instead classically on states) with the natural embedding relation between runs to give the first upper bound for the complexity of the reachability problem in Petri nets. They established that the decomposition produced by the complex reachability algorithm is, in fact, *"the ideal decomposition of the set of runs, using the natural embedding relation between runs as well quasi ordering. In a second part, we apply recent results on the complexity of termination thanks to well quasi orders and well orders to obtain a cubic*

Ackermann upper bound for the decomposition algorithms, thus providing the first known upper bounds for general VAS reachability."

- Lazić and Schmitz studied in *The Ideal View on Rackoff's Coverability Technique* [LS15a] the well-known Rackoff coverability algorithm and they renewed the study by using the ideals framework: *We take a dual view on the backward coverability algorithm, by considering successively the sets of configurations that do not cover y in* $0, 1, 2, \dots$ *or fewer steps. Such sets are downwards-closed, and enjoy a (usually effective) canonical representation as finite unions of ideals. We show that, in the case of VAS, this dual view exhibits an additional structural property of* ω*-monotonicity, which allows to derive the desired doubly-exponential bound.*

- Lazić and Schmitz proved in *The Complexity of Coverability in* ν*-Petri Nets* [LS16a] that coverability for ν-Petri nets is complete for "double Ackermann" time by using the ideals framework with the multiset ordering. They proved that the ν-Petri nets are ideally effective and they studied the length of controlled descending chains of downwards-closed sets which are finite unions of ideals. The proof deeply relies on ideals.

- Hofman, Lasota, Lazić, Leroux, Schmitz and Totzke studied in *Coverability Trees for Petri Nets with Unordered Data* [HLL+16] *"an extension of classical Petri nets where tokens carry values from a countable data domain, that can be tested for equality upon firing transitions. These Unordered Data Petri Nets (UDPN) are well-structured and therefore allow generic decision procedures for several verification problems including coverability and boundedness. We show how to construct a finite representation of the coverability set in terms of its ideal decomposition."*.

- Blondin, Finkel and McKenzie studied in *Handling Infinitely Branching Well-structured Transition Systems* [BFM14,BFM16b] coverability, termination and boundedness for infinitely branching WSTS. *"Here we develop tools to handle infinitely branching WSTS by exploiting the crucial property that in the (ideal) completion of a well-quasi-ordered set, downward-closed sets are* finite *unions of ideals. Then, using these tools, we derive decidability results and we delineate the undecidability frontier in the case of the termination, the maintainability and the coverability problems. Coverability and boundedness under new effectiveness conditions are shown decidable."*

Other applications of ideals arrive: Goubault-Larrecq and Schmitz showed using effective representations for tree ideals that it entails the decidability of piecewise testable separability when the input languages are regular. [GLS16]

4.2 Decomposition of Downward Closed Sets into Ideals

Even if it was observed that any downward closed subset of a well-quasi-ordered set is equal to a finite union of ideals, here, we stress the fact that such finite decompositions also exist in quasi-ordered sets with *no infinite antichain*. The existence of such a decomposition has been proved numerous times (for partial orderings instead of quasi-orderings) in the order theory community under different terminologies, and is a particular case of a more general result of Erdös & Tarski [ET43]. But, to the best of our knowledge, this has never been remarked neither used in the verification community.

Theorem 3 ([ET43,Bon75,Fra86,BFM16a]). *A countable quasi-ordered set X contains no infinite antichain if, and only if, every downward closed subset of X is equal to a finite union of ideals.*

We give a self-contained proof of this result in [BFM16a].

Theorem 3 allows us, as in [BFM14], to define a canonical finite decomposition of a downward closed subset $D \subseteq X$, that is, the (finite) set $\mathsf{IdealDecomp}(D)$ of maximal ideals contained in D under inclusion.

4.3 Well Behaved Transition Systems

Since downward closed sets decompose in finitely ideals, we may use the forward coverability algorithm and then we are motivated to define a new class of monotonic transition systems.

Definition 2 ([BFM16a]). *A* Well Behaved Transition System (WBTS) *is a monotonic transition system $\mathfrak{S} = (S, \rightarrow, \leq)$ such that $(S, \leq)$ contains no infinite antichain.*

Every WSTS is trivially a WBTS but, for example, a one counter automaton on $\mathbb{Z}$ is a WBTS but it is not a WSTS, for the usual ordering.

We describe effectiveness hypotheses that allow manipulating downward closed sets in WBTS.

Definition 3 ([BFM16a]). *A class C of WBTS $\mathfrak{S}$ is* ideally effective *if*

- *the function mapping the encoding of a state s of an ordered transition system to the encoding of the ideal $\downarrow s$ is computable;*
- *inclusion of ideals is decidable;*
- *the downward closure $\downarrow post(I)$ expressed as a finite union of ideals is computable from the ideal I.*

Let us emphasize that an ideally effective WBTS is effective and post-effective: S embeds into $\mathsf{Ideals}(S)$ hence S is also decidable; the inequation $s \leq t$ is equivalent to $\downarrow s \subseteq \downarrow t$ hence it is decidable; and computing $post(s)$ boils down to computing $post(\downarrow s)$.

Remark 1. Enforcing WBTS to be ideally effective is not an issue for virtually all useful models. Indeed, a large scope of WBTS are ideally effective [FG09a]: ideally effective WSTS, Petri nets, VASS and their extensions (with resets, transfers, affine functions), lossy channel systems and extensions with data.

We recently proved in [BFM16a] that coverability is decidable for ideally effective Well Behaved Transition Systems.

Theorem 4 ([BFM16a]). *Coverability is decidable for ideally effective Well Behaved Transition Systems.*

5 Completion of WSTS and Accelerations [3]

5.1 Completion of a WSTS

The ideal completion of a WSTS is useful to define lub-accelerations (that are defined in the completed set of states, i.e., in the set S completed with lubs) ; then one may design coverability set procedures like abstracted Karp Miller procedures working on states and lubs, i.e., "limits of states". Let us recall that S is canonically included in $\mathsf{Ideals}(S)$, that $\mathsf{Ideals}(S)$ is a continuous dcpo, and in the case of continuous dcpos, the set S with its set of lubs, is isomorphic to $\mathsf{Ideals}(S)$. The following definition extends [FG09b] to non-functional WSTS and uses the ideal completion instead of the more complex sober topological completion.

Definition 4 ([BFM14,FG09b]). *The* completion $\widehat{\mathfrak{S}}$ *of a WSTS* $\mathfrak{S} = (S, \rightarrow, \leq)$ *is the ordered transition system* $\widehat{\mathfrak{S}} = (\widehat{S}, \rightsquigarrow, \subseteq)$ *where* $\widehat{S} = \mathit{Ideals}(S)$ *and* $I \rightsquigarrow J$ *if* $J \in \mathit{IdealDecomp}(\downarrow Post_{\widehat{\mathfrak{S}}}(I))$.

It would seem clear that the construction of the completion $\widehat{\mathfrak{S}} = (\widehat{S}, \rightsquigarrow, \subseteq)$ of a WSTS $\mathfrak{S} = (S, \rightarrow, \leq)$ be, again, a WSTS. We shall recall that this is not the case. The only missing ingredient to show that $\widehat{\mathfrak{S}}$ is a WSTS is to check that $\widehat{S}$ is well-ordered by inclusion. And this is not the case, the Rado wqo is a well known example.

When is $\widehat{X}$ well-ordered by inclusion? We shall see that there is a definite answer: when X is ω^2-wqo. Hence, when the original wqo $\leq$ is also a ω^2-wqo, the ordered set $(\mathsf{Ideals}(S), \subseteq)$ is also a wqo and then the completion of a such WSTS would be still a WSTS.

In fact, the completion can be extended to *WBTS* since the completion only needs a quasi ordering without infinite antichains.

Definition 5 ([BFM16a]). *The* completion $\widehat{\mathfrak{S}}$ *of a WBTS* $\mathfrak{S} = (S, \rightarrow, \leq)$ *is the ordered transition system* $\widehat{\mathfrak{S}} = (\widehat{S}, \rightsquigarrow, \subseteq)$ *where* $\widehat{S} = \mathit{Ideals}(S)$ *and* $I \rightsquigarrow J$ *if* $J \in \mathit{IdealDecomp}(\downarrow Post_{\widehat{\mathfrak{S}}}(I))$.

Let us remark that the completion of a WBTS is not necessarily a WBTS. Take X to be Rado's structure X_{Rado} [Rad54], i.e., $\{(m,n) \in \mathbb{N}^2 \mid m < n\}$, ordered by $\leq_{\mathrm{Rado}}$: $(m,n) \leq_{\mathrm{Rado}} (m',n')$ iff $m = m'$ and $n \leq n'$, or $n < m'$. It is well-known that $\leq_{\mathrm{Rado}}$ is a well quasi-ordering, hence without infinite antichains.

[3] The text of sections 5.2, 5.3, 5.4 and 5.5 is drawn from the paper [FG12].

Since the completion of the Rado ordering contains the infinite set of the ω_i [Section 5.3, Lemma 1], which is an infinite antichain, we conclude that the completion of a WBTS is not necessary a WBTS

5.2 Lub-accelerations

A subset U of a dcpo X is (Scott-)*open* iff U is upward-closed, and for any directed subset D of X such that $\text{lub}(D) \in U$, some element of D is already in U. A *partial ω-continuous* map $f : X \to X$, where $(X, \leq)$ is a dcpo, is a partial map whose domain $\text{dom}\, f$ is upward-closed, and such that for every directed subset D *in* $\text{dom}\, f$, $\text{lub}(f(D)) = f(\text{lub}(D))$. The composition of two partial ω-continuous maps again yields a partial ω-continuous map. This is all we require when we define accelerations. The *closed* sets are the complements of open sets. Every closed set is downward-closed. On a dcpo, the closed subsets are the subsets B that are both downward-closed and *inductive*, i.e., such that $\text{Lub}(B) = B$. An inductive subset of X is none other than a sub-dcpo of X. The *closure* $cl(A)$ of $A \subseteq X$ is the smallest closed set containing A. This should not be confused with the *inductive closure* $\text{Ind}(A)$ of A, which is obtained as the smallest inductive subset B containing A. In general, $\downarrow A \subseteq \text{Lub}(\downarrow A) \subseteq \text{Ind}(\downarrow A) \subseteq cl(A)$, and all inclusions can be strict. All this nitpicking is irrelevant when X is a *continuous* dcpo, and A is downward-closed in X. In this case indeed, $\text{Lub}(A) = \text{Ind}(A) = cl(A)$. This is well-known, see e.g., [FG09a, Proposition 3.5], and will play an important role in our constructions. As a matter in fact, the fact that $\text{Lub}(A) = cl(A)$, in the particular case of continuous dcpos, is required for lub-accelerations to ever reach the closure of the set of states that are reachable in a transition system.

In [FG12], we illustrate that ω^2-wqo are crucial to establish a *progress* property that consists to make infinitely often lub-accelerations.

The reasons why the original Karp-Miller procedure terminates on (ordinary) Petri nets are two-fold. First, when $\widehat{X} = \mathbb{N}^k_\omega$, one cannot lub-accelerate more than k times, because each lub-acceleration introduces a new ω component to the label of the produced state, which will not disappear in later node extensions. This is specific to Petri nets, and already fails for reset Petri nets, where ω components do disappear. The second reason is of more general applicability: $\widehat{X} = \mathbb{N}^k_\omega$ is wpo, and this implies that along every infinite branch of the tree thus constructed, case (*) will eventually happen, and in fact will happen infinitely many times. Call this *progress*: along any infinite path, one will lub-accelerate infinitely often. In the original Karp-Miller procedure for Petri nets, this will entail termination.

As we have already announced, for WSTS other than Petri nets, termination cannot be ensured. But at least we would like to ensure progress. The argument above shows that progress is obtained provided $\widehat{X}$ is wqo. *This* is our main motivation in characterizing those wqos X such that $\widehat{X}$ is wqo again.

5.3 The Rado Structure

We now return to the purpose of this section: showing that $\widehat{X}$ is well-ordered iff X is ω^2-wqo. We start by showing that, in some cases, $\widehat{X}$ is indeed *not* well-ordered.

Recall that X is Rado's structure X_{Rado} [Rad54], i.e., $\{(m,n) \in \mathbb{N}^2 \mid m < n\}$, ordered by $\leq_{\text{Rado}}$: $(m,n) \leq_{\text{Rado}} (m',n')$ iff $m = m'$ and $n \leq n'$, or $n < m'$. It is well-known that $\leq_{\text{Rado}}$ is a well quasi-ordering, and that $\mathbb{P}(X_{\text{Rado}})$ is not well-quasi-ordered by $\leq^\sharp_{\text{Rado}}$, defined as $A \leq^\sharp_{\text{Rado}} B$ iff for every $y \in B$, there is a $x \in A$ such that $x \leq_{\text{Rado}} y$ [Jan99]. (Equivalently, $A \leq^\sharp_{\text{Rado}} B$ iff $\uparrow B \subseteq \uparrow A$.)

Consider indeed $\omega_i = \{(i,n) \mid n \geq i+1\} \cup \{(m,n) \in X_{\text{Rado}} \mid n \leq i-1\}$, for each $i \in \mathbb{N}$. This is pictured as the dark blue (or dark grey) region in Figure 2, and arises naturally in Lemma 1 below. Note that ω_i is downward-closed in $\leq_{\text{Rado}}$. Consider the complement $\overline{\omega}_i$ of ω_i, and note that $\overline{\omega}_i \leq^\sharp_{\text{Rado}} \overline{\omega}_j$ iff $\uparrow \overline{\omega}_j \subseteq \uparrow \overline{\omega}_i$, iff $\overline{\omega}_j \subseteq \overline{\omega}_i$ (since $\overline{\omega}_i$ is upward-closed), iff $\omega_i \subseteq \omega_j$. However, when $i < j$, (i,j) is in ω_i but not in ω_j, so $\overline{\omega}_i \not\leq^\sharp_{\text{Rado}} \overline{\omega}_j$. So $(\overline{\omega}_i)_{i\in\mathbb{N}}$ is an infinite sequence of $\mathbb{P}(X_{\text{Rado}})$ from which one cannot extract any infinite ascending chain. Hence $\mathbb{P}(X_{\text{Rado}})$ is indeed not wqo. Since $\widehat{X_{\text{Rado}}} = \mathsf{Ideals}(X_{\text{Rado}})$, let us examine the structure of directed subsets of X_{Rado}.

Lemma 1 ([FG12]). *The downward-closed directed subsets of X_{Rado}, apart from those of the form $\downarrow(m,n)$, are of the form $\omega_i = \{(i,n) \mid n \geq i+1\} \cup \{(m,n) \in X_{Rado} \mid n \leq i-1\}$, or $\omega = X_{Rado}$.*

See Figure 2 for a pictorial representation of ω_i.

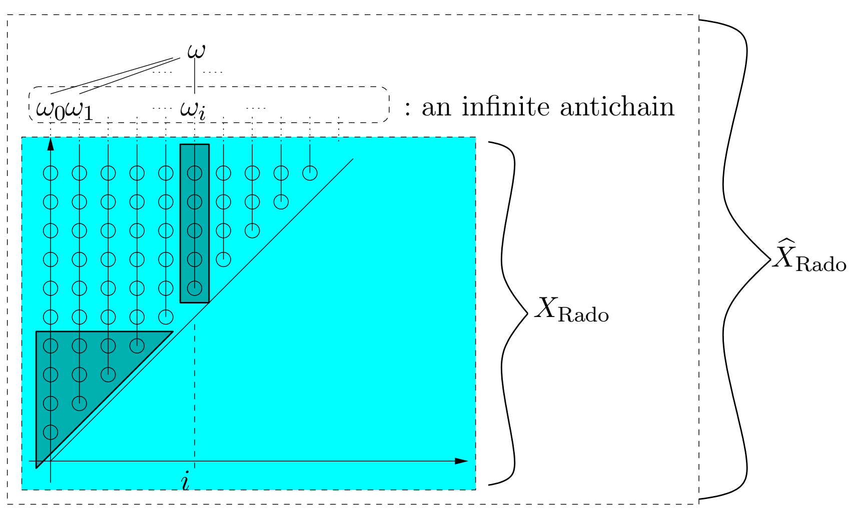

Fig. 2. Ideals in Rado's Structure

5.4 ω^2-WSTS

Recall here the working definition in [Jan99]: a well-quasi-order X is ω^2-*wqo* if and only if it does not contain an (isomorphic copy of) X_{Rado}; here we use Jančar's definition, as it is more tractable than the complex definition of [Mar94]. Jančar proved that X is ω^2-*wqo* iff $(\mathbb{P}(X), \leq^\sharp)$ is wqo (where $A \leq^\sharp B$ iff for every $b \in B$, there is an $a \in A$ such that $a \leq b$ or equivalently iff $\uparrow B \subseteq \uparrow A$ iff $B \subseteq \uparrow A$). We have shown that the above is the only case that can go bad:

Proposition 1 ([FG09b]). *Let S be a well-quasi-order. Then $\widehat{S}$ is well-quasi-ordered by inclusion iff S is ω^2-wqo.*

Let an ω^2-*WSTS* be any WSTS whose underlying poset is ω^2-wqo. It follows:

Theorem 5 ([FG09b]). *Let $\mathfrak{S} = (S, \to, \leq)$ be a WSTS. Then $\widehat{\mathfrak{S}}$ is a WSTS iff $\mathfrak{S}$ is an ω^2-WSTS.* □

Note that $\widehat{S} = \mathsf{Ideals}(S)$ is an algebraic dcpo [AJ94], whence $\widehat{S}$ is a continuous dcwo as soon as S is ω^2-wqo.

5.5 Are ω^2-wqos Ubiquitous?

X_{Rado} is an example of a wqo that is not ω^2-wqo. It is natural to ask whether this is the norm or an exception. We claim that all wqos used in the verification literature are in fact ω^2-wpo.

Consider the following grammar of datatypes, which extends that of [FG09a, Section 5] with the case of finite trees (last line):

$$\begin{array}{rll} D ::= & \mathbb{N} & \text{natural numbers} \\ \mid & A_{\leq} & \text{finite set } A\text{, ordered by } \leq \\ \mid & D_1 \times \ldots \times D_k & \text{finite product} \\ \mid & D_1 + \ldots + D_k & \text{finite, disjoint sum} \\ \mid & D^* & \text{finite words} \\ \mid & D^\circledast & \text{finite multisets} \\ \mid & \mathcal{T}(D) & \text{finite trees} \end{array} \tag{1}$$

Proposition 2 ([FG09a,FG09b]). *Every datatype defined in (1) is ω^2-wqo, and in fact bqo.*

In fact, all naturally occurring wqos are bqos, perhaps to the notable exception of finite graphs quasi-ordered by the graph minor relation, which are wqo [RS04] but not known to be bqo.

6 A Conceptual Karp-Miller Procedure [4]

An argument in favor of computing clovers is Emerson and Namjoshi's [EN98] approach to model-checking *liveness* properties of WSTS, which uses a finite

[4] The content of Section 6 is mainly drawn from the paper [FG12].

(coverability) graph based on the clover. Since WSTS enjoy the finite path property ([EN98], Definition 7), model-checking liveness properties is decidable for WSTS for which the clover is computable. This motivate us to *try* to compute the clover for classes of WSTS, even though it is not computable in general. The key to designing some form of a Karp-Miller procedure, such as the $\mathbf{Clover}_{\mathfrak{S}}$ procedure below is being able to *compute* lub-accelerations. To define and to compute lub-accelerations, one will use functional WSTS and one will accelerate compositions of functions. Complete WSTS is the framework to define and compute lub-accelerations.

Definition 6 (Complete WSTS [FG12]). *A* complete *transition system is a functional transition system* $\mathfrak{S} = (S, \xrightarrow{F}, \leq)$ *where* $(S, \leq)$ *is a continuous dcwo and every function in* F *is partial* ω*-continuous. A* complete WSTS *is a functional WSTS that is complete as a functional transition system.*

Let us remark that complete WSTS are strongly monotonic and that $\widehat{S} = \mathsf{Ideals}(S)$ is always a continuous dcpo [AJ94, Proposition 2.2.22], hence the completion of a WSTS (resp. a WBTS) is a complete WSTS (resp. a WBTS).

The point in complete WSTS is that one can *accelerate* loops:

Definition 7 (Lub-acceleration [FG12]). *Let* $(X, \leq)$ *be a dcpo,* $f : X \to X$ *be partial* ω*-continuous. The* lub-acceleration $f^\infty : X \to X$ *is defined by:* $\mathrm{dom}\, f^\infty = \mathrm{dom}\, f$*, and for any* $x \in \mathrm{dom}\, f$*, if* $x < f(x)$ *then* $f^\infty(x) = \mathrm{lub}\{f^n(x) \mid n \in \mathbb{N}\}$*, else* $f^\infty(x) = f(x)$*.*

Note that if $x \leq f(x)$, then $f(x) \in \mathrm{dom}\, f$, and $f(x) \leq f^2(x)$. By induction, we can show that $\{f^n(x) \mid n \in \mathbb{N}\}$ is an increasing sequence, so that the definition makes sense.

Remark 2. In [FG09b], we define, only for complete WSTS, the *clover* as the finite set (not necessarily computable) of maximal elements of the least upper bounds of the cover: more precisely, the *clover* $Clover_{\mathfrak{S}}(s_0)$ of the state $s_0 \in S$ is $\mathrm{Max}\,\mathrm{Lub}(Cover_{\mathfrak{S}}(s_0))$. Now we may extend the previous definition of *Clover* to any WBTS as follows: $Clover_{\mathfrak{S}}(s_0) \stackrel{\mathrm{def}}{=} \mathsf{IdealDecomp}(Cover_{\mathfrak{S}}(s_0))$ where

$\mathsf{IdealDecomp}(Cover_{\mathfrak{S}}(s_0))$ is the canonical ideal decomposition of $Cover_{\mathfrak{S}}(s_0)$. Then each maximal element in $\mathrm{Lub}(Cover_{\mathfrak{S}}(s_0))$ can be identified with a maximal ideal in $\mathsf{IdealDecomp}(Cover_{\mathfrak{S}}(s_0))$. Lub-accelerations in WBTS could be defined for functional *complete* WBTS.

Definition 8 (∞-Effective [FG12]). *An effective complete functional WSTS* $\mathfrak{S} = (S, \xrightarrow{F}, \leq)$ *is* ∞-effective *iff every function* g^∞ *is computable, for every* $g \in F^*$*, where* F^* *is the set of all compositions of maps in* F*.*

E.g., the completion of a Petri net is ∞-effective: not only is $\mathbb{N}^k_\omega$ a wpo, but every composition of transitions $g \in F^*$ is of the form $g(\boldsymbol{x}) = \boldsymbol{x} + \delta$, where $\delta \in \mathbb{Z}^k$. If $\boldsymbol{x} < g(\boldsymbol{x})$ then $\delta \in \mathbb{N}^k \setminus \{0\}$. Write $\boldsymbol{x}_i$ the ith component of $\boldsymbol{x}$, it follows that $g^\infty(\boldsymbol{x})$ is the tuple whose ith component is $\boldsymbol{x}_i$ if $\delta_i = 0$, ω otherwise.

Let $\mathfrak{S}$ be an ∞-effective WSTS, and write $A \leq^\flat B$ iff $\downarrow A \subseteq \downarrow B$, i.e., iff every element of A is below some element of B. The following is a simple procedure which computes the clover of its input $s_0 \in S$ (when it terminates):

Procedure Clover$_{\mathfrak{S}}(s_0)$:
1. $A \leftarrow \{s_0\}$;
2. **while** $Post_{\mathfrak{S}}(A) \not\leq^\flat A$ **do**
   (a) Choose fairly (see below) $(g, a) \in F^* \times A$ such that $a \in \mathrm{dom}\, g$;
   (b) $A \leftarrow A \cup \{g^\infty(a)\}$;
3. **return** $\mathrm{Max}\, A$;

The reader will find in [FG12] arguments showing that **Clover**$_{\mathfrak{S}}$ is well-defined and all its lines are computable by assumption, provided we make clear what we mean by fair choice in line (a).

We use a *fixpoint test* (line 2) that is not in the Karp-Miller algorithm; and this improvement allows **Clover**$_{\mathfrak{S}}$ to terminate in *more cases* than the Karp-Miller procedure when it is used for extended Petri nets (for reset Petri nets for instance, which are a special case of the affine maps above). To decide whether the current set A, which is always an under-approximation of $Clover_{\mathfrak{S}}(s_0)$, is the clover, it is enough to decide whether $Post_{\mathfrak{S}}(A) \leq^\flat A$.

7 Conclusion and Perspectives

We have made a (partial) survey on WSTS among the tens of papers related to WSTS. Then we have presented the new (for the verification community) framework of ideals and we have shown how it has been used in recent papers concerning decidability and complexity of different Petri nets extensions. We have also presented the new definition of Well Behaved Transition Systems, which extends WSTS, and where coverability is still decidable [BFM16a]. We have recalled the framework in [FG12] of *complete WSTS*, and of *completions* of WSTS, on which forward reachability analyses can be conducted, using the *clover*, i.e., the set of maximal elements of the cover. For complete WSTS, the clover is finite, describes the cover exactly and it is computed by a simple procedure, **Clover**$_{\mathfrak{S}}$, for ∞-effective complete WSTS $\mathfrak{S}$.

From [BFM16a], one could extend the Clover's definition and the procedure **Clover**$_{\mathfrak{S}}$ to WBTS. In the future, it would be interesting to investigate all of the previous questions for *WBTS* instead of WSTS.

Acknowledgement

I would like to thank Michael Blondin, Jean Goubault-Larrecq and Pierre McKenzie for fruitful discussions and for having allowed me to use some parts of common papers.

References

[ABJ98] Parosh Abdulla, Ahmed Bouajjani, and Bengt Jonsson. On-the-fly analysis of systems with unbounded, lossy Fifo channels. In *Proc. 10th CAV'98*, pages 305–318, Vancouver, Canada, June 1998. Springer Verlag LNCS 1427.

[ACABJ04a] Parosh Aziz Abdulla, Aurore Collomb-Annichini, Ahmed Bouajjani, and Bengt Jonsson. Using forward reachability analysis for verification of lossy channel systems. *Formal Methods in System Design*, 25(1):39–65, 2004.

[ACABJ04b] Parosh Aziz Abdulla, Aurore Collomb-Annichini, Ahmed Bouajjani, and Bengt Jonsson. Using forward reachability analysis for verification of lossy channel systems. *Formal Methods in System Design*, 25(1):39–65, 2004.

[AČJT00] Parosh Aziz Abdulla, Karlis Čerāns, Bengt Jonsson, and Yih-Kuen Tsay. Algorithmic analysis of programs with well quasi-ordered domains. *Information and Computation*, 160(1–2):109–127, 2000.

[AČJYK96] Parosh Aziz Abdulla, Karlis Čerāns, Bengt Jonsson, and Tsay Yih-Kuen. General decidability theorems for infinite-state systems. In *11th LICS*, pages 313–321, 1996.

[ADB07] Parosh Aziz Abdulla, Giorgio Delzanno, and Laurent Van Begin. Comparing the expressive power of well-structured transition systems. In Jacques Duparc and Thomas A. Henzinger, editors, *21st CSL 2007, Lausanne, Switzerland, September 11-15, 2007, Proceedings*, volume 4646 of *LNCS*, pages 99–114. Springer, 2007.

[ADMN04] Parosh Aziz Abdulla, Johann Deneux, Pritha Mahata, and Aletta Nylén. Forward reachability analysis of timed Petri nets. In *FORMATS/FTRTFT*, pages 343–362. Springer Verlag LNCS 3253, 2004.

[AJ93] Parosh Abdulla and Bengt Jonsson. Verifying programs with unreliable channels. In *Proceedings of the 8th LICS*, pages 160–170, 1993.

[AJ94] Samson Abramsky and Achim Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3, pages 1–168. Oxford University Press, 1994.

[AL78] André Arnold and Michel Latteux. Recursivite et cones rationnels fermes par intersection. *CALCOLO*, 15(4):381–394, 1978.

[AN00] Parosh Aziz Abdulla and Aletta Nylén. Better is better than well: On efficient verification of infinite-state systems. In *Proc. 14th IEEE Symp. LICS'00*, pages 132–140, 2000.

[BDK+12] Nathalie Bertrand, Giorgio Delzanno, Barbara König, Arnaud Sangnier, and Jan Stückrath. On the decidability status of reachability and coverability in graph transformation systems. In Ashish Tiwari, editor, *23rd RTA 2012, May 28 - June 2, 2012, Nagoya, Japan*, volume 15 of *LIPIcs*, pages 101–116. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2012.

[BFHR11] Rémi Bonnet, Alain Finkel, Serge Haddad, and Fernando Rosa-Velardo. Ordinal theory for expressiveness of well structured transition systems. In Martin Hofmann, editor, *FOSSACS 2011, Saarbrücken, Germany, March 26-April 3, 2011. Proceedings*, volume 6604 of *LNCS*, pages 153–167. Springer, 2011.

[BFM14] Michael Blondin, Alain Finkel, and Pierre McKenzie. Handling infinitely branching WSTS. In *Proc. 41^{st} ICALP*, pages 13–25, 2014.

[BFM16a] Michael Blondin, Alain Finkel, and Pierre McKenzie. Well behaved transition systems. (in preparation), 2016.

[BFM16b] Michael Blondin, Alain Finkel, and Pierre McKenzie. Handling infinitely branching well-structured transition systems. *Information and Computation*, (submitted), 2016.

[BG14] Paolo Baldan and Daniele Gorla, editors. *25th CONCUR 2014, Rome, Italy, September 2-5, 2014. Proceedings*, volume 8704 of *LNCS*. Springer, 2014.

[BHM15] Eric Badouel, Loïc Hélouët, and Christophe Morvan. Petri nets with structured data. In Raymond R. Devillers and Antti Valmari, editors, *36th International Conference, PETRI NETS 2015, Brussels, Belgium, June 21-26, 2015, Proceedings*, volume 9115 of *LNCS*, pages 212–233. Springer, 2015.

[BLP15] Mikolaj Bojanczyk, Slawomir Lasota, and Igor Potapov, editors. *Reachability Problems - 9th International Workshop, RP 2015, Warsaw, Poland, September 21-23, 2015, Proceedings*, volume 9328 of *LNCS*. Springer, 2015.

[Bon75] R. Bonnet. On the cardinality of the set of initial intervals of a partially ordered set. *Infinite and finite sets: to Paul Erdös on his* 60^{th} *birthday*, pages 189–198, 1975.

[CFS11] Pierre Chambart, Alain Finkel, and Sylvain Schmitz. Forward analysis and model checking for trace bounded WSTS. In Lars Michael Kristensen and Laure Petrucci, editors, *32nd International Conference, PETRI NETS 2011, Newcastle, UK, June 20-24, 2011. Proceedings*, volume 6709 of *Lecture Notes in Computer Science*, pages 49–68. Springer, 2011.

[DFS98] Catherine Dufourd, Alain Finkel, and Philippe Schnoebelen. Reset nets between decidability and undecidability. In *25th ICALP'98*, pages 103–115. Springer Verlag LNCS 1443, 1998.

[DJS99] Catherine Dufourd, Petr Jancar, and Philippe Schnoebelen. Boundedness of reset P/T nets. In Jirí Wiedermann, Peter van Emde Boas, and Mogens Nielsen, editors, *26th ICALP'99, Prague, Czech Republic, July 11-15, 1999, Proceedings*, volume 1644 of *LNCS*, pages 301–310. Springer, 1999.

[EFM99] Javier Esparza, Alain Finkel, and Richard Mayr. On the verification of broadcast protocols. In *14th LICS*, pages 352–359, 1999.

[EN98] E. Allen Emerson and Kedar S. Namjoshi. On model-checking for non-deterministic infinite-state systems. In *13th LICS*, pages 70–80, 1998.

[ET43] P. Erdös and A. Tarski. On families of mutually exclusive sets. *Ann. of Math.*, 2(44):315–329, 1943.

[FFSS11] Diego Figueira, Santiago Figueira, Sylvain Schmitz, and Philippe Schnoebelen. Ackermannian and primitive-recursive bounds with dickson's lemma. In *26th Annual IEEE LICS 2011, June 21-24, 2011, Toronto, Ontario, Canada*, pages 269–278. IEEE Computer Society, 2011.

[FG09a] Alain Finkel and Jean Goubault-Larrecq. Forward analysis for WSTS, part I: Completions. In Susanne Albers and Jean-Yves Marion, editors, *26th Annual STACS'09*, volume 3 of *Leibniz International Proceedings in Informatics*, pages 433–444, Freiburg, Germany, February 2009. Leibniz-Zentrum für Informatik.

[FG09b] Alain Finkel and Jean Goubault-Larrecq. Forward analysis for wsts, part II: complete WSTS. In Susanne Albers, Alberto Marchetti-Spaccamela, Yossi Matias, Sotiris E. Nikoletseas, and Wolfgang Thomas, editors, *36th ICALP 2009, Rhodes, Greece, July 5-12, 2009, Proceedings, Part II*, volume 5556 of *LNCS*, pages 188–199. Springer, 2009.

[FG12] Alain Finkel and Jean Goubault-Larrecq. Forward analysis for WSTS, part II: Complete WSTS. *Logical Methods in Computer Science*, 8(3:28), 2012.

[Fin87] Alain Finkel. A generalization of the procedure of Karp and Miller to well structured transition systems. In *13th Intl. ICALP'87*, pages 499–508. Springer Verlag LNCS 267, 1987.

[Fin90] Alain Finkel. Reduction and covering of infinite reachability trees. *Information and Computation*, 89(2):144–179, 1990.

[FMP04] Alain Finkel, Pierre McKenzie, and Claudine Picaronny. A well-structured framework for analysing Petri net extensions. *Information and Computation*, 195(1-2):1–29, 2004.

[Fra86] R. Fraïssé. Theory of relations. *Studies in Logic and the Foundations of Mathematics*, 118:1–456, 1986.

[FS01] Alain Finkel and Philippe Schnoebelen. Well-structured transition systems everywhere! *Theoretical Computer Science*, 256(1–2):63–92, 2001.

[GHK+03] Gerhard Gierz, Karl Heinrich Hofmann, Klaus Keimel, Jimmie D. Lawson, Michael Mislove, and Dana S. Scott. Continuous lattices and domains. In *Encyclopedia of Mathematics and its Applications*, volume 93. Cambridge University Press, 2003.

[GLS16] Jean Goubault-Larrecq and Sylvain Schmitz. Deciding Piecewise Testable Separability for Regular Tree Languages. In *ICALP 2016*, Rome, Italy, July 2016.

[GRB04] Gilles Geeraerts, Jean-François Raskin, and Laurent Van Begin. Expand, enlarge, and check: New algorithms for the coverability problem of WSTS. In Kamal Lodaya and Meena Mahajan, editors, *FSTTCS 2004: 24th International Conference, Chennai, India, December 16-18, 2004, Proceedings*, volume 3328 of *LNCS*, pages 287–298. Springer, 2004.

[GRB07] Gilles Geeraerts, Jean-François Raskin, and Laurent Van Begin. Well-structured languages. *Acta Inf.*, 44(3-4):249–288, 2007.

[GRvB06a] Pierre Ganty, Jean-François Raskin, and Laurent van Begin. A complete abstract interpretation framework for coverability properties of WSTS. In E. Allen Emerson and Kedar S. Namjoshi, editors, *Proc. 7th VMCAI'06*, pages 49–64. Springer Verlag LNCS 3855, 2006.

[GRvB06b] Gilles Geeraerts, Jean-François Raskin, and Laurent van Begin. Expand, enlarge and check: New algorithms for the coverability problem of WSTS. *J. Comp. and System Sciences*, 72(1):180–203, 2006.

[GRvB07] Gilles Geeraerts, Jean-François Raskin, and Laurent van Begin. On the efficient computation of the minimal coverability set for Petri nets. In *Proc. 5th ATVA'05*, pages 98–113. Springer LNCS 4762, 2007.

[HLL+16] Piotr Hofman, Slawomir Lasota, Ranko Lazić, Jérôme Leroux, Sylvain Schmitz, and Patrick Totzke. Coverability trees for petri nets with unordered data. In Bart Jacobs and Christof Löding, editors, *19th FOSSACS 2016, Eindhoven, The Netherlands, April 2-8, 2016, Proceedings*, volume 9634 of *LNCS*, pages 445–461. Springer, 2016.

[HMM14] Reiner Hüchting, Rupak Majumdar, and Roland Meyer. Bounds on mobility. In Baldan and Gorla [BG14], pages 357–371.

[HP07] Serge Haddad and Denis Poitrenaud. Recursive petri nets. *Acta Inf.*, 44(7-8):463–508, 2007.

[HSS12] Serge Haddad, Sylvain Schmitz, and Philippe Schnoebelen. The ordinal-recursive complexity of timed-arc petri nets, data nets, and other enriched nets. In *27th Annual IEEE LICS 2012, Dubrovnik, Croatia, June 25-28, 2012*, pages 355–364. IEEE Computer Society, 2012.

[Jan99] Petr Jančar. A note on well quasi-orderings for powersets. *Information Processing Letters*, 72(5–6):155–160, 1999.

[KM69] R. M. Karp and R. E. Miller. Parallel program schemata. *J. Comp. and System Sciences*, 3(2):147–195, 1969.

[KS96] Olga Kouchnarenko and Philippe Schnoebelen. A model for recursive-parallel programs. *Electr. Notes Theor. Comput. Sci.*, 5:30, 1996.

[KS14] Barbara König and Jan Stückrath. A general framework for well-structured graph transformation systems. In Baldan and Gorla [BG14], pages 467–481.

[Las16] Slawomir Lasota. Decidability border for petri nets with data: WQO dichotomy conjecture. In Fabrice Kordon and Daniel Moldt, editors, *37th International Conference, PETRI NETS 2016, Toruń, Poland, June 19-24, 2016. Proceedings*, volume 9698 of *LNCS*, pages 20–36. Springer, 2016.

[Laz13] Ranko Lazić. The reachability problem for vector addition systems with a stack is not elementary. *CoRR*, abs/1310.1767, 2013.

[LNO+07] Ranko Lazić, Thomas Christopher Newcomb, Joël Ouaknine, A. W. Roscoe, and James Worrell. Nets with tokens which carry data. In *ICATPN*, volume 4546 of *LNCS*, pages 301–320. Springer, 2007.

[LS15a] Ranko Lazić and Sylvain Schmitz. The ideal view on rackoff's coverability technique. In Bojanczyk et al. [BLP15], pages 76–88.

[LS15b] Jérôme Leroux and Sylvain Schmitz. Demystifying reachability in vector addition systems. In *30th Annual ACM/IEEE LICS 2015, Kyoto, Japan, July 6-10, 2015*, pages 56–67. IEEE Computer Society, 2015.

[LS16a] Ranko Lazić and Sylvain Schmitz. The Complexity of Coverability in Î½-Petri Nets. In *LICS 2016*, New York, United States, 2016. ACM Press.

[LS16b] Jérôme Leroux and Sylvain Schmitz. Ideal decompositions for vector addition systems (invited talk). In Nicolas Ollinger and Heribert Vollmer, editors, *33rd STACS 2016, February 17-20, 2016, Orléans, France*, volume 47 of *LIPIcs*, pages 1:1–1:13, 2016.

[LST15a] Jérôme Leroux, Grégoire Sutre, and Patrick Totzke. On boundedness problems for pushdown vector addition systems. In Bojanczyk et al. [BLP15], pages 101–113.

[LST15b] Jérôme Leroux, Grégoire Sutre, and Patrick Totzke. On the coverability problem for pushdown vector addition systems in one dimension. In Magnús M. Halldórsson, Kazuo Iwama, Naoki Kobayashi, and Bettina Speckmann, editors, *42nd ICALP 2015, Kyoto, Japan, July 6-10, 2015, Proceedings, Part II*, volume 9135 of *LNCS*, pages 324–336. Springer, 2015.

[Mar94] Alberto Marcone. Foundations of BQO theory. *Trans. Amer. Math. Soc.*, 345(2):641–660, 1994.

[May00] Richard Mayr. Process rewrite systems. *Inf. Comput.*, 156(1-2):264–286, 2000.

[Mey08] Roland Meyer. On boundedness in depth in the pi-calculus. In Giorgio Ausiello, Juhani Karhumäki, Giancarlo Mauri, and C.-H. Luke Ong, editors, *Fifth IFIP - TCS 2008, IFIP 20th World Computer Congress, TC 1, Foundations of Computer Science, September 7-10, 2008, Milano, Italy*, volume 273 of *IFIP*, pages 477–489. Springer, 2008.

[Rad54] Richard Rado. Partial well-ordering of sets of vectors. *Mathematika*, 1:89–95, 1954.

[RdF07] Fernando Rosa-Velardo and David de Frutos-Escrig. Name creation vs. replication in petri net systems. In Jetty Kleijn and Alexandre Yakovlev, editors, *ICATPN 2007, Siedlce, Poland, June 25-29, 2007, Proceedings*, volume 4546 of *LNCS*, pages 402–422. Springer, 2007.

[RHB+10] James Rogers, Jeffrey Heinz, Gil Bailey, Matt Edlefsen, Molly Visscher, David Wellcome, and Sean Wibel. On languages piecewise testable in the strict sense. In *The Mathematics of Language*, pages 255–265. Springer, 2010.

[RMdF11] Fernando Rosa-Velardo, María Martos-Salgado, and David de Frutos-Escrig. Accelerations for the coverability set of petri nets with names. *Fundam. Inform.*, 113(3-4):313–341, 2011.

[RS04] Neil Robertson and P.D. Seymour. Graph minors. XX. Wagner's conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325–357, 2004.

[SS11] Sylvain Schmitz and Philippe Schnoebelen. Multiply-recursive upper bounds with higman's lemma. In Luca Aceto, Monika Henzinger, and Jirí Sgall, editors, *ICALP 2011, Zurich, Switzerland, July 4-8, 2011, Proceedings, Part II*, volume 6756 of *LNCS*, pages 441–452. Springer, 2011.

[Val78] Rüdiger Valk. Self-modidying nets, a natural extension of Petri nets. In *5th ICALP'78*, pages 464–476. Springer Verlag LNCS 62, 1978.

[WZH10] Thomas Wies, Damien Zufferey, and Thomas A. Henzinger. Forward analysis of depth-bounded processes. In C.-H. Luke Ong, editor, *FOSSACS*, volume 6014 of *LNCS*, pages 94–108. Springer, 2010.

[ZWH12] Damien Zufferey, Thomas Wies, and Thomas A. Henzinger. Ideal abstractions for well-structured transition systems. In *VMCAI*, pages 445–460, 2012.